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A BOUND FOR THE A_α-SPECTRAL RADIUS OF A CONNECTED GRAPH AFTER VERTEX DELETION

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Abstract. G is a simple connected graph with adjacency matrix $A(G)$ and degree diagonal matrix $D(G)$. The signless Laplacian matrix of G is defined as $Q(G) = D(G) + A(G)$. In 2017, Nikiforov [1] defined the matrix $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$ for $\alpha \in [0, 1]$. The A_α-spectral radius of G is the maximum eigenvalue of $A_\alpha(G)$. In 2019, Liu *et al.* [2] defined the matrix $\Theta_k(G)$ as $\Theta_k(G) = kD(G) + A(G)$, for $k \in \mathbb{R}$. In this paper, we present a new type of lower bound for the A_α-spectral radius of a graph after vertex deletion. Furthermore, we deduce some corollaries on $\Theta_k(G), A(G), Q(G)$ matrices.

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1. Introduction and preliminaries

We consider non-empty simple connected graph G with vertex set $V(G)$ and edge set $E(G)$ throughout this paper. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. If any pair of vertices v_i and v_j are adjacent, then we write $v_iv_j \in E(G)$ or $v_i \sim v_j$. For a vertex $v_k \in V(G)$, the neighborhood of v_k is the set $N(v_k) = N_G(v_k) = \{w \in V(G) : w \sim v_k\}$, and $d_G(v_k)$ denotes the degree of v_k with $d_G(v_k) = |N(v_k)|$. Let $d_k = d_G(v_k)$ if there is no ambiguity. Let V_m be any fixed subset of $V(G)$ containing m vertices. For $V_m \subseteq V(G)$ with $|V_m| = m$, let $G[V_m]$ be the subgraph of G induced by V_m, $G - V_m$ be the subgraph induced by $V(G) - V_m$. Let $G \vee H$ denote the graph obtained from the disjoint union $G + H$ by adding all edges between graph G and graph H. A regular graph with vertices of degree r is called a r-regular graph. Let $K_n, K_{s,t}$ denote the clique and complete bipartite graph respectively and $K_{1,n-1}$ be the star of order n.

$A(G)$ denotes the adjacency matrix and $D(G)$ denotes the diagonal matrix of the degrees of G. The signless Laplacian matrix of G is defined as $Q(G) = D(G) + A(G)$. In 2017, Nikiforov [1] proposed the matrix $A_\alpha(G)$ of a graph G

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G),$$

for $\alpha \in [0, 1]$, which successfully extends the theories of $A(G)$ and $Q(G)$. Let $A_\alpha = A_\alpha(G)$ if there is no ambiguity. It is not hard to see that A_0 is the adjacency matrix and $2A_{\frac{1}{2}}$ is the signless Laplacian matrix. In 2019, Liu *et al.* [2] defined the matrix $\Theta_k(G)$ as $\Theta_k(G) = kD(G) + A(G)$, for $k \in \mathbb{R}$.

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Denote the eigenvalues of $n \times n$ symmetric matrix M by $\rho_1(M) \geq \rho_2(M) \geq \ldots \geq \rho_n(M)$. The largest eigenvalue $\rho_\alpha(G) := \rho_1(A_\alpha(G))$ of the A_α-matrix is defined as the A_α-spectral radius of G. Similarly, we can define the Θ_k-spectral radius and Q-spectral radius of G. Let $\rho_\alpha = \rho_\alpha(G)$ if there is no ambiguity. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be a unit nonnegative eigenvector of ρ_α corresponding to the vertex set $\{v_1, v_2, \ldots, v_n\}$. Let $G_m = (G - V_m) \bigcup mK_1$. For other undefined notations and terminologies, refer to [3].

Many scholars already succeeded in finding bounds for the A_α-spectral radius. For more results in this direction, readers can refer to a survey [1] by Nikiforov and some other articles [4–9]. In 2019, Guo *et al.* [10] and Sun *et al.* [11] presented a relation between $\rho_1(G)$ and $\rho_1(G - v_k)$ for adjacency matrices $A(G)$ and $A(G - v_k)$, where v_k is a vertex of G. The better bound given by [11] is shown as follow

$$\rho_1(A(G)) \leq \sqrt{\rho_1^2(A(G - v_k)) + 2d_k - 1}. \tag{1}$$

Since (1) is well used in analyzing the graph structure (see [12]), we try to extend the above inequation to more matrices, such as $A_\alpha(G), \Theta_k(G), Q(G)$. Using some different methods from [10, 11], we get the results in this paper. As far as the authors know, this topic has not been explored elsewhere.

From the following proposition given by Nikiforov, we know that there exists a positive eigenvector $\mathbf{x}$ corresponding to ρ_α if $\alpha \in [0, 1)$.

Proposition 1.1 (Proposition 13 of [1]). *Let $\alpha \in [0, 1)$, let G be a connected graph, let $\boldsymbol{x}$ be a nonnegative eigenvector to $\rho_1(A_\alpha(G))$, and let H be a proper subgraph of G, then*
(i) $\boldsymbol{x}$ is positive and is unique up to scaling;
(ii) $\rho_1(A_\alpha(H)) < \rho_1(A_\alpha(G))$.

The A_α-spectrum of K_n and $K_{1,n-1}$ are given by Nikiforov as follows:

Proposition 1.2 (Proposition 36 of [1]). *The eigenvalues of $A_\alpha(K_n)$ are*

$$\begin{aligned} \rho_1(A_\alpha(K_n)) &= n - 1, \\ \rho_k(A_\alpha(K_n)) &= \alpha n - 1 \quad for \quad 1 < k \leq n. \end{aligned}$$

Proposition 1.3 (Proposition 38 of [1]). *The eigenvalues of $A_\alpha(K_{1,n-1})$ are*

$$\begin{aligned} \rho_1(A_\alpha(K_{1,n-1})) &= \frac{1}{2}\big(\alpha n + \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)}\,\big), \\ \rho_n(A_\alpha(K_{1,n-1})) &= \frac{1}{2}\big(\alpha n - \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)}\,\big), \\ \rho_k(A_\alpha(K_{1,n-1})) &= \alpha \quad for \quad 1 < k < n. \end{aligned}$$

Proposition 1.4 (Corollary 1 of [10]). *Let G be a connected graph with n vertices. Then*

$$\rho_1(A(G)) \geq max\{\sqrt{d(v)}, v \in V(G)\},$$

with equality holding if and only if $G = K_{1,n-1}$.

2. Main results

Theorem 2.1. *Let $\alpha \in [0, 1)$ and let v_k be a vertex of a connected graph G with degree d_k. Then*

$$\rho_\alpha(G - v_k) \geq \rho_\alpha - \alpha - \frac{(1-\alpha)^2 d_k}{\rho_\alpha - \alpha d_k} \tag{2}$$

with equality holding if and only if G is the join of the vertex v_k and a regular graph of order $n - 1$.

Theorem 2.1 will be proved in Section 3.

Corollary 2.2. *Let $k \in [0, +\infty)$ and let v be a vertex of a connected graph G with degree $d(v)$. Then*

$$\lambda_1(\Theta_k(G-v)) \geq \lambda_1(\Theta_k(G)) - k - \frac{d(v)}{\lambda_1(\Theta_k(G)) - kd(v)}$$

with equality holding if and only if G is the join of the vertex v and a regular graph of order $n-1$.

By setting $\alpha = 0$ in (2) we obtain the following corollary.

Corollary 2.3. *Let v_k be a vertex of a connected graph G with degree d_k. Then*

$$\rho_1(A(G-v_k)) \geq \rho_1(A(G)) - \frac{d_k}{\rho_1(A(G))} \tag{3}$$

with equality holding if and only if G is the join of the vertex v_k and a regular graph of order $n-1$.

When $d_k > \rho_1(A(G))$, Corollary 2.3 gives a lower bound for $\rho_\alpha(G - v_k)$ which is better than $\sqrt{\rho^2(A(G)) - 2d_k + 1}$ deducible from (1). In fact, by Proposition 1.4, we have $\rho_1(A(G)) - \frac{d_k}{\rho_1(A(G))} > 0$. By Corollary 2.3, we have

$$\begin{aligned} \rho_1(A(G-v_k)) &\geq \rho_1(A(G)) - \frac{d_k}{\rho_1(A(G))} \\ &= \sqrt{\rho_1^2(A(G)) - 2d_k + \frac{d_k^2}{\rho_1^2(A(G))}} \\ &> \sqrt{\rho^2(A(G)) - 2d_k + 1}. \end{aligned}$$

By setting $\alpha = \frac{1}{2}$ in (2) we obtain the following corollary.

Corollary 2.4. *Let v_k be a vertex of a connected graph G with degree d_k, and let $Q(G)$ be the signless Laplacian matrix of G. Then*

$$\rho_1(Q(G-v_k)) \geq \rho_1(Q(G)) - 1 - \frac{d_k}{\rho_1(Q(G)) - d_k} \tag{4}$$

with equality holding if and only if G is the join of the vertex v_k and a regular graph of order $n-1$.

Define $\Phi(\alpha, r, n) = \frac{1}{2}(\alpha n + r) + \frac{1}{2}\sqrt{(\alpha n + r)^2 - 4(r\alpha + 2\alpha - 1)(n-1)}$.

Corollary 2.5. *Let G be the join of the vertex v_k and a r-regular graph of order $n-1$, $\alpha \in [0,1)$. Then*

$$\begin{aligned} &(i) \quad \rho_\alpha = \frac{1}{2}(\alpha n + r) + \frac{1}{2}\sqrt{(\alpha n + r)^2 - 4(r\alpha + 2\alpha - 1)(n-1)}; \\ &(ii) \quad \rho_1(A(G)) = \frac{r}{2} + \frac{1}{2}\sqrt{r^2 + 4(n-1)}; \\ &(iii) \quad \rho_1(Q(G)) = \frac{n}{2} + r + \sqrt{(\frac{n}{2} + r)^2 - 2r(n-1)}. \end{aligned}$$

Proof. (i) Since $G - v_k$ is a r-regular graph, it is obvious that $\rho_\alpha(G - v_k) = r$. By Theorem 2.1, we have

$$r = \rho_\alpha(G - v_k) = \rho_\alpha - \alpha - \frac{(1-\alpha)^2 d_k}{\rho_\alpha - \alpha d_k} = \rho_\alpha - \alpha - \frac{(1-\alpha)^2(n-1)}{\rho_\alpha - \alpha(n-1)}.$$

Hence

$$\rho_\alpha^2 - (\alpha n + r)\rho_\alpha + (r\alpha + 2\alpha - 1)(n-1) = 0.$$
$$\rho_\alpha = \frac{1}{2}(\alpha n + r) + \frac{1}{2}\sqrt{(\alpha n + r)^2 - 4(r\alpha + 2\alpha - 1)(n-1)} = \Phi(\alpha, r, n).$$

(ii) $\rho_1(A(G)) = \Phi(0, r, n) = \frac{r}{2} + \frac{1}{2}\sqrt{r^2 + 4(n-1)}$.
(iii) $\rho_1(Q(G)) = 2\Phi(\frac{1}{2}, r, n) = \frac{n}{2} + r + \sqrt{(\frac{n}{2} + r)^2 - 2r(n-1)}$. □

Remark 2.6. In particular, ρ_α values of $K_{1,n-1}$(star), $K_1 \vee \frac{n-1}{2}K_2$(friendship graph), $K_1 \vee C_{n-1}$(wheel graph), $K_1 \vee (C_{n_1} + C_{n_2} + \ldots + C_{n_k})$(multi-wheel graph), K_n(complete graph) are $\Phi(\alpha, 0, n)$, $\Phi(\alpha, 1, n)$, $\Phi(\alpha, 2, n)$, $\Phi(\alpha, 2, n)$, $\Phi(\alpha, n-1, n)$ respectively.

We can also calculate ρ_α by equitable quotient matrices for the graphs in Corollary 2.5.

3. Proof of Theorem 2.1

Lemma 3.1. *Let $\alpha \in [0, 1)$ and let v_k be a vertex of a connected graph G with degree d_k, then $\rho_\alpha > \alpha d_k$.*

Proof. Let x_i be the ith component of the positive eigenvector $\mathbf{x}$ corresponding to ρ_α. Since $x_k > 0$, $x_i > 0$, $1 - \alpha > 0$ and

$$\rho_\alpha x_k = \sum_{i=1}^{n} a_{ki}x_i = (1-\alpha)\sum_{v_i \in N(v_k)} x_i + \alpha d_k x_k,$$

the result immediately follows. □

Lemma 3.2. *Let $\alpha \in [0, 1)$, $V_m \subset V(G)$ with $|V_m| = m$ and $c_s = |N_G(v_s) \bigcap V_m|$ for $v_s \notin V_m$. Suppose a_{ij} is the (i)th row (j)th column element of A_α, and x_i is the ith component of the positive unit eigenvector $\boldsymbol{x}$ corresponding to ρ_α. Then*

$$\sum_{v_k \in V_m} x_k^2 \le \frac{1}{2} + \frac{1}{2\rho_\alpha}\Big(\sum_{v_i \in V_m}\sum_{v_j \in V_m} a_{ij}x_i x_j - \alpha \sum_{v_s \notin V_m} c_s x_s^2\Big) \tag{5}$$

with equality holding if and only if G_m is an empty graph.

Proof. Since $A_\alpha = \alpha D(G) + (1-\alpha)A(G)$, we have

$$a_{ij} = \begin{cases} 1-\alpha & if \quad i \neq j \quad and \quad v_i \in N_G(v_j), \\ 0 & if \quad i \neq j \quad and \quad v_i \notin N_G(v_j), \\ \alpha d_i & if \quad i = j. \end{cases}$$

Suppose a'_{ij} is the (i)th row (j)th column element of $A_\alpha(G_m)$, then

$$a'_{ij} = \begin{cases} 0 & if \quad v_i \in V_m \quad or \quad v_j \in V_m, \\ a_{ij} & if \quad i \neq j\ , \quad v_i \notin V_m \quad and \quad v_j \notin V_m, \\ a_{ii} - \alpha c_i & if \quad i = j \quad and \quad v_i \notin V_m. \end{cases}$$

Let $\mathbf{a}_k$ be the column vector $(a_{k1}, a_{k2}, \ldots, a_{kn})^T$ and $\mathbf{e}_k$ be the kth basis column vector $(0, \ldots, 0, 1, 0, \ldots, 0)^T$, where only the kth component is 1. We have

$$\begin{aligned}
&\mathbf{x}^T(A_\alpha - A_\alpha(G_m))\mathbf{x} \\
&= \mathbf{x}^T\Big(\sum_{v_k \in V_m} (\mathbf{a}_k\mathbf{e}_k^T + \mathbf{e}_k\mathbf{a}_k^T) - \sum_{v_i \in V_m}\sum_{v_j \in V_m} a_{ij}\mathbf{e}_i\mathbf{e}_j^T\Big)\mathbf{x} + \sum_{v_s \notin V_m} \alpha c_s x_s^2 \\
&= \sum_{v_k \in V_m} (\mathbf{x}^T\mathbf{a}_k\mathbf{e}_k^T\mathbf{x} + \mathbf{x}^T\mathbf{e}_k\mathbf{a}_k^T\mathbf{x}) - \sum_{v_i \in V_m}\sum_{v_j \in V_m} a_{ij}\mathbf{x}^T\mathbf{e}_i\mathbf{e}_j^T\mathbf{x} + \sum_{v_s \notin V_m} \alpha c_s x_s^2 \\
&= 2\sum_{v_k \in V_m} x_k \sum_{i=1}^{n} x_i a_{ki} - \sum_{v_i \in V_m}\sum_{v_j \in V_m} a_{ij}x_i x_j + \alpha \sum_{v_s \notin V_m} c_s x_s^2 \\
&= 2\rho_\alpha \sum_{v_k \in V_m} x_k^2 - \sum_{v_i \in V_m}\sum_{v_j \in V_m} a_{ij}x_i x_j + \alpha \sum_{v_s \notin V_m} c_s x_s^2 \,.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbf{x}^T A_\alpha(G_m)\mathbf{x} &= \mathbf{x}^T A_\alpha \mathbf{x} - \mathbf{x}^T(A_\alpha - A_\alpha(G_m))\mathbf{x} \\
&= \rho_\alpha - 2\rho_\alpha \sum_{v_k \in V_m} x_k^2 + \sum_{v_i \in V_m}\sum_{v_j \in V_m} a_{ij}x_i x_j - \alpha \sum_{v_s \notin V_m} c_s x_s^2 \\
&\geq 0 \,,
\end{aligned}$$

which implies

$$\sum_{v_k \in V_m} x_k^2 \leq \frac{1}{2} + \frac{1}{2\rho_\alpha}\Big(\sum_{v_i \in V_m}\sum_{v_j \in V_m} a_{ij}x_i x_j - \alpha \sum_{v_s \notin V_m} c_s x_s^2\Big) \,.$$

Note that $\mathbf{x}$ is a positive unit eigenvector, we can see that the equality holds if and only if G_m is an empty graph. □

Lemma 3.3. *Let $\alpha \in [0, 1)$, let $\boldsymbol{x}$ be the positive eigenvector of $A_\alpha(G)$ corresponding to ρ_α with $\boldsymbol{x}^T\boldsymbol{x} = 1$, let x_k be the kth component of $\boldsymbol{x}$, and let v_k be the kth vertex of G. Then*

$$\rho_\alpha(G - v_k) \geq \rho_\alpha - \alpha - \frac{x_k^2}{1 - x_k^2}(\rho_\alpha - \alpha d_k). \tag{6}$$

Proof. Let $V_m = V(G) - v_k$, then G_m is an empty graph. By Lemma 3.2, we have

$$\sum_{v_t \in V_m} x_t^2 = \frac{1}{2} + \frac{1}{2\rho_\alpha}\Big(\sum_{v_i \in V_m}\sum_{v_j \in V_m} a_{ij}x_i x_j - \alpha \sum_{v_s \notin V_m} c_s x_s^2\Big).$$

Let $\hat{\mathbf{x}} = \{x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n\}^T$, whose kth component is 0.
Let $G_1 = (G - v_k)\bigcup K_1$. We get

$$\begin{aligned}
1 - x_k^2 &= \sum_{v_t \in V_m} x_t^2 \\
&= \frac{1}{2} + \frac{1}{2\rho_\alpha}\Big(\sum_{v_i \in V_m}\sum_{v_j \in V_m} a_{ij}x_i x_j - \alpha d_k x_k^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} + \frac{1}{2\rho_\alpha}\Big(\mathbf{x}^T A_\alpha(G_1)\,\mathbf{x} + \alpha \sum_{v_i \in N(v_k)} x_i^2 - \alpha d_k x_k^2\Big) \\
&= \frac{1}{2} + \frac{1}{2\rho_\alpha}\Big(\hat{\mathbf{x}}^T A_\alpha(G_1)\,\hat{\mathbf{x}} + \alpha \sum_{v_i \in N(v_k)} x_i^2 - \alpha d_k x_k^2\Big) \\
&\le \frac{1}{2} + \frac{1}{2\rho_\alpha}\Big(\rho_\alpha(G_1)\,\hat{\mathbf{x}}^T\hat{\mathbf{x}} + \alpha - \alpha x_k^2 - \alpha d_k x_k^2\Big) \qquad (7)\\
&= \frac{1}{2} + \frac{1}{2\rho_\alpha}\Big(\rho_\alpha(G - v_k)\,(1 - x_k^2) + \alpha - \alpha x_k^2 - \alpha d_k x_k^2\Big),
\end{aligned}
$$

which implies

$$\rho_\alpha(G - v_k) \ge \rho_\alpha - \alpha - \frac{x_k^2}{1 - x_k^2}(\rho_\alpha - \alpha d_k).$$

□

Lemma 3.4. *Let $\alpha \in [0, 1)$, let $\boldsymbol{x}$ be the positive eigenvector of $A_\alpha(G)$ corresponding to ρ_α with $\boldsymbol{x}^T\boldsymbol{x} = 1$, let x_k be the kth component of $\boldsymbol{x}$, and let v_k be the kth vertex of G. Then*

$$x_k^2 \le \frac{(1-\alpha)^2 d_k}{(\rho_\alpha - \alpha d_k)^2 + (1-\alpha)^2 d_k}, \qquad (8)$$

with equality holding if and only if G is the join of the vertex v_k and a regular graph of order $n - 1$.

Proof. Since

$$\rho_\alpha x_k = \sum_{i=1}^{n} a_{ki} x_i = (1-\alpha) \sum_{v_i \in N(v_k)} x_i + \alpha d_k x_k,$$

we have

$$
\begin{aligned}
(\rho_\alpha - \alpha d_k)^2 x_k^2 &= (1-\alpha)^2 \Big(\sum_{v_i \in N(v_k)} x_i\Big)^2 \\
&\le (1-\alpha)^2 d_k \sum_{v_i \in N(v_k)} x_i^2 \qquad (9)\\
&\le (1-\alpha)^2 d_k (1 - x_k^2). \qquad (10)
\end{aligned}
$$

That is

$$x_k^2 \le \frac{(1-\alpha)^2 d_k}{(\rho_\alpha - \alpha d_k)^2 + (1-\alpha)^2 d_k}.$$

Equality in (10) holds if and only if $N(v_k) = V(G) - v_k$. On this basis, the equality in (9) holds if and only if all the x_i is same where $i \in \{1, 2, \dots, n\}$ and $i \ne k$, if and only if $G - v_k$ is a regular graph. □

Proof of Theorem 2.1. Let $\mathbf{x}$ be the positive eigenvector of $A_\alpha(G)$ corresponding to ρ_α with $\mathbf{x}^T\mathbf{x} = 1$, and let x_k be the kth component of $\mathbf{x}$. By Lemma 3.3, we obtain

$$\rho_\alpha(G - v_k) \ge \rho_\alpha - \alpha - \frac{x_k^2}{1 - x_k^2}(\rho_\alpha - \alpha d_k). \qquad (11)$$

By Lemma 3.4 we get

$$x_k^2 \le \frac{(1-\alpha)^2 d_k}{(\rho_\alpha - \alpha d_k)^2 + (1-\alpha)^2 d_k}. \tag{12}$$

Since $h(x_k) = \frac{x_k^2}{1-x_k^2}$ is a increasing function on $x_k \in (0,1)$, $\rho_\alpha - \alpha d_k > 0$ if $\alpha \in [0,1)$ (by Lem. 3.1), combining (11)with (12), we have

$$\begin{aligned}\rho_\alpha(G - v_k) &\ge \rho_\alpha - \alpha - \frac{x_k^2}{1-x_k^2}(\rho_\alpha - \alpha d_k)\\ &\ge \rho_\alpha - \alpha - \frac{(1-\alpha)^2 d_k}{(\rho_\alpha - \alpha d_k)^2}(\rho_\alpha - \alpha d_k)\\ &= \rho_\alpha - \alpha - \frac{(1-\alpha)^2 d_k}{\rho_\alpha - \alpha d_k}.\end{aligned}$$

This completes the proof of inequality (2).

On one hand, the equality in (2) holds implying the equality in (12) holds. By Lemma 3.4, G is the join of the vertex v_k and a regular graph of order $n-1$.

On the other hand, if G is the join of the vertex v_k and a regular graph of order $n-1$, then the equality in (7) holds, subsequently the equalities in (6) and (11) hold. Combining with Lemma 3.4, we get that the equalities in (8) and (12) hold, and the equality in (2) holds. □

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