A BOUND FOR THE $A_{\alpha}$-SPECTRAL RADIUS OF A CONNECTED GRAPH AFTER VERTEX DELETION

CHUNXIANG WANG and TAO SHE

Abstract. $G$ is a simple connected graph with adjacency matrix $A(G)$ and degree diagonal matrix $D(G)$. The signless Laplacian matrix of $G$ is defined as $Q(G) = D(G) + A(G)$. In 2017, Nikiforov [1] defined the matrix $A_{\alpha}(G) = \alpha D(G) + (1 - \alpha)A(G)$ for $\alpha \in [0, 1]$. The $A_{\alpha}$-spectral radius of $G$ is the maximum eigenvalue of $A_{\alpha}(G)$. In 2019, Liu et al. [2] defined the matrix $\Theta_k(G)$ as $\Theta_k(G) = kD(G) + A(G)$, for $k \in \mathbb{R}$. In this paper, we present a new type of lower bound for the $A_{\alpha}$-spectral radius of a graph after vertex deletion. Furthermore, we deduce some corollaries on $\Theta_k(G), A(G), Q(G)$ matrices.

Mathematics Subject Classification. 05C50, 15A18.

Received March 14, 2022. Accepted October 4, 2022.

1. INTRODUCTION AND PRELIMINARIES

We consider non-empty simple connected graph $G$ with vertex set $V(G)$ and edge set $E(G)$ throughout this paper. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. If any pair of vertices $v_i$ and $v_j$ are adjacent, then we write $v_i v_j \in E(G)$ or $v_i \sim v_j$. For a vertex $v_k \in V(G)$, the neighborhood of $v_k$ is the set $N(v_k) = N_G(v_k) = \{w \in V(G) : w \sim v_k\}$, and $d_G(v_k)$ denotes the degree of $v_k$ with $d_G(v_k) = |N(v_k)|$. Let $d_k = d_G(v_k)$ if there is no ambiguity. Let $V_m$ be any fixed subset of $V(G)$ containing $m$ vertices. For $V_m \subseteq V(G)$ with $|V_m| = m$, let $G[V_m]$ be the subgraph of $G$ induced by $V_m$, $G - V_m$ be the subgraph induced by $V(G) - V_m$. Let $G \vee H$ denote the graph obtained from the disjoint union $G + H$ by adding all edges between graph $G$ and graph $H$. A regular graph with vertices of degree $r$ is called a $r$-regular graph. Let $K_n, K_{s,t}$ denote the clique and complete bipartite graph respectively and $K_{1,n-1}$ be the star of order $n$.

$A(G)$ denotes the adjacency matrix and $D(G)$ denotes the diagonal matrix of the degrees of $G$. The signless Laplacian matrix of $G$ is defined as $Q(G) = D(G) + A(G)$. In 2017, Nikiforov [1] proposed the matrix $A_{\alpha}(G)$ of a graph $G$

$$A_{\alpha}(G) = \alpha D(G) + (1 - \alpha)A(G),$$

for $\alpha \in [0, 1]$, which successfully extends the theories of $A(G)$ and $Q(G)$. Let $A_\alpha = A_{\alpha}(G)$ if there is no ambiguity. It is not hard to see that $A_\alpha$ is the adjacency matrix and $2A_{1/2}$ is the signless Laplacian matrix. In 2019, Liu et al. [2] defined the matrix $\Theta_k(G)$ as $\Theta_k(G) = kD(G) + A(G)$, for $k \in \mathbb{R}$.

Keywords. Spectral radius, eigenvalue, eigenvector, adjacency matrix.

School of Mathematics and Statistics, Central China Normal University, Wuhan, P.R. China.

*Corresponding author: she_tao@163.com

© The authors. Published by EDP Sciences, ROADEF, SMAI 2022

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
Denote the eigenvalues of \( n \times n \) symmetric matrix \( M \) by \( \rho_1(M) \geq \rho_2(M) \geq \ldots \geq \rho_n(M) \). The largest eigenvalue \( \rho_\alpha(G) := \rho_1(A_\alpha(G)) \) of the \( A_\alpha \)-matrix is defined as the \( A_\alpha \)-spectral radius of \( G \). Similarly, we can define the \( \Theta_k \)-spectral radius and \( Q \)-spectral radius of \( G \). Let \( \rho_\alpha = \rho_\alpha(G) \) if there is no ambiguity. Let \( x = (x_1, x_2, \ldots, x_n)^T \) be a unit nonnegative eigenvector of \( \rho_\alpha \) corresponding to the vertex set \( \{v_1, v_2, \ldots, v_n\} \). Let \( G_m = (G - V_m) \cup mK_1 \). For other undefined notations and terminologies, refer to [3].

Many scholars already succeeded in finding bounds for the \( A_\alpha \)-spectral radius. For more results in this direction, readers can refer to a survey [1] by Nikiforov and some other articles [4-9]. In 2019, Guo et al. [10] and Sun et al. [11] presented a relation between \( \rho_1(G) \) and \( \rho_1(G - v_k) \) for adjacency matrices \( A(G) \) and \( A(G - v_k) \), where \( v_k \) is a vertex of \( G \). The better bound given by [11] is shown as follow

\[
\rho_1(A(G)) \leq \sqrt{\rho_1^2(A(G - v_k)) + 2d_k - 1}.
\]

Since (1) is well used in analyzing the graph structure (see [12]), we try to extend the above inequation to more matrices, such as \( A_\alpha(G), \Theta_k(G), Q(G) \). Using some different methods from [10, 11], we get the results in this paper. As far as the authors know, this topic has not been explored elsewhere.

From the following proposition given by Nikiforov, we know that there exists a positive eigenvector \( x \) corresponding to \( \rho_\alpha \) if \( \alpha \in [0, 1) \).

**Proposition 1.1** (Proposition 13 of [1]). Let \( \alpha \in [0, 1) \), let \( G \) be a connected graph, let \( x \) be a nonnegative eigenvector to \( \rho_1(A_\alpha(G)) \), and let \( H \) be a proper subgraph of \( G \), then

(i) \( x \) is positive and is unique up to scaling;

(ii) \( \rho_1(A_\alpha(H)) < \rho_1(A_\alpha(G)) \).

The \( A_\alpha \)-spectrum of \( K_n \) and \( K_{1,n-1} \) are given by Nikiforov as follows:

**Proposition 1.2** (Proposition 36 of [1]). The eigenvalues of \( A_\alpha(K_n) \) are

\[
\begin{align*}
\rho_1(A_\alpha(K_n)) & = n - 1, \\
\rho_k(A_\alpha(K_n)) & = \alpha n - 1 \quad \text{for} \quad 1 < k \leq n.
\end{align*}
\]

**Proposition 1.3** (Proposition 38 of [1]). The eigenvalues of \( A_\alpha(K_{1,n-1}) \) are

\[
\begin{align*}
\rho_1(A_\alpha(K_{1,n-1})) & = \frac{1}{2}(\alpha n + \sqrt{\alpha^2 n^2 + 4(n - 1)(1 - 2\alpha)}), \\
\rho_n(A_\alpha(K_{1,n-1})) & = \frac{1}{2}(\alpha n - \sqrt{\alpha^2 n^2 + 4(n - 1)(1 - 2\alpha)}), \\
\rho_k(A_\alpha(K_{1,n-1})) & = \alpha \quad \text{for} \quad 1 < k < n.
\end{align*}
\]

**Proposition 1.4** (Corollary 1 of [10]). Let \( G \) be a connected graph with \( n \) vertices. Then

\[
\rho_1(A(G)) \geq \max\{\sqrt{d(v)}, v \in V(G)\},
\]

with equality holding if and only if \( G = K_{1,n-1} \).

![Image](image.png)

2. Main results

**Theorem 2.1.** Let \( \alpha \in [0, 1) \) and let \( v_k \) be a vertex of a connected graph \( G \) with degree \( d_k \). Then

\[
\rho_\alpha(G - v_k) \geq \rho_\alpha - \alpha - \frac{(1 - \alpha)^2 d_k}{\rho_\alpha - \alpha d_k}
\]

with equality holding if and only if \( G \) is the join of the vertex \( v_k \) and a regular graph of order \( n - 1 \).
Theorem 2.1 will be proved in Section 3.

**Corollary 2.2.** Let \( k \in [0, +\infty) \) and let \( v \) be a vertex of a connected graph \( G \) with degree \( d(v) \). Then

\[
\lambda_1(\Theta_k(G - v)) \geq \lambda_1(\Theta_k(G)) - k - \frac{d(v)}{\lambda_1(\Theta_k(G)) - kd(v)}
\]

with equality holding if and only if \( G \) is the join of the vertex \( v \) and a regular graph of order \( n - 1 \).

By setting \( \alpha = 0 \) in (2) we obtain the following corollary.

**Corollary 2.3.** Let \( v_k \) be a vertex of a connected graph \( G \) with degree \( d_k \). Then

\[
\rho_1(A(G - v_k)) \geq \rho_1(A(G)) - \frac{d_k}{\rho_1(A(G))}
\]

with equality holding if and only if \( G \) is the join of the vertex \( v_k \) and a regular graph of order \( n - 1 \).

When \( d_k \geq \rho_1(A(G)) \), Corollary 2.3 gives a lower bound for \( \rho_\alpha(G - v_k) \) which is better than \( \sqrt{\rho^2(A(G)) - 2d_k + 1} \) deducible from (1). In fact, by Proposition 1.4, we have \( \rho_1(A(G)) - \frac{d_k}{\rho_1(A(G))} > 0 \). By Corollary 2.3, we have

\[
\rho_1(A(G - v_k)) \geq \rho_1(A(G)) - \frac{d_k}{\rho_1(A(G))} = \sqrt{\rho_1^2(A(G)) - 2d_k + \frac{d_k^2}{\rho_1^2(A(G))}} > \sqrt{\rho^2(A(G)) - 2d_k + 1}.
\]

By setting \( \alpha = \frac{1}{2} \) in (2) we obtain the following corollary.

**Corollary 2.4.** Let \( v_k \) be a vertex of a connected graph \( G \) with degree \( d_k \), and let \( Q(G) \) be the signless Laplacian matrix of \( G \). Then

\[
\rho_1(Q(G - v_k)) \geq \rho_1(Q(G)) - 1 - \frac{d_k}{\rho_1(Q(G)) - d_k}
\]

with equality holding if and only if \( G \) is the join of the vertex \( v_k \) and a regular graph of order \( n - 1 \).

Define \( \Phi(\alpha, r, n) = \frac{1}{2}(\alpha n + r) + \frac{1}{2} \sqrt{(\alpha n + r)^2 - 4(r\alpha + 2\alpha - 1)(n - 1)} \).

**Corollary 2.5.** Let \( G \) be the join of the vertex \( v_k \) and a \( r \)-regular graph of order \( n - 1 \), \( \alpha \in [0, 1) \). Then

\( i \) \hspace{1cm} \( \rho_\alpha = \frac{1}{2}(\alpha n + r) + \frac{1}{2} \sqrt{(\alpha n + r)^2 - 4(r\alpha + 2\alpha - 1)(n - 1)} \);

\( ii \) \hspace{1cm} \( \rho_1(A(G)) = \frac{r}{2} + \frac{1}{2} \sqrt{r^2 + 4(n - 1)} \);

\( iii \) \hspace{1cm} \( \rho_1(Q(G)) = \frac{n}{2} + r + \sqrt{(\frac{n}{2} + r)^2 - 2r(n - 1)} \).

**Proof.** \( i \) \hspace{1cm} Since \( G - v_k \) is a \( r \)-regular graph, it is obvious that \( \rho_\alpha(G - v_k) = r \). By Theorem 2.1, we have

\[
r = \rho_\alpha(G - v_k) = \rho_\alpha - \alpha - \frac{(1 - \alpha)^2 d_k}{\rho_\alpha - \alpha d_k} = \rho_\alpha - \alpha - \frac{(1 - \alpha)^2 (n - 1)}{\rho_\alpha - \alpha (n - 1)}.
\]
Hence
\[
\rho_{\alpha}^2 - (an + r)\rho_{\alpha} + (ra + 2\alpha - 1)(n - 1) = 0.
\]
\[
\rho_{\alpha} = \frac{1}{2}(an + r) + \frac{1}{2}\sqrt{(an + r)^2 - 4(ra + 2\alpha - 1)(n - 1)} = \Phi(\alpha, r, n).
\]

(iii) \( \rho_1(A(G)) = \Phi(0, r, n) = \frac{r}{2} + \frac{1}{2}\sqrt{r^2 + 4(n - 1)}. \)

Remark 2.6. In particular, \( \rho_{\alpha} \) values of \( K_{1,n-1}(\text{star}), K_1 \vee \frac{n-1}{2}K_2(\text{friendship graph}), K_1 \vee C_{n-1}(\text{wheel graph}), K_1 \vee (C_{n_1} + C_{n_2} + \ldots + C_{n_k})(\text{multi-wheel graph}), K_n(\text{complete graph}) \) are \( \Phi(\alpha, 0, n), \Phi(\alpha, 1, n), \Phi(\alpha, 2, n), \Phi(\alpha, 2, n), \Phi(\alpha, n - 1, n) \) respectively.

We can also calculate \( \rho_{\alpha} \) by equitable quotient matrices for the graphs in Corollary 2.5.

3. PROOF OF THEOREM 2.1

Lemma 3.1. Let \( \alpha \in [0, 1) \) and let \( v_k \) be a vertex of a connected graph \( G \) with degree \( d_k \), then \( \rho_{\alpha} > \alpha d_k \).

Proof. Let \( x_i \) be the \( i \)th component of the positive eigenvector \( \mathbf{x} \) corresponding to \( \rho_{\alpha} \). Since \( x_k > 0, x_i > 0, 1 - \alpha > 0 \) and
\[
\rho_{\alpha}x_k = \sum_{i=1}^{n} a_{ki}x_i = (1 - \alpha) \sum_{v_i \in N(v_k)} x_i + \alpha d_k x_k,
\]
the result immediately follows. \( \square \)

Lemma 3.2. Let \( \alpha \in [0, 1), V_m \subset V(G) \) with \( |V_m| = m \) and \( c_s = |N_G(v_s) \cap V_m| \) for \( v_s \notin V_m \). Suppose \( a_{ij} \) is the \((i)\)th row \((j)\)th column element of \( A_{\alpha} \), and \( x_i \) is the \( i \)th component of the positive unit eigenvector \( \mathbf{x} \) corresponding to \( \rho_{\alpha} \). Then
\[
\sum_{v_k \in V_m} x_k^2 \leq \frac{1}{2} + \frac{1}{2\rho_{\alpha}} \left( \sum_{v_i \in V_m} \sum_{v_j \in V_m} a_{ij}x_i x_j - \alpha \sum_{v_s \notin V_m} c_s x_s^2 \right) \tag{5}
\]
with equality holding if and only if \( G_m \) is an empty graph.

Proof. Since \( A_{\alpha} = \alpha D(G) + (1 - \alpha)A(G) \), we have
\[
a_{ij} = \begin{cases} 
1 - \alpha & \text{if } i \neq j \text{ and } v_i \in N_G(v_j), \\
0 & \text{if } i \neq j \text{ and } v_i \notin N_G(v_j), \\
ad_i & \text{if } i = j.
\end{cases}
\]
Suppose \( a'_{ij} \) is the \((i)\)th row \((j)\)th column element of \( A_{\alpha}(G_m) \), then
\[
a'_{ij} = \begin{cases} 
0 & \text{if } v_i \in V_m \text{ or } v_j \in V_m, \\
a_{ij} & \text{if } i \neq j, v_i \notin V_m \text{ and } v_j \notin V_m, \\
av_{ii} - \alpha c_i & \text{if } i = j \text{ and } v_i \notin V_m.
\end{cases}
\]
Let $\mathbf{a}_k$ be the column vector $(a_{k1}, a_{k2}, \ldots, a_{kn})^T$ and $\mathbf{e}_k$ be the $k$th basis column vector $(0, \ldots, 0, 1, \ldots, 0)^T$, where only the $k$th component is 1. We have
\[
(\mathbf{x}^T - A_\alpha(G_m))\mathbf{x} = \mathbf{x}^T(-\sum_{v_i \in V_N} \sum_{v_j \in V_N} a_{ij} \mathbf{e}_i \mathbf{e}_j^T) + \sum_{v_i \in V_N} \alpha c_s x_s^2
\]
\[
\mathbf{x}^T A_\alpha(G_m)\mathbf{x} = \mathbf{x}^T A_\alpha\mathbf{x} - \mathbf{x}^T(-\sum_{v_i \in V_N} \sum_{v_j \in V_N} a_{ij} \mathbf{e}_i \mathbf{e}_j^T) + \sum_{v_i \in V_N} \alpha c_s x_s^2
\]

Hence
\[
\mathbf{x}^T A_\alpha(G_m)\mathbf{x} = \mathbf{x}^T A_\alpha\mathbf{x} - \mathbf{x}^T(-\sum_{v_i \in V_N} \sum_{v_j \in V_N} a_{ij} \mathbf{e}_i \mathbf{e}_j^T) + \sum_{v_i \in V_N} \alpha c_s x_s^2
\]

which implies
\[
\sum_{v_i \in V_N} x_i^2 \leq \frac{1}{2} + \frac{1}{2\rho_\alpha} \left( \sum_{v_i \in V_N} \sum_{v_j \in V_N} a_{ij} x_i x_j - \alpha \sum_{v_i \in V_N} c_s x_s^2 \right).
\]

Note that $\mathbf{x}$ is a positive unit eigenvector, we can see that the equality holds if and only if $G_m$ is an empty graph. \hfill \Box

**Lemma 3.3.** Let $\alpha \in [0, 1]$, let $\mathbf{x}$ be the positive eigenvector of $A_\alpha(G)$ corresponding to $\rho_\alpha$ with $\mathbf{x}^T \mathbf{x} = 1$, let $x_k$ be the $k$th component of $\mathbf{x}$, and let $v_k$ be the $k$th vertex of $G$. Then
\[
\rho_\alpha(G - v_k) \geq \rho_\alpha - \alpha - \frac{x_k^2}{1 - x_k^2} (\rho_\alpha - \alpha d_k).
\]

**Proof.** Let $V_m = V(G) - v_k$, then $G_m$ is an empty graph. By Lemma 3.2, we have
\[
\sum_{v_i \in V_N} x_i^2 = \frac{1}{2} + \frac{1}{2\rho_\alpha} \left( \sum_{v_i \in V_N} \sum_{v_j \in V_N} a_{ij} x_i x_j - \alpha \sum_{v_i \in V_N} c_s x_s^2 \right).
\]

Let $\mathbf{x} = (x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n)^T$, whose $k$th component is 0.

Let $G_1 = (G - v_k) \cup K_1$. We get
\[
1 - x_k^2 = \sum_{v_i \in V_m} x_i^2
\]

\[
\frac{1}{2} + \frac{1}{2\rho_\alpha} \left( \sum_{v_i \in V_m} \sum_{v_j \in V_m} a_{ij} x_i x_j - \alpha d_k x_k^2 \right)
\]

\[
\begin{align*}
&= \frac{1}{2} + \frac{1}{2\rho_\alpha} \left( x^T A_\alpha(G_1) x + \alpha \sum_{v_i \in N(v_k)} x_i^2 - \alpha x_k x_k^2 \right) \\
&= \frac{1}{2} + \frac{1}{2\rho_\alpha} \left( \hat{x}^T A_\alpha(G_1) \hat{x} + \alpha \sum_{v_i \in N(v_k)} x_i^2 - \alpha x_k x_k^2 \right) \\
&\leq \frac{1}{2} + \frac{1}{2\rho_\alpha} \left( \rho_\alpha(G_1) \hat{x}^T \hat{x} + \alpha - \alpha x_k^2 - \alpha x_k x_k^2 \right) \\
&= \frac{1}{2} + \frac{1}{2\rho_\alpha} \left( \rho_\alpha(G - v_k) (1 - x_k^2) + \alpha - \alpha x_k^2 - \alpha x_k x_k^2 \right), \\
\end{align*}
\]
which implies
\[
\rho_\alpha(G - v_k) \geq \rho_\alpha - \frac{x_k^2}{1 - x_k^2} (\rho_\alpha - \alpha d_k).
\]

\[\square\]

**Lemma 3.4.** Let \( \alpha \in [0, 1] \), let \( x \) be the positive eigenvector of \( A_\alpha(G) \) corresponding to \( \rho_\alpha \) with \( x^T x = 1 \), let \( x_k \) be the \( k \)th component of \( x \), and let \( v_k \) be the \( k \)th vertex of \( G \). Then
\[
x_k^2 \leq \frac{(1 - \alpha)^2 d_k}{(\rho_\alpha - \alpha d_k)^2 + (1 - \alpha)^2 d_k},
\]
with equality holding if and only if \( G \) is the join of the vertex \( v_k \) and a regular graph of order \( n - 1 \).

**Proof.** Since
\[
\rho_\alpha x_k = \sum_{i=1}^{n} a_{ki} x_i = (1 - \alpha) \sum_{v_i \in N(v_k)} x_i + \alpha d_k x_k,
\]
we have
\[
(\rho_\alpha - \alpha d_k)^2 x_k^2 = (1 - \alpha)^2 \left( \sum_{v_i \in N(v_k)} x_i \right)^2 \\
\leq (1 - \alpha)^2 d_k \sum_{v_i \in N(v_k)} x_i^2 \\
\leq (1 - \alpha)^2 d_k (1 - x_k^2),
\]
That is
\[
x_k^2 \leq \frac{(1 - \alpha)^2 d_k}{(\rho_\alpha - \alpha d_k)^2 + (1 - \alpha)^2 d_k}.
\]
Equality in (10) holds if and only if \( N(v_k) = V(G) - v_k \). On this basis, the equality in (9) holds if and only if all the \( x_i \) is same where \( i \in \{1, 2, \ldots, n\} \) and \( i \neq k \), if and only if \( G - v_k \) is a regular graph. \(\square\)

**Proof of Theorem 2.1.** Let \( x \) be the positive eigenvector of \( A_\alpha(G) \) corresponding to \( \rho_\alpha \) with \( x^T x = 1 \), and let \( x_k \) be the \( k \)th component of \( x \). By Lemma 3.3, we obtain
\[
\rho_\alpha(G - v_k) \geq \rho_\alpha - \frac{x_k^2}{1 - x_k^2} (\rho_\alpha - \alpha d_k).
\]
By Lemma 3.4 we get
\[ x_k^2 \leq \frac{(1 - \alpha)^2 d_k}{(\rho_\alpha - \alpha d_k)^2 + (1 - \alpha)^2 d_k}. \] 
(12)

Since \( h(x_k) = \frac{x_k^2}{1-x_k^2} \) is a increasing function on \( x_k \in (0, 1) \), \( \rho_\alpha - \alpha d_k > 0 \) if \( \alpha \in [0, 1) \) (by Lem. 3.1), combining (11) with (12), we have

\[
\rho_\alpha(G - v_k) \geq \rho_\alpha - \alpha - \frac{x_k^2}{1-x_k^2} (\rho_\alpha - \alpha d_k) \geq \rho_\alpha - \alpha - \frac{(1 - \alpha)^2 d_k}{(\rho_\alpha - \alpha d_k)^2} (\rho_\alpha - \alpha d_k) = \rho_\alpha - \alpha - \frac{(1 - \alpha)^2 d_k}{\rho_\alpha - \alpha d_k}.
\]

This completes the proof of inequality (2).

On one hand, the equality in (2) holds implying the equality in (12) holds. By Lemma 3.4, \( G \) is the join of the vertex \( v_k \) and a regular graph of order \( n - 1 \).

On the other hand, if \( G \) is the join of the vertex \( v_k \) and a regular graph of order \( n - 1 \), then the equality in (7) holds, subsequently the equalities in (6) and (11) hold. Combining with Lemma 3.4, we get that the equalities in (8) and (12) hold, and the equality in (2) holds. \( \square \)

Acknowledgements. The work was partially supported by the National Natural Science Foundation of China under Grants 11771172,12061039.

Data Availability Statements. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

This journal is currently published in open access under a Subscribe-to-Open model (S2O). S2O is a transformative model that aims to move subscription journals to open access. Open access is the free, immediate, online availability of research articles combined with the rights to use these articles fully in the digital environment. We are thankful to our subscribers and sponsors for making it possible to publish this journal in open access, free of charge for authors.

Please help to maintain this journal in open access!

Check that your library subscribes to the journal, or make a personal donation to the S2O programme, by contacting subscribers@edpsciences.org

More information, including a list of sponsors and a financial transparency report, available at: https://www.edpsciences.org/en/maths-s2o-programme