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ON r-HUED COLORING OF PRODUCT GRAPHS

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Abstract. A (k,r)-coloring of a graph G is a proper coloring with k colors such that for every vertex v with degree $d(v)$ in G, the color number of the neighbors of v is at least $\min\{d(v),r\}$. The smallest integer k such that G has a (k,r)-coloring is called the r-hued chromatic number and denoted by $\chi_r(G)$. In Kaliraj *et al.* [*Taibah Univ. Sci.* **14** (2020) 168–171], it is determined the 2-hued chromatic numbers of Cartesian product of complete graph and star graph. In this paper, we extend its result and determine the r-hued chromatic number of Cartesian product of complete graph and star graph.

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1. Introduction

All graphs are simple and finite, with undefined terminologies and notion begins referred to [1] in this paper. As in [1], $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ denote the vertex set, the edge set, the maximum degree and the minimum degree of a graph G, respectively. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to v in G, and $d_G(v) = |N_G(v)|$. For positive integers k and r, a (k,r)-coloring of a graph G is a mapping $c : V(G) \to \{1,2,3,4,\ldots,k\}$, satisfying both of the following conditions:

$(C1)$: $c(u) \neq c(v)$ for every edge $uv \in E(G)$;
$(C2)$: $|c(N_G(v))| \geq \min\{d_G(v), r\}$ for any $v \in V(G)$.

Following [1], a mapping $c : V(G) \to \{1,2,3,4,\ldots,k\}$ satisfying $(C1)$ only is a proper k-coloring of G. The chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that G has a proper k-coloring. The *r-hued chromatic number* of G, denoted by $\chi_r(G)$, is the smallest integer k such that G has a (k,r)-coloring. The notion of r-hued coloring was first introduced in [7,9], where $\chi_2(G)$ is called the dynamic number of graph G, and the corresponding chromatic number is denoted $\chi_d(G)$. In [2], Brooks' Theorem stated that a connected graph G satisfies $\chi(G) \leq \Delta(G) + 1$, where the equality holds if and only if G is an odd cycle or a complete graph. In [7], Lai *et al.* proved the best possible upper bounds of $\chi_2(G)$ as an analogue to Brooks' Theorem.

Theorem 1.1. *Let G be a connected graph.*

(i) *If $\Delta(G) \leq 3$, then $\chi_2(G) \leq 4$, unless $G = C_5$, in which case $\chi_2(C_5) = 5$ [7].*
(ii) *If $\Delta(G) \geq 4$, then $\chi_2(G) \leq \Delta(G) + 1$ [7].*

Keywords. (k,r)-coloring, r-hued chromatic number, Cartesian product.

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(iii) *If G is planar graph with $G \neq C_5$, then $\chi_2(G) \leq 4$ [5].*

In [10], Lai *et al.* proved that if G is a planar graph and $r \geq 8$, then $\chi_r(G) \leq 2r + 16$. Earlier Brooks type upper bounds of the r-hued chromatic number can be found in [3,6,8].

Theorem 1.2. *Let G be a connected graph, and $r \geq 2$ be an integer.*

(i) *If $\Delta(G) \leq r$, then $\chi_r(G) \leq \Delta(G) + r^2 - r + 1$ [6].*
(ii) *$\chi_r(G) \leq \Delta^2(G) + 1$, where the equality holds if and only if G is a Moore graph [3].*
(iii) *$\chi_r(G) \leq r\Delta(G) + 1$, with equality if and only if G is r-regular with diameter 2 and girth 5 [8].*

A lower bound for r-hued chromatic number of G as follows.

Theorem 1.3 ([6], Prop. 2.1)**.** *Let G be a graph, and $r \geq 2$ be an integer. Then $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$, and this lower bound is sharp.*

Let G and H be two graphs. The Cartesian product of G and H, denoted by $G \square H$, is a graph with the vertex set $V(G) \times V(H)$ such that two vertices (u,v) and (x,y) are adjacent if and only if $u = x$ and $vy \in E(H)$ or $v = y$ and $ux \in E(G)$. It follows by definition that $\Delta(G \square H) = \Delta(G) + \Delta(H)$.

Kaliraj *et al.* [4] studied 2-hued chromatic numbers of Cartesian product of complete graph and star graph, for positive integers $s \geq 2$ and n,

$$\chi_2\left(K_n \square K_{1,s}\right) = \begin{cases} 3, & \text{if } n = 1; \\ 4, & \text{if } n = 2; \\ n, & \text{otherwise.} \end{cases}$$

In this paper, we extend the above result, and prove the following theorem.

Theorem 1.4. *For all fixed positive integers r, r-hued chromatic number of Cartesian product of complete graph and star graph as follows:*

$$\chi_r(K_n \square K_{1,s}) = \begin{cases} n, & \text{if } r < n; \\ \max\{2n, \min\{r+1, n+s\}\}, & \text{if } r \geq n. \end{cases}$$

2. Proofs of the main results

Throughout this section, $n \geq 2$, $s \geq 1$ are integers, and we always devote $V(K_n) = \{a_1, a_2, \ldots, a_n\}$, $V(K_{1,s}) = \{w, v_1, \ldots, v_s\}$, where w is the only vertex with $d(w) = s$ in $V(K_{1,s})$. By the definition of Cartesian products,

$$V(K_n \square K_{1,s}) = \bigcup_{i=1}^{n} \{a_i w\} \cup \bigcup_{i=1}^{n} \{a_i v_j : 1 \leq j \leq s\}.$$

For presentational purpose, we also write

$$V(K_n \square K_{1,s}) = \begin{bmatrix} a_1 w & a_1 v_1 & a_1 v_2 & \cdots & a_1 v_s \\ a_2 w & a_2 v_1 & a_2 v_2 & \cdots & a_2 v_s \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} w & a_{n-1} v_1 & a_{n-1} v_2 & \cdots & a_{n-1} v_s \\ a_n w & a_n v_1 & a_n v_2 & \cdots & a_n v_s \end{bmatrix}_{n \times (s+1)}$$

By the definition of $K_n \square K_{1,s}$, we have the following observations.

$$N_{K_n \square K_{1,s}}(a_i w) = \bigcup_{j=1}^{s} \{a_i v_j\} \cup \bigcup_{k=1, k \neq i}^{n} \{a_k w\} \tag{2.1}$$

$$N_{K_n \square K_{1,s}}(a_i v_j) = \bigcup_{k=1, k \neq i}^{n} \{a_k v_j\} \cup \{a_i w\}. \tag{2.2}$$

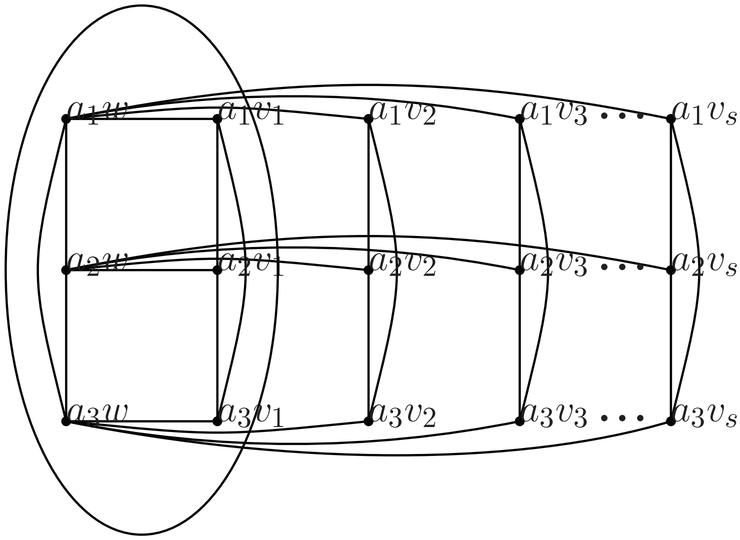

FIGURE 1. $K_3\square K_{1,s}$; the circle is $K_3\square K_2$.

For a fixed positive integer $r \geq n$, we first determine a lower bound for r-hued chromatic number of Cartesian product of complete graphs K_n and $K_{1,s}$, which is useful for the proof of Theorems 2.3 and 2.4.

Lemma 2.1. *If $r \geq n$, then $\chi_r(K_n\square K_{1,s}) \geq 2n$.*

Proof. We prove $\chi_r(K_n\square K_{1,s}) \geq 2n$ by contradiction. Suppose that $\chi_r(K_n\square K_{1,s}) \leq 2n-1$. We assume that $c_0 : V(K_n\square K_{1,s}) \to \{1, 2, 3, \dots, 2n-1\}$ is a $(2n-1, r)$-coloring. As $s \geq 1$, $K_{1,s}$ contains a subgraph isomorphic to K_2, and so $K_n\square K_{1,s}$ always contains an induced subgraph $H = K_n\square K_2$ (see Fig. 1 for an illustration, where $K_3\square K_{1,s}$ contains $K_3\square K_2$ as a subgraph).

Since $|V(H)| = 2n$, there always exist two vertices in H which are colored with the same color. Without loss of generality, we assume that $c_0(a_iw) = c_0(a_jv_1)$, where $i \neq j$. For the vertex a_iv_1, by (2.1), we have $\{a_iw, a_jv_1\} \subseteq N_{K_n\square K_{1,s}}(a_iv_1)$. Since $r \geq n$, $|c_0(N_{K_n\square K_{1,s}}(a_iv_1))| \leq n-1 < \min\{r, n\} = n$, which contradicts to that c_0 is a $(2n-1, r)$-coloring. Hence $\chi_r(K_n\square K_{1,s}) \geq 2n$. □

Corollary 2.2. *If $r \geq n$, then $\chi_r(K_n\square K_2) = 2n$.*

Proof. Let $V(K_n) = \{a_1, a_2, a_3, \dots, a_n\}$, and $V(K_2) = \{v_1, v_2\}$. By the definition of Cartesian products, $V(K_n\square K_2) = \bigcup_{i=1}^{n}\{a_iv_j : 1 \leq j \leq 2\}$. The order of $K_n\square K_2$ is $|V(K_n\square K_2)| = 2n$. On the one hand, $\chi_r(K_n\square K_2) \leq |V(K_n\square K_2)|$, then $\chi_r(K_n\square K_2) \leq 2n$. On the other hand, by Lemma 2.1, let $s = 1$, then $\chi_r(K_n\square K_2) \geq 2n$, so $\chi_r(K_n\square K_2) = 2n$. □

We first prove the case when $s \geq r$ for Theorem 1.4.

Theorem 2.3. *Let $K_n\square K_{1,s}$ be a Cartesian product graph. If $s \geq r$, then*

$$\chi_r(K_n\square K_{1,s}) = \begin{cases} r+1, & \text{if } r \geq 2n; \\ 2n, & \text{if } n \leq r < 2n; \\ n, & \text{if } r < n. \end{cases}$$

Proof. Since $\Delta(K_n) = n-1$, $\Delta(K_{1,s}) = s$, then $\Delta(K_n\square K_{1,s}) = \Delta(K_n) + \Delta(K_{1,s}) = (n-1) + s$. As $n \geq 1$, $\Delta(K_n\square K_{1,s}) = n-1+s \geq s \geq r$. We consider the following three cases to prove this theorem, and we shall use $n \times (s+1)$ matrix to present a coloring of $V(K_n\square K_{1,s})$.

Case 1. $r \geq 2n$.

By Theorem 1.3, we have $\chi_r(K_n \square K_{1,s}) \geq \min\{\Delta(K_n \square K_{1,s}), r\} + 1 = \min\{n-1+s, r\} + 1 = r+1$. To show that $\chi_r(K_n \square K_{1,s}) \leq r+1$, we define $c_1 : V(K_n \square K_{1,s}) \to \{1,2,3,\ldots,r,r+1\}$ as follows. Since $r \geq 2n$, $r-n+1 \geq 2n-n+1 = n+1$, $r-n+1 > n$. Let $A = (a_{ij})_{n\times(s+1)}$ be a $n \times (s+1)$ matrix as follows,

$$A = \begin{bmatrix} r-n+2 & 1 & 2 & \cdots & r-n-1 & r-n & r-n+1 & \cdots & r-n+1 \\ r-n+3 & 2 & 3 & \cdots & r-n & r-n+1 & 1 & \cdots & 1 \\ r-n+4 & 3 & 4 & \cdots & r-n+1 & 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ r+1 & n & n+1 & \cdots & n-3 & n-2 & n-1 & \cdots & n-1 \end{bmatrix}_{n\times(s+1)}$$

where the s entries of ith row are $\{r-n+1+i, i, i+1, \ldots, r-n, r-n+1, 1, 2, \ldots, i-1, i-1, \ldots, i-1\} \subseteq \{1,2,3,\ldots,r+1\}$ when $1 \leq i \leq n$, and $a_{i,j} = a_{i,r-n+2}$ when $r-n+3 \leq j \leq s+1$. Define $c_1(V(K_n \square K_{1,s})) = A$. For $1 \leq i \leq n$, $1 \leq j \leq s$, $c_1(a_iv_j) = a_{i,j+1}$, and so $\{c_1(a_iv_j) | 1 \leq i \leq n, 1 \leq j \leq s\} = \{1,2,3,\ldots,r-n+1\}$. For $1 \leq i \leq n$, $c_1(a_iw) = a_{i1} = r-n+1+i$, and so $\{c_1(a_iw) | 1 \leq i \leq n\} = \{r-n+1+1, r-n+1+2, r-n+1+3, \ldots, r-n+1+n\} = \{r-n+2, r-n+3, r-n+4, \ldots, r+1\}$. It follows that, if $k \neq i$, then $c_1(a_iw) \neq c_1(a_iv_j)$, $c_1(a_iw) \neq c_1(a_kw)$ and $c_1(a_iv_j) \neq c_1(a_kv_j)$. As $r \geq 2n$, every entry a_{ij} in A satisfies $1 \leq a_{ij} \leq r+1$, and so c_1 is a proper $(r+1)$-coloring of $K_n \square K_{1,s}$.

Next we need to show c_1 satisfies $(C2)$. For a vertex of the form a_iw, by (2.1), we have $d(a_iw) = |N_{K_n \square K_{1,s}}(a_iw)| = n-1+s$. Since $c_1(N_{K_n \square K_{1,s}}(a_iw)) = \{1,2,3,\ldots,r+1\} \backslash \{r-n+1+i\}$, $|c_1(N_{K_n \square K_{1,s}}(a_iw))| = r = \min\{n-1+s, r\}$. For a vertex of the form a_iv_j, by (2.2), we have $d(a_iv_j) = |N_{K_n \square K_{1,s}}(a_iv_j)| = n$. By matrix A, $c_1(N_{K_n \square K_{1,s}}(a_iv_j))$ contains $n-1$ different colors of $\{1,2,3,\ldots,r-n+1\}$ and one color $c_1(a_iw) = r-n+1+i$, so $|c_1(N_{K_n \square K_{1,s}}(a_iv_j))| = n = \min\{d(a_iv_j), r\}$. Thus c_1 is a $(r+1,r)$-coloring of $K_n \square K_{1,s}$, hence $\chi_r(K_n \square K_{1,s}) \leq r+1$. To sum up, $\chi_r(K_n \square K_{1,s}) = r+1$.

Case 2. $n \leq r < 2n$.

By Lemma 2.1, we have $\chi_r(K_n \square K_{1,s}) \geq 2n$. Since $1 \leq n \leq r \leq s$, $1 \leq r-n+1 \leq r \leq s$, $r-n+1 \leq s$, and as $n \leq r < 2n$, $1 \leq r-n+1 < n+1$, so $r-n+1 \leq n$. To show that $\chi_r(K_n \square K_{1,s}) \leq 2n$, we define $c_2 : V(K_n \square K_{1,s}) \to \{1,2,3,\ldots,n,n+1,\ldots,2n\}$ and a $n \times (s+1)$ matrix $B = (b_{ij})_{n\times(s+1)}$ as follows,

$$B = \begin{bmatrix} n+1 & 1 & 2 & 3 & \cdots & r-n+1 & r-n+1 & \cdots & r-n+1 \\ n+2 & 2 & 3 & 4 & \cdots & r-n+2 & r-n+2 & \cdots & r-n+2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ n+n-2 & n-2 & n-1 & n & \cdots & r-n-2 & r-n-2 & \cdots & r-n-2 \\ n+n-1 & n-1 & n & 1 & \cdots & r-n-1 & r-n-1 & \cdots & r-n-1 \\ 2n & n & 1 & 2 & \cdots & r-n & r-n & \cdots & r-n \end{bmatrix}_{n\times(s+1)}$$

where the n entries of jth column are $\{j-1, j, j+1, j+2, \ldots, n-1, n, 1, 2, \ldots, j-2, \ldots, j-2\} \subseteq \{1,2,3,\ldots,n-1,n\}$ when $2 \leq j \leq r-n+2$, and $b_{ij} = b_{i,r-n+2}$ when $r-n+3 \leq j \leq s+1$, $2 \leq i \leq n$. Define $c_2(V(K_n \square K_{1,s})) = B$. For $1 \leq i \leq n$, $1 \leq j \leq s$, $c_2(a_iv_j) = b_{i,j+1}$, and so $\{c_2(a_iv_j) | 1 \leq i \leq n, 1 \leq j \leq s\} = \{1,2,3,\ldots,n\}$. For $1 \leq i \leq n$, $c_2(a_iw) = b_{i1} = n+i$, and so $\{c_2(a_iw) | 1 \leq i \leq n\} = \{n+1, n+2, n+3, \ldots, n+n\} = \{n+1, n+2, n+3, \ldots, 2n\}$. It follows that, if $k \neq i$, then $c_2(a_iw) \neq c_2(a_iv_j)$, $c_2(a_iw) \neq c_2(a_kw)$ and $c_2(a_iv_j) \neq c_2(a_kv_j)$. As $n \leq r < 2n$, every entry b_{ij} in B satisfies $1 \leq b_{ij} \leq 2n$, and so c_2 is a proper $2n$-coloring of $K_n \square K_{1,s}$.

Next we need to show c_2 satisfies $(C2)$. For a vertex of the form a_iw, by (2.1), we have $d(a_iw) = |N_{K_n \square K_{1,s}}(a_iw)| = n-1+s$. Since $c_2(N_{K_n \square K_{1,s}}(a_iw)) = \{1,2,3,\ldots,r-n+1\} \cup \{n+1, n+2, n+3, \ldots, 2n\} \setminus \{n+i\}$, then $|c_2(N_{K_n \square K_{1,s}}(a_iw))| = r = \min\{n-1+s, r\}$. For a vertex of the form a_iv_j, by (2.2), we have $d(a_iv_j) = |N_{K_n \square K_{1,s}}(a_iv_j)| = n$. By matrix B, the color set $c_2(N_{K_n \square K_{1,s}}(a_iv_j))$ contains $n-1$ different colors of $\{1,2,3,\ldots,n\}$ and one color $c_2(a_iw) = n+i$, we have $|c_2(N_{K_n \square K_{1,s}}(a_iv_j))| = n$, then $|c_2(N_{K_n \square K_{1,s}}(a_iv_j))| = n = \min\{d(a_iv_j), r\}$. Thus c_2 is a $(2n,r)$-coloring of $K_n \square K_{1,s}$, and so $\chi_r(K_n \square K_{1,s}) \leq 2n$. To sum up, $\chi_r(K_n \square K_{1,s}) = 2n$.

Case 3. $r < n$.

Since $K_n \square K_{1,s}$ always contains an induced subgraph K_n, $\chi_r(K_n \square K_{1,s}) \geq n$. To show that $\chi_r(K_n \square K_{1,s}) \leq n$, we define $c_3 : V(K_n \square K_{1,s}) \to \{1, 2, 3, \ldots, n\}$ and a $n \times (s+1)$ matrix $C = (c_{ij})_{n\times(s+1)}$ as follows,

$$C = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 3 & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ n & n-1 & n-1 & \cdots & n-1 \\ 1 & n & n & \cdots & n \end{bmatrix}_{n\times(s+1)}$$

Define $c_3(V(K_n \square K_{1,s})) = C$. For $1 \leq i \leq n$, $1 \leq j \leq s$, $c_3(a_i v_j) = c_{i,j+1}$, and so $\{c_3(a_i v_j) | 1 \leq i \leq n, 1 \leq j \leq s\} = \{1, 2, 3, \ldots, n\}$. For $1 \leq i \leq n-1$, $c_3(a_i w) = i+1$, and $c_3(a_n w) = 1$, so $\{c_3(a_i w) | 1 \leq i \leq n\} = \{1, 2, 3, \ldots, n\}$. It follows that, if $k \neq i$, then $c_3(a_i w) \neq c_3(a_i v_j)$, $c_3(a_i w) \neq c_3(a_k w)$ and $c_3(a_i v_j) \neq c_3(a_k v_j)$. Since every entry c_{ij} in C satisfies $1 \leq c_{ij} \leq n$, so c_3 is a proper n-coloring of $K_n \square K_{1,s}$.

Next we need to show c_3 satisfies $(C2)$. For a vertex of the form $a_i w$, by (2.1), we have $|N_{K_n \square K_{1,s}}(a_i w)| = n-1+s$, so $d(a_i w) = n-1+s \geq n > r$. For $1 \leq i \leq n-1$, $c_3(N_{K_n \square K_{1,s}}(a_i w)) = \{1, 2, \ldots, n\} \backslash \{i+1\}$, and for $i = n$, $c_3(N_{K_n \square K_{1,s}}(a_n w)) = \{2, \ldots, n\}$, then $|c_3(N_{K_n \square K_{1,s}}(a_i w))| = n-1 \geq \min\{d(a_i w), r\} = \min\{n-1+s, r\} = r$. For a vertex of the form $a_i v_j$, by (2.2), we have $|N_{K_n \square K_{1,s}}(a_i v_j)| = n$, so $d(a_i w) = n > r$. Since $c_3(N_{K_n \square K_{1,s}}(a_i v_j)) = \{1, 2, \ldots, n\} \backslash \{i\}$, then $|c_3(N_{K_n \square K_{1,s}}(a_i v_j))| = n-1 \geq \min\{d(a_i v_j), r\} = \min\{n, r\} = r$. Thus c_3 is a (n, r)-coloring of $K_n \square K_{1,s}$, hence $\chi_r(K_n \square K_{1,s}) \leq n$. To sum up, $\chi_r(K_n \square K_{1,s}) = n$. □

In the following, we prove the case $s < r$ for Theorem 1.4.

Theorem 2.4. *Let $K_n \square K_{1,s}$ be a Cartesian product graph. If $s < r$, then*

$$\chi_r(K_n \square K_{1,s}) = \begin{cases} \max(n+s, 2n), & \text{if } r \geq n \text{ and } n-1+s \leq r; \\ \max(2n, r+1), & \text{if } r \geq n \text{ and } n-1+s > r; \\ n, & \text{if } r < n. \end{cases}$$

Proof. We consider the following three cases to prove this theorem, and we shall use $n \times (s+1)$ matrix to present a coloring of $V(K_n \square K_{1,s})$.

Case 1. $r \geq n$ and $n-1+s \leq r$.

Since $\Delta(K_n) = n-1$, $\Delta(K_{1,s}) = s$, then $\Delta(K_n \square K_{1,s}) = \Delta(K_n) + \Delta(K_{1,s}) = (n-1) + s \leq r$. We consider the following two subcases.

Subcase 1.1. $n \leq s$.

By Theorem 1.3, we have $\chi_r(K_n \square K_{1,s}) \geq \min\{\Delta(K_n \square K_{1,s}), r\} + 1 = \min\{n-1+s, r\} + 1 = n-1+s+1 = n+s$. To show that $\chi_r(K_n \square K_{1,s}) \leq n+s$, we define $c_4 : V(K_n \square K_{1,s}) \to \{1, 2, 3, \ldots, n+s\}$ and a $n \times (s+1)$ matrix $D = (d_{ij})_{n\times(s+1)}$ as follows,

$$D = \begin{bmatrix} s+1 & 1 & 2 & \cdots & s-2 & s-1 & s \\ s+2 & 2 & 3 & \cdots & s-1 & s & 1 \\ s+3 & 3 & 4 & \cdots & s & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ s+n & n & n+1 & \cdots & n-3 & n-2 & n-1 \end{bmatrix}_{n\times(s+1)}$$

where the $s+1$ entries of ith row are $\{s+i, i, i+1, \ldots, s-1, s, 1, 2, \ldots, i-1\} \subseteq \{1, 2, \cdots, s+n\}$ when $1 \leq i \leq n$. Define $c_4(V(K_n \square K_{1,s})) = D$. For $1 \leq i \leq n$, $1 \leq j \leq s$, $c_4(a_i v_j) = d_{i,j+1}$, and so $\{c_4(a_i v_j) | 1 \leq i \leq n, 1 \leq j \leq s\} = \{1, 2, 3, \ldots, s\}$. For $1 \leq i \leq n$, $c_4(a_i w) = d_{i1} = s+i$, and so $\{c_4(a_i w) | 1 \leq i \leq n\} = \{s+1, s+2, s+3, \ldots, s+n\}$. It follows that, if $k \neq i$, then $c_4(a_i w) \neq c_4(a_i v_j)$,

$c_4(a_iw) \neq c_4(a_kw)$ and $c_4(a_iv_j) \neq c_4(a_kv_j)$. As $n \leq s$, every entry d_{ij} in D satisfies $1 \leq d_{ij} \leq n+s$, and so c_4 is a proper $(n+s)$-coloring of $K_n\Box K_{1,s}$.

Next we need to show c_4 satisfies $(C2)$. For a vertex of the form a_iw, by (2.1), we have $d(a_iw) = |N_{K_n\Box K_{1,s}}(a_iw)| = n-1+s$. Since $c_4(N_{K_n\Box K_{1,s}}(a_iw)) = \{1,2,3,\ldots,s\} \cup \{s+1, s+2,\ldots, s+n\} \backslash \{s+i\}$, then $|c_4(N_{K_n\Box K_{1,s}}(a_iw))| = n+s-1$, $|c_4(N_{K_n\Box K_{1,s}}(a_iw))| = \min\{d(a_iw), r\} = \min\{n-1+s, r\}$. For a vertex of the form a_iv_j, by (2.2), we have $d(a_iv_j) = |N_{K_n\Box K_{1,s}}(a_iv_j)| = n$. By matrix D, the color set $c_4(N_{K_n\Box K_{1,s}}(a_iv_j))$ always contains $n-1$ different colors of $\{1,2,3,\ldots,s\}$ and one color $c_4(a_iw) = s+i$, so $|c_4(N_{K_n\Box K_{1,s}}(a_iv_j))| = n = \min\{d(a_iv_j), r\} = \min\{n, r\}$. Thus c_4 is a $(n+s, r)$-coloring of $K_n\Box K_{1,s}$, then $\chi_r(K_n\Box K_{1,s}) \leq n+s$. To sum up, $\chi_r(K_n\Box K_{1,s}) = n+s$.

Subcase 1.2. $n > s$.

By Lemma 2.1, we have $\chi_r(K_n\Box K_{1,s}) \geq 2n$. To show that $\chi_r(K_n\Box K_{1,s}) \leq 2n$, we define $c_5 : V(K_n\Box K_{1,s}) \to \{1,2,3,\ldots,2n\}$ and a $n \times (s+1)$ matrix $E = (e_{ij})_{n\times(s+1)}$ as follows,

$$E = \begin{bmatrix} n+1 & 1 & 2 & 3 & \cdots & s \\ n+2 & 2 & 3 & 4 & \cdots & s+1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ n+n-2 & n-2 & n-1 & n & \cdots & s-3 \\ n+n-1 & n-1 & n & 1 & \cdots & s-2 \\ 2n & n & 1 & 2 & \cdots & s-1 \end{bmatrix}_{n\times(s+1)}$$

where the n entries of jth column are $\{j-1, j, j+1, \ldots, n-1, n, 1, 2, \ldots, j-2\} \subseteq \{1,2,\ldots,n\}$ when $2 \leq j \leq s+1$. Define $c_5(V(K_n\Box K_{1,s})) = E$. For $1 \leq i \leq n$, $1 \leq j \leq s$, $c_5(a_iv_j) = e_{i,j+1}$, and so $\{c_5(a_iv_j) | 1 \leq i \leq n, 1 \leq j \leq s\} = \{1,2,3,\ldots,n\}$. For $1 \leq i \leq n$, $c_5(a_iw) = e_{i1} = n+i$, and so $\{c_5(a_iw) | 1 \leq i \leq n\} = \{n+1, n+2, n+3, \ldots, 2n\}$. It follows that, if $k \neq i$, then $c_5(a_iw) \neq c_5(a_iv_j)$, $c_5(a_iw) \neq c_5(a_kw)$ and $c_5(a_iv_j) \neq c_5(a_kv_j)$. As $n > s$, every entry e_{ij} in E satisfies $1 \leq e_{ij} \leq 2n$, and so c_5 is a proper $2n$-coloring of $K_n\Box K_{1,s}$.

Next we need to show c_5 satisfies $(C2)$. For a vertex of the form a_iw, by (2.1), we have $d(a_iw) = |N_{K_n\Box K_{1,s}}(a_iw)| = n-1+s$. Since $c_5(N_{K_n\Box K_{1,s}}(a_iw)) = \{1,2,\ldots,n,n+1,n+2,\ldots,2n\} \backslash \{n+i\}$, then $|c_5(N_{K_n\Box K_{1,s}}(a_iw))| = 2n-1 \geq \min\{d(a_iw), r\} = \min\{n-1+s, r\} = n-1+s$. For a vertex of the form a_iv_j, by (2.2), we have $d(a_iv_j) = |N_{K_n\Box K_{1,s}}(a_iv_j)| = n$. By matrix E, the color set $c_5(N_{K_n\Box K_{1,s}}(a_iv_j))$ always contains $n-1$ different colors of $\{1,2,3,\ldots,n\}$ and one color $c_5(a_iw) = n+i$, so $|c_5(N_{K_n\Box K_{1,s}}(a_iv_j))| = n$, then $|c_5(N_{K_n\Box K_{1,s}}(a_iv_j))| = n = \min\{d(a_iv_j), r\} = \min\{n, r\}$. Thus c_5 is a $(2n, r)$-coloring of $K_n\Box K_{1,s}$, so $\chi_r(K_n\Box K_{1,s}) \leq 2n$. To sum up, $\chi_r(K_n\Box K_{1,s}) = 2n$.

By Subcases 1.1 and 1.2, we can conclude that $\chi_r(K_n\Box K_{1,s}) = \max(n+s, 2n)$, where $r \geq n$ and $n-1+s \leq r$.

Case 2. $r \geq n$ and $n-1+s > r$.

Now, we consider the following two subcases.

Subcase 2.1. $r-n+1 \leq n$.

By Lemma 2.1, we have $\chi_r(K_n\Box K_{1,s}) \geq 2n$. To show that $\chi_r(K_n\Box K_{1,s}) \leq 2n$, we define $c_6 : V(K_n\Box K_{1,s}) \to \{1,2,3,\ldots,n,n+1,\ldots,2n\}$ and a $n \times (s+1)$ matrix $F = (f_{ij})_{n\times(s+1)}$ as follows,

$$F = \begin{bmatrix} n+1 & 1 & 2 & 3 & \cdots & r-n+1 & \cdots & r-n+1 \\ n+2 & 2 & 3 & 4 & \cdots & r-n+2 & \cdots & r-n+2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ n+n-2 & n-2 & n-1 & n & \cdots & r-n-2 & \cdots & r-n-2 \\ n+n-1 & n-1 & n & 1 & \cdots & r-n-1 & \cdots & r-n-1 \\ 2n & n & 1 & 2 & \cdots & r-n & \cdots & r-n \end{bmatrix}_{n\times(s+1)}$$

where the n entries of jth column are $\{j-1, j, j+1, j+2, \ldots, n, 1, 2, \ldots, j-2\} \subseteq \{1,2,\ldots,n\}$ when $2 \leq j \leq r-n+2$, and $f_{ij} = f_{i,r-n+2}$ when $r-n+3 \leq j \leq s+1$, $2 \leq i \leq n$. Define $c_6(V(K_n\Box K_{1,s})) = F$. For $1 \leq i \leq n$, $1 \leq j \leq s$, $c_6(a_iv_j) = f_{i,j+1}$, and so $\{c_6(a_iv_j) | 1 \leq i \leq n, 1 \leq j \leq s\} = \{1,2,3,\ldots,n\}$. For

$1 \leq i \leq n$, $c_6(a_iw) = f_{i1} = n+i$, and so $\{c_6(a_iw)|1 \leq i \leq n\} = \{n+1, n+2, n+3, \ldots, 2n\}$. It follows that, if $k \neq i$, then $c_6(a_iw) \neq c_6(a_iv_j)$, $c_6(a_iw) \neq c_6(a_kw)$ and $c_6(a_iv_j) \neq c_6(a_kv_j)$. As $r-n+1 \leq n$, every entry f_{ij} in F satisfies $1 \leq f_{ij} \leq 2n$, and so c_6 is a proper $2n$-coloring of $K_n \square K_{1,s}$.

Next we need to show c_6 satisfies $(C2)$. For a vertex of the form a_iw, by (2.1), we have $d(a_iw) = |N_{K_n \square K_{1,s}}(a_iw)| = n-1+s$. Since $c_6(N_{K_n \square K_{1,s}}(a_iw)) = \{1, 2, 3, \ldots, r-n+1\} \cup \{n+1, n+2, \ldots, 2n\} \backslash \{n+i\}$, then $|c_6(N_{K_n \square K_{1,s}}(a_iw))| = r = \min\{d(a_iw), r\} = \min\{n-1+s, r\}$. For a vertex of the form a_iv_j, by (2.2), we have $d(a_iv_j) = |N_{K_n \square K_{1,s}}(a_iv_j)| = n$. By matrix F, the color set $c_6(N_{K_n \square K_{1,s}}(a_iv_j))$ always contains $n-1$ different colors of $\{1, 2, 3, \ldots, n\}$ and one color $c_6(a_iw) = n+i$, so $|c_6(N_{K_n \square K_{1,s}}(a_iv_j))| = n = \min\{d(a_iv_j), r\} = \min\{n, r\}$. Thus c_6 is a $(2n, r)$-coloring of $K_n \square K_{1,s}$, so $\chi_r(K_n \square K_{1,s}) \leq 2n$. To sum up, $\chi_r(K_n \square K_{1,s}) = 2n$.

Subcase 2.2. $r-n+1 > n$.

By Theorem 1.3, we have $\chi_r(K_n \square K_{1,s}) \geq \min\{\Delta(K_n \square K_{1,s}), r\} + 1 = \min\{n-1+s, r\} + 1 = r+1$. To show that $\chi_r(K_n \square K_{1,s}) \leq r+1$, we define $c_7 : V(K_n \square K_{1,s}) \to \{1, 2, 3, \ldots, r, r+1\}$ and a $n \times (s+1)$ matrix $P = (p_{ij})_{n \times (s+1)}$ as follows,

$$P = \begin{bmatrix} r-n+2 & 1 & 2 & \cdots & r-n & r-n+1 & \cdots & r-n+1 \\ r-n+3 & 2 & 3 & \cdots & r-n+1 & 1 & \cdots & 1 \\ r-n+4 & 3 & 4 & \cdots & 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ r+1 & n & n+1 & \cdots & n-2 & n-1 & \cdots & n-1 \end{bmatrix}_{n \times (s+1)}$$

where the s entries of ith row are $\{r-n+1+i, i, i+1, i+2, \ldots, r-n+1, 1, 2, \ldots, i-1, \ldots, i-1\} \subseteq \{1, 2, \ldots, r+1\}$ when $1 \leq i \leq n$, and $p_{i,j} = p_{i,r-n+2}$ when $r-n+3 \leq j \leq s+1$. Define $c_7(V(K_n \square K_{1,s})) = P$. For $1 \leq i \leq n$, $1 \leq j \leq s$, $c_7(a_iv_j) = p_{i,j+1}$, and so $\{c_7(a_iv_j)|1 \leq i \leq n, 1 \leq j \leq s\} = \{1, 2, 3, \ldots, r-n+1\}$. For $1 \leq i \leq n$, $c_7(a_iw) = p_{i1} = r-n+1+i$, and so $\{c_7(a_iw)|1 \leq i \leq n\} = \{r-n+2, r-n+3, r-n+4, \ldots, r-n+1+n\} = \{r-n+2, r-n+3, r-n+4, \ldots, r+1\}$. It follows that, if $k \neq i$, then $c_7(a_iw) \neq c_7(a_iv_j)$, $c_7(a_iw) \neq c_7(a_kw)$ and $c_7(a_iv_j) \neq c_7(a_kv_j)$. As $r-n+1 > n$, every entry p_{ij} in P satisfies $1 \leq p_{ij} \leq r+1$, and so c_7 is a proper $(r+1)$-coloring of $K_n \square K_{1,s}$.

Next we need to show c_7 satisfies $(C2)$. For a vertex of the form a_iw, by (2.1), we have $d(a_iw) = |N_{K_n \square K_{1,s}}(a_iw)| = n-1+s$. Since $c_7(N_{K_n \square K_{1,s}}(a_iw)) = \{1, 2, 3, \ldots, r+1\} \backslash \{r-n+1+i\}$, then $|c_7(N_{K_n \square K_{1,s}}(a_iw))| = r = \min\{d(a_iw), r\} = \min\{n-1+s, r\}$. For a vertex of the form a_iv_j, by (2.2), we have $d(a_iv_j) = |N_{K_n \square K_{1,s}}(a_iv_j)| = n$. By matrix P, the color set $c_7(N_{K_n \square K_{1,s}}(a_iv_j))$ contains $n-1$ different colors of $\{1, 2, 3, \ldots, r-n+1\}$ and one color $c_7(a_iw) = r-n+1+i$, so $|c_7(N_{K_n \square K_{1,s}}(a_iv_j))| = n = \min\{d(a_iv_j), r\} = \min\{n, r\}$. Thus c_7 is a $(r+1, r)$-coloring of $K_n \square K_{1,s}$, so $\chi_r(K_n \square K_{1,s}) \leq r+1$. To sum up, $\chi_r(K_n \square K_{1,s}) = r+1$.

By Subcase 2.1 and Subcase 2.2, we can conclude that $\chi_r(K_n \square K_{1,s}) = \max(2n, r+1)$, where $r \geq n$ and $n-1+s > r$.

Case 3. $r < n$.

The proof in this case is the same as in case 3 of Theorem 2.3. □

By Theorems 2.3 and 2.4, we can get Theorem 1.4.

3. Conclusion

In this paper, we considered the r-hued chromatic number of Cartesian product of complete graph K_n and star graph $K_{1,s}$. Firstly, we classify the positive integer r according to its different values, and then combine with the properties of chromatic number of graph G, we get a lower bound of r-hued chromatic number of $K_n \square K_{1,s}$. Secondly, we find a (k, r)-coloring of $K_n \square K_{1,s}$, so we get an upper bound of r-hued chromatic number of $K_n \square K_{1,s}$. Finally, we determine the r-hued chromatic number of Cartesian product of complete graph and star graph.

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