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ISOLATED TOUGHNESS VARIANT AND FRACTIONAL k-FACTOR

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Abstract. Isolated toughness is a crucial parameter considered in network security which characterizes the vulnerability of the network from the perspective of graph topology. $I'(G)$ is the unique variant of isolated toughness which was introduced in 2001. This work investigates the correlation of $I'(G)$ and the existence of fractional factor. It is proved that a graph G with $\delta(G) \geq k$ admits fraction k-factor if $I'(G) > 2k-1$, where $k \geq 2$ is an integer. A counterexample is presented to show the sharpness of $I'(G)$ bound.

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1. Introduction

This work only considers simple and finite graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote $d_G(v)$ and $N_G(v)$ (simply by $d(v)$ and $N(v)$) as the degree and the neighborhood of $v \in V(G)$, respectively. For any $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, and set $G-S = G[V(G) \setminus S]$. Set $\delta(G) = \min_{v \in V(G)}\{d(v)\}$ as the minimum degree of G. The notations and terminologies used but undefined in this paper can be found in Bondy and Mutry [1].

Let k be a positive integer and $h : E(G) \to [0,1]$ be an indicator function defined on the edge set. A *fractional k-factor* is a spanning subgraph induced by $E_h = \{e \in E(G) | h(e) > 0\}$ if $d_G^h(v) = \sum_{v' \in N(v)} h(vv') = k$ for each vertex v. We say graph G admits a fractional factor if such indictor function h exists.

Inspired by the idea of toughness, Yang *et al.* [8] introduced the notion of *isolated toughness* which is formalized as follows: $I(G) = +\infty$ if G is a complete graph; otherwise,

$$I(G) = \min\left\{\frac{|S|}{i(G-S)}\Big| S \subset V(G), i(G-S) \geq 2\right\},$$

where $i(G-S)$ is the number of isolated vertices in $G-S$. A variant of isolated toughness was introduced by Zhang and Liu [10] which is formulated as

$$I'(G) = \min\left\{\frac{|S|}{i(G-S)-1}\Big| S \subset V(G), i(G-S) \geq 2\right\}$$

if G is not a complete graph, and otherwise $I'(G) = +\infty$.

Keywords. graph, isolated toughness, fractional k-factor.

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Due to the theoretical importance and significant application of such parameters in specific fields, the investigation of isolation toughness in the setting of extended fractional factors (e.g., fractional (g, f)-factor, all fractional factors, component factor), and in the setting of fractional deleted graph and fractional critical graph has attracted the attention from scholars. Ma and Liu [7] confirmed that a graph G admits a fractional k-factor if $\delta(G) \geq k$ and $I(G) \geq k$. Gao and Wang [2] determined an $I(G)$ bound for fractional (g, f, n)-critical graphs. Gao *et al.* [3] studied the $I(G)$ condition for a graph which admits the component factor when the given numbers of edges are missing. Gao *et al.* [4] considered the isolated toughness parameter in 5-dimensional space, and computed the expression of detailed space structures. Zhou *et al.* [14] investigated the relationship between isolated toughness and path factors. More results on this topic and other extensions can be referred to [9,11–13].

However, these extant results almost focus on original isolated toughness $I(G)$, and there are few advances on $I'(G)$. Early studies found that $I(G)$ and $I'(G)$ have obvious differences in parameter characteristics, while it is observed that most of the previously confirmed results for $I(G)$ are still open when considering $I'(G)$ variant. For instance, the sharp $I(G)$ bound for a graph with fractional k-factor was completely solved in 2006, and unfortunately, the tight $I'(G)$ condition for the existence of fractional k-factor is open till now. This tragic situation motivates us to do further in-depth research on $I'(G)$.

In this paper, we study the correlation between $I'(G)$ and fractional k-factor. Our main result can be formalized in the following theorem.

Theorem 1. *Let G be a graph and $k \geq 2$ be an integer. If $\delta(G) \geq k$ and $I'(G) > 2k-1$, then G has a fractional k-factor.*

Obviously, $\delta(G) \geq k$ is tight for the existence of fractional k-factor in terms of its definition. The following example reveals the sharpness of $I'(G)$ bound in Theorem 1. Consider $G = (2K_k) \vee K_1$ where " $\vee$ " means a vertex in K_1 adjacent to all vertices in $2K_k$. Thus, we infer

$$I'(G) = 2k - 1.$$

Set $S = V(K_1)$ and $T = V(2K_k)$. We verify

$$k|S| - k|T| + \sum_{x \in T} d_{G-S}(x) = k - k(2k) + 2k(k-1) = -k < 0,$$

which implies that G has no fractional k-factor in view of Lemma 1.

To prove Theorem 1, the following lemma which characterises the necessary and sufficient condition of fractional k-factor is required.

Lemma 1. (Liu and Zhang [5]) *Let $k \geq 1$ be an integer. Then G has a fractional k-factor if and only if*

$$k|S| - k|T| + \sum_{x \in T} d_{G-S}(x) \geq 0$$

holds for any $S \subseteq V(G)$, where $T = \{x \in V(G) - S | d_{G-S}(x) \leq k\}$.

Obviously, for a given subset S of $V(G)$, the subset T in Lemma 1 can be equivalently stated by $T = \{x \in V(G) - S | d_{G-S}(x) \leq k-1\}$. It is worthy to emphasize that Lemma 1 has its equal statement as follows.

Lemma 2. (Liu and Zhang [5]) *Let $k \geq 1$ be an integer. Then G has a fractional k-factor if and only if*

$$k|S| - k|T| + \sum_{x \in T} d_{G-S}(x) \geq 0$$

holds for any disjoint subsets $S, T \subseteq V(G)$.

The following two lemmas illustrate the properties of independent sets and covering sets in the specific conditions, which play a key role in the proof of the main theorem.

Lemma 3. (Liu and Zhang [6]) *Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \dots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that*

$$\sum_{j=1}^{k-1}(k-j)c_j \leq \sum_{j=1}^{k-1}(k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for $j = 1, \dots, k-1$.

The following lemma is obtained by slightly modifying the Lemma 2.2 in [6] according to its proving process.

Lemma 4. (Liu and Zhang [6]) *Let G be a graph and let $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$ and $k \geq 2$. Then there exists an independent set I and the covering set $C = V(H) - I$ of H satisfying*

$$|V(H)| \leq \sum_{i=1}^{k}(k-i+1)\left|I^{(i)}\right| - \frac{\left|I^{(1)}\right|}{2}$$

and

$$|C| \leq \sum_{i=1}^{k}(k-i)\left|I^{(i)}\right| - \frac{\left|I^{(1)}\right|}{2}$$

where $I^{(i)} = \{x \in I, d_H(x) = k-i\}$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k}|I^{(i)}| = |I|$.

2. Proof of main result

If G is complete, the result is directly yielded by means of $\delta(G) \geq k$. In what follows, we always assume that G is not complete. Suppose that G satisfies the conditions of Theorem 1, but has no fractional k-factor. By Lemma 2, there exist disjoint subsets S and T of $V(G)$ satisfying

$$k|S| - k|T| + \sum_{x\in T} d_{G-S}(x) = k|S| + \sum_{x \in T}(d_{G-S}(x) - k) \leq -1. \tag{1}$$

We select S and T such that $|T|$ is minimum. Thus, we immediately get $T \neq \emptyset$, and $d_{G-S}(x) \leq k-1$ for any $x \in T$.

Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_k and let $T_0 = \{x \in V(H') | d_{G-S}(x) = 0\}$. Let H be the subgraph inferred from $H' - T_0$ by deleting those l components isomorphic to K_k. Let S' be a set of vertices that contains exactly $k-1$ vertices in each component of K_k in H'.

If $|V(H)| = 0$, then from (1) we obtain $|S| < |T_0| + l$ (or $|S| \leq |T_0| + l - 1$). We verify $|T_0| + l \geq 1$ due to $|T| \neq 0$. If $|T_0| + l = 1$, then $d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq k$ and $d_{G-S}(x) \geq k - |S| > k-1$, which contradicts to $d_{G-S}(x) \leq k-1$ for any $x \in T$. Hence, we deduce $i(G - S \cup S') \geq |T_0| + l \geq 2$ and

$$\begin{aligned} I'(G) &\leq \frac{|S \cup S'|}{i(G-S-S')-1} \leq \frac{|T_0| + l - 1 + l(k-1)}{|T_0| + l - 1} = 1 + \frac{l(k-1)}{|T_0| + l - 1} \\ &\leq 1 + \frac{(l + |T_0|)(k-1)}{|T_0| + l - 1} = 1 + \frac{(l + |T_0| - 1)(k-1)}{|T_0| + l - 1} + \frac{k-1}{|T_0| + l - 1} \end{aligned}$$

$$= k + \frac{k-1}{|T_0| + l - 1} \leq k + \frac{k-1}{2-1} = 2k - 1.$$

This contradicts with $I'(G) > 2k - 1$. It implies $|V(H)| > 0$.

Let $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(v) = k - 1$ for each vertex $v \in V(H_1)$ and $H_2 = H - H_1$. By means of Lemma 4, H_1 has a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ such that

$$|V(H_1)| \leq \sum_{i=1}^{k} (k - i + 1) \left| I^{(i)} \right| - \frac{\left| I^{(1)} \right|}{2}, \tag{2}$$

and

$$|C_1| \leq \sum_{i=1}^{k} (k - i) \left| I^{(i)} \right| - \frac{\left| I^{(1)} \right|}{2}, \tag{3}$$

where $I^{(i)} = \{v \in I_1, d_{H_1}(v) = k - i\}$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k} |I^{(i)}| = |I_1|$. Let $T_j = \{v \in V(H_2) | d_{G-S}(v) = j\}$ for $1 \leq j \leq k - 1$. Using the definitions of H and H_2, we verify that each component of H_2 has a vertex of degree at most $k - 2$ in $G - S$. According to Lemma 3, H_2 has a maximal independent set I_2 and the covering set $C_2 = V(H_2) - I_2$ such that

$$\sum_{j=1}^{k-1} (k - j) c_j \leq \sum_{j=1}^{k-1} (k - 2)(k - j) i_j, \tag{4}$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for every $j = 1, \ldots, k - 1$. Set $W = V(G) - S - T$ and $U = S \cup S' \cup C_1 \cup (N_G(I_1) \cap W)) \cup C_2 \cup (N_G(I_2) \cap W)$. We yield

$$|U| \leq |S| + l(k - 1) + |C_1| + \sum_{j=1}^{k-1} j i_j + \sum_{i=1}^{k} (i - 1) \left| I^{(i)} \right| \tag{5}$$

and

$$i(G - U) \geq t_0 + l + |I_1| + \sum_{j=1}^{k-1} i_j, \tag{6}$$

where $t_0 = |T_0|$. When $i(G - U) \geq 2$, using the definition of $I'(G)$, we have

$$|U| \geq I'(G) i(G - U) - I'(G). \tag{7}$$

If $i(G - U) = 1$, then $G[T]$ is a clique and $|T| < k$. Let $d_{cl} = \min_{v \in T}\{d_{G-S}(v)\}$ and set $d_{G-S}(v_{cl}) = d_{cl}$. Thus, $|T| - 1 \leq d_{cl} \leq k - 1$. In view of (1), we deduce

$$|S| \leq \frac{k|T| - d_{G-S}(T) - 1}{k} \leq \frac{k|T| - d_{cl}|T| - 1}{k}$$

and

$$d_{cl} = d_{G-S}(v_{cl}) \geq \delta(G) - |S| \geq k - |S| \geq k - \frac{k|T| - d_{cl}|T| - 1}{k}.$$

Therefore,

$$0 \geq k^2 - k|T| + d_{cl}(|T| - k) + 1 \geq k^2 - k|T| + (k - 1)(|T| - k) + 1 = k - |T| + 1 \geq 2,$$

a contradiction. Therefore, (7) always established.

Followed from (5)–(7), we yield

$$|S|+|C_1| \geq \sum_{j=1}^{k-1}(I'(G)-j)i_j + I'(G)(t_0+l) + I'(G)|I_1| - \sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| - l(k-1) - I'(G). \tag{8}$$

In light of $k|T| - d_{G-S}(T) \geq k|S|+1$, we have

$$kt_0 + kl + |V(H_1)| + \sum_{j=1}^{k-1}(k-j)i_j + \sum_{j=1}^{k-1}(k-j)c_j \geq k|S|+1.$$

Combining with (8), we derive

$$\begin{aligned}|V(H_1)| &+ \sum_{j=1}^{k-1}(k-j)c_j + k|C_1| \\ &\geq \sum_{j=1}^{k-1}(kI'(G)-kj-k+j)i_j + (kI'(G)-k)(t_0+l) + kI'(G)|I_1| \\ &- k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| - lk(k-1) - kI'(G) + 1.\end{aligned} \tag{9}$$

In view of (2) and (3), we get

$$|V(H_1)| + k|C_1| \leq \sum_{i=1}^{k}\left(k^2-ki+k-i+1\right)\left|I^{(i)}\right| - \frac{(k+1)\left|I^{(1)}\right|}{2}. \tag{10}$$

By means of (4), (9) and (10), we have

$$\begin{aligned}\sum_{j=1}^{k-1}(k-2)(k-j)i_j &+ \sum_{i=1}^{k}\left(k^2-ki+k-i+1\right)\left|I^{(i)}\right| \\ &\geq \sum_{j=1}^{k-1}(kI'(G)-kj-k+j)i_j + (kI'(G)-k)(t_0+l) + kI'(G)|I_1| \\ &+ \frac{(k+1)\left|I^{(1)}\right|}{2} - k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| - lk(k-1) - kI'(G) + 1.\end{aligned} \tag{11}$$

The following discussion is divided into two cases in terms of whether $t_0 + l$ is zero.

Case 1. $t_0 + l \geq 1$. In this case, by (11) and

$$\begin{aligned}&(kI'(G)-k)(t_0+l) - lk(k-1) - kI'(G) + 1 \\ &\geq (kI'(G)-k)(t_0+l) - (l+t_0)k(k-1) - kI'(G) + 1 \\ &= \left(kI'(G)-k^2\right)(t_0+l) - kI'(G) + 1 \\ &\geq kI'(G) - k^2 - kI'(G) + 1 = -k^2+1,\end{aligned}$$

we have

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j+\sum_{i=1}^{k}\left(k^2-ki+k-i+1\right)\left|I^{(i)}\right| \tag{12}$$
$$\geq \sum_{j=1}^{k-1}(kI'(G)-kj-k+j)i_j+kI'(G)|I_1|+\frac{(k+1)\left|I^{(1)}\right|}{2}-k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right|-k^2+1.$$

In particular, if $t_0+l\geq 2$, then

$$(kI'(G)-k)(t_0+l)-lk(k-1)-kI'(G)+1\geq 2(kI'(G)-k^2)-kI'(G)+1=kI'(G)-2k^2+1$$

and

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j+\sum_{i=1}^{k}\left(k^2-ki+k-i+1\right)\left|I^{(i)}\right| \tag{13}$$
$$\geq \sum_{j=1}^{k-1}(kI'(G)-kj-k+j)i_j+kI'(G)|I_1|+\frac{(k+1)\left|I^{(1)}\right|}{2}-k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right|$$
$$+kI'(G)-2k^2+1. \tag{14}$$

Claim 1. If $t_0+l\geq 1$, then $|I_2|\neq 0$.

Proof. Suppose $|I_2|=0$. Then $|I_1|\neq 0$ by $|V(H)|>0$.
If $t_0+l\geq 2$, then (13) becomes

$$\sum_{i=1}^{k}\left(k^2-i+1\right)\left|I^{(i)}\right|-kI'(G)|I_1|-\frac{(k+1)\left|I^{(1)}\right|}{2}-kI'(G)+2k^2-1\geq 0$$

and thus using $k\geq 2$ and $I'(G)>2k-1$, we deduce

$$\begin{aligned}0&<\sum_{i=1}^{k}\left(k^2-i+1\right)\left|I^{(i)}\right|-\left(2k^2-k\right)|I_1|-\frac{\left|I^{(1)}\right|(k+1)}{2}-k(2k-1)+2k^2-1\\&=\sum_{i=1}^{k}\left(-k^2+k-i+1\right)\left|I^{(i)}\right|-\frac{\left|I^{(1)}\right|(k+1)}{2}+k-1\\&<-k^2+k+k-1<0,\end{aligned}$$

a contradiction.
Now, consider $t_0+l=1$ and (12) becomes

$$\sum_{i=1}^{k}\left(k^2-i+1\right)\left|I^{(i)}\right|-kI'(G)|I_1|-\frac{(k+1)\left|I^{(1)}\right|}{2}+k^2-1\geq 0. \tag{15}$$

If $|I_1|=1$, then $|V(H_1)|\leq k-1$ and we consider the following two circumstances.

- $t_0=1$ and $l=0$. Then $|V(T)|=|V(H_1)|+1$, $k|S|\leq k|T|-d_{G-S}(T)-1=k+|V(H_1)|-1$ and $|S|\leq\frac{2k-2}{k}$. Hence $k\leq\delta(G)\leq 0+|S|\leq\frac{2k-2}{k}$, which contradicts to $k\geq 2$.

– $t_0 = 0$ and $l = 1$. Then $|V(T)| = |V(H_1)| + k$, $k|S| \le k|T| - d_{G-S}(T) - 1 = |V(T)| - 1$ and $|S| \le \frac{2k-2}{k}$ (which implies $|S| \le 1$). Hence

$$I'(G) \le \frac{|S \cup S' \cup N_{G-S}(I_1)|}{i(G - S \cup S' \cup N_{G-S}(I_1)) - 1} = \frac{1 + (k-1) + (k-1)}{2-1} = 2k-1,$$

which contradicts to $I'(G) > 2k - 1$. It implies that $|I_1| \ge 2$ if $t_0 + l = 1$.

In light of (15), $|I_1| \ge 2$, $k \ge 2$ and $I'(G) > 2k-1$, we have

$$\begin{aligned} 0 &\le \sum_{i=1}^{k}\big(k^2 - i + 1\big)\Big|I^{(i)}\Big| - kI'(G)|I_1| - \frac{(k+1)\big|I^{(1)}\big|}{2} + k^2 - 1 \\ &< \sum_{i=1}^{k}\big(k^2 - i + 1\big)\Big|I^{(i)}\Big| - \big(2k^2 - k\big)|I_1| - \frac{\big|I^{(1)}\big|(k+1)}{2} + k^2 - 1 \\ &= \sum_{i=1}^{k}\big(-k^2 + k - i + 1\big)\Big|I^{(i)}\Big| - \frac{\big|I^{(1)}\big|(k+1)}{2} + k^2 - 1 < 0. \end{aligned}$$

The last inequality can be derived by discussing $\big|I^{(1)}\big| = 0$ and $\big|I^{(1)}\big| \ge 1$ respectively. $\square$

Claim 2. If $t_0 + l \ge 1$, then $|I_1| \ne 0$.

Proof. Suppose $|I_1| = 0$. We yield $|I_2| \ne 0$ by $|V(H)| > 0$, and hence $k \ge 3$.

If $t_0 + l \ge 2$, then (13) becomes

$$\begin{aligned} &\sum_{j=1}^{k-1}(k-2)(k-j)i_j \\ &\ge \sum_{j=1}^{k-1}(kI'(G) - kj - k + j)i_j + kI'(G) - 2k^2 + 1 \\ &> \sum_{j=1}^{k-1}(kI'(G) - kj - k + j)i_j - k + 1. \end{aligned}$$

Since

$$\begin{aligned} &(k-2)(k-j) - kI'(G) + kj + k - j \\ &< (k-2)(k-j) - 2k^2 + kj + 2k - j \\ &= -k^2 + j \le -k^2 + k - 1, \end{aligned}$$

we get $-k^2 + 2k - 2 > 0$, contradicting to $k \ge 3$.

Now, consider $t_0 + l = 1$ and (12) becomes

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j \ge \sum_{j=1}^{k-1}(kI'(G) - kj - k + j)i_j - k^2 + 1. \tag{16}$$

Set $d_{\min} = \min\{d_{G-S}(x)|x \in V(H_2)\}$, then $d_{\min} \in \{1, \dots, k-2\}$. Let $z \in V(H_2)$ such that $d_{G-S}(z) = d_{\min}$. If $|I_2| = 1$, then we consider the following two circumstances.

– $t_0 = 1$ and $l = 0$. Then $|V(T)| = |V(H_2)| + 1$, $k|S| \le k|T| - d_{G-S}(T) - 1 \le k + |V(H_2)|(k - d_{\min}) - 1$ and $|S| \le \frac{k+|V(H_2)|(k-d_{\min})-1}{k} \le \frac{k+|I_2|(k-1)(k-d_{\min})-1}{k} \le \frac{k+(k-1)(k-1)-1}{k} = k-1$. Hence $k \le \delta(G) \le 0+|S| \le k-1$, a contradiction.
– $t_0 = 0$ and $l = 1$. Then $|V(T)| = |V(H_2)| + k$, $k|S| \le k|T| - d_{G-S}(T) - 1 \le k + |V(H_2)|(k - d_{\min}) - 1$ and

$$\begin{aligned}|S| &\le \frac{k + |V(H_2)|(k - d_{\min}) - 1}{k} \le \frac{k + (d_{\min} + 1)(k - d_{\min}) - 1}{k} \\ &= -\frac{1}{k}d_{\min}^2 + \left(1 - \frac{1}{k}\right)d_{\min} + 2 - \frac{1}{k}.\end{aligned}$$

Hence, we acquire

$$\begin{aligned}I'(G) &\le \frac{|S \cup S' \cup N_{G-S}(z)|}{i(G - S \cup S' \cup N_{G-S}(z)) - 1} = \frac{d_{\min} - \frac{d_{\min}^2 + d_{\min}}{k} + 2 - \frac{1}{k} + (k-1) + d_{\min}}{2-1} \\ &= -\frac{1}{k}d_{\min}^2 + \left(2 - \frac{1}{k}\right)d_{\min} + k - \frac{1}{k} + 1.\end{aligned}$$

Set

$$\Psi(d_{\min}) = -\frac{1}{k}d_{\min}^2 + \left(2 - \frac{1}{k}\right)d_{\min} + k - \frac{1}{k} + 1.$$

Thus, $\max\{\Psi(d_{\min})\} = \Psi\big(k - \frac{1}{2}\big)$ and actually $\max\{\Psi(d_{\min})\} = \Psi(k-2)$ due to the range of variable $d_{\min}$. When $d_{\min} = k-2$, we infer

$$|S| \le -\frac{1}{k}d_{\min}^2 + \left(1 - \frac{1}{k}\right)d_{\min} + 2 - \frac{1}{k} = -\frac{1}{k}(k-2)^2 + \left(1 - \frac{1}{k}\right)(k-2) + 2 - \frac{1}{k} = 3 - \frac{3}{k}.$$

Due to $k \ge 3$, we acquire $|S| \le 2$, and then

$$I'(G) \le \frac{|S \cup S' \cup N_{G-S}(z)|}{i(G - S \cup S' \cup N_{G-S}(z)) - 1} \le \frac{2 + (k-1) + (k-2)}{2-1} = 2k - 1,$$

which contradicts to $I'(G) > 2k-1$. It's summarized that $|I_2| \ge 2$ if $t_0 + l = 1$.

Hence, we get

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j - \sum_{j=1}^{k-1}(kI'(G) - kj - k + j)i_j < -2k^2 + 2k - 2,$$

which contradicts to (16). □

From Claims 1 and 2, we can see that $|I_1| > 0$ and $|I_2| > 0$. Applying $|I_2| \ge 1$ yields

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j - \sum_{j=1}^{k-1}(kI'(G) - kj - k + j)i_j + k^2 - k + 1 < 0,$$

we infer

$$\sum_{i=1}^{k}\left(k^2 - ki + k - i + 1\right)\left|I^{(i)}\right| > kI'(G)|I_1| + \frac{(k+1)\left|I^{(1)}\right|}{2} - k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| - k + 2$$

or

$$\sum_{i=1}^{k}(k^2 - ki + k - i + 1)\left|I^{(i)}\right| - kI'(G)|I_1| - \frac{(k+1)\left|I^{(1)}\right|}{2} + k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| + k - 2 > 0. \tag{17}$$

In light of (17), $k \geq 2$ and $I'(G) > 2k-1$, we obtain

$$0 < \sum_{i=1}^{k}\left(k^2 - ki + k - i + 1\right)\left|I^{(i)}\right| - (2k^2-k)|I_1| - \frac{(k+1)\left|I^{(1)}\right|}{2} + k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| + k - 2$$
$$= \sum_{i=1}^{k}\left(-k^2 + k - i + 1\right)\left|I^{(i)}\right| - \frac{(k+1)\left|I^{(1)}\right|}{2} + k - 2 < 0,$$

a contradiction.

Case 2. $t_0 + l = 0$. In this case, by (11) we deduce,

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j + \sum_{i=1}^{k}\left(k^2 - ki + k - i + 1\right)\left|I^{(i)}\right| \geq \sum_{j=1}^{k-1}(kI'(G) - kj - k + j)i_j + kI'(G)|I_1| + \frac{(k+1)\left|I^{(1)}\right|}{2} - k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| - kI'(G) + 1. \quad (18)$$

Claim 3. If $t_0 + l = 0$, then $|I_2| \neq 0$.

Proof. Suppose $|I_2| = 0$. Then we infer $|I_1| \neq 0$, $|V(T)| = |V(H_1)|$ and $k|S| \leq k|T| - d_{G-S}(T) - 1 = |T| - 1$. If $|I_1| = 1$, then $|T| \leq k-1$ and $|S| \leq \frac{|T|-1}{k} \leq 1 - \frac{2}{k}$. Thus, $k \leq \delta(G) \leq |S| + (k-1) \leq k - \frac{2}{k}$, a contradiction.

If $|I_1| = 2$, then $|T| \leq 2k$ and $|S| \leq \frac{|T|-1}{k} \leq 2 - \frac{1}{k}$. Thus, from $k \leq \delta(G) \leq |S| + (k-1) \leq k + 1 - \frac{1}{k}$, we verify $\delta(G) = k$ and $|S| = 1$. In this case, $i(G-U) = 2$ where $U = S \cup C_1 \cup (N_G(I_1) \cap W)$, and

$$|U| \leq |S| + |C_1| + \sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| \leq 1 + \sum_{i=1}^{k}(k-i)\left|I^{(i)}\right| - \frac{\left|I^{(1)}\right|}{2} + \sum_{i=1}^{k}(i-1)\left|I^{(i)}\right|$$
$$= 1 + (k-1)|I_1| - \frac{\left|I^{(1)}\right|}{2} = 2k - 1 - \frac{\left|I^{(1)}\right|}{2} \leq 2k-1.$$

Hence,

$$I'(G) \leq \frac{|U|}{i(G-U)-1} \leq 2k-1,$$

which contradicts to $I'(G) > 2k-1$. Thus, we have $|I_1| \geq 3$.

Using (18), we derive

$$\sum_{i=1}^{k}\left(k^2 - ki + k - i + 1\right)\left|I^{(i)}\right| - kI'(G)|I_1| - \frac{(k+1)\left|I^{(1)}\right|}{2} + k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| + kI'(G) - 1 \geq 0. \quad (19)$$

In light of (19), $I'(G) > 2k-1$, $k \geq 2$ and $|I_1| \geq 3$, we get

$$0 \leq \sum_{i=1}^{k}\left(k^2 - i + 1\right)\left|I^{(i)}\right| - kI'(G)|I_1| - \frac{(k+1)\left|I^{(1)}\right|}{2} + kI'(G) - 1$$
$$< \sum_{i=1}^{k}\left(k^2 - i + 1\right)\left|I^{(i)}\right| - \left(2k^2 - k\right)(|I_1| - 1) - \frac{(k+1)\left|I^{(1)}\right|}{2} - 1$$
$$= \sum_{i=1}^{k}\left(-k^2 + k - i + 1\right)\left|I^{(i)}\right| - \frac{(k+1)\left|I^{(1)}\right|}{2} + 2k^2 - k - 1 < 0,$$

a contradiction. ◻

Claim 4. If $t_0 + l = 0$, then $|I_1| \neq 0$.

Proof. Suppose $|I_1| = 0$. Then $|I_2| \neq 0$ using $|V(H)| > 0$, and thus $k \geq 3$. In terms of (18), we infer

$$\begin{aligned}
&\sum_{j=1}^{k-1}(k-2)(k-j)i_j \\
&\geq \sum_{j=1}^{k-1}(kI'(G) - kj - k + j)i_j - kI'(G) + 1 \\
&= \sum_{j=1}^{k-1}(-kj - k + j)i_j + kI'(G)(|I_2| - 1) + 1 \\
&> \sum_{j=1}^{k-1}(-kj - k + j)i_j + \left(2k^2 - k\right)(|I_2| - 1) + 1 \\
&= \sum_{j=1}^{k-1}\left(2k^2 - kj - 2k + j\right)i_j - 2k^2 + k + 1.
\end{aligned}$$

Note that $(k-2)(k-j) - 2k^2 + kj + 2k - j = -k^2 + j \leq -k^2 + k - 1$.

Set $d_{\min}$ and $z \in V(H_2)$ as in Claim 2, thus $d_{\min} \in \{1, \ldots, k-2\}$ and $d_{G-S}(z) = d_{\min}$. If $|I_2| = 1$, then $k|S| \leq k|T| - d_{G-S}(T) - 1 \leq |T|(k - d_{\min}) - 1$ and $|S| \leq \frac{|T|(k-d_{\min})-1}{k} \leq \frac{|I_2|(k-1)(k-d_{\min})-1}{k} = \frac{(k-1)(k-d_{\min})-1}{k}$. Hence $k \leq \delta(G) \leq d_{\min} + |S| \leq d_{\min} + \frac{(k-1)(k-d_{\min})-1}{k} = k - 1 + \frac{d_{\min}}{k} - \frac{1}{k} \leq k - 1 + \frac{k-2}{k} - \frac{1}{k} = k - \frac{3}{k}$, a contradiction.

Hence, we get $|I_2| \geq 2$ and

$$\begin{aligned}
\left(-k^2 + k - 1\right)|I_2| &= \sum_{j=1}^{k-1}\left(-k^2 + k - 1\right)i_j \geq \sum_{j=1}^{k-1}\left(-k^2 + j\right)i_j \\
&= \sum_{j=1}^{k-1}(k-2)(k-j)i_j - \sum_{j=1}^{k-1}(2k^2 - kj - 2k + j)i_j \\
&> -2k^2 + k + 1.
\end{aligned}$$

If $|I_2| \geq 3$, then $-3k^2 + 3k - 3 > -2k^2 + k + 1$, i.e., $-k^2 + 2k - 4 > 0$, a contradiction. Hence, $|I_2| = 2$ and we set d_1 and d_2 as the degrees in $G - S$ of these two vertices in I_2 (assume that $d_1 \leq d_2$ and thus $d_1 \leq k-2$). We get $(-k^2 + d_1) + (-k^2 + d_2) = \sum_{j=1}^{k-1}(-k^2 + j)i_j > -2k^2 + k + 1$. Hence, we infer $d_1 + d_2 \geq k + 2$.

We check that $|S| \leq \frac{(d_1+1)(k-d_1)+(d_2+1)(k-d_2)-1}{k}$, $i(G-U) = 2$ where $U = S \cup C_2 \cup (N_G(I_2) \cap W)$, and

$$\begin{aligned}
|U| &\leq |S| + |C_2| + |N_G(I_2) \cap W| \\
&\leq \frac{(d_1+1)(k-d_1) + (d_2+1)(k-d_2) - 1}{k} + \sum_{j=1}^{k-1} j i_j \\
&= \frac{(d_1+1)(k-d_1) + (d_2+1)(k-d_2) - 1}{k} + d_1 + d_2 \\
&= \left(-\frac{1}{k}d_1^2 + d_1\left(2 - \frac{1}{k}\right)\right) + \left(-\frac{1}{k}d_2^2 + d_2\left(2 - \frac{1}{k}\right)\right) + 2 - \frac{1}{k}\cdot
\end{aligned}$$

Set

$$\Upsilon(d_1, d_2) = \left(-\frac{1}{k}d_1^2 + d_1\left(2 - \frac{1}{k}\right)\right) + \left(-\frac{1}{k}d_2^2 + d_2\left(2 - \frac{1}{k}\right)\right) + 2 - \frac{1}{k}\cdot$$

Clearly, $\max \Upsilon(d_1, d_2) = \Upsilon(k-2, k-1)$. When $(d_1, d_2) = (k-2, k-1)$, we get $|S| \le \frac{2(k-1)+k-1}{k} = 3 - \frac{3}{k}$. Thus, $|S| \le 2$ due to $k \ge 3$.

Therefore, $|U| \le |S| + |C_2| + |N_G(I_2) \cap W| \le 2 + (k-2) + (k-1) = 2k-1$ and

$$2k - 1 < I'(G) \le \frac{|U|}{i(G-U)-1} \le 2k-1,$$

which leads to a contradiction. □

From Claims 3 and 4, we can see that $|I_1| \ge 1$ and $|I_2| \ge 1$.

Claim 5. $|I_1| + |I_2| \ge 3$.

Proof. Otherwise, we have $|I_1| + |I_2| = 2$, i.e., $|I_1| = |I_2| = 1$. We get $|T| \le (k-1)+(k-1) = 2k-2$ and $k|S| \le k|T| - d_{G-S}(T) - 1 \le (k-1) + (d_3+1)(k-d_3) - 1$, where $I_2 = \{z'\}$ and $d_3 = d_{G-S}(z')$ ($d_3 \in \{1, \dots, k-2\}$). Moreover, $|S| \le \frac{k+(d_3+1)(k-d_3)-2}{k}$ and

$$\begin{aligned} k \le \delta(G) \le |S| + d_3 &\le \frac{k + (d_3+1)(k-d_3) - 2}{k} + d_3 \\ &= -\frac{1}{k}d_3^2 + \left(2 - \frac{1}{k}\right)d_3 + 2 - \frac{2}{k} \\ &\le -\frac{1}{k}(k-2)^2 + \left(2 - \frac{1}{k}\right)(k-2) + 2 - \frac{2}{k} = k + 1 - \frac{4}{k}, \end{aligned}$$

we have $k \ge 4$.

We acquire $i(G-U) = 2$ where $U = S \cup C_1 \cup (N_G(I_1) \cap W) \cup C_2 \cup (N_G(I_2) \cap W)$, and

$$\begin{aligned} |U| &\le |S| + |C_1| + \sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| + \sum_{j=1}^{k-1} j i_j \\ &\le \frac{k + (d_3+1)(k-d_3) - 2}{k} + (k-1) + d_3 = -\frac{1}{k}d_3^2 + \left(2 - \frac{1}{k}\right)d_3 + k + 1 - \frac{2}{k}. \end{aligned}$$

Set

$$\Xi(d_3) = -\frac{1}{k}d_3^2 + \left(2 - \frac{1}{k}\right)d_3 + k + 1 - \frac{2}{k}.$$

Then $\max\{\Xi(d_3)\} = \Xi\left(k - \frac{1}{2}\right)$ and actually $\max\{\Xi(d_3)\} = \Xi(k-2)$ due to the range d_3. When $d_3 = k-2$, we have

$$|S| \le \frac{k + (d_3+1)(k-d_3) - 2}{k} = \frac{k + 2(k-1) - 2}{k} = 3 - \frac{4}{k},$$

and thus $|S| \le 2$ due to $k \ge 4$.

Therefore,

$$|U| \le |S| + |C_1| + \sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| + \sum_{j=1}^{k-1} j i_j \le 2 + (k-1) + (k-2) = 2k-1$$

and

$$I'(G) \leq \frac{|U|}{i(G-U)-1} \leq 2k-1,$$

which contradicts to $I'(G) > 2k-1$. □

The final discussion is divided into two subcases.

Case 2.1. $|I_2| \geq 2$.

In terms of $|I_2| \geq 2$, we deduce

$$\sum_{j=1}^{k-1}(kI'(G)-kj-k+j)i_j > \sum_{j=1}^{k-1}(k-2)(k-j)i_j + 2k^2 - 2k + 2. \tag{20}$$

Combining (18) and (20), one has

$$\begin{aligned}&\sum_{i=1}^{k}\left(k^2-ki+k-i+1\right)\left|I^{(i)}\right| \\ &> kI'(G)|I_1| + \frac{(k+1)\left|I^{(1)}\right|}{2} - k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| - kI'(G) + 2k^2 - 2k + 3.\end{aligned} \tag{21}$$

In light of (21), $|I_1| \geq 1$ and $I'(G) > 2k-1$, we have

$$\begin{aligned}0 &< \sum_{i=1}^{k}\left(k^2-i+1\right)\left|I^{(i)}\right| - (k+1)\frac{\left|I^{(1)}\right|}{2} - kI'(G)(|I_1|-1) - 2k^2 + 2k - 3 \\ &< \sum_{i=1}^{k}\left(k^2-i+1\right)\left|I^{(i)}\right| - (k+1)\frac{\left|I^{(1)}\right|}{2} - \left(2k^2-k\right)(|I_1|-1) - 2k^2 + 2k - 3 \\ &= \sum_{i=1}^{k}\left(-k^2+k-i+1\right)\left|I^{(i)}\right| - (k+1)\frac{\left|I^{(1)}\right|}{2} + k - 3 < 0,\end{aligned}$$

a contradiction.

Case 2.2. $|I_1| \geq 2$.

In terms of $|I_2| \geq 1$, we deduce

$$\sum_{j=1}^{k-1}(kI'(G)-kj-k+j)i_j > \sum_{j=1}^{k-1}(k-2)(k-j)i_j + k^2 - k + 1.$$

Combining the above inequality into (18), one gets

$$\begin{aligned}&\sum_{i=1}^{k}\left(k^2-ki+k-i+1\right)\left|I^{(i)}\right| \\ &> kI'(G)|I_1| + \frac{(k+1)\left|I^{(1)}\right|}{2} - k\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| - kI'(G) + k^2 - k + 2.\end{aligned} \tag{22}$$

In light of (22), $|I_1| \geq 2$ and $I'(G) > 2k-1$, we derive

$$
\begin{aligned}
0 &< \sum_{i=1}^{k} \left(k^2 - i + 1\right)\left|I^{(i)}\right| - (k+1)\frac{\left|I^{(1)}\right|}{2} - kI'(G)(|I_1| - 1) - k^2 + k - 2 \\
&< \sum_{i=1}^{k} \left(k^2 - i + 1\right)\left|I^{(i)}\right| - (k+1)\frac{\left|I^{(1)}\right|}{2} - \left(2k^2 - k\right)(|I_1| - 1) - k^2 + k - 2 \\
&= \sum_{i=1}^{k} \left(-k^2 + k - i + 1\right)\left|I^{(i)}\right| - (k+1)\frac{\left|I^{(1)}\right|}{2} + k^2 - 2 < 0,
\end{aligned}
$$

a contradiction

Therefore, we complete the proof of the desired result. □

3. Conclusion and discussion

In this contribution, we obtain the tight $I'(G)$ bound for a graph to admit fractional k-factor. Since isolation toughness plays a key role in network security and the fractional factor is a characterization of fractional flow in data transmission networks, we believe that the theoretical conclusion determined in our paper has certain guiding significance for the practical application of network engineering. Furthermore, Theorem 1 has potential to be generalized in other fractional factor settings, as well as fractional deleted graph and fractional critical graph frameworks. Therefore, we propose the following open problems (the explanation of these concepts can be found in the relevant literatures).

Problem 1. What is the tight $I'(G)$ bound for fractional (g, f, n)-critical graphs?

Problem 2. What is the tight $I'(G)$ bound for fractional (g, f, m)-deleted graphs?

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References

[1] J.A. Bondy and U.S.R. Mutry, Graph Theory. Springer, Berlin (2008).
[2] W. Gao and W. Wang, New isolated toughness condition for fractional (g, f, n)-critical graphs. *Colloq. Math.* **147** (2017) 55-66.
[3] W. Gao, Y. Chen and Y. Wang, Network vulnerability parameter and results on two surfaces, *Int. J. Intell. Syst.* **36** (2021) 4392–4414.
[4] W. Gao, Y. Chen and Y. Zhang, Viewing the network parameters and $\mathcal{H}$-factors from the perspective of geometry. *Int. J. Intell. Syst.* **37** (2022) 6686–6728.
[5] G. Liu and L. Zhang, Fractional (g, f)-factors of graphs. *Acta Math. Sci.* **21** (2001) 541–545.
[6] G. Liu and L. Zhang, Toughness and the existence of fractional k-factors of graphs. *Discrete Math.* **308** (2008) 1741–1748.
[7] Y. Ma and G. Liu, Fractional factors and isolated toughness of graphs. *Math. Appl.* **19** (2006) 188–194.
[8] J. Yang, Y. Ma and G. Liu, Fractional (g, f)-factors in graphs. *Appl. Math. J. Chinese Univ. Ser. A* **16** (2001) 385–390.
[9] S. Zhou, Remarks on restricted fractional (g, f)-factors in graphs. *Discrete Appl. Math.* https://doi.org/10.1016/j.dam.2022.07.020.
[10] L. Zhang and G. Liu, Fractional k-factor of graphs. *J. Syst. Sci. Math. Sci.* **21** (2001) 88–92.
[11] S. Zhou, H. Liu and Y. Xu, A note on fractional ID-$[a, b]$-factor-critical covered graphs. *Discrete Appl. Math.* **319** (2022) 511–516.
[12] S. Zhou, J. Wu and Q. Bian, On path-factor critical deleted (or covered) graphs. *Aequ. Math.* **96** (2022) 795–802.
[13] S. Zhou, J. Wu and H. Liu, Independence number and connectivity for fractional (a, b, k)- critical covered graphs. *RAIRO:OR* **56** (2022) 2535–2542.
[14] S. Zhou, J. Wu and Y. Xu, Toughness, isolated toughness and path factors in graphs. *Bull. Aust. Math. Soc.* **106** (2022) 195–202.