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ON 2-MATCHING COVERED GRAPHS AND 2-MATCHING DELETED GRAPHS

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Abstract. For a family of connected graphs $\mathcal{A}$, a spanning subgraph H of a graph G is called an $\mathcal{A}$-factor of G if each component of H is isomorphic to some graph in $\mathcal{A}$. A graph G has a perfect 2-matching if G has a spanning subgraph H such that each component of H is either an edge or a cycle, *i.e.*, H is a $\{P_2, C_i | i \geq 3\}$-factor of G. A graph G is said to be 2-matching covered if, for every edge $e \in E(G)$, there is a perfect 2-matching M_e of G such that e belongs to M_e. A graph G is called a 2-matching deleted graph if, for every edge $e \in E(G)$, $G - e$ possesses a perfect 2-matching. In this paper, we first obtain respective new characterizations for 2-matching covered graphs in bipartite and non-bipartite graphs by new proof technologies, distinct from Hetyei's or Berge's classical results. Secondly, we give a necessary and sufficient condition for a graph to be a 2-matching deleted graph. Thirdly, we we prove that planar graphs with minimum degree at least 4 and $K_{1,r}$-free graphs ($r \geq 3$) with minimum degree at least $r + 1$ are 2-matching deleted graphs, respectively.

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1. Introduction

All graphs in this paper are finite and simple. We refer to [5] for notation and terminologies not defined here. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, we use $d_G(v)$ and $N_G(v)$ to denote the degree of v and the set of vertices adjacent to v in G, respectively. For $S \subseteq V(G)$, we write $N_G(S) = \cup_{v \in S} N_G(v)$. We use $\delta(G)$ to denote the minimum degree of a graph G. We use $\omega(G)$, $i(G)$ to denote the number of components and isolated vertices of a graph G, respectively.

For a connected graph G, its *toughness*, denoted by $\tau(G)$, was first introduced by Chvátal [6] as follows. If G is complete, then $\tau(G) = +\infty$; otherwise,

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G), \omega(G-S) \geq 2\right\}.$$

The *binding number* is introduced by Woodall [19] and defined as

$$\mathrm{bind}(G) = \min\left\{\frac{|N_G(S)|}{|S|} : \emptyset \neq S \subseteq V(G), N_G(S) \neq V(G)\right\}.$$

Keywords. Graph theory, perfect 2-matching, $\{P_2, C_i | i \geq 3\}$-factor, 2-matching covered graphs, 2-matching deleted graphs.

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The *isolated toughness*, denoted by $I_t(G)$, was first introduced by Yang, Ma and Liu [20] as follows. If G is complete, then $I_t(G) = +\infty$; otherwise,

$$I_t(G) = \min\left\{\frac{|S|}{i(G-S)} : S \subseteq V(G), i(G-S) \geq 2\right\}.$$

For $X \subseteq V(G)$, let $G[X]$ be the subgraph of G induced by X, and define $G - X := G[V(G) - X]$. For convenience, we use $G - x$ to denote the graph $G - \{x\}$. Let $K_{1,r}$ denote the complete bipartite with partite sets of size one and r. For an integer $r \geq 3$, a graph G is said to be $K_{1,r}$-free if G does not contain an induced subgraph isomorphic to $K_{1,r}$.

Let $\mathcal{A}$ be a family of connected graphs. If G has a spanning subgraph H such that every component of H is isomorphic to some graph in $\mathcal{A}$, then H is said to be an $\mathcal{A}$-factor of G. A graph G has a perfect 2-matching if G has a spanning subgraph H such that each component of H is either an edge or a cycle, *i.e.*, H is a $\{P_2, C_i | i \geq 3\}$-factor of G. A graph G is said to be 2-matching covered if there is a perfect 2-matching of G including any given edge $e \in E(G)$. A graph G is called a 2-matching deleted graph if G possesses a perfect 2-matching excluding any given edge $e \in E(G)$.

A spanning subgraph H of graph G is called 1-factor (perfect matching) if $d_H(x) = 1$ holds for any $x \in V(G)$. Since Tutte proposed the well known Tutte 1-factor theorem [17], there are many results on graph factors [2, 7–9, 11–13, 18] and path-factors in claw-free graphs and cubic graphs [3, 10, 14, 15]. More results on graph factors can be found in the survey papers and books in [1, 16, 21].

For matchings in bipartite graphs, König (1931) and Hall (1935) obtained the so-called König-Hall Theorem (sometimes, known as Hall's Theorem), respectively.

Theorem 1.1. (König-Hall [5]) *Let $G = (X, Y)$ be a connected bipartite graph such that $|X| = |Y|$. Then G has a perfect matching if and only if $|N_G(S)| \geq |S|$ for any subset $S \subseteq X$.*

In 1953, Tutte proved the following characterization for the existence of perfect 2-matchings in a graph.

Theorem 1.2. (Tutte [17]) *A graph G has a perfect 2-matching if and only if $i(G - S) \leq |S|$ for any subset $S \subseteq V(G)$.*

The equivalence as following is due mostly to Hetyei (see also Akiyama and Kano [1]).

Theorem 1.3. (Hetyei [1]) *If G is a connected bipartite graph with partition (U, W), then each edge of G is contained in a 1-factor if and only if $|U| = |W|$ and $|N_G(X)| > |X|$ for any non-empty proper subset $X \subseteq U$.*

A graph G is 2-matching covered if and only if G is "regularizable", where a graph is regularizable if it can be transformed into a regular multigraph by giving each edge some positive multiplicity. Regularizable graphs were introduced and studied by Berge.

Theorem 1.4. (Berge [4]) *For a connected graph G that is not a bipartite graph with partite sets of equal size, the following conditions are equivalent:*

(a) G is regularizable,

(b) for each edge e of G, there exists a perfect 2-matching of G covering e,

(c) for every non-empty independent set X of vertices, $|N_G(X)| > |X|$.

2. 2-matching covered graph

Lemma 2.1. *Let G be a graph such that $V(G) = X \cup Y, X \cap Y = \emptyset$ and $|X| = |Y|$. If Y is an independent set in G, then G has a 1-factor if and only if $i(G - S) \leq |S|$ for any $S \subseteq X$.*

Proof. Necessity: Let M be a 1-factor of G, then M is a perfect 2-matching of G obviously. It follows from Theorem 1.2 that $i(G-S) \leq |S|$ for any subset $S \subseteq V(G)$. Hence, $i(G-S) \leq |S|$ holds for any $S \subseteq X$.

Sufficiency: Choose $S := \emptyset$, then $i(G) = i(G-S) \leq |S| = 0$. Let $G' = G - E(X)$, then there is no isolated vertex in G'. Otherwise, there exists a isolated vertex $u \in V(G')$ such that $u \in X$. By choosing $S := X \setminus \{u\}$, it follows that $i(G-S) = |Y \cup \{u\}| = |Y| + 1 = |X| + 1 > |S|$, a contradiction. On the one hand, it is obviously $I(G-S) \subseteq I(G'-S)$. On the other hand, all the isolated vertices of $G'-S$ are in Y since $i(G') = i(G) = 0$. Note that every edge in $E(X)$ is not adjacent to any vertex in Y. This together with $I(G'-S) \subseteq Y$ implies that $I(G'-S) \subseteq I(G-S)$. Therefore, $i(G'-S) = i(G-S) \leq |S|$ for any $S \subseteq X$. It follows immediately that $|N'_G(T)| \geq |T|$ for any $T \subseteq Y$ since $N'_G(T) \subseteq X$ and $|T| \leq i(G - N'_G(T)) \leq |N'_G(T)|$. By Theorem 1.1, G' has a 1-factor. Hence, G has a 1-factor. □

Theorem 2.2. *(i) A connected bipartite graph $G = (X, Y)$ is 2-matching covered if and only if $|X| = |Y|$ and $i(G-S) \leq |S| - 1$ for any non-empty proper subset $S \subseteq X$; (ii) A connected non-bipartite graph G is 2 matching covered if and only if $i(G-S) \leq |S| - 1$ for any non-empty proper subset $S \subseteq V(G)$.*

Proof. (i) *Necessity:* Let $G = (X, Y)$ be a connected 2-matching covered bipartite graph. Then, by Theorem 1.3, $|X| = |Y|$ and $|N_G(T)| \geq |T| + 1$ for any non-empty proper subset $T \subseteq Y$. For any non-empty proper subset $S \subseteq X$, every isolated vertex of $G-S$ belongs to Y, denoted by $T = I(G-S) \subseteq Y$. On the one hand, $|N_G(T)| \geq |T| + 1 = i(G-S) + 1$. On the other hand, $|N_G(T)| \leq |S|$ since $T = I(G-S)$. Hence, $i(G-S) \leq |N_G(T)| - 1 \leq |S| - 1$ for any non-empty proper subset $S \subseteq X$.

Sufficiency: Let $G = (X, Y)$ be a connected bipartite graph with $|X| = |Y|$ such that $i(G-S) \leq |S|-1$ for any non-empty proper subset $S \subseteq X$. Then we argue that $|N_G(T)| \geq |T|+1$ for any non-empty proper subset $T \subseteq Y$. Otherwise, there exists proper subset $\emptyset \neq T \subseteq Y$ such that $|N_G(T)| \leq |T|$. Note that $|N_G(T)| \neq \emptyset$ since G is connected. Choose $S := |N_G(T)|$. As $|N_G(T)| \leq |T| < |Y| = |X|$, we have that $N_G(T) \neq X$ and thus $\emptyset \neq S \subseteq X$. It follows that $i(G-S) \leq |S| - 1$. On the other hand, $i(G-S) = i(G - N_G(T)) \geq |T| \geq |N_G(T)| = |S|$, a contradiction. Hence, $|N_G(T)| \geq |T| + 1$ for any non-empty proper subset $T \subseteq Y$. It follows from Theorem 1.3 that G is a 2-matching covered graph.

(ii) *Necessity:* By way of contradiction, assume that there exists non-empty proper subset $S \subseteq V(G)$ such that $i(G-S) \geq |S|$. Since G is a 2-matching covered graph, G has a perfect 2-matching, and thus $i(G-S') \leq |S'|$ for any $S' \subseteq V(G)$. This together with $i(G-S) \geq |S|$ implies that $i(G-S) = |S| > 0$. As G is a nonbipartite graph, $E(S) \neq \emptyset$ or there exists a nontrivial component of $G-S$.

Case 1. $E(S) \neq \emptyset$.

Suppose $e = xy \in E(S)$ such that $x, y \in S$. There must exist a perfect 2-matching F covering e since G is a 2-matching covered graph. Denote the component of F containing e by C. Note that every component of F is either an edge or a cycle by the definition of perfect 2-matchings.

- If C is an edge, then $G' = G - \{x, y\}$ has a perfect 2-matching $F - \{x, y\}$. Choose $S' := S \setminus \{x, y\}$, then $i(G'-S') = i(G-S) = |S| = |S'| + 2 > |S'|$. By Theorem 1.2, G' has no perfect 2-matching, a contradiction.
- If C is a cycle and $|C| \geq 3$, then $G'' = G - V(C)$ has a perfect 2-matching $F - V(C)$. As $e \in E(S)$, we have that $|S \cap V(C)| > |I(G-S) \cap V(C)|$. Choose $S'' := S - V(C)$, then

$$\begin{aligned} i(G''-S'') &\geq |I(G-S) - I(G-S) \cap V(C)| \\ &= i(G-S) - |I(G-S) \cap V(C)| \\ &> i(G-S) - |S \cap V(C)| \\ &\geq |S| - |S \cap V(C)| \\ &= |S''|. \end{aligned}$$

By Theorem 1.2, G'' has no perfect 2-matching, a contradiction.

Case 2. $E(S) = \emptyset$.

In this case, $G - S$ has a nontrivial component D, and there exists an edge $e' = uv$ connecting D and S such that $u \in D, v \in S$. Then G has a perfect 2-matching covering e', denoted by F', since G is a 2-matching covered graph. Denote the component of F' containing e' by C'.

– If C' is an edge, then $G_1 = G - \{u, v\}$ has a perfect 2-matching $F' - \{u, v\}$. Choose $S_1 := S \setminus \{u\}$, then $i(G_1 - S_1) \geq i(G - S) \geq |S| > |S_1|$. By Theorem 1.2, G_1 has no perfect 2-matching, a contradiction.
– If C' is a cycle and $|C'| \geq 3$, then $G_2 = G - V(C)$ has a perfect 2-matching $F' - V(C')$. As C' is a cycle containing e', we have that $|S \cap V(C')| > |I(G - S) \cap V(C')|$. Choose $S_2 := S - V(C')$, then

$$\begin{aligned} i(G_2 - S_2) &\geq |I(G - S) - I(G - S) \cap V(C')| \\ &= i(G - S) - |I(G - S) \cap V(C')| \\ &> i(G - S) - |S \cap V(C')| \\ &\geq |S| - |S \cap V(C')| \\ &= |S_2|. \end{aligned}$$

By Theorem 1.2, G_2 has no perfect 2-matching, a contradiction.

Sufficiency: For any edge $e = xy$, $G^* := G - \{x, y\}$ has at most one isolated vertex since $i(G^*) = i(G - \{x, y\}) \leq |\{x, y\}| - 1 = 1$.

Case 1. $i(G^*) = 1$.

Let u be the isolated vertex of G^*, and $C_1, C_2, \ldots, C_k$ be the other connected components of G^*. We first argue that C_i has a perfect 2-matching F_i for any $1 \leq i \leq k$. Otherwise, by Theorem 1.2, there exists $S_i \subseteq V(C_i)$ such that $i(C_i - S_i) \geq |S_i| + 1$. Choose $S := S_i \cup \{x, y\}$, then S is a non-empty proper subset of G, and $i(G - S) = |\{u\}| + i(C_i - S_i) \geq 1 + (|S_i| + 1) = |S|$, a contradiction. It is easy to find that there exists no pendant vertex in G since $i(G - S) \leq |S| - 1$ for any proper subset $\emptyset \neq S \subseteq V(G)$. Hence, $u \in N_G(x) \cap N_G(y)$, *i.e.*, $xuyx$ is a cycle in G. Then, $xuyx \cup (\bigcup_{i=1}^{k} F_i)$ is a perfect 2-matching of G containing e, *i.e.*, G is a 2-matching covered graph.

Case 2. $i(G^*) = 0$.

Let $C_1, C_2, \ldots, C_m$ be the connected components of G^*, where $m \geq 1$. If every C_i has a perfect 2-matching F_i for $1 \leq i \leq m$, then $\{xy, F_1, F_2, \ldots, F_m\}$ is a perfect 2-matching of G containing e, *i.e.*, G is a 2-matching covered graph. If there exist $C_i, C_j (1 \leq i \neq j \leq m)$ such that both C_i and C_j has no perfect 2-matching, then, by Theorem 1.2, there exist $S_i \subseteq V(C_i), S_j \subseteq V(C_j)$ respectively such that $i(C_i - S_i) \geq |S_i| + 1$ and $i(C_j - S_j) \geq |S_j| + 1$. Choose $S := S_i \cup S_j \cup \{x, y\}$ which is a non-empty proper subset of G, then $i(G - S) = i(C_i - S_i) + i(C_j - S_j) \geq |S_i| + 1 + |S_j| + 1 = |S|$, a contradiction. Thus, there is exactly one element of $\{C_1, C_2, \ldots, C_m\}$ which has no perfect 2-matching. Without of generality, assume C_1 has no perfect 2-matching and every C_t has a perfect 2-matching F_t for $2 \leq t \leq m$. Then it is sufficient to show that

$$C_1' := G[V(C_1) \cup \{x, y\}] \text{ has a perfect } 2-\text{matching covering } e. \tag{1}$$

On the one hand, since C_1 has no perfect 2-matching, by Theorem 1.2, there exists $S' \subseteq V(C_1)$ such that $i(C_1 - S') \geq |S'| + 1$. On the other hand, if $i(C_1 - S') \geq |S'| + 2$, then $S := S' \cup \{x, y\}$ is a non-empty proper subset of G and $i(G - S) = i(C_1 - S') \geq |S'| + 2 = |S|$, a contradiction. Hence, we have that $i(C_1 - S') = |S'| + 1$. Note that S' is a non-empty set since $i(G^*) = 0$. We assume that S' is a minimal barrier set of $V(C_1)$, *i.e.*, $i(C_1 - S'') \leq |S''|$ holds for any proper subset $S'' \subseteq S'$.

Let $W := \{x_1, x_2, \ldots, x_d\}$ be the set of isolated vertices of $C_1 - S'$, where $d \geq 2$. We argue that

$$N_G(x) \cap W \neq \emptyset. \tag{2}$$

Otherwise, $i(G-(S'\cup\{y\}))\geq i(C_1-S')=|S'|+1=|S'\cup\{y\}|$, a contradiction. Similarly, we can obtain that

$$N_G(y)\cap W\neq\emptyset. \tag{3}$$

Moreover, we also argue that every nontrivial component C_1^j of $i(C_1-S')$ has a perfect 2-matching F_1^j ($j=1,2,\ldots,p$). Otherwise, suppose C_1^1 has no perfect 2-matching, then there exists $S^1\subseteq V(C_1^1)$ such that $i(C_1^1-S^1)\geq|S^1|+1$ by Theorem 1.2. Choose $S:=S^1\cup S'\cup\{x,y\}$, then S is a non-empty proper subset of G and $i(G-S)=i(C_1-S')+i(C_1^1-S^1)\geq|S'|+1+|S^1|+1=|S^1\cup S'\cup\{x,y\}|=|S|$, a contradiction.

Claim 2.3. *$\overline{G}:=G[S'\cup\{x,y\}\cup W]$ has a perfect 2-matching containing e.*

Proof. We first argue that for any $1\leq i\leq d$, $G_i:=\overline{G}-\{x,y,x_i\}$ has a 1-factor F_i'. Suppose there is no 1-factor in G_i, then by Lemma 2.1, there exists $S''\subseteq S'$ such that $i(G_i-S'')\geq|S''|+1$. On the one hand, by the arguments similar to Lemma 2.1, we have that $I(G_i-S'')\subseteq I(C_1-S'')$, and thus $i(C_1-S'')\geq i(G_i-S'')\geq|S''|+1$. On the other hand, it is obviously S'' is a proper subset of S', then $i(C_1-S'')\leq|S''|$ since S' is a minimal barrier set of $V(C_1)$, a contradiction. According to (2) and (3), we will distinguish two cases below to show that $\overline{G}$ has perfect 2-matchings containing e.

- If there exists $u\in N_G(x)\cap N_G(y)\cap W$, then without of generality, assume $u=x_1$. Since $G_1:=\overline{G}-\{x,y,x_1\}$ has a 1-factor F_1', $xx_1yx\cup F_1'$ is a perfect 2-matching of $\overline{G}$ containing e.
- If $N_G(x)\cap N_G(y)\cap W=\emptyset$, then we assume that $x_1\in N_G(x), x_d\in N_G(y)$. By the arguments similar to Lemma 2.1, both F_1' and F_d' has no edge in $E(G[S'])$. Note that, due to structural properties of 1-factors, there is an alternating path P from x_1 to x_d whose edges are alternately in $E(F_1')$ and $E(F_d')$. Then $F_1'':=F_1'-V(P)$ or $F_d''=F_d'-V(P)$ is a 1-factor of $\overline{G}-\{x,y\}-V(P)$. Thus, $xPyx\cup F_1''$ or $xPyx\cup F_d''$ is a perfect 2-matching of $\overline{G}$ containing e.

Hence, Claim 2.3 is true. □

Due to Claim 2.3, let $\overline{F}$ be a perfect 2-matching of $\overline{G}$ containing e. Then $\overline{F}\cup(\bigcup_{i=1}^{p}F_1^i)$ is a perfect 2-matching of C_1' containing e, *i.e.*, the argument (1) holds. Thus, $(\bigcup_{j=2}^{m}F_j)\cup\overline{F}\cup(\bigcup_{i=1}^{p}F_1^i)$ is a perfect 2-matching of G containing e, *i.e.*, G is a 2-matching covered graph. □

3. 2-MATCHING DELETED GRAPH

Theorem 3.1. *Let G be a connected graph. Then G is a 2-matching deleted graph if and only if $i(G-S)\leq|S|-\varepsilon(S)$ for all $S\subseteq V(G)$, where $\varepsilon(S)$ is defined by*

$$\varepsilon(S)=\begin{cases}2 & \textit{if there exists a component of } G-S \textit{ containing exactly two vertices};\\ 1 & \textit{if there exists a component } C \textit{ of } G-S \textit{ with pendant edges and } |V(C)|\geq 3;\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. Necessity: Let G be a 2-matching deleted graph. Obviously, G has a perfect 2-matching. Then, by Theorem 1.2, $i(G-S)\leq|S|$ for any $S\subseteq V(G)$. If $i(G-S)\leq|S|-2$, then $i(G-S)\leq|S|-\varepsilon(S)$ by the definition of $\varepsilon(S)$; If $i(G-S)=|S|-1$, then we argue that $\varepsilon(S)\leq 1$. Otherwise, $\varepsilon(S)=2$, and $G-S$ has a component with exactly one edge, denoted by e. It follows that $i(G-e-S)=i(G-S)+2=|S|+1>|S|$, and thus $G-e$ has no perfect 2-matching, a contradiction. Hence, $\varepsilon(S)\leq 1$ and $i(G-S)=|S|-1\leq|S|-\varepsilon(S)$; Now, we may assume that $i(G-S)=|S|$. We argue that $\varepsilon(S)=0$, otherwise $\varepsilon(S)=1,2$, then $G-S$ has a component with pendant edges. It is easy to find that $i(G-e-S)\geq i(G-S)+1=|S|+1$, and thus $G-e$ has no perfect 2-matching, a contradiction. Hence, $\varepsilon(S)=0$ and $i(G-S)=|S|\leq|S|-\varepsilon(S)$.

Sufficiency: For any given $S\subseteq V(G)$, $i(G-S)\leq|S|-\varepsilon(S)$. Now, it suffices to show that $i(G-e-S)\leq|S|$ for any edge $e\in E(G)$ by Theorem 1.2. If e belongs to some component of $G-S$ containing exactly two vertices, then $\varepsilon(S)=2$, and thus $i(G-e-S)=i(G-S)+2\leq|S|-\varepsilon(S)+2=|S|$; If e is a pendant edge belongs to a component C of $G-S$ such that $|C| \geq 3$, then $\varepsilon(S) \geq 1$, and thus $i(G-e-S) = i(G-S)+1 \leq |S|-\varepsilon(S)+1 \leq |S|$; Otherwise, $i(G-e-S) = i(G-S)$, and thus $i(G-e-S) = i(G-S) \leq |S| - \varepsilon(S) \leq |S|$ by the definition of $\varepsilon(S)$. Therefore, $i(G-e-S) \leq |S|$ holds for any $S \subseteq V(G)$ and $e \in E(G)$. This completes the proof of Theorem 3.1. □

Corollary 3.2. *Let G be a connected graph of order $n \geq 3$. Then G is a 2-matching deleted graph if one of the following statements holds: (i) $\tau(G) > 1$; (ii) $bind(G) > 3/2$; (iii) $I_t(G) > 2$.*

Proof. Suppose, to the contrary, that G is not a 2-matching deleted graph. By Theorem 3.1, there exists $S \subseteq V(G)$ such that $i(G-S) > |S| - \varepsilon(S)$. Due to the integrality, $i(G-S) \geq |S| - \varepsilon(S) + 1$.

(i) If G has a pendant edge xy such that $d_G(y) = 1$, then $\tau(G) \leq \frac{|\{x\}|}{\omega(G-x)} \leq \frac{1}{2}$, a contradiction. Hence, G has no pendant edge. We argue that $S \neq \emptyset$, otherwise we have that $\varepsilon(S) = 0$, and thus $i(G) = i(G-S) > |S|-\varepsilon(S) = 0$, a contradiction.

- If $\varepsilon(S) = 0$, then $i(G-S) \geq |S| - \varepsilon(S) + 1 = |S| + 1 \geq 2$. Then by the definition of $\tau(G)$, we obtain $\tau(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{i(G-S)} \leq \frac{|S|}{|S|+1} \leq 1$, a contradiction.
- If $\varepsilon(S) \in \{1, 2\}$, then there is a nontrivial component of $G-S$. It follows that $i(G-S) \geq |S|-\varepsilon(S)+1 \geq |S|-1$ and $\omega(G-S) \geq i(G-S) + 1 \geq |S|$. Then by the definition of $\tau(G)$, we obtain $\tau(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{|S|} = 1$, a contradiction.

(ii) If G has a pendant vertex u, then $bind(G) \leq \frac{|N_G(u)|}{|\{u\}|} = 1$, a contradiction. Hence, G has no pendant edge. We argue that $S \neq \emptyset$, otherwise we have that $\varepsilon(S) = 0$, and thus $i(G) = i(G-S) > |S| - \varepsilon(S) = 0$, a contradiction.

- If $\varepsilon(S) \in \{0, 1\}$, then $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S|$. Note that $I(G-S) \neq \emptyset$ and $N_G(I(G-S)) \neq V(G)$. Let $X := I(G-S)$. Then by the definition of $bind(G)$, we obtain $bind(G) \leq \frac{|N_G(X)|}{|X|} \leq \frac{|S|}{i(G-S)} \leq 1$, a contradiction.
- If $\varepsilon(S) = 2$, then there is a component C of $G-S$ containing exactly two vertices. It follows that $i(G-S) \geq |S| - \varepsilon(S) + 1 = |S| - 1$. Let $Y := I(G-S) \cup V(C)$. By the definition of $bind(G)$, we obtain $bind(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{|S \cup V(C)|}{i(G-S)+|C|} \leq \frac{|S|+2}{|S|+1} = 1 + \frac{1}{|S|+1} \leq \frac{3}{2}$, a contradiction.

(iii) If G has a pendant edge xy such that $d_G(y) = 1$, then $I_t(G) \leq \frac{|\{x\}|}{i(G-x)} \leq 1$, a contradiction. Hence, G has no pendant edge. We argue that $S \neq \emptyset$, otherwise we have that $\varepsilon(S) = 0$, and thus $i(G) = i(G-S) > |S| - \varepsilon(S) = 0$, a contradiction.

- If $\varepsilon(S) = 0$, then $i(G-S) \geq |S| - \varepsilon(S) + 1 = |S| + 1 \geq 2$. Then by the definition of $I_t(G)$, we obtain $I_t(G) \leq \frac{|S|}{i(G-S)} \leq \frac{|S|}{|S|+1} \leq 1$, a contradiction.
- If $\varepsilon(S) = 1$, then $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S|$. Then by the definition of $I_t(G)$, we obtain $I_t(G) \leq \frac{|S|}{i(G-S)} \leq 1$, a contradiction.
- If $\varepsilon(S) = 2$, then there is a component of $G-S$ containing exactly two vertices, denoted by $\{u, v\}$. It follows that $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S| - 1$. Let $S' := S \cup \{u\}$. Then by the definition of $I_t(G)$, we obtain $I_t(G) \leq \frac{|S'|}{i(G-S')} = \frac{|S|+1}{i(G-S)+1} \leq \frac{|S|+1}{|S|} = 1 + \frac{1}{|S|} \leq 2$, a contradiction. □

Next, we study the relationship between planar graphs or $K_{1,r}$-free graphs and 2-matching deleted graphs, and obtain a minimum degree condition for a planar graph or a $K_{1,r}$-free graph being a 2-matching deleted graph, respectively. To prove our results, we will use an important lemma as following.

Lemma 3.3. *[5] Let G be a connected planar graph with at least three vertices. If G does not contain triangles, then $|E(G)| \leq 2|G| - 4$.*

Corollary 3.4. *Let G be a connected graph of order $n \geq 3$. Then G is a 2-matching deleted graph if G is one of the following two special classes of graphs:*

(i) planar graphs with $\delta(G) \geq 4$;

(ii) $K_{1,r}$-free graphs with $\delta(G) \geq r+1$, where $r \geq 3$.

Proof. Suppose G is not a 2-matching deleted graph. By Theorem 3.1, there exists a subset $S \subseteq V(G)$ such that $i(G-S) > |S| - \varepsilon(S)$. According to the integrality of $i(G-S)$, we obtain that $i(G-S) \geq |S| - \varepsilon(S) + 1$. It is obviously that G has no pendant edge since $\delta(G) \geq 2$. We argue that $S \neq \emptyset$, otherwise we have that $\varepsilon(S) = 0$, and thus $i(G) = i(G-S) > |S| - \varepsilon(S) = 0$, a contradiction.

(i) Set $|S| = s \geq 1$. We denote by $I(G-S)$ the set of isolated vertices in $G-S$. Then we construct a simple bipartite graph $H := H[S, \overline{S}]$ as follows.

- If $\varepsilon(S) = 0$, then $i(G-S) \geq |S| - \varepsilon(S) + 1 = |S| + 1$. Choose $\overline{S} \subseteq I(G-S)$ such that $|\overline{S}| = s+1$. For any $x \in S$ and $y \in \overline{S}$, $xy \in E(H)$ if and only if $xy \in E(G)$. As G is a connected planar graph, it is easy to see that H is also a connected planar graph. On the one hand, as $\delta(G) \geq 4$, it is clear that for each $y \in \overline{S}$, we have $|N_H(y)| \geq 4$. Hence, $|H| \geq s + (s+1) = 2s+1 \geq 3$ and $|E(H)| \geq 4 \times |\overline{S}| = 4s+4$. On the other hand, according to the fact that a bipartite graph does not contain any odd cycles, Lemma 3.3 implies that $|E(H)| \leq 2|H| - 4 = 2 \times (2s+1) - 4 = 4s - 2$, a contradiction.
- If $\varepsilon(S) = 1$, then there is a nontrivial component C of $G-S$ with pendant vertex u and $|C| \geq 3$. It follows that $i(G-S) \geq |S| - \varepsilon(S) + 1 = |S|$. Choose $\overline{S} := S' \cup \{u\}$, where $S' \subseteq I(G-S)$ such that $|S'| = s$. For any $x \in S$ and $y \in \overline{S}$, $xy \in E(H)$ if and only if $xy \in E(G)$. As G is a connected planar graph, it is easy to see that H is also a connected planar graph. On the one hand, as $\delta(G) \geq 4$, it is clear that $|N_H(u)| \geq 3$ and $|N_H(y)| \geq 4$ holds for each $y \in S'$. Hence, $|H| \geq s + s + 1 = 2s+1 \geq 3$ and $|E(H)| \geq 4 \times |S'| + 3 = 4s+3$. On the other hand, according to the fact that a bipartite graph does not contain any odd cycles, Lemma 3.3 implies that $|E(H)| \leq 2|H| - 4 = 2 \times (2s+1) - 4 = 4s-2$, a contradiction.
- If $\varepsilon(S) = 2$, then there is a component of $G-S$ containing exactly two vertices, denoted by $\{u, v\}$. It follows that $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S| - 1$. Choose $\overline{S} := S'' \cup \{u, v\}$, where $S'' \subseteq I(G-S)$ such that $|S''| = s-1$. For any $x \in S$ and $y \in \overline{S}$, $xy \in E(H)$ if and only if $xy \in E(G)$. As G is a connected planar graph, it is easy to see that H is also a connected planar graph. On the one hand, as $\delta(G) \geq 4$, it is clear that $|N_H(u)| \geq 3, |N_H(v)| \geq 3$ and $|N_H(y)| \geq 4$ holds for each $y \in S''$. Hence, $|H| \geq s + (s-1) + 2 = 2s+1 \geq 3$ and $|E(H)| \geq 4 \times |S''| + 3 \times 2 = 4(s-1) + 6 = 4s+2$. On the other hand, according to the fact that a bipartite graph does not contain any odd cycles, Lemma 3.3 implies that $|E(H)| \leq 2|H| - 4 = 2 \times (2s+1) - 4 = 4s-2$, a contradiction.

(ii) Set $|S| = s \geq 1$. Note that $\delta(G) \geq r+1 \geq 4$. We denote by $I(G-S)$ the set of isolated vertices in $G-S$.

- If $\varepsilon(S) \in \{0, 1\}$, then $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S|$. Then we construct a bipartite subgraph $F := F[S, I(G-S)]$ of G such that $xy \in E(F)$ if and only if $xy \in E(G)$ for any $x \in S, y \in I(G-S)$. Note that for any $y \in I(G-S)$, we have $d_F(y) \geq \delta(G)$. Thus, $|E(F)| = \sum_{y \in I(G-S)} d_F(y) \geq \delta(G) \times i(G-S) \geq \delta(G) \times |S|$. It follows immediately that $\frac{|E(F)|}{|S|} \geq \frac{\delta(G) \times |S|}{|S|} = \delta(G) \geq r+1 > r$. This together with pigeonhole principle implies that there exists $x \in S$ such that $d_F(x) \geq r$. Then $G[\{x\} \cup N_F(x)]$ has a subgraph isomorphic to $K_{1,r}$, a contradiction.
- If $\varepsilon(S) = 2$, then there is a component of $G-S$ containing exactly two vertices, denoted by $\{u, v\}$. It follows that $|S| \geq |N_G(u)| - 1 \geq \delta(G) - 1 \geq r \geq 3$, and thus $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S| - 1$. Let $\overline{S} := I(G-S) \cup \{u\}$. Then we construct a bipartite subgraph $F := F[S, \overline{S}]$ of G such that $xy \in E(F)$ if and only if $xy \in E(G)$ for any $x \in S, y \in \overline{S}$. It is clear that $|N_F(u)| \geq \delta(G) - 1$ and $|N_F(y)| \geq \delta(G)$ holds for each $y \in I(G-S)$. Thus, $|E(F)| = d_F(u) + \sum_{y \in I(G-S)} d_F(y) \geq (\delta(G) - 1) + \delta(G) \times i(G-S) \geq \delta(G) \times |S| - 1$. It follows immediately that $\frac{|E(F)|}{|S|} \geq \frac{\delta(G) \times |S| - 1}{|S|} = \delta(G) - \frac{1}{|S|} \geq r + 1 - \frac{1}{3} > r$. This together with pigeonhole principle implies that there exists $x \in S$ such that $d_F(x) \geq r$. Then $G[\{x\} \cup N_F(x)]$ has a subgraph isomorphic to $K_{1,r}$, a contradiction. $\square$

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References

[1] J. Akiyama and M. Kano, Factors and factorizations of graphs, in *Lecture Notes in Mathematics*. Springer, Berlin, 2031 (2011) 1–347.
[2] A. Amahashi and M. Kano, Factors with given components. *Discrete Math.* **42** (1982) 1–6.
[3] K. Ando, Y. Egawa, A. Kaneko, K.I. Kawarabayashi and H. Matsuda, Path factors in claw-free graphs. *Discrete Math.* **243** (2002) 195–200.
[4] C. Berge, Regularizable graphs I. *Discrete Math.* **23** (1978) 85–89.
[5] J.A. Bondy and U.S.R. Murty, Graph theory with applications. Macmillan, London (1976).
[6] V. Chvátal, Tough graphs and Hamiltonian Circuits. *Discrete Math.* **5** (1973) 215–228.
[7] G. Dai, Remarks on component factors in graphs. *RAIRO:RO* **56** (2022) 721–730.
[8] G. Dai, The existence of path-factor covered graphs. *Discuss. Math. Graph Theory* **43** (2023) 5–16.
[9] G. Dai and Z. Hu, P_3-factors in the square of a tree. *Graphs Combin.* **36** (2020) 1913–1925.
[10] G. Dai, Z. Zhang, Y. Hang and X. Zhang, Some degree conditions for $P_{\geq 3}$-factor covered graphs. *RAIRO:RO* **55** (2021) 2907–2913.
[11] G. Dai, Y. Hang, X. Zhang, Z. Zhang and W. Wang, Sufficient conditions for graphs with $\{P_2, P_5\}$-factor. *RAIRO:RO* **56** (2022) 2895–2901.
[12] Y. Egawa, M. Furuya and K. Ozeki, Sufficient conditions for the existence of a path-factor which are related to odd components. *J. Graph Theory* **89** (2018) 327–340.
[13] A. Kaneko, A. Kelmans and T. Nishimura, On packing 3-vertex paths in a graph. *J. Graph Theory* **36** (2001) 175–197.
[14] M. Kano, C. Lee and K. Suzuki, Path and cycle factors of cubic bipartite graphs. *Discuss. Math. Graph Theory* **28** (2008) 551–556.
[15] K. Kawarabayashi, H. Matsuda, Y. Oda and K. Ota, Path factors in cubic graphs. *J. Graph Theory* **39** (2002) 188–193.
[16] M.D. Plummer, Perspectives: Graph factors and factorization: 1985–2003: A survey. *Discrete Math.* **307** (2007) 791–821.
[17] W.T. Tutte, The factors of graphs. *Canad. J. Math.* **4** (1952) 314–328.
[18] H. Wang, Path factors of bipartite graphs. *J. Graph Theory* **18** (1994) 161–167.
[19] D.R. Woodall, The binding number of a graph and its Anderson number. *J. Combin. Theory Ser. B* **15** (1973) 225–255.
[20] J. Yang, Y. Ma and G. Liu, Fractional $(g, f)-$factors in graphs. *Appl. Math. J. Chinese Univ. Ser. A* **16** (2001) 385–390.
[21] Q.R. Yu and G.Z. Liu, Graph Factors and Matching Extensions. Higher Education Press, Beijing (2009).