RAIRO-Oper. Res. 56 (2022) 3257–3265
https://doi.org/10.1051/ro/2022144

**RAIRO Operations Research**
www.rairo-ro.org

# SHARP CONDITIONS ON FRACTIONAL ID-$(g, f)$-FACTOR-CRITICAL COVERED GRAPHS

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**Abstract.** Combining the concept of a fractional $(g, f)$-covered graph with that of a fractional ID-$(g, f)$-factor-critical graph, we define the concept of a fractional ID-$(g, f)$-factor-critical covered graph. This paper reveals the relationship between some graph parameters and the existence of fractional ID-$(g, f)$-factor-critical covered graphs. A sufficient condition for a graph being a fractional ID-$(g, f)$-factor-critical covered graph is presented. In addition, we demonstrate the sharpness of the main result in this paper by constructing a special graph class. Furthermore, the relationship between other graph parameters(such as binding number, toughness, sun toughness and neighborhood union) and fractional ID-$(g, f)$-factor-critical covered graphs can be studied further.

**Mathematics Subject Classification.** 05C70.

Received March 25, 2022. Accepted August 16, 2022.

## 1. Introduction

It is well known that the fractional factor problem is a relaxation problem of the cardinality matching problem, which is widely used in operations research and computer networks. The fractional factor is mainly used in network design, scheduling, combinatorial polyhedron and other fields. We look upon the network as a graph. The positions match the vertices, and the channels match the edges of the graph. In communication networks, large data packets are sent to different destinations *via* channels. In order to effectively increase the workload, scientists model the feasible data allocation problem and divide the large data packets into small data packets, thus, the available assignments of data packets is described as a fractional flow problem.

The fractional factor in fractional graph theory can be used to measure the tolerance of a network. Based on the graph theory model, the problem can be transformed into a fractional factor problem. In the research of data transmission feasibility analysis algorithm, as an important factor in the design stage, the network engineer should consider the vulnerability of the network, which requires that the transmission of data is guaranteed when multiple stations and transmission channels are destroyed in a network. In the graph representation model of the network, the fractional ID-critical deleted graph is a graph that requires the presence of the fractional factor under the combined frame of deleting vertices and edges, which corresponds to data transmissibility under the situation where multiple stations and transmission channels are destroyed in the network. Similarly,

*Keywords.* Independence number, minimum degree, fractional $(g, f)$-covered graph, fractional ID-$(g, f)$-factor-critical graph, fractional ID-$(g, f)$-factor-critical covered graph.

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the existence of fractional ID-factor-critical covered graph is equivalent to the possibility of data transmission in a communication network under the situation when some nonadjacent nodes with each other are damaged and a special channel is assigned. At the same time, graph parameters (such as: toughness, neighborhood union, neighbor set, binding number and minimum degree) are used to measure the robustness of network attacks. Analysis of the feasibility of data transmission over the network helps to identify lost links in the recovery or realignment of the network. These theories can be extended to other problem areas, such as protein interactions, social networks, and so on. The study on the existence of fractional ID-factor-critical covered graph can help scientists to design and construct networks with high data transmission.

The graphs considered in this paper will be finite undirected graphs without loops or multiple edges. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(x)$ the degree of a vertex $x$ in $G$ and $N_G(x)$ the set of vertices adjacent to $x$ in $G$, respectively. We use $N_G[x]$ to denote $N_G(x) \cup \{x\}$. The minimum degree of $G$ is denoted by $\delta(G)$. For a subset $X \subseteq V(G)$, let $N_G(X)$ denote the union of $N_G(x)$ for every $x \in X$ and $G[X]$ the subgraph of $G$ induced by $X$, respectively. We write $G - X$ for the subgraph obtained from $G$ by deleting the vertices in $X$ together with the edges incident to the vertices in $X$. If $G[X]$ has no edges, $X$ is called an independent set of $G$. $\alpha(G)$ denotes the independence number of a graph $G$, which is the maximum cardinality of any subset $X \subseteq V(G)$ such that $X$ is an independent set.

Functions $g(x)$ and $f(x)$ are integer-valued and defined on $V(G)$ satisfying $0 \le g(x) \le f(x)$ for every $x \in V(G)$. We call a spanning subgraph $F$ of $G$ a $(g, f)$-factor if $g(x) \le d_F(x) \le f(x)$ holds for any $x \in V(G)$. Function $h$ is defined on $E(G)$ satisfying $h(e) \in [0, 1]$ for every edge $e \in E(G)$. A fractional $(g, f)$-factor is regarded as a score function $h$ which maps to every element in $E(G)$ a real number belonging to $[0, 1]$, namely, for each vertex $x$ we have $g(x) \le d_G^h(x) \le f(x)$, where $d_G^h(x) = \sum_{e\in E_x} h(e)$ is the fractional degree of $x \in V(G)$ satisfying $E_x = \{e : e = xy \in E(G)\}$. Set $E_h = \{e : e \in E(G) \text{ and } h(e) \neq 0\}$. If $G_h$ is a spanning subgraph of $G$ with $E(G_h) = E_h$, then we call $G_h$ a fractional $(g, f)$-factor of $G$ with indicator function $h$. If $h(e) \in \{0, 1\}$ for any edge $e \in E(G)$, then a fractional $(g, f)$-factor is just a $(g, f)$-factor. If a graph $G$ contains a $(g, f)$-factor, it is clear that it also contains a fractional $(g, f)$-factor.

If for every edge $e \in E(G)$, $G$ possesses a fractional $(g, f)$-factor $F$, with indicator function $h_F$, such that $h_F(e) = 1$, we call graph $G$ a fractional $(g, f)$-covered graph. Let $a$ and $b$ be two nonnegative integers with $a \le b$. A fractional $(g, f)$-covered graph is called a fractional $[a, b]$-covered graph if $g(x) = a$ and $f(x) = b$ for every $x \in V(G)$. So an $[a, b]$-covered graph is a special case of fractional $(g, f)$-covered graph. A fractional $[k, k]$-covered graph is simply called a fractional $k$-covered graph.

If for any independent set $I$ of $G$, $G-I$ admits a fractional $(g, f)$-factor, then a graph $G$ is called a fractional ID-$(g, f)$-factor-critical graph. A fractional ID-$(g, f)$-factor-critical graph is called a fractional ID-$[a, b]$-factor-critical graph if $g(x) = a$ and $f(x) = b$ for every $x \in V(G)$. A fractional ID-$[k, k]$-factor-critical graph is simply called a fractional ID- $k$-factor-critical graph.

If for any independent set $I$ of $G$, $G - I$ is a fractional $(g, f)$-covered graph, then a graph $G$ is called a fractional ID-$(g, f)$-factor-critical covered graph. A fractional ID-$(g, f)$-factor-critical covered graph is called a fractional ID-$[a, b]$-factor-critical covered graph if $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$. A fractional ID-$[k, k]$-factor-critical covered graph is simply called a fractional ID-$k$-factor-critical covered graph.

Haghparast and Ozeki [6], and Holub *et al.* [7] investigated the existence of 2-factors in graphs. Shiu and Liu [13] presented some sufficient conditions for a graph admitting $k$-factors in regular graphs. Wang and Zhang [16], Liu [10], Zhou [20], Zhou *et al.* [22,23,25,27,28] characterized a graph with a $[1, 2]$-factor, and put forword some sufficient conditions for graphs to have $[1, 2]$-factors. Matsuda [12] derived some results in terms of the neighborhood condition for graphs to admit $[a, b]$-factors. Egawa and Kano [2] posed some sufficient conditions for graphs to have $(g, f)$-factors. Wang and Zhang [15], and Zhou and Liu [24] showed the properties of edge-disjoint factors in graphs. Gao *et al.* [4], Katerinis [8], Liu and Zhang [11] discussed the existence of fractional $k$-factors. Cai, Wang and Yan [1] investigated the existence of fractional $f$-factors in random graphs. Zhou [17,18], Zhou, Wu and Liu [29] obtained some sufficient conditions for the existence of fractional $[a, b]$-factors in graphs. Gao *et al.* [5], Wang and Zhang [14] demonstrated some sufficient conditions for the existences of fractional $(g, f)$-factors. Zhou [19,21] got some results for graphs to possess restricted fractional $(g, f)$-factors.

Gao *et al.* [3] discussed the existence of fractional ID-$k$-factor-critical graph, and established a relationship between the degree sum condition and fractional ID-$k$-factor-critical graph. Zhou, Liu and Xu [26] investigated the existence of fractional ID-$[a, b]$-factor-critical covered graphs, and posed a neighborhood of independent set and minimum degree condition for a graph to be fractional ID-$[a, b]$-factor-critical covered. In this article, we investigate fractional ID-$(g, f)$-factor-critical covered graphs, and verify a minimum degree and independence number condition for the existence of fractional ID-$(g, f)$-factor-critical covered graphs, which is shown in the following.

**Theorem 1.1.** *Let $a, b$ and $m$ be three nonnegative integers with $b - m \geq a \geq 2$, let $G$ be a graph of order $p$ with $p \geq \frac{(2a+b+m-1)(a+b-2)+a+b+1}{a+m}$ , and let $g$ and $f$ be two integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) - m \leq b - m$ for any $x \in V(G)$. Suppose that $G$ satisfies*

$$\delta(G) \geq \frac{(a+b-2)p + 2\alpha(G) + 2}{2a+b+m-2}$$

*and*

$$\delta(G) \geq \frac{(a+b-1)p + a + b + 1}{2a+b+m-1}.$$

*Then $G$ is a fractional ID-$(g, f)$-factor-critical covered graph.*

If $m = 0$ in Theorem 1.1, then we have the following corollary.

**Corollary 1.2.** *Let $a, b$ be two nonnegative integers with $b \geq a \geq 2$, let $G$ be a graph of order $p$ such that $p \geq \frac{(2a+b-1)(a+b-2)+a+b+1}{a}$ , and let $g$ and $f$ be two integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) \leq b$ for any $x \in V(G)$. Suppose that $G$ satisfies*

$$\delta(G) \geq \frac{(a+b-2)p + 2\alpha(G) + 2}{2a+b-2}$$

*and*

$$\delta(G) \geq \frac{(a+b-1)p + a + b + 1}{2a+b-1}.$$

*Then $G$ is a fractional ID-$(g, f)$-factor-critical covered graph.*

We easily obtain the following corollary by setting $a = b = k$ in Corollary 1.2.

**Corollary 1.3.** *Let $k$ be a nonnegative integer with $k \geq 2$, let $G$ be a graph of order $p$ such that $p \geq \frac{2(3k-1)(k-1)+2k+1}{k}$. Suppose that $G$ satisfies*

$$\delta(G) \geq \frac{2(k-1)p + 2\alpha(G) + 2}{3k-2}$$

*and*

$$\delta(G) \geq \frac{(2k-1)p + 2k + 1}{3k-1}.$$

*Then $G$ is a fractional ID-$k$-factor-critical covered graph.*

## 2. Proof of Theorem 1

The following criterion for a graph being a fractional $(g, f)$-covered graph is due to Li, Yan and Zhang [9], known as fractional $(g, f)$-covered graph theorem, which is vital for the proof of Theorem 1.1.

**Theorem 2.1.** *Let $G$ be a graph, $g, f : V(G) \to \{0, 1, 2, 3, \dots\}$ be two functions such that $g(x) \le f(x)$ for all $x \in V(G)$. Then a graph $G$ is fractional $(g, f)$-covered if and only if for all $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S,$ and $d_{G-S}(x) \le g(x)\}$,*

$$\gamma_G(S,T) = d_{G-S}(T) - g(T) + f(S) \ge \varepsilon(S)$$

where $\varepsilon(S)$ is defined as follows.

(1) $\varepsilon(S) = 2$, if $S$ is not independent.
(2) $\varepsilon(S) = 1$, if $S$ is independent and there is an edge joining $S$ and $V(G) \setminus S \cup T$, or there is an edge $e = uv$ joining $S$ and $T$ such that $v \in T, d_{G-S}(v) = g(v)$.
(3) $\varepsilon(S) = 0$, if neither (1) nor (2) holds.

*Proof of Theorem 1.* We verify the theorem by contradiction. Let $I$ be any independent set of $G$, we write $H = G - I$. It suffices to present that $H$ is fractional $(g, f)$-covered. Assume, to the contrary, that H is not fractional $(g, f)$-covered. Then by Theorem 2.1, we obtain

$$\gamma_H(S,T) = d_{H-S}(T) - g(T) + f(S) \le \varepsilon(S) - 1 \tag{2.1}$$

for some subset $S \subseteq V(H)$, where $T = \{x : x \in V(H) \setminus S,$ and $d_{H-S}(x) \le g(x)\}$, Note that $\varepsilon(S) \le |S|$. If $T = \emptyset$, then it follows from (1)

$$|S| \le (a+m)|S| \le f(S) \le \varepsilon(S) - 1 \le |S| - 1,$$

a contradiction. Thus, $T \ne \emptyset$. Define $l = \min\{d_{H-S}(x) \mid x \in T\}$.

By the definition of $T$, we know $0 \le l \le b - m$. Note that $I$ is an independent set of $G$. Hence, we admit

$$|I| \le p - \delta(G). \tag{2.2}$$

We examine a vertex of $T$, it can possess neighbors in $S$, $I$ and at most $l$ additional neighbors. This shows the upper bound on $\delta(G)$, namely, $\delta(G) \le |S| + |I| + l$. Hence, $|S| \ge \delta(G) - |I| - l$.

**Case 1.** $l = 0$.

We write $T_0 = \{x : x \in T, d_{H-S}(x) = 0\}$, $T_1 = \{x : x \in T, d_{H-S}(x) = 1\}$, $T_2 = \{x : x \in T_1, N_{H-S}(x) \subseteq T\}$ and $T_3 = T_1 \setminus T_2$. Obviously the graph induced by $T_2$ in $H - S$ admits maximum degree at most 1. Let $P$ denote a maximum independent set of this graph. It is clear that $|P| \ge \frac{1}{2}|T_2|$. Note that $T_0 \cup P \cup T_3$ is an independent set of $H$. Hence,

$$\alpha(H) \ge |T_0| + |P| + |T_3| \ge |T_0| + \frac{1}{2}|T_2| + \frac{1}{2}|T_3| = |T_0| + \frac{1}{2}|T_1|. \tag{2.3}$$

In view of $H = G - I$, it follows that $\alpha(G) \ge \alpha(H)$. Combining this with (3), we obtain

$$\alpha(G) \ge |T_0| + \frac{1}{2}|T_1|. \tag{2.4}$$

In light of (1), (2), (4), $\varepsilon(S) \le 2$, $p \ge |S| + |T| + |I|$, $b \ge a \ge 2$, $l = 0$ and $|S| \ge \delta(G) - |I| - l$, we yield

$$\begin{aligned}
1 \ge \varepsilon(S) - 1 \ge \gamma_H(S,T) &= d_{H-S}(T) - g(T) + f(S) \\
&\ge 2|T \setminus (T_0 \cup T_1)| + |T_1| - (b-m)|T| + (a+m)|S| \\
&= (a+m)|S| - (b-2-m)|T| - 2(|T_0| + \frac{1}{2}|T_1|)) \\
&\ge (a+m)|S| - (b-2-m)|T| - 2\alpha(G) \\
&\ge (a+m)|S| - (b-2-m)(p - |S| - |I|) - 2\alpha(G) \\
&= (a+b-2)|S| + (b-2-m)|I| - 2\alpha(G) - (b-2-m)p \\
&\ge (a+b-2)(\delta(G) - |I| - l) + (b-2-m)|I| - 2\alpha(G) - (b-2-m)p \\
&= (a+b-2)\delta(G) - (a+m)|I| - 2\alpha(G) - (b-2-m)p \\
&\ge (a+b-2)\delta(G) - (a+m)(p - \delta(G)) - 2\alpha(G) - (b-2-m)p \\
&= (2a+b-2+m)\delta(G) - 2\alpha(G) - (a+b-2)p,
\end{aligned}$$

which implies

$$\delta(G) \le \frac{(a+b-2)p + 2\alpha(G) + 1}{2a+b+m-2}.$$

This contradicts

$$\delta(G) \ge \frac{(a+b-2)p + 2\alpha(G) + 2}{2a+b+m-2}.$$

**Case 2.** $1 \le l \le b - m$.

From (2), $|S| + |T| + |I| \le p$, $\delta(G) \ge \frac{(a+b-1)p+a+b+1}{2a+b+m-1}$, and $|S| \ge \delta(G) - |I| - l$, we obtain

$$\begin{aligned}
\gamma_H(S,T) &= d_{H-S}(T) - g(T) + f(S) \\
&\ge l|T| - (b-m)|T| + (a+m)|S| \\
&= (a+m)|S| - (b-m-l)|T| \\
&\ge (a+m)|S| - (b-m-l)(p - |S| - |I|) \\
&= (a+b-l)|S| - (b-m-l)p + (b-m-l)|I| \\
&\ge (a+b-l)(\delta(G) - |I| - l) - (b-m-l)p + (b-m-l)|I| \\
&= (a+b-l)\delta(G) - (a+m)|I| - (a+b-l)l - (b-m-l)p \\
&\ge (a+b-l)\delta(G) - (a+m)(p - \delta(G)) - (a+b-l)l - (b-m-l)p \\
&= (2a+b+m-l)\delta(G - (a+b-l)l - (a+b-l)p \\
&\ge (2a+b+m-l)\frac{(a+b-1)p+a+b+1}{2a+b+m-1} - (a+b-l)l - (a+b-l)p,
\end{aligned}$$

Let $Q(l) = (2a+b+m-l)\frac{(a+b-1)p+a+b+1}{2a+b+m-1} - (a+b-l)l - (a+b-l)p$. According to $1 \le l \le b-m$ and $p \ge \frac{(2a+b+m-1)(a+b-2)+a+b+1}{a+m}$,

$$\begin{aligned}
Q'(l) &= \frac{-(a+b-1)p - a - b - 1}{2a+b+m-1} - (a+b) + 2l + p \\
&= \frac{(a+m)p - a - b - 1}{2a+b+m-1} - (a+b) + 2l \\
&\ge a + b - 2 - a - b + 2l \\
&= 2l - 2 \\
&\ge 0.
\end{aligned}$$

Hence, $Q(l)$ is an increasing function at $1 \le l \le b-m$, and attains its minimum value at $l = 1$. Then

$$\gamma_H(S,T) \ge Q(l) \ge Q(1). \tag{2.5}$$

In view of (1), (5), $\varepsilon(S) \le 2$,

$$\begin{aligned} 1 &\ge \varepsilon(S) - 1 \ge \gamma_H(S,T) \ge Q(1) \\ &= (2a+b+m-1)\frac{(a+b-1)p+a+b+1}{2a+b+m-1} - (a+b-1) - (a+b-1)p \\ &= 2, \end{aligned}$$

which is a contradiction. This completes the proof of Theorem 1.1. □

## 3. Remark

The conditions proposed in Theorem 1.1 are tight. Neither

$$\delta(G) \ge \frac{(a+b-2)p+2\alpha(G)+2}{2a+b+m-2}$$

nor

$$\delta(G) \ge \frac{(a+b-1)p+a+b+1}{2a+b+m-1}$$

is substitute for

$$\delta(G) \ge \frac{(a+b-2)p+2\alpha(G)+2}{2a+b+m-2} - 1$$

$$\delta(G) \ge \frac{(a+b-1)p+a+b+1}{2a+b+m-1} - 1.$$

We construct a graph $G = (dK_2) \vee ((2d-1)K_1 \vee K_{\frac{2d(b-m-1)+1}{a+m}})$, where $a, b$ and $d$ be integers with $2 \le a = b-m$, $a+b-1 < 2d < 3a+2b+m-3$ and $\frac{2d(b-m-1)+1}{a+m}$ is an integer. Note that $(2d-1)K_1$ is maximum independent set of $G$. Then $\alpha(G) = 2d-1$ and

$$\begin{aligned} p &= 2d + 2d - 1 + \frac{2d(b-m-1)+1}{a+m} \\ &= 4d - 1 + \frac{2d(b-m-1)+1}{a+m} \\ &= \frac{2d(2a+b+m-1)-(a+m)+1}{a+m} \\ &\ge \frac{(a+b)(2a+b+m-1)-(a+m)+1}{a+m} \\ &= \frac{(a+b-2)(2a+b+m-1)+3a+2b+m-1}{a+m} \\ &\ge \frac{(2a+b+m-1)(a+b-2)+a+b+1}{a+m}. \end{aligned}$$

We have

$$\begin{aligned}\delta(G) &= p - 2d + 1\\ &= \frac{(2a+b+m-1)(p-2d+1)}{2a+b+m-1}\\ &= \frac{(a+b-1)p + (a+m)p - (2a+b+m-1)(2d-1)}{2a+b+m-1}\\ &= \frac{(a+b-1)p + (4d-1)(a+m) + 2d(b-m-1) + 1 - (2a+b+m-1)(2d-1)}{2a+b+m-1}\\ &= \frac{(a+b-1)p + a + b}{2a+b+m-1}.\end{aligned}$$

and

$$\begin{aligned}\delta(G) &= p - 2d + 1\\ &= \frac{(2a+b+m-2)(p-2d+1)}{2a+b+m-2}\\ &= \frac{(a+b-2)p + (a+m)p - (2a+b+m-2)(2d-1)}{2a+b+m-2}\\ &= \frac{(a+b-2)p + (4d-1)(a+m) + 2d(b-m-1) + 1 - (2a+b+m-2)(2d-1)}{2a+b+m-2}\\ &= \frac{(a+b-2)p + 2d + a + b - 1}{2a+b+m-2}\\ &< \frac{(a+b-2)p + 2d + 2d}{2a+b+m-2}\\ &= \frac{(a+b-2)p + 2\alpha(G) + 2}{2a+b+m-2},\end{aligned}$$

$$\begin{aligned}\delta(G) &= \frac{(a+b-2)p + 2d + a + b - 1}{2a+b+m-2}\\ &= \frac{(a+b-2)p + 2d + 3a + 2b + m - 3 - (2a+b+m-2)}{2a+b+m-2}\\ &> \frac{(a+b-2)p + 2d + 2d - (2a+b+m-2)}{2a+b+m-2}\\ &> \frac{(a+b-2)p + 2\alpha(G) + 2}{2a+b+m-2} - 1.\end{aligned}$$

Hence, we obtain

$$\frac{(a+b-1)p + a + b + 1}{2a+b+m-1} - 1 < \delta(G) < \frac{(a+b-1)p + a + b + 1}{2a+b+m-1}$$

and

$$\frac{(a+b-2)p + 2\alpha(G) + 2}{2a+b+m-2} - 1 < \delta(G) < \frac{(a+b-2)p + 2\alpha(G) + 2}{2a+b+m-2}.$$

Let $g$ and $f$ be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) = a$, $f(x) = b$ for any $x \in V(G)$. We choose $I = V((2d-1)K_1)$, and so $I$ is an independent set of $G$. Set $H = G - I$, $S = V(K_{\frac{2d(b-m-1)+1}{a+m}})$, $T = (dK_2)$. Thus, $|S| = \frac{2d(b-m-1)+1}{a+m}$, $|T| = 2d$, $d_{H-S}(T) = 2d$ and $\varepsilon(S) = 2$. Hence, we get

$$\begin{aligned}\gamma_H(S,T) &= d_{H-S}(T) - g(T) + f(S)\\ &= 2d - a\cdot 2d + b\cdot \frac{2d(b-m-1)+1}{a+m}\\ &= 2d - a\cdot 2d + 2d(a-1)+1\\ &= 1 < 2 = \varepsilon(S).\end{aligned}$$

According to Theorem 2.1, $H$ is not a fractional $(g,f)$-covered graph, and so $G$ is not a fractional ID-$(g,f)$-factor-critical covered graph.

*Acknowledgements.* The author is grateful to the anonymous referees for their careful reading and many valuable suggestions.

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