ON 2-MATCHING COVERED GRAPHS AND 2-MATCHING DELETED GRAPHS

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Abstract. For a family of connected graphs \( \mathcal{A} \), a spanning subgraph \( H \) of a graph \( G \) is called an \( \mathcal{A} \)-factor of \( G \) if each component of \( H \) is isomorphic to some graph in \( \mathcal{A} \). A graph \( G \) has a perfect 2-matching if \( G \) has a spanning subgraph \( H \) such that each component of \( H \) is either an edge or a cycle, i.e., \( H \) is a \( \{P_2, C_i | i \geq 3 \} \)-factor of \( G \). A graph \( G \) is said to be 2-matching covered if, for every edge \( e \in E(G) \), there is a perfect 2-matching \( M_e \) of \( G \) such that \( e \) belongs to \( M_e \). A graph \( G \) is called a 2-matching deleted graph if, for every edge \( e \in E(G) \), \( G - e \) possesses a perfect 2-matching. In this paper, we first obtain respective new characterizations for 2-matching covered graphs in bipartite and non-bipartite graphs by new proof technologies, distinct from Hetyei’s or Berge’s classical results. Secondly, we give a necessary and sufficient condition for a graph to be a 2-matching deleted graph. Thirdly, we prove that planar graphs with minimum degree at least 4 and \( K_{1,r} \)-free graphs (\( r \geq 3 \)) with minimum degree at least \( r + 1 \) are 2-matching deleted graphs, respectively.

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1. Introduction

All graphs in this paper are finite and simple. We refer to [5] for notation and terminologies not defined here. Let \( G = (V(G), E(G)) \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). For \( v \in V(G) \), we use \( d_G(v) \) and \( N_G(v) \) to denote the degree of \( v \) and the set of vertices adjacent to \( v \) in \( G \), respectively. For \( S \subseteq V(G) \), we write \( N_G(S) = \cup_{v \in S} N_G(v) \). We use \( \delta(G) \) to denote the minimum degree of a graph \( G \). We use \( \omega(G) \), \( i(G) \) to denote the number of components and isolated vertices of a graph \( G \), respectively.

For a connected graph \( G \), its toughness, denoted by \( \tau(G) \), was first introduced by Chvátal [6] as follows. If \( G \) is complete, then \( \tau(G) = +\infty \); otherwise,

\[
\tau(G) = \min \left\{ \frac{|S|}{\omega(G - S)} : S \subseteq V(G), \omega(G - S) \geq 2 \right\}.
\]

The binding number is introduced by Woodall [19] and defined as

\[
\text{bind}(G) = \min \left\{ \frac{|N_G(S)|}{|S|} : \emptyset \neq S \subseteq V(G), N_G(S) \neq V(G) \right\}.
\]

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The isolated toughness, denoted by $I_t(G)$, was first introduced by Yang, Ma and Liu [20] as follows. If $G$ is complete, then $I_t(G) = +\infty$; otherwise,

$$I_t(G) = \min \left\{ \frac{|S|}{i(G - S)} : S \subseteq V(G), i(G - S) \geq 2 \right\}.$$

For $X \subseteq V(G)$, let $G[X]$ be the subgraph of $G$ induced by $X$, and define $G - X := G[V(G) - X]$. For convenience, we use $G - x$ to denote the graph $G - \{x\}$. Let $K_{1,r}$ denote the complete bipartite with partite sets of size one and $r$. For an integer $r \geq 3$, a graph $G$ is said to be $K_{1,r}$-free if $G$ does not contain an induced subgraph isomorphic to $K_{1,r}$.

Let $\mathcal{A}$ be a family of connected graphs. If $G$ has a spanning subgraph $H$ such that every component of $H$ is isomorphic to some graph in $\mathcal{A}$, then $H$ is said to be an $\mathcal{A}$-factor of $G$. A graph $G$ has a perfect 2-matching if $G$ has a spanning subgraph $H$ such that each component of $H$ is either an edge or a cycle, i.e., $H$ is a $\{P_2, C_i | i \geq 3\}$-factor of $G$. A graph $G$ is said to be 2-matching covered if there is a perfect 2-matching of $G$ including any given edge $e \in E(G)$. A graph $G$ is called a 2-matching deleted graph if $G$ possesses a perfect 2-matching excluding any given edge $e \in E(G)$.

A spanning subgraph $H$ of graph $G$ is called 1-factor (perfect matching) if $d_H(x) = 1$ holds for any $x \in V(G)$. Since Tutte proposed the well known Tutte 1-factor theorem [17], there are many results on graph factors [2, 7–9, 11–13, 18] and path-factors in claw-free graphs and cubic graphs [3, 10, 14, 15]. More results on graph factors can be found in the survey papers and books in [1, 16, 21].

For matchings in bipartite graphs, König (1931) and Hall (1935) obtained the so-called König-Hall Theorem (sometimes, known as Hall’s Theorem), respectively.

**Theorem 1.1.** (König-Hall [5]) Let $G = (X, Y)$ be a connected bipartite graph such that $|X| = |Y|$. Then $G$ has a perfect matching if and only if $|N_G(S)| \geq |S|$ for any subset $S \subseteq X$.

In 1953, Tutte proved the following characterization for the existence of perfect 2-matchings in a graph.

**Theorem 1.2.** (Tutte [17]) A graph $G$ has a perfect 2-matching if and only if $i(G - S) \leq |S|$ for any subset $S \subseteq V(G)$.

The equivalence as following is due mostly to Hetyei (see also Akiyama and Kano [1]).

**Theorem 1.3.** (Hetyei [1]) If $G$ is a connected bipartite graph with partition $(U, W)$, then each edge of $G$ is contained in a 1-factor if and only if $|U| = |W|$ and $|N_G(X)| > |X|$ for any non-empty proper subset $X \subseteq U$.

A graph $G$ is 2-matching covered if and only if $G$ is “regularizable”, where a graph is regularizable if it can be transformed into a regular multigraph by giving each edge some positive multiplicity. Regularizable graphs were introduced and studied by Berge.

**Theorem 1.4.** (Berge [4]) For a connected graph $G$ that is not a bipartite graph with partite sets of equal size, the following conditions are equivalent:

(a) $G$ is regularizable,

(b) for each edge $e$ of $G$, there exists a perfect 2-matching of $G$ covering $e$,

(c) for every non-empty independent set $X$ of vertices, $|N_G(X)| > |X|$.

### 2. 2-MATCHING COVERED GRAPH

**Lemma 2.1.** Let $G$ be a graph such that $V(G) = X \cup Y$, $X \cap Y = \emptyset$ and $|X| = |Y|$. If $Y$ is an independent set in $G$, then $G$ has a 1-factor if and only if $i(G - S) \leq |S|$ for any $S \subseteq X$. 
Proof. Necessity: Let $M$ be a 1-factor of $G$, then $M$ is a perfect 2-matching of $G$ obviously. It follows from Theorem 1.2 that $i(G - S) \leq |S|$ for any subset $S \subseteq V(G)$. Hence, $i(G - S) \leq |S|$ holds for any $S \subseteq X$. 

Sufficiency: Choose $S := \emptyset$, then $i(G) = i(G - S) \leq |S| = 0$. Let $G' = G - E(X)$, then there is no isolated vertex in $G'$. Otherwise, there exists a isolated vertex $u \in V(G')$ such that $u \in X$. By choosing $S := X \setminus \{u\}$, it follows that $i(G - S) = |Y \cup \{u\}| = |Y| + 1 = |X| + 1 > |S|$, a contradiction. On the one hand, it is obviously $I(G - S) \subseteq I(G' - S)$. On the other hand, all the isolated vertices of $G' - S$ are in $Y$ since $i(G') = i(G) = 0$. Note that every edge in $E(X)$ is not adjacent to any vertex in $Y$. This together with $I(G' - S) \subseteq Y$ implies that $I(G' - S) \subseteq I(G - S)$. Therefore, $i(G' - S) = i(G - S) \leq |S|$ for any $S \subseteq X$. It follows immediately that $|N'_G(T)| \geq |T|$ for any $T \subseteq Y$ since $N'_G(T) \subseteq X$ and $|T| \leq i(G - N'_G(T)) \leq |N'_G(T)|$. By Theorem 1.1, $G'$ has a 1-factor. Hence, $G$ has a 1-factor. □

Theorem 2.2. (i) A connected bipartite graph $G = (X, Y)$ is 2-matching covered if and only if $|X| = |Y|$ and $i(G - S) \leq |S| - 1$ for any non-empty proper subset $S \subseteq X$; (ii) A connected non-bipartite graph $G$ is 2 matching covered if and only if $i(G - S) \leq |S| - 1$ for any non-empty proper subset $S \subseteq V(G)$.

Proof. (i) Necessity: Let $G = (X, Y)$ be a connected 2-matching covered bipartite graph. Then, by Theorem 1.3, $|X| = |Y|$ and $|N_G(T)| \geq |T| + 1$ for any non-empty proper subset $T \subseteq Y$. For any non-empty proper subset $S \subseteq X$, every isolated vertex of $G - S$ belongs to $Y$, denoted by $T = I(G - S) \subseteq Y$. On the one hand, $|N_G(T)| \geq |T| + 1 = i(G - S) + 1$. On the other hand, $|N_G(T)| \leq |S|$ since $T = I(G - S)$. Hence, $i(G - S) \leq |N_G(T)| - 1 \leq |S| - 1$ for any non-empty proper subset $S \subseteq X$.

Sufficiency: Let $G = (X, Y)$ be a connected bipartite graph with $|X| = |Y|$ such that $i(G - S) \leq |S| - 1$ for any non-empty proper subset $S \subseteq X$. Then we argue that $|N_G(T)| \geq |T| + 1$ for any non-empty proper subset $T \subseteq Y$. Otherwise, there exists proper subset $T \subseteq Y$ such that $|N_G(T)| \leq |T|$. Note that $|N_G(T)| \neq 0$ since $G$ is connected. Choose $S := |N_G(T)|$. As $|N_G(T)| \leq |T| < |Y| = |X|$, we have that $N_G(T) \neq X$ and thus $\emptyset \neq S \subseteq X$. It follows that $i(G - S) \leq |S| - 1$. On the other hand, $i(G - S) = i(G - N_G(T)) = |T| \geq |N_G(T)| = |S|$, a contradiction. Hence, $|N_G(T)| \geq |T| + 1$ for any non-empty proper subset $T \subseteq Y$. It follows from Theorem 1.3 that $G$ is a 2-matching covered graph.

(ii) Necessity: By way of contradiction, assume that there exists non-empty proper subset $S \subseteq V(G)$ such that $i(G - S) \geq |S|$. Since $G$ is a 2-matching covered graph, $G$ has a perfect 2-matching, and thus $i(G - S') \leq |S'|$ for any $S' \subseteq V(G)$. This together with $i(G - S) \geq |S|$ implies that $i(G - S) = |S| > 0$. As $G$ is a nonbipartite graph, $E(S) \neq \emptyset$ or there exists a nontrivial component of $G - S$.

Case 1. $E(S) \neq \emptyset$.

Suppose $e = xy \in E(S)$ such that $x, y \in S$. There must exist a perfect 2-matching $F$ covering $e$ since $G$ is a 2-matching covered graph. Denote the component of $F$ containing $e$ by $C$. Note that every component of $F$ is either an edge or a cycle by the definition of perfect 2-matchings.

- If $C$ is an edge, then $G' = G - \{x, y\}$ has a perfect 2-matching $F - \{x, y\}$. Choose $S' := S \setminus \{x, y\}$, then $i(G' - S') = i(G - S) = |S| = |S'| + 2 > |S'|$. By Theorem 1.2, $G'$ has no perfect 2-matching, a contradiction.

- If $C$ is a cycle and $|C| \geq 3$, then $G'' = G - V(C)$ has a perfect 2-matching $F - V(C)$. As $e \in E(S)$, we have that $|S \cap V(C)| > |I(G - S) \cap V(C)|$. Choose $S'' := S - V(C)$, then

$$i(G'' - S'') \geq |I(G - S) - I(G - S) \cap V(C)| = i(G - S) - |I(G - S) \cap V(C)| > i(G - S) - |S \cap V(C)| \geq |S| - |S \cap V(C)| = |S'|.$$ 

By Theorem 1.2, $G''$ has no perfect 2-matching, a contradiction.
Case 2. $E(S) = \emptyset$.

In this case, $G - S$ has a nontrivial component $D$, and there exists an edge $e' = uv$ connecting $D$ and $S$ such that $u \in D, v \in S$. Then $G$ has a perfect 2-matching covering $e'$, denoted by $F'$, since $G$ is a 2-matching covered graph. Denote the component of $F'$ containing $e'$ by $C'$.

- If $C'$ is an edge, then $G_1 = G - \{u, v\}$ has a perfect 2-matching $F' - \{u, v\}$. Choose $S_1 := S \setminus \{u\}$, then $i(G_1 - S_1) \geq i(G - S) \geq |S| > |S_1|$. By Theorem 1.2, $G_1$ has no perfect 2-matching, a contradiction.

- If $C'$ is a cycle and $|C'| \geq 3$, then $G_2 = G - V(C)$ has a perfect 2-matching $F' - V(C')$. As $C'$ is a cycle containing $e'$, we have that $|S \cap V(C')| > |I(G - S) \cap V(C')|$. Choose $S_2 := S - V(C')$, then

$$i(G_2 - S_2) \geq |I(G - S) - I(G - S) \cap V(C')|$$
$$= i(G - S) - |I(G - S) \cap V(C')|$$
$$> i(G - S) - |S \cap V(C')|$$
$$\geq |S| - |S \cap V(C')|$$
$$= |S_2|.$$  

By Theorem 1.2, $G_2$ has no perfect 2-matching, a contradiction.

**Sufficiency:** For any edge $e = xy$, $G^* := G - \{x, y\}$ has at most one isolated vertex since $i(G^*) = i(G - \{x, y\}) \leq |\{x, y\}| - 1 = 1$.

**Case 1.** $i(G^*) = 1$.

Let $u$ be the isolated vertex of $G^*$, and $C_1, C_2, \ldots, C_k$ be the other connected components of $G^*$. We first argue that $C_i$ has a perfect 2-matching $F_i$ for any $1 \leq i \leq k$. Otherwise, by Theorem 1.2, there exists $S_i \subseteq V(C_i)$ such that $i(C_i - S_i) \geq |S_i| + 1$. Choose $S := S_i \cup \{u, y\}$, then $S$ is a non-empty proper subset of $G$, and $i(G - S) = |\{u\}| + i(C_i - S_i) \geq 1 + (|S_i| + 1) = |S|$, a contradiction. It is easy to find that there exists no pendant vertex in $G$ since $i(G - S) \leq |S| - 1$ for any proper subset $\emptyset \neq S \subseteq V(G)$. Hence, $u \in N_G(x) \cap N_G(y)$, i.e., $uxy$ is a cycle in $G$. Then, $uxy \cup \bigcup_{i=1}^k F_i$ is a perfect 2-matching of $G$ containing $e$, i.e., $G$ is a 2-matching covered graph.

**Case 2.** $i(G^*) = 0$.

Let $C_1, C_2, \ldots, C_m$ be the connected components of $G^*$, where $m \geq 1$. If every $C_i$ has a perfect 2-matching $F_i$ for $1 \leq i \leq m$, then $\{xy, F_1, F_2, \ldots, F_m\}$ is a perfect 2-matching of $G$ containing $e$, i.e., $G$ is a 2-matching covered graph. If there exist $C_i, C_j (1 \leq i \neq j \leq m)$ such that both $C_i$ and $C_j$ has no perfect 2-matching, then, by Theorem 1.2, there exist $S_i \subseteq V(C_i), S_j \subseteq V(C_j)$ respectively such that $i(C_i - S_i) \geq |S_i| + 1$ and $i(C_j - S_j) \geq |S_j| + 1$. Choose $S := S_i \cup S_j \cup \{x, y\}$ which is a non-empty proper subset of $G$, then $i(G - S) = i(C_i - S_i) + i(C_j - S_j) \geq |S_i| + 1 + |S_j| + 1 = |S|$, a contradiction. Thus, there is exactly one element of $\{C_1, C_2, \ldots, C_m\}$ which has no perfect 2-matching. Without of generality, assume $C_1$ has no perfect 2-matching and every $C_i$ has a perfect 2-matching $F_i$ for $2 \leq t \leq m$. Then it is sufficient to show that

$$C'_1 := G[V(C_1) \cup \{x, y\}]$$
has a perfect 2-matching covering $e$.  

On the one hand, since $C_1$ has no perfect 2-matching, by Theorem 1.2, there exists $S' \subseteq V(C_1)$ such that $i(C_1 - S') \geq |S'| + 1$. On the other hand, if $i(C_1 - S') \geq |S'| + 2$, then $S := S' \cup \{x, y\}$ is a non-empty proper subset of $G$ and $i(G - S) = i(C_1 - S') \geq |S'| + 2 = |S|$, a contradiction. Hence, we have that $i(C_1 - S') = |S'| + 1$. Note that $S'$ is a non-empty set since $i(G^*) = 0$. We assume that $S'$ is a minimal barrier set of $V(C_1)$, i.e., $i(C_1 - S'') \leq |S''|$ holds for any proper subset $S'' \subseteq S'$.

Let $W := \{x_1, x_2, \ldots, x_d\}$ be the set of isolated vertices of $C_1 - S'$, where $d \geq 2$. We argue that

$$N_G(x) \cap W \neq \emptyset.$$  

(2)
Otherwise, \( i(G - (S' \cup \{y\})) \geq i(C_1 - S') = |S'| + 1 = |S' \cup \{y\}| \), a contradiction. Similarly, we can obtain that

\[
N_G(y) \cap W \neq \emptyset. \tag{3}
\]

Moreover, we also argue that every nontrivial component \( C_i \) of \( i(C_1 - S') \) has a perfect 2-matching \( F_i^j \) \((j = 1, 2, \ldots, p)\). Otherwise, suppose \( C_i \) has no perfect 2-matching, then there exists \( S^1 \subseteq V(C_i) \) such that \( i(C_1 - S^1) \geq |S^1| + 1 \) by Theorem 1.2. Choose \( S := S^1 \cup S' \cup \{x, y\} \), then \( S \) is a non-empty proper subset of \( G \) and \( i(G - S) = i(C_1 - S') + i(C_i - S^1) \geq |S'| + |S^1| + 1 = |S^1 \cup S' \cup \{x, y\}| = |S| \), a contradiction.

**Claim 2.3.** \( \overline{G} := G[S' \cup \{x, y\} \cup W] \) has a perfect 2-matching containing \( e \).

**Proof.** We first argue that for any \( 1 \leq i \leq d \), \( G_i := \overline{G} - \{x, y, x_i\} \) has a 1-factor \( F_i \). Suppose there is no 1-factor in \( G_i \), then by Lemma 2.1, there exists \( S'' \subseteq S' \) such that \( i(G_i - S'') \geq |S''| + 1 \). On the one hand, by the arguments similar to Lemma 2.1, we have that \( I(G_i - S'') \subseteq I(C_1 - S'') \), and thus \( i(C_i - S'') \geq i(G_i - S'') \geq |S''| + 1 \). On the other hand, it is obviously \( S'' \) is a proper subset of \( S' \), then \( i(C_i - S'' \cup S^1) \leq |S''| \), since \( S'' \) is a minimal barrier set of \( V(C_1) \), a contradiction. According to (2) and (3), we will distinguish two cases below to show that \( \overline{G} \) has perfect 2-matchings containing \( e \).

- If there exists \( u \in N_G(x) \cap N_G(y) \cap W \), then without of generality, assume \( u = x_1 \). Since \( G_1 := \overline{G} - \{x, y, x_1\} \) has a 1-factor \( F_1 \), \( x_1yx_1 \cup F_1 \) is a perfect 2-matching of \( \overline{G} \) containing \( e \).
- If \( N_G(x) \cap N_G(y) \cap W = \emptyset \), then we assume that \( x_1 \in N_G(x), x_d \in N_G(y) \). By the arguments similar to Lemma 2.1, both \( F_1 \) and \( F_d \) has no edge in \( E(G[S']) \). Note that, due to structural properties of 1-factors, there is an alternating path \( P \) from \( x_1 \) to \( x_d \) whose edges are alternately in \( E(F_1) \) and \( E(F_d) \). Then \( F_1 := F_1 - V(P) \) or \( F_d := F_d - V(P) \) is a 1-factor of \( \overline{G} - \{x, y\} - V(P) \). Thus, \( xyP \cup F_1 \) or \( xyP \cup F_d \) is a perfect 2-matching of \( \overline{G} \) containing \( e \).

Hence, Claim 2.3 is true. \( \square \)

Due to Claim 2.3, let \( \overline{F} \) be a perfect 2-matching of \( \overline{G} \) containing \( e \). Then \( \overline{F} \cup (\bigcup_{i=1}^p F_i) \) is a perfect 2-matching of \( C_i \) containing \( e \), i.e., the argument (1) holds. Thus, \( (\bigcup_{i=2}^p F_i) \cup \overline{F} \cup (\bigcup_{i=1}^p F_i) \) is a perfect 2-matching of \( G \) containing \( e \), i.e., \( G \) is a 2-matching covered graph. \( \square \)

### 3. 2-MATCHING DELETED GRAPH

**Theorem 3.1.** Let \( G \) be a connected graph. Then \( G \) is a 2-matching deleted graph if and only if \( i(G - S) \leq |S| - \varepsilon(S) \) for all \( S \subseteq V(G) \), where \( \varepsilon(S) \) is defined by

\[
\varepsilon(S) = \begin{cases} 
2 & \text{if there exists a component of } G - S \text{ containing exactly two vertices;} \\
1 & \text{if there exists a component } C \text{ of } G - S \text{ with pendant edges and } |V(C)| \geq 3; \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** 

**Necessity:** Let \( G \) be a 2-matching deleted graph. Obviously, \( G \) has a perfect 2-matching. Then, by Theorem 1.2, \( i(G - S) \leq |S| \) for any \( S \subseteq V(G) \). If \( i(G - S) \leq |S| - 2 \), then \( i(G - S) \leq |S| - \varepsilon(S) \) by the definition of \( \varepsilon(S) \); if \( i(G - S) = |S| - 1 \), then we argue that \( \varepsilon(S) \leq 1 \). Otherwise, \( \varepsilon(S) = 2 \), and \( G - S \) has a component with exactly one edge, denoted by \( e \). It follows that \( i(G - e - S) = i(G - S) + 2 = |S| + 1 > |S| \), and thus \( G - e \) has no perfect 2-matching, a contradiction. Hence, \( \varepsilon(S) \leq 1 \) and \( i(G - S) = |S| - 1 \leq |S| - \varepsilon(S) \). Now, we may assume that \( i(G - S) = |S| \). We argue that \( \varepsilon(S) = 0 \), otherwise \( \varepsilon(S) = 1, 2 \), then \( G - S \) has a component with pendant edges. It is easy to find that \( i(G - e - S) \geq i(G - S) + 1 = |S| + 1 \), and thus \( G - e \) has no perfect 2-matching, a contradiction. Hence, \( \varepsilon(S) = 0 \) and \( i(G - S) = |S| \leq |S| - \varepsilon(S) \).

**Sufficiency:** For any given \( S \subseteq V(G) \), \( i(G - S) \leq |S| - \varepsilon(S) \). Now, it suffices to show that \( i(G - e - S) \leq |S| \) for any edge \( e \in E(G) \) by Theorem 1.2. If \( e \) belongs to some component of \( G - S \) containing exactly two vertices, then \( \varepsilon(S) = 2 \), and thus \( i(G - e - S) = i(G - S) + 2 \leq |S| - \varepsilon(S) + 2 = |S| \). If \( e \) is a pendant edge belongs to a
component $C$ of $G - S$ such that $|C| \geq 3$, then $\varepsilon(S) \geq 1$, and thus $i(G-e-S) = i(G-S) + 1 \leq |S| - \varepsilon(S) + 1 \leq |S|$; Otherwise, $i(G - e - S) = i(G - S)$, and thus $i(G - e - S) = i(G - S) \leq |S| - \varepsilon(S) \leq |S|$ by the definition of $\varepsilon(S)$. Therefore, $i(G - e - S) \leq |S|$ holds for any $S \subseteq V(G)$ and $e \in E(G)$. This completes the proof of Theorem 3.1.

Corollary 3.2. Let $G$ be a connected graph of order $n \geq 3$. Then $G$ is a 2-matching deleted graph if one of the following statements holds: (i) $\tau(G) > 1$; (ii) $\text{bind}(G) > 3/2$; (iii) $I_t(G) > 2$.

Proof. Suppose, to the contrary, that $G$ is not a 2-matching deleted graph. By Theorem 3.1, there exists $S \subseteq V(G)$ such that $i(G - S) > |S| - \varepsilon(S)$. Due to the integrality, $i(G - S) \geq |S| - \varepsilon(S) + 1$.

(i) If $G$ has a pendant edge $xy$ such that $d_G(y) = 1$, then $\tau(G) \leq \frac{|\{x\}|}{\omega(G-xy)} \leq 1$, a contradiction. Hence, $G$ has no pendant edge.

We argue that $S \neq \emptyset$, otherwise we have that $\varepsilon(S) = 0$, and thus $i(G) = i(G - S) > |S| - \varepsilon(S) = 0$, a contradiction.

- If $\varepsilon(S) = 0$, then $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S| + 1 \geq 2$. Then by the definition of $\tau(G)$, we obtain $\tau(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{i(G-S)} \leq \frac{|S|}{|S|+1} \leq 1$, a contradiction.

- If $\varepsilon(S) \in \{1,2\}$, then there is a nontrivial component of $G-S$. It follows that $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S| - 1$ and $\omega(G-S) \geq i(G-S) + 1 \geq |S|$. Then by the definition of $\tau(G)$, we obtain $\tau(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{|S|} = 1$, a contradiction.

(ii) If $G$ has a pendant vertex $u$, then $\text{bind}(G) \leq \frac{|N_G(u)|}{|N_G(u)|} = 1$, a contradiction. Hence, $G$ has no pendant edge.

We argue that $S \neq \emptyset$, otherwise we have that $\varepsilon(S) = 0$, and thus $i(G) = i(G - S) > |S| - \varepsilon(S) = 0$, a contradiction.

- If $\varepsilon(S) \in \{0,1\}$, then $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S|$. Note that $I(G-S) \neq \emptyset$ and $N_G(I(G-S)) \neq V(G)$.

Let $X := I(G-S)$. Then by the definition of $\text{bind}(G)$, we obtain $\text{bind}(G) \leq \frac{|N_G(X)|}{|X|} \leq \frac{|S|}{i(G-S)} \leq 1$, a contradiction.

- If $\varepsilon(S) = 2$, then there is a component $C$ of $G - S$ containing exactly two vertices. It follows that $i(G - S) \geq |S| - \varepsilon(S) + 1 = |S| - 1$. Let $Y := I(G - S) \cup V(C)$. By the definition of $\text{bind}(G)$, we obtain $\text{bind}(G) \leq \frac{|N_G(Y)|}{|V|} \leq \frac{|S| + |V(C)|}{\omega(G-S) + |V|} \leq \frac{|S| + 2}{|S|+1} = 1 + \frac{1}{|S|+1} \leq \frac{3}{2}$, a contradiction.

(iii) If $G$ has a pendant edge $xy$ such that $d_G(y) = 1$, then $I_t(G) \leq \frac{|\{x\}|}{\omega(G-xy)} \leq 1$, a contradiction. Hence, $G$ has no pendant edge.

We argue that $S \neq \emptyset$, otherwise we have that $\varepsilon(S) = 0$, and thus $i(G) = i(G - S) > |S| - \varepsilon(S) = 0$, a contradiction.

- If $\varepsilon(S) = 0$, then $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S| + 1 \geq 2$. Then by the definition of $I_t(G)$, we obtain $I_t(G) \leq \frac{|S|}{i(G-S)} \leq \frac{|S|}{|S|+1} \leq 1$, a contradiction.

- If $\varepsilon(S) = 1$, then $i(G-S) \geq |S| - \varepsilon(S) + 1 \geq |S|$. Then by the definition of $I_t(G)$, we obtain $I_t(G) \leq \frac{|S|}{i(G-S)} \leq 1$, a contradiction.

- If $\varepsilon(S) = 2$, then there is a component of $G - S$ containing exactly two vertices, denoted by $\{u,v\}$. It follows that $i(G - S) \geq |S| - \varepsilon(S) + 1 \geq |S| - 1$. Let $S' := S \cup \{u\}$. Then by the definition of $I_t(G)$, we obtain $I_t(G) \leq \frac{|S'|}{i(G-S')} = \frac{|S'|+1}{i(G-S')+1} \leq \frac{|S'|+1}{|S|} = 1 + \frac{1}{|S|} \leq 2$, a contradiction. □

Next, we study the relationship between planar graphs or $K_{1,r}$-free graphs and 2-matching deleted graphs, and obtain a minimum degree condition for a planar graph or a $K_{1,r}$-free graph being a 2-matching deleted graph, respectively. To prove our results, we will use an important lemma as following.

Lemma 3.3. [5] Let $G$ be a connected planar graph with at least three vertices. If $G$ does not contain triangles, then $|E(G)| \leq 2|G| - 4$. 

Corollary 3.4. Let $G$ be a connected graph of order $n \geq 3$. Then $G$ is a 2-matching deleted graph if $G$ is one of the following two special classes of graphs:

(i) planar graphs with $\delta(G) \geq 4$;

(ii) $K_{1,r}$-free graphs with $\delta(G) \geq r + 1$, where $r \geq 3$.

Proof. Suppose $G$ is not a 2-matching deleted graph. By Theorem 3.1, there exists a subset $S \subseteq V(G)$ such that $i(G - S) \geq |S| - \varepsilon(S)$. According to the integrality of $i(G - S)$, we obtain that $i(G - S) \geq |S| - \varepsilon(S) + 1$. It is obviously that $G$ has no pendant edge since $\delta(G) \geq 2$. We argue that $S \neq \emptyset$, otherwise we have that $\varepsilon(S) = 0$, and thus $i(G) = i(G - S) > |S| - \varepsilon(S) = 0$, a contradiction.

(i) Set $|S| = s \geq 1$. We denote by $I(G - S)$ the set of isolated vertices in $G - S$. Then we construct a simple bipartite graph $H := H[S, \overline{S}]$ as follows.

- If $\varepsilon(S) = 0$, then $i(G - S) \geq |S| - \varepsilon(S) + 1 = |S| + 1$. Choose $\overline{S} \subseteq I(G - S)$ such that $|\overline{S}| = s + 1$. For any $x \in S$ and $y \in \overline{S}$, $xy \in E(H)$ if and only if $xy \in E(G)$. As $G$ is a connected planar graph, it is easy to see that $H$ is also a connected planar graph. On the one hand, as $\delta(G) \geq 4$, it is clear that for each $y \in \overline{S}$, we have $|N_H(y)| \geq 4$. Hence, $|H| \geq s + (s + 1) = 2s + 1 \geq 3$ and $|E(H)| \geq 4 \times |\overline{S}| = 4s + 4$. On the other hand, according to the fact that a bipartite graph does not contain any odd cycles, Lemma 3.3 implies that $|E(H)| \leq 2|H| - 4 = 2 \times (2s + 1) - 4 = 4s - 2$, a contradiction.

- If $\varepsilon(S) = 1$, then there is a nontrivial component $C$ of $G - S$ with pendant vertex $u$ and $|C| \geq 3$. It follows that $i(G - S) \geq |S| - \varepsilon(S) + 1 = |S|$. Choose $\overline{S} := S'' \cup \{u\}$, where $S'' \subseteq I(G - S)$ such that $|S''| = s$. For any $x \in S$ and $y \in \overline{S}$, $xy \in E(H)$ if and only if $xy \in E(G)$. As $G$ is a connected planar graph, it is easy to see that $H$ is also a connected planar graph. On the one hand, as $\delta(G) \geq 4$, it is clear that $|N_H(u)| \geq 3$ and $|N_H(y)| \geq 4$ holds for each $y \in S''$. Hence, $|H| \geq s + (s + 1) = 2s + 1 \geq 3$ and $|E(H)| \geq 4 \times |S''| + 3 \times 2 = 4(s - 1) + 6 = 4s + 2$. On the other hand, according to the fact that a bipartite graph does not contain any odd cycles, Lemma 3.3 implies that $|E(H)| \leq 2|H| - 4 = 2 \times (2s + 1) - 4 = 4s - 2$, a contradiction.

(ii) Set $|S| = s \geq 1$. Note that $\delta(G) \geq r + 1 \geq 4$. We denote by $I(G - S)$ the set of isolated vertices in $G - S$.

- If $\varepsilon(S) \in \{0, 1\}$, then $i(G - S) \geq |S| - \varepsilon(S) + 1 \geq |S|$. Then we construct a bipartite subgraph $F := F[S, I(G - S)]$ of $G$ such that $xy \in E(F)$ if and only if $xy \in E(G)$ for any $x \in S$, $y \in I(G - S)$. Note that for any $y \in I(G - S)$, we have $d_F(y) \geq \delta(G)$. Thus, $|E(F)| = \sum_{g \in I(G - S)} d_F(y) \geq \delta(G) \times i(G - S) \geq \delta(G) \times |S|$. It follows immediately that $\frac{|E(F)|}{|S|} \geq \frac{\delta(G) \times |S|}{|S|} = \delta(G) \geq r + 1 > r$. This together with pigeonhole principle implies that there exists $x \in S$ such that $d_F(x) \geq r$. Then $G[x \cup N_F(x)]$ has a subgraph isomorphic to $K_{1,r}$, a contradiction.

- If $\varepsilon(S) = 2$, then there is a component of $G - S$ containing exactly two vertices, denoted by $\{u, v\}$. It follows that $|S| \geq |N_G(u)| - 1 \geq \delta(G) - 1 \geq r + 3$, and thus $i(G - S) \geq |S| - \varepsilon(S) + 1 \geq |S| - 1$. Let $\overline{S} := I(G - S) \cup \{u, v\}$. Then we construct a bipartite subgraph $F := F[S, \overline{S}]$ of $G$ such that $xy \in E(F)$ if and only if $xy \in E(G)$ for any $x \in S$, $y \in \overline{S}$. It is clear that $|N_F(u)| \geq \delta(G) - 1$ and $|N_F(y)| \geq \delta(G)$ holds for each $y \in I(G - S)$. Thus, $|E(F)| = d_F(u) + \sum_{g \in I(G - S)} d_F(y) \geq (\delta(G) - 1) + \delta(G) \times i(G - S) \geq \delta(G) \times |S| - 1$. It follows immediately that $\frac{|E(F)|}{|S|} \geq \frac{\delta(G) \times |S| - 1}{|S|} = \delta(G) - \frac{1}{|S|} \geq r + 1 - \frac{1}{r} > r$. This together with pigeonhole principle implies that there exists $x \in S$ such that $d_F(x) \geq r$. Then $G[x \cup N_F(x)]$ has a subgraph isomorphic to $K_{1,r}$, a contradiction. □
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References