ON \textit{r}-HUED COLORING OF PRODUCT GRAPHS

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Abstract. A \((k,r)\)-coloring of a graph \(G\) is a proper coloring with \(k\) colors such that for every vertex \(v\) with degree \(d(v)\) in \(G\), the color number of the neighbors of \(v\) is at least \(\min\{d(v), r\}\). The smallest integer \(k\) such that \(G\) has a \((k,r)\)-coloring is called the \(r\)-hued chromatic number and denoted by \(\chi_r(G)\). In Kaliraj et al. \textit{Taibah Univ. Sci.} 14 (2020) 168–171, it is determined the 2-hued chromatic numbers of Cartesian product of complete graph and star graph. In this paper, we extend its result and determine the \(r\)-hued chromatic number of Cartesian product of complete graph and star graph.

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1. INTRODUCTION

All graphs are simple and finite, with undefined terminologies and notion begins referred to [1] in this paper. As in [1], \(V(G)\), \(E(G)\), \(\Delta(G)\) and \(\delta(G)\) denote the vertex set, the edge set, the maximum degree and the minimum degree of a graph \(G\), respectively. For \(v \in V(G)\), let \(N_G(v)\) denote the set of vertices adjacent to \(v\) in \(G\), and \(d_G(v) = |N_G(v)|\). For positive integers \(k\) and \(r\), a \((k,r)\)-coloring of a graph \(G\) is a mapping \(c : V(G) \to \{1, 2, 3, 4, \ldots, k\}\), satisfying both of the following conditions:

\((C1)\): \(c(u) \neq c(v)\) for every edge \(uv \in E(G)\); \n\((C2)\): \(|c(N_G(v))| \geq \min\{d_G(v), r\}\) for any \(v \in V(G)\).

Following [1], a mapping \(c : V(G) \to \{1, 2, 3, 4, \ldots, k\}\) satisfying \((C1)\) only is a proper \(k\)-coloring of \(G\). The chromatic number of \(G\), denoted by \(\chi(G)\), is the smallest integer \(k\) such that \(G\) has a proper \(k\)-coloring. The \(r\)-hued chromatic number of \(G\), denoted by \(\chi_r(G)\), is the smallest integer \(k\) such that \(G\) has a \((k,r)\)-coloring. The notion of \(r\)-hued coloring was first introduced in [7,9], where \(\chi_2(G)\) is called the dynamic number of graph \(G\), and the corresponding chromatic number is denoted \(\chi_d(G)\). In [2], Brooks’ Theorem stated that a connected graph \(G\) satisfies \(\chi(G) \leq \Delta(G) + 1\), where the equality holds if and only if \(G\) is an odd cycle or a complete graph. In [7], Lai et al. proved the best possible upper bounds of \(\chi_2(G)\) as an analogue to Brooks’ Theorem.

Theorem 1.1. Let \(G\) be a connected graph.

(i) If \(\Delta(G) \leq 3\), then \(\chi_2(G) \leq 4\), unless \(G = C_5\), in which case \(\chi_2(C_5) = 5\) [7].

(ii) If \(\Delta(G) \geq 4\), then \(\chi_2(G) \leq \Delta(G) + 1\) [7].

Keywords. \((k,r)\)-coloring, \(r\)-hued chromatic number, Cartesian product.

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(iii) If $G$ is planar graph with $G \neq C_5$, then $\chi_2(G) \leq 4$ [5].

In [10], Lai et al. proved that if $G$ is a planar graph and $r \geq 8$, then $\chi_r(G) \leq 2r + 16$. Earlier Brooks type upper bounds of the $r$-hued chromatic number can be found in [3, 6, 8].

**Theorem 1.2.** Let $G$ be a connected graph, and $r \geq 2$ be an integer.

(i) If $\Delta(G) \leq r$, then $\chi_r(G) \leq \Delta(G) + r^2 - r + 1$ [6].

(ii) $\chi_r(G) \leq \Delta^2(G) + 1$, where the equality holds if and only if $G$ is a Moore graph [3].

(iii) $\chi_r(G) \leq r\Delta(G) + 1$, with equality if and only if $G$ is $r$-regular with diameter 2 and girth 5 [8].

A lower bound for $r$-hued chromatic number of $G$ as follows.

**Theorem 1.3** ([6], Prop. 2.1). Let $G$ be a graph, and $r \geq 2$ be an integer. Then $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$, and this lower bound is sharp.

Let $G$ and $H$ be two graphs. The Cartesian product of $G$ and $H$, denoted by $G \square H$, is a graph with the vertex set $V(G) \times V(H)$ such that two vertices $(u, v)$ and $(x, y)$ are adjacent if and only if $u = x$ and $vy \in E(H)$ or $v = y$ and $ux \in E(G)$. It follows by definition that $\Delta(G \square H) = \Delta(G) + \Delta(H)$.

Kaliraj et al. [4] studied 2-hued chromatic numbers of Cartesian product of complete graph and star graph, for positive integers $s \geq 2$ and $n$,

$$\chi_2(K_n \square K_{1,s}) = \begin{cases} 3, & \text{if } n = 1; \\ 4, & \text{if } n = 2; \\ n, & \text{otherwise.} \end{cases}$$

In this paper, we extend the above result, and prove the following theorem.

**Theorem 1.4.** For all fixed positive integers $r$, $r$-hued chromatic number of Cartesian product of complete graph and star graph as follows:

$$\chi_r(K_n \square K_{1,s}) = \begin{cases} n, & \text{if } r < n; \\ \max\{2n, \min\{r + 1, n + s\}\}, & \text{if } r \geq n. \end{cases}$$

## 2. Proofs of the Main Results

Throughout this section, $n \geq 2$, $s \geq 1$ are integers, and we always devote $V(K_n) = \{a_1, a_2, \ldots, a_n\}$, $V(K_{1,s}) = \{w, v_1, \ldots, v_s\}$, where $w$ is the only vertex with $d(w) = s$ in $V(K_{1,s})$. By the definition of Cartesian products,

$$V(K_n \square K_{1,s}) = \bigcup_{i=1}^{n} \{a_i w\} \cup \bigcup_{i=1}^{n} \{a_i v_j : 1 \leq j \leq s\}.$$ 

For presentational purpose, we also write

$$V(K_n \square K_{1,s}) = \begin{bmatrix} a_1 w & a_1 v_1 & a_1 v_2 & \cdots & a_1 v_s \\ a_2 w & a_2 v_1 & a_2 v_2 & \cdots & a_2 v_s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} w & a_{n-1} v_1 & a_{n-1} v_2 & \cdots & a_{n-1} v_s \\ a_n w & a_n v_1 & a_n v_2 & \cdots & a_n v_s \end{bmatrix}_{n \times (s+1)}.$$ 

By the definition of $K_n \square K_{1,s}$, we have the following observations.

$$N_{K_n \square K_{1,s}}(a_i w) = \bigcup_{j=1}^{s} \{a_i v_j\} \cup \bigcup_{k=1, k \neq i}^{n} \{a_k w\} \quad (2.1)$$

$$N_{K_n \square K_{1,s}}(a_i v_j) = \bigcup_{k=1, k \neq i}^{n} \{a_k v_j\} \cup \{a_i w\} \quad (2.2)$$
We prove

Proof. Lemma 2.1. product of complete graphs $K_c$ to $c$

loss of generality, we assume that

Let

Corollary 2.2. If

Since $\Delta(K_n) = \min\{r, n\}$, we first determine a lower bound for $r$-hued chromatic number of Cartesian product graphs $K_n$ and $K_1,s$, which is useful for the proof of Theorems 2.3 and 2.4.

Lemma 2.1. If $r \geq n$, then $\chi_r(K_n \boxtimes K_1,s) \geq 2n$.

Proof. We prove $\chi_r(K_n \boxtimes K_1,s) \geq 2n$ by contradiction. Suppose that $\chi_r(K_n \boxtimes K_1,s) \leq 2n - 1$. We assume that $c_0 : V(K_n \boxtimes K_1,s) \rightarrow \{1, 2, 3, \ldots, 2n - 1\}$ is a $(2n - 1, r)$-coloring. As $s \geq 1$, $K_1,s$ contains a subgraph isomorphic to $K_2$, and so $K_n \boxtimes K_1,s$ always contains an induced subgraph $H = K_n \boxtimes K_2$ (see Fig. 1 for an illustration, where $K_3 \boxtimes K_1,s$ contains $K_3 \boxtimes K_2$ as a subgraph).

Since $|V(H)| = 2n$, there always exist two vertices in $H$ which are colored with the same color. Without loss of generality, we assume that $c_0(a_iw) = c_0(a_jv_1)$, where $i \neq j$. For the vertex $a_i v_1$, by (2.1), we have $\{a_iw, a_jv_1\} \subseteq N_{K_n \boxtimes K_1,s}(a_i v_1)$. Since $r \geq n$, $|c_0(N_{K_n \boxtimes K_1,s}(a_i v_1))| \leq n - 1 < \min\{r, n\} = n$, which contradicts to that $c_0$ is a $(2n - 1, r)$-coloring. Hence $\chi_r(K_n \boxtimes K_1,s) \geq 2n$.

Corollary 2.2. If $r \geq n$, then $\chi_r(K_n \boxtimes K_2) = 2n$.

Proof. Let $V(K_n) = \{a_1, a_2, a_3, \ldots, a_n\}$, and $V(K_2) = \{v_1, v_2\}$. By the definition of Cartesian products, $V(K_n \boxtimes K_2) = \bigcup_{j=1}^{n} \{a_jv_1 : 1 \leq j \leq 2\}$. The order of $K_n \boxtimes K_2$ is $|V(K_n \boxtimes K_2)| = 2n$. On the one hand, $\chi_r(K_n \boxtimes K_2) \leq |V(K_n \boxtimes K_2)| = 2n$. On the other hand, by Lemma 2.1, let $s = 1$, then $\chi_r(K_n \boxtimes K_2) \geq 2n$, so $\chi_r(K_n \boxtimes K_2) = 2n$.

We first prove the case when $s \geq r$ for Theorem 1.4.

Theorem 2.3. Let $K_n \boxtimes K_1,s$ be a Cartesian product graph. If $s \geq r$, then

$$\chi_r(K_n \boxtimes K_1,s) = \begin{cases} r + 1, & \text{if } r \geq 2n; \\ 2n, & \text{if } n \leq r < 2n; \\ n, & \text{if } r < n. \end{cases}$$

Proof. Since $\Delta(K_n) = n - 1$, $\Delta(K_1,s) = s$, then $\Delta(K_n \boxtimes K_1,s) = \Delta(K_n) + \Delta(K_1,s) = (n - 1) + s$. As $n \geq 1$, $\Delta(K_n \boxtimes K_1,s) = n - 1 + s \geq s \geq r$. We consider the following three cases to prove this theorem, and we shall use $n \times (s + 1)$ matrix to present a coloring of $V(K_n \boxtimes K_1,s)$. 

![Figure 1. $K_3 \boxtimes K_1,s$: the circle is $K_3 \boxtimes K_2$.](image-url)
Case 1. $r \geq 2n$.

By Theorem 1.3, we have $\chi_r(K_\infty K_1, s) \geq \min\{\Delta(K_\infty K_1, s) + 1, \min\{n - 1 + s, r\} + 1\} = r + 1$. To show that $\chi_r(K_\infty K_1, s) \leq r + 1$, we define $c_1 : V(K_\infty K_1, s) \rightarrow \{1, 2, 3, \ldots, r + 1\}$ as follows. Since $r \geq 2n$, $r - n + 1 \geq 2n - n + 1 = n + 1$, $r - n + 1 > n$. Let $A = (a_{ij})_{n \times (s + 1)}$ be a $n \times (s + 1)$ matrix as follows,

$$A = \begin{bmatrix}
  r - n + 2 & 1 & 2 & \ldots & r - n - 1 & r - n & r - n + 1 & \ldots & r - n + 1 \\
  r - n + 3 & 2 & 3 & \ldots & r - n & r - n - 1 & 1 & \ldots & 1 \\
  r - n + 4 & 3 & 4 & \ldots & r - n + 1 & 1 & 2 & \ldots & 2 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  r + 1 & n & n - 1 & \ldots & n - 3 & n - 2 & n - 1 & \ldots & n - 1
\end{bmatrix}_{n \times (s + 1)}$$

where the $s$ entries of $i$th row are $\{r - n + 1 + i, i, i + 1, \ldots, r - n, r - n + 1, 1, 2, \ldots, i - 1, i - 1, \ldots, i - 1\} \subseteq \{1, 2, 3, \ldots, r + 1\}$ when $1 \leq i \leq n$, and $a_{i,j} = a_{i-r,n+2}$ when $r - n + 3 \leq j \leq s + 1$. Define $c_1(V(K_\infty K_1, s)) = A$. For $1 \leq i \leq n$, $a_{i,j} = a_{i,j+1}$, and so $\{c_1(a_{ij}) : 1 \leq i \leq n, 1 \leq j \leq s\} = \{1, 2, 3, \ldots, r - n + 1\}$. For $1 \leq i \leq n$, $c_1(a_{i,w}) = a_{i+1} - r - n + i$, and so $\{c_1(a_{i,w}) : 1 \leq i \leq n\} = \{r - n + 1 + 1, r - n + 1 + 2, n - n + 1, \ldots, n - 1, n\}. It follows that, if $k \neq i$, then $c_1(a_{i,w}) \neq c_1(a_{j,v})$, and $c_1(a_{i,w}) \neq c_1(a_{k,w})$ and $c_1(a_{j,v}) \neq c_1(a_{k,v})$. As $r \geq 2n$, every entry $a_{i,j}$ in $A$ satisfies $1 \leq a_{i,j} \leq r + 1$, and so $c_1$ is a proper $(r + 1)$-coloring of $K_\infty K_1, s$.

Next we need to show $c_1$ satisfies (C2). For a vertex of the form $a_iw$, by (2.1), we have $d(a_{i,w}) = |N_{K_\infty K_1, s}(a_{i,w})| - n - 1$. Since $c_1(N_{K_\infty K_1, s}(a_{i,w})) = \{1, 2, 3, \ldots, r + 1\} \setminus \{r - n + 1 + i\}$, $\chi_r(K_\infty K_1, s) \leq r + 1$. Thus $c_1$ is a $(r + 1)$-coloring of $K_\infty K_1, s$.

Case 2. $n \leq r < 2n$.

By Lemma 2.1, we have $\chi_r(K_\infty K_1, s) \geq 2n$. Since $1 \leq n \leq r \leq s, 1 \leq r - n + 1 \leq r \leq s, r - n + 1 \leq s$, and as $n \leq r < 2n$, $1 \leq r - n + 1 < n + 1$, so $r - n + 1 \leq s$. To show that $\chi_r(K_\infty K_1, s) \leq 2n$, we define $c_2 : V(K_\infty K_1, s) \rightarrow \{1, 2, 3, \ldots, n, n + 1, 2n\}$ and a $n \times (s + 1)$ matrix $B = (b_{ij})_{n \times (s + 1)}$ as follows,

$$B = \begin{bmatrix}
  n + 1 & 1 & 2 & 3 & \ldots & r - n + 1 & r - n + 1 & \ldots & r - n + 1 \\
  n + 2 & 2 & 3 & 4 & \ldots & r - n + 2 & r - n + 2 & \ldots & r - n + 2 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  n + n - 2 & n - 2 & n - 1 & n & \ldots & r - n + 2 & r - n + 2 & \ldots & r - n + 2 \\
  n + n - 1 & n - 1 & n & \ldots & r - n + 1 & r - n + 1 & \ldots & r - n + 1 \\
  2n & n - 1 & n & \ldots & r - n & r - n & \ldots & r - n
\end{bmatrix}_{n \times (s + 1)}$$

where the $n$ entries of $j$th column are $\{j - 1, j + 1, j + 2, \ldots, n - 1, n, 1, 2, \ldots, j - 2, \ldots, j - 2\} \subseteq \{1, 2, 3, \ldots, n - 1, n\}$ when $2 \leq j \leq r - n + 2$, and $b_{ij} = b_{i-r,n+2}$ when $r - n + 3 \leq j \leq s + 1, 2 \leq i \leq n$. Define $c_2(V(K_\infty K_1, s)) = B$. For $1 \leq i \leq n$, $1 \leq j \leq s, c_2(a_{i,v}) = a_{i+j+1}$, and so $\{c_2(a_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq s\} = \{1, 2, 3, \ldots, n\}$. For $1 \leq i \leq n$, $c_2(a_{i,w}) = a_{i+1} - n + i$, and so $\{c_2(a_{i,w}) : 1 \leq i \leq n\} = \{n + 1, n + 2, n + 3, \ldots, 2n\}. It follows that, if $k \neq i$, then $c_2(a_{i,w}) \neq c_2(a_{j,v})$, $c_2(a_{i,w}) \neq c_2(a_{k,w})$ and $c_2(a_{j,v}) \neq c_2(a_{k,v})$. As $n \leq r < 2n$, every entry $b_{ij}$ in $B$ satisfies $1 \leq b_{ij} \leq 2n$, and so $c_2$ is a proper $2n$-coloring of $K_\infty K_1, s$.

Next we need to show $c_2$ satisfies (C2). For a vertex of the form $a_iw$, by (2.1), we have $d(a_{i,w}) = |N_{K_\infty K_1, s}(a_{i,w})| = n - 1 + s$. Since $c_2(N_{K_\infty K_1, s}(a_{i,w})) = \{1, 2, 3, \ldots, n - 1 + s\} \cup \{n + 1, n + 2, n + 3, \ldots, 2n\}$, then $\chi_r(K_\infty K_1, s) \leq 2n$. By matrix $B$, the color set $c_2(N_{K_\infty K_1, s}(a_{i,w}))$ contains $n - 1$ different colors of $\{1, 2, 3, \ldots, n\}$ and one color $c_2(a_{i,w}) = n + i$, we have $\{c_2(N_{K_\infty K_1, s}(a_{i,w}))\} = n$, then $\chi_r(K_\infty K_1, s) \leq 2n$. To sum up, $\chi_r(K_\infty K_1, s) \leq 2n.$
Case 3. $r < n$.

Since $K_n \Box K_{1,s}$ always contains an induced subgraph $K_n$, $\chi_r(K_n \Box K_{1,s}) \geq n$. To show that $\chi_r(K_n \Box K_{1,s}) \leq n$, we define $c_3 : V(K_n \Box K_{1,s}) \rightarrow \{1, 2, 3, \ldots, n\}$ and a $n \times (s+1)$ matrix $C = (c_{ij})_{n \times (s+1)}$ as follows,

$$C = \begin{bmatrix}
2 & 1 & 1 & \cdots & 1 \\
3 & 2 & 2 & \cdots & 2 \\
\vdots & \vdots & \vdots & & \vdots \\
n & n-1 & n-1 & \cdots & n-1 \\
1 & n & n & \cdots & n
\end{bmatrix}_{n \times (s+1)}$$

Define $c_3(V(K_n \Box K_{1,s})) = C$. For $1 \leq i \leq n$, $1 \leq j \leq s$, $c_3(a_{ij}) = c_{ij} + 1$, and so $\{c_3(a_{ij}) | 1 \leq i \leq n, 1 \leq j \leq s\} = \{1, 2, 3, \ldots, n\}$. For $1 \leq i \leq n - 1$, $c_3(a_iw) = i + 1$, and $c_3(a_{nw}) = 1$, so $\{c_3(a_iw) | 1 \leq i \leq n\} = \{1, 2, 3, \ldots, n\}$. It follows that, if $k \neq i$, then $c_3(a_iw) \neq c_3(a_{ivj})$, $c_3(a_iw) \neq c_3(a_kw)$ and $c_3(a_{ivj}) \neq c_3(a_{kvj})$. Since every entry $c_{ij}$ in $C$ satisfies $1 \leq c_{ij} \leq n$, so $c_3$ is a proper $n$-coloring of $K_n \Box K_{1,s}$.

Next we need to show $c_3$ satisfies $(C2)$. For a vertex of the form $a_iw$, by (2.1), we have $|N_{K_n \Box K_{1,s}}(a_iw)| = n - 1 + s$, so $d(a_iw) = n - 1 + s \geq n > r$. For $1 \leq i \leq n - 1$, $c_3(N_{K_n \Box K_{1,s}}(a_iw)) = \{1, 2, 3, \ldots, n\} \{i+1\}$, and for $i = n$, $c_3(N_{K_n \Box K_{1,s}}(a_iw)) = \{n\}$. For a vertex of the form $a_{ivj}$, by (2.2), we have $|N_{K_n \Box K_{1,s}}(a_{ivj})| = n$, so $d(a_{ivj}) = n > r$. Since $c_3(N_{K_n \Box K_{1,s}}(a_{ivj})) = \{1, 2, 3, \ldots, n\} \{i\}$, then $c_3(N_{K_n \Box K_{1,s}}(a_{ivj})) = n - 1 \geq n - 1 + s, r$. Thus $c_3$ is a $(n, r)$-coloring of $K_n \Box K_{1,s}$, hence $\chi_r(K_n \Box K_{1,s}) \leq n$. To sum up, $\chi_r(K_n \Box K_{1,s}) = n$. \hfill $\Box$

In the following, we prove the case $s < r$ for Theorem 1.4.

**Theorem 2.4.** Let $K_n \Box K_{1,s}$ be a Cartesian product graph. If $s < r$, then

$$\chi_r(K_n \Box K_{1,s}) = \begin{cases} 
\max(n+s, 2n), & \text{if } r \geq n \text{ and } n - 1 + s \leq r; \\
\max(2n, r + 1), & \text{if } r \geq n \text{ and } n - 1 + s > r; \\
n, & \text{if } r < n.
\end{cases}$$

**Proof.** We consider the following three cases to prove this theorem, and we shall use $n \times (s+1)$ matrix to present a coloring of $V(K_n \Box K_{1,s})$.

**Case 1.** $r \geq n$ and $n - 1 + s \leq r$.

Since $\Delta(K_n) = n - 1$, $\Delta(K_{1,s}) = s$, then $\Delta(K_n \Box K_{1,s}) = \Delta(K_n) + \Delta(K_{1,s}) = (n - 1) + s \leq r$. We consider the following two subcases.

**Subcase 1.1.** $n \leq s$.

By Theorem 1.3, we have $\chi_r(K_n \Box K_{1,s}) \geq \min\{\Delta(K_n \Box K_{1,s}), r\} + 1 = \min\{n-1+s, r\} + 1 = n-1+s+1 = n+s$. To show that $\chi_r(K_n \Box K_{1,s}) \leq n+s$, we define $c_4 : V(K_n \Box K_{1,s}) \rightarrow \{1, 2, 3, \ldots, n+s\}$ and a $n \times (s+1)$ matrix $D = (d_{ij})_{n \times (s+1)}$ as follows,

$$D = \begin{bmatrix}
s+1 & 1 & 2 & \cdots & s-2 & s-1 & s \\
s+2 & 2 & 3 & \cdots & s-1 & s & 1 \\
s+3 & 3 & 4 & \cdots & s & 1 & 2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
s+n & n & n+1 & \cdots & n-3 & n-2 & n-1
\end{bmatrix}_{n \times (s+1)}$$

where the $s+1$ entries of $i$th row are $\{s+i, i+1, \ldots, s-1, s, i, 2, \ldots, i-1\} \subseteq \{1, 2, \ldots, s+n\}$ when $1 \leq i \leq n$. Define $c_4(V(K_n \Box K_{1,s})) = D$. For $1 \leq i \leq n$, $1 \leq j \leq s$, $c_4(a_{ij}) = d_{ij}+1$, and so $\{c_4(a_{ij}) | 1 \leq i \leq n, 1 \leq j \leq s\} = \{1, 2, 3, \ldots, s\}$. For $1 \leq i \leq n$, $c_4(a_iw) = d_{1i} = s+i$, and so $\{c_4(a_iw) | 1 \leq i \leq n\} = \{s+1, s+2, s+3, \ldots, s+n\}$. It follows that, if $k \neq i$, then $c_4(a_iw) \neq c_4(a_{ivj})$,
Now, we consider the following two subcases.

Next we need to show $c_4$ satisfies (C2). For a vertex of the form $a_i w$, by (2.1), we have $d(a_i w) = |N_{K_n □ K_1}(a_i)| = n + 1 - s$. Since $c_1(N_{K_n □ K_1}(a_i)) = \{1, 2, 3, \ldots, s\} \cup \{s + 1, s + 2, \ldots, s + n\} \setminus \{s + i\}$, then $|c_4(N_{K_n □ K_1}(a_i))| = n + s + 1$, $|c_4(N_{K_n □ K_1}(a_i))| = \min\{d(a_i w), r\} = \min\{n - 1 + s, r\}$. For a vertex of the form $a_i v_j$, by (2.2), we have $d(a_i v_j) = |N_{K_n □ K_1}(a_i v_j)| = n$. By matrix $D$, the color set $c_4(N_{K_n □ K_1}(a_i v_j))$ always contains $n - 1$ different colors of $\{1, 2, 3, \ldots, s\}$ and one color $c_4(a_i w) = s + i$, so $|c_4(N_{K_n □ K_1}(a_i v_j))| = n = \min\{d(a_i v_j), r\} = \min\{n, r\}$. Thus $c_4$ is a $(n + s, r)$-coloring of $K_n □ K_1$, then $\chi_r(K_n □ K_1) ≤ n + s$. To sum up, $\chi_r(K_n □ K_1) = n + s$.

**Subcase 1.2.** $n > s$.

By Lemma 2.1, we have $\chi_r(K_n □ K_1) ≥ 2n$. To show that $\chi_r(K_n □ K_1) ≤ 2n$, we define $c_5 : V(K_n □ K_1, s) → \{1, 2, 3, \ldots, 2n\}$ and a $n \times (s + 1)$ matrix $E = (e_{i j})_{n \times (s + 1)}$ as follows,

$$E = \begin{bmatrix}
  n + 1 & 1 & 2 & 3 & \cdots & s \\
  n + 2 & 2 & 3 & 4 & \cdots & s + 1 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  n + n - 2 & n - 2 & n - 3 & \cdots & 1 & \cdots & 3 \\
  n + n - 1 & n - 1 & n & 1 & \cdots & 2 \\
  2n & n & 1 & 2 & \cdots & s + 1 \\
\end{bmatrix}_{n \times (s + 1)}$$

where the $n$ entries of $j$th column are $\{j - 1, j, j + 1, \ldots, n - 1, n, 1, 2, \ldots, j - 2\} \subseteq \{1, 2, \ldots, n\}$ when $2 \leq j \leq s + 1$. Define $c_5(V(K_n □ K_1)) = E$. For $1 \leq i \leq n$, $1 \leq j \leq s$, $c_5(a_i v_j) = e_{i j + 1}$, and so $\{c_5(a_i v_j)\}_1 \leq i \leq n, 1 \leq j \leq s = \{1, 2, 3, \ldots, n\}$. For $1 \leq i \leq n, c_5(a_i w) = e_{i 1} = n + i$, and so $\{c_5(a_i w)\}_1 \leq i \leq n = \{n + 1, n + 2, n + 3, \ldots, 2n\}$. It follows that, if $k \neq i$, then $c_5(a_i w) \neq c_5(a_i v_j)$, $c_5(a_i w) \neq c_5(a_k v_j)$ and $c_5(a_i v_j) \neq c_5(a_k v_j)$. As $n > s$, every entry $e_{i j}$ in $E$ satisfies $1 \leq e_{i j} \leq 2n$, and so $c_5$ is a proper $2n$-coloring of $K_n □ K_1$.

Next we need to show $c_5$ satisfies (C2). For a vertex of the form $a_i w$, by (2.1), we have $d(a_i w) = |N_{K_n □ K_1}(a_i)| = n + 1 - s$. Since $c_5(N_{K_n □ K_1}(a_i)) = \{1, 2, \ldots, n + 1, n, 2, \ldots, 2n\} \setminus \{n + i\}$, then $|c_5(N_{K_n □ K_1}(a_i))| = 2n - 1 ≥ \min\{d(a_i w), r\} = \min\{n - 1 + s, r\} = n - 1 + s$. For a vertex of the form $a_i v_j$, by (2.2), we have $d(a_i v_j) = |N_{K_n □ K_1}(a_i v_j)| = n$. By matrix $E$, the color set $c_5(N_{K_n □ K_1}(a_i v_j))$ always contains $n - 1$ different colors of $\{1, 2, 3, \ldots, n\}$ and one color $c_5(a_i w) = n + i$, so $|c_5(N_{K_n □ K_1}(a_i v_j))| = 2n = \min\{d(a_i v_j), r\} = \min\{n, r\}$. Thus $c_5$ is a $(2n, r)$-coloring of $K_n □ K_1$, so $\chi_r(K_n □ K_1) ≤ 2n$. To sum up, $\chi_r(K_n □ K_1) = 2n$.

By Subcases 1.1 and 1.2, we can conclude that $\chi_r(K_n □ K_1) = \max(n + s, 2n)$, where $r ≥ n$ and $n - 1 + s ≤ r$.

**Case 2.** $r ≥ n$ and $n - 1 + s > r$.

Now, we consider the following two subcases.

**Subcase 2.1.** $r - n + 1 ≤ n$.

By Lemma 2.1, we have $\chi_r(K_n □ K_1) ≥ 2n$. To show that $\chi_r(K_n □ K_1) ≤ 2n$, we define $c_6 : V(K_n □ K_1) → \{1, 2, 3, \ldots, n + 1, \ldots, 2n\}$ and a $n \times (s + 1)$ matrix $F = (f_{i j})_{n \times (s + 1)}$ as follows,

$$F = \begin{bmatrix}
  n + 1 & 1 & 2 & 3 & \cdots & r - n + 1 & \cdots & r - n + 1 \\
  n + 2 & 2 & 3 & 4 & \cdots & r - n + 2 & \cdots & r - n + 2 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  n + n - 2 & n - 2 & n - 3 & \cdots & 1 & \cdots & 3 \\
  n + n - 1 & n - 1 & n & 1 & \cdots & 2 \\
  2n & n & 1 & 2 & \cdots & r - n + 1 & \cdots & r - n + 1 \\
\end{bmatrix}_{n \times (s + 1)}$$

where the $n$ entries of $j$th column are $\{j - 1, j, j + 1, j + 2, \ldots, n - 1, n, 1, 2, \ldots, j - 2\} \subseteq \{1, 2, \ldots, n\}$ when $2 \leq j \leq r - n + 2$, and $f_{i j} = f_{i r - n + 2}$ when $r - n + 3 ≤ j ≤ s + 1$, $2 \leq i \leq n$. Define $c_6(V(K_n □ K_1)) = F$. For $1 \leq i \leq n, 1 \leq j \leq s$, $c_6(a_i v_j) = f_{i j + 1}$, and so $\{c_6(a_i v_j)\}_1 \leq i \leq n, 1 \leq j \leq s = \{1, 2, 3, \ldots, n\}$. For
1 ≤ i ≤ n, c₀(aᵢ,w) = fᵢ₁ = n + i, and so \{c₀(aᵢ,w)|1 ≤ i ≤ n\} = \{n + 1, n + 2, n + 3, \ldots, 2n\}. It follows that, if k ≠ i, then c₀(aᵢ,w) ≠ c₀(aᵢ⁺⁰, w), c₀(aᵢ,w) ≠ c₀(aᵢwj) and c₀(aᵢ⁺⁰, w) ≠ c₀(aᵢwj). As r - n + 1 ≤ n, every entry fᵢᵢ in F satisfies 1 ≤ fᵢᵢ ≤ 2n, and so c₀ is a proper 2n-coloring of \(K_n \square K_{1,s}\).

Next we need to show c₀ satisfies (C2). For a vertex of the form aᵢ,w, by (2.1), we have \(d(aᵢ,w) = |N\{aᵢ,w\}| = n - 1 + s\). Since \(c₀(N\{aᵢ,w\}) = \{1, 2, 3, \ldots, r - n + 1\} \cup \{n + 1, n + 2, \ldots, 2n\} \setminus \{n + i\}\), then \(c₀(N\{aᵢ,w\}) = r = \min\{d(aᵢ,w), r\} = \min\{n - 1 + s, r\}\). For a vertex of the form aᵢ⁺⁰, w, by (2.2), we have \(d(aᵢ⁺⁰, w) = |N\{aᵢ⁺⁰, w\}| = n\). By matrix F, the color set \(c₀(N\{aᵢ⁺⁰, w\})\) always contains n - 1 different colors of \(\{1, 2, 3, \ldots, n\}\) and one color c₀(aᵢ⁺⁰, w) = n + i, so \(c₀(N\{aᵢ⁺⁰, w\}) = n = \min\{d(aᵢ⁺⁰, w), r\} = \min\{n, r\}\). Thus c₀ is a \((2n, r)\)-coloring of \(K_n \square K_{1,s}\), so \(χ_r(K_n \square K_{1,s}) ≤ 2n\). To sum up, \(χ_r(K_n \square K_{1,s}) = 2n\).

Subcase 2.2. \(r - n + 1 > n\).

By Theorem 1.3, we have \(χ_r(K_n \square K_{1,s}) ≥ \min\{Δ(K_n \square K_{1,s}), r\} + 1 = \min\{n - 1 + s, r\} + 1 = r + 1\). To show that \(χ_r(K_n \square K_{1,s}) ≤ r + 1\), we define \(c₇ : V(K_n \square K_{1,s}) → \{1, 2, 3, \ldots, r + 1\}\) and a \(n \times (s + 1)\) matrix \(P = (pᵢⱼ)_{n \times (s + 1)}\) as follows,

\[
P = \begin{bmatrix}
    r - n + 2 & 1 & 2 & \cdots & r - n & r - n + 1 & \cdots & r - n + 1 \\
    r - n + 3 & 2 & 3 & \cdots & r - n + 1 & 1 & \cdots & 1 \\
    r - n + 4 & 3 & 4 & \cdots & 1 & 2 & \cdots & 2 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    r + 1 & n & n + 1 & \cdots & n - 2 & n - 1 & \cdots & n - 1 \\
\end{bmatrix}_{n \times (s + 1)}
\]

where the \(s\) entries of \(i\)th row are \(\{r - n + 1 + i, i, i + 1, i + 1, i + 2, \ldots, r - n + 1, 1, 2, \ldots, i - 1, i - 1\}\) \(\subseteq \{1, 2, \ldots, r + 1\}\) when \(1 ≤ i ≤ n\), and \(pᵢᵢ = pᵢᵢ+n+2\) when \(r - n + 3 \leq s + 1\). Define \(c₇(V(K_n \square K_{1,s})) = P\). For \(1 ≤ i ≤ n, 1 ≤ j ≤ s\), \(c₇(aᵢ⁺⁰, w) = pᵢᵢ+j+1\), and so \(c₇(aᵢ⁺⁰, w)|1 ≤ i ≤ n, 1 ≤ j ≤ s\} = \{1, 2, 3, \ldots, r - n + 1\}. For \(1 ≤ i ≤ n, c₇(aᵢ,w) = pᵢᵢ = r - n + 1 + i\), and so \(c₇(aᵢ,w)|1 ≤ i ≤ n\} = \{r - n + 2, r - n + 3, r - n + 4, \ldots, r - n + 1\}\). It follows that, if \(k ≠ i\), then \(c₇(aᵢ,w) ≠ c₇(aᵢ⁺⁰, w)\). As \(r - n + 1 > n\), every entry \(pᵢᵢ+j\) in \(P\) satisfies \(1 ≤ pᵢᵢ+j ≤ r + 1\), and so \(c₇\) is a proper \((r + 1)\)-coloring of \(K_n \square K_{1,s}\).

Next we need to show \(c₇\) satisfies (C2). For a vertex of the form aᵢ⁺⁰, w, by (2.1), we have \(d(aᵢ⁺⁰, w) = |N\{aᵢ⁺⁰, w\}| = n - 1 + s\). Since \(c₇(N\{aᵢ⁺⁰, w\}) = \{1, 2, 3, \ldots, r + 1\} \setminus \{r - n + 1 + i\}\), then \(c₇(N\{aᵢ⁺⁰, w\}) = r = \min\{d(aᵢ⁺⁰, w), r\} = \min\{n - 1 + s, r\}\). For a vertex of the form aᵢ⁺⁰, w, by (2.2), we have \(d(aᵢ⁺⁰, w) = |N\{aᵢ⁺⁰, w\}| = n\). By matrix \(P\), the color set \(c₇(N\{aᵢ⁺⁰, w\})\) always contains \(n - 1\) different colors of \(\{1, 2, 3, \ldots, n\}\) and one color c₀(aᵢ⁺⁰, w) = n + i, so \(c₇(N\{aᵢ⁺⁰, w\}) = n = \min\{d(aᵢ⁺⁰, w), r\} = \min\{n, r\}\). Thus \(c₇\) is a \((r + 1, r)\)-coloring of \(K_n \square K_{1,s}\), so \(χ_r(K_n \square K_{1,s}) ≤ r + 1\).

By Subcase 2.1 and Subcase 2.2, we can conclude that \(χ_r(K_n \square K_{1,s}) = \max\{2n, r + 1\}\), where \(r ≥ n\) and \(n - 1 + s > r\).

Case 3. \(r < n\).

The proof in this case is the same as in case 3 of Theorem 2.3.

By Theorems 2.3 and 2.4, we can get Theorem 1.4.

3. Conclusion

In this paper, we considered the \(r\)-hued chromatic number of Cartesian product of complete graph \(K_n\) and star graph \(K_{1,s}\). Firstly, we classify the positive integer \(r\) according to its different values, and then combine with the properties of chromatic number of graph \(G\), we get a lower bound of \(r\)-hued chromatic number of \(K_n \square K_{1,s}\). Secondly, we find a \((k, r)\)-coloring of \(K_n \square K_{1,s}\), so we get an upper bound of \(r\)-hued chromatic number of \(K_n \square K_{1,s}\). Finally, we determine the \(r\)-hued chromatic number of Cartesian product of complete graph and star graph.
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