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ROLE OF FLEXIBLE DATA IN EVALUATION PRODUCTIVITY AND COST EFFICIENCY USING DATA ENVELOPMENT ANALYSIS

M. Shahkooeei[1], F. Rezai Balf[2,*], M. Rabbani[1] and M. Fallah Jelodar[3]

Abstract. In decision management science, recognizing the inputs and outputs of an organization is very important to evaluate its performance. In particular, it becomes more important when costs are incurred for the organization's inputs. In this paper, we evaluate the cost efficiency of a set of decision-making units (DMUs) so that some of its indices can appear as flexibly in the input or output role. Since, if flexible indices are evident in the input index, then they play an important role in costs, it will be important to identify the performance of the units. However, in this paper, using data envelopment analysis (DEA) models, we determine the cost efficiency and productivity of a set of decision-making units with multiple inputs and multiple outputs in the presence of flexible indices. Finally, we present an example that shows the effect of the flexible index on cost efficiency, and also with an application example, we will determine the cost efficiency and productivity of 40 branches of the banks. The obtained result is compared with one of the other methods.

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1. Introduction

One of the economic issues that has attracted the attention of managers and industry owners is to achieve the lowest cost. They seek appropriate management strategies and tools in order to maximize revenue due to restricted resources such as capital, energy, and manpower [13]. Meanwhile, DEA is a suitable non-parametric method is based on mathematical programming problems and several essential hypothesis [4, 18]. It measures the relative efficiency of a set of DMUs with multiple inputs and multiple outputs. DEA was first suggested by Charens *et al.* [6] and later developed by other researchers. However, measuring the efficiency of a set of DMUs in terms of economic production has a long history, for example, the impact of financial technology on China's banking industry: An application of the metafrontier cost Malmquist productivity index [7].

Cost efficiency is an important issue that has a critical role in achieving economic growth and sustainable development [7, 15, 28].The presented method by Cook and Zhu [8] assumes the capability of the ability DMU to produce the current output concerning its prices with minimal cost. These concepts are explored in the Farrell paper [12], which is the idea of evaluating the efficiency of many generating units. The Farrell model

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[1] Department of Mathematics, Sari Branch, Islamic Azad University, Sari, Iran.
[2] Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran.
[3] Department of Mathematics, Ayatollah Amoli Branch, Islamic Azad University, Amol, Iran.
*Corresponding author: fr.balf@qaemiau.ac.ir

[12] has been developed as a mathematical planning model for evaluating the cost efficiency of DMUs by Färe *et al.* [11]. The cost efficiency and cost Malmquist productivity has been used in many application problems [14, 19, 23–26, 29].

Productivity is another concept that can show the progress and regress of a production unit in a particular period [12, 34]. In 1953, the Malmquist productivity index (MPI) was introduced by Malmquist, which did not have the drawbacks of the previous methods [20]. On the other hand, there was no need for information about the prices of inputs and outputs. The same factor caused this method to meet a high turnout on the other side, to the point where the index developed by Fare *et al.* [10, 11]. He combined it with the efficiency of DEA that was introduced by Farrell to measure productivity [12]. Therefore, the Malmquist productivity index has consisted of two components;

1) Efficiency change, which shows the relative change in efficiency between the periods.
2) Changes in technology or frontier productivity index that shows the relative distance between the frontiers, *i.e.*, measures the change of frontiers between two periods.

According to the economic crises in the world, some researchers expanded MPI and merged it with various concepts. Subal *et al.* [28] used cost Malmquist models for the allocative efficiency and effects of input price change should also be used. Hosseinzade *et al.* presented the cost Malmquist productivity model for price uncertainty and interval data [16, 17].

Traditional DEA models assume that input and output data are known, but in some cases, there is data whose role in input or output is disputed by some DMs. Therefore, this type of data is as flexible data by Cook and Zhu due to the uncertainty of their role [8]. In such a case, the previous models can no longer be used and should be used as flexible models [1, 29]. To better describe the issue in evaluating banks, deposits play a flexible data role. A deposit is an input because it is an investment in economic growth or production, and on the other hand it is an output, because workforce, advertising, and money have been spent to obtain it. In 2007, Cook and Zhu [8] proposed a model for classifying flexible variables using the CCR multiplier model. Amirteimoori *et al.* [2, 3] and Toloo [31–33] developed models on inputs and outputs indices and also flexible data and proposed a model for determining the nature of flexible data using the CCR envelopment form. The presence of flexible data in some cases made difficulties and differences in decisions and calculations. Martin [22] proposed a comment on Classifying flexible measures in data envelopment analysis according to a slacks-based measure.

The importance of input indices is obvious in the discussion related to the cost efficiency of decision-making units (DMUs) in the empirical application [35]. Some problems arise in decision-making if the inputs index of the units is not precisely recognized [9]. Especially, some indices have flexibility. In this case, previous models of data envelopment analysis will have difficulty in evaluating cost efficiency. Therefore, in this paper, we introduce a model for determining cost efficiency so that our model can determine the cost efficiency and cost productivity of DMUs in the presence of flexible indices.

This paper unfolds as follows. Section 2 discusses the background of the research. Section 3 introduces the methodology and two education examples. In Section 4, we illustrate the application of 40 Tehran banking industry. Some concluding remarks follow in Section 5.

2. The background of research

DEA was introduced by Charnes, Cooper, and Rhodes (CCR) [6] as a powerful tool for assessing the relative efficiency of DMUs. In DEA, efficient units accept an efficiency value of one, and in the case of inefficiency, values less than one. Efficient DMUs are referred to as benchmark DMUs for inefficient units. In today's world, the correct and optimal use of resources requires careful management solutions that prevent people's capital and investors so each unit of production and economics needs to be used to manage their current costs at any time, using the appropriate tool such as DEA and specifically the cost efficiency in this field.

2.1. The cost efficiency

Cost efficiency is one of the important application subjects in decision sciences in order to evaluate the performance of DMUs [16]. Consider n DMUs (DMUj, $j = 1, \ldots, n$.) each uses m to inputs to produce s outputs. Also $x_j = (x_{1j}, \ldots, x_{mj})$ and $y_j = (y_{1j}, \ldots, y_{sj})$ are the input and output vectors related to DMUj, respectively, while the cost vector $c = (c_1, \ldots, c_m)$ corresponds to the input. The cost efficiency model aims to find the unit that uses the lowest cost to purchase inputs less than the desired inputs, to produce outputs equal to the outputs of the unit under evaluation. The cost efficiency model for evaluating DMUo is as follows [15,16]:

$$\begin{aligned} \min \omega = & \sum_{i=1}^{m} c_i x_i \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, \ldots, s, \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n, \end{aligned} \tag{1}$$

where $x_i, i = 1, \ldots, m$, are the variables of the model (1) and express the amount of input required to produce a given output. The total cost efficiency ratio is defined as the ratio of the minimum cost to the current cost, i.e;

$$CE_o = \frac{\omega^*}{cx_o} \leq 1. \tag{2}$$

Therefore, it can be easily found that the value is always less than or equal to one. As we know, cost efficiency can be divided into two parts: One is the input radial technical efficiency (TE) and the other is the input allocation efficiency (AE) [19,28,29]. It's that, $CE = TE \times AE$.

2.2. The Malmquist productivity index

Productivity growth is one of the major sources of economic development and a thorough understanding of factors affecting productivity is very important [28, 30, 34]. The Malmquist index was first suggested by Malmquist [20] as a quantity index for use in the analysis of consumption of inputs. Färe *et al.* [11] combined ideas on the measurement of efficiency Farrell [12] for the measurement of productivity to construct a Malmquist productivity index directly from input and output data using DEA. This DEA-based Malmquist productivity index has proven itself to be a good tool for measuring the productivity change of DMUs. The index is usually applied to the measurement of productivity change over time, and can be multiplicatively decomposed into an efficiency change index and a technological change index [5,30]. To introduce the concept of a distance function, consider that in a time period t, the DMUs are using inputs $x \in \mathbb{R}^m_+$ to produce outputs $y \in \mathbb{R}^s_+$. The two single period measures can be obtained by using the CCR DEA model for evaluation DMUo as follows [6]:

$$\begin{aligned} D^t_o(x^t_o, y^t_o) = & \min \theta_o \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x^t_{ij} \leq \theta_o x^t_{io}, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y^t_{rj} \geq y^t_{ro}, \quad r = 1, \ldots, s, \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n, \end{aligned} \tag{3}$$

Model (3) evaluates DMUo at time t. If in the optimality the value objective function accepts the value of 1, then DMUo is efficient, and otherwise, it gives inefficiency of the unit o with the value less than one at time t.

The optimal value of the objective function (3) shows how much can be reduced the input value of the under-evaluation unit, DMUo, to produce at least its output level. Also, by replacing $t+1$ instead to t in model (3), we obtain the CCR model at the period $t+1$, so that it determines the technical efficiency of $D_o^{t+1}(x_o^{t+1}, y_o^{t+1})$ for DMUo at time $t+1$.

In order to calculate changes in technology between two periods t and $t+1$, model (4) is introduced for evaluating DMUo at the period t relative to the technology frontier at the time period $t+1$ with the technical efficiency of $D_o^{t+1}(x_o^t, y_0^t)$, as follows:

$$
\begin{aligned}
D_o^{t+1}(x_o^t, y_o^t) = \min \theta_o \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^{t+1} \leq \theta_o x_{io}^t, \quad && i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{t+1} \geq y_{ro}^t, \quad && r = 1, \ldots, s, \\
& \lambda_j \geq 0, \quad && j = 1, \ldots, n,
\end{aligned}
\tag{4}
$$

By replacing $t+1$ instead to t in the model (4), we evaluate the period $t+1$ relative to the technology frontier at the period t with the optimal value $D_o^t(x_o^{t+1}, y_o^{t+1})$. Now, the Malmquist index is defined in order to evaluate the productivity DMU o between two periods t and $t+1$ as follows [11]:

$$
MPI_o = \left[\frac{D_o^t(x_o^{t+1}, y_o^{t+1})}{D_o^t(x_o^t, y_o^t)} \times \frac{D_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{D_o^{t+1}(x_o^t, y_o^t)} \right]^{\frac{1}{2}}. \tag{5}
$$

In relation (5), $D_o^t(x_o^t, y_o^t)$ $D_o^{t+1}(x_o^{t+1}, y_o^{t+1})$ denote the technical efficiencies and $D_o^t(x_o^{t+1}, y_o^{t+1})$ $D_o^{t+1}(x_o^t, y_o^t)$ show the technology changes or the efficiency frontier shift at the time t and $t+1$. According to Färe hypothesis, if the return to scale remains unchanged, the Malmquist index can be obtained as an index of technical efficiency change (TEC) and an efficiency frontier shift (FS) level index. This index can be rewritten as follows [11]:

$$
MPI_o = TEC_o \times FS_o = \frac{D_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{D_o^t(x_o^t, y_o^t)} \left[\frac{D_o^t(x_o^{t+1}, y_o^{t+1})}{D_o^{t+1}(x_o^{t+1}, y_o^{t+1})} \times \frac{D_o^t(x_o^t, y_o^t)}{D_o^{t+1}(x_o^t, y_o^t)} \right]^{\frac{1}{2}}. \tag{6}
$$

As regards the equation (6), it can be concluded, if $FS_o > 1$, then the production possibility tends to include efficiency improvements DMU_o, otherwise it indicates a reduction in its efficiency. On the other hand, $TEC_o > 1$, indicates that DMU_o tends to the frontier efficiency and in this case, it will result in the increasing of efficiency and otherwise it shows less efficiency. In the following sections, we will present the proposed model using the Malmquist index and discuss its results.

3. The methodology

In this section, we focus on two subjects flexible cost efficiency and cost Malmquist productivity index in the presence of flexible index in data.

3.1. The flexible cost efficiency model

Traditional DEA models assume that all DMUs are homogeneous and their inputs and outputs indices are known. But in some application problems, due to some management policies, input or output indices play a dual role. Such data whose input or output role is not clear is called flexible data. Traditional DEA models cannot evaluate the efficiency of DMUs in the presence of flexible data. Cook and Zhu [8] introduced an output-oriented model for classifying flexible data as follows:

$$\begin{aligned}
\min\ \ & \phi_o = \sum_{i=1}^{m} v_i x_{io} + \sum_{l=1}^{L} \gamma_l z_{lo} \\
\text{s.t.}\ \ & \sum_{r=1}^{s} \mu_r y_{ro} + \sum_{l=1}^{L} \delta_l z_{lo} = 1, \\
& \sum_{i=1}^{m} v_i x_{ij} + \sum_{l=1}^{L} \gamma_l z_{lj} - \sum_{r=1}^{s} \mu_r y_{rj} - \sum_{l=1}^{L} \delta_l z_{lj} \geq 0, j = 1, \ldots, n, \\
& \delta_l \leq M d_1, \qquad l = 1, \ldots, L, \\
& \gamma_l \leq M(1 - d_1), \qquad l = 1, \ldots, L, \\
& v_i \geq 0, \qquad i = 1, \ldots, m, \\
& \mu_r \geq 0, \qquad r = 1, \ldots, s, \\
& d_l \in \{0, 1\}, \delta_l, \gamma_l \geq 0, \qquad l = 1, \ldots, L,
\end{aligned} \tag{7}$$

where, in model (7) n DMUs, each of them has m input index, s output index and l flexible index. Also, γ_l and δ_l are weights of the inputs and outputs of z_l for $l = 1, \ldots, L$, and M is a large positive number.

In order to obtain a measure of flexible cost efficiency for DMUs with several inputs and outputs in presence flexible index, we introduce the following model (8) [21].

$$\begin{aligned}
\min C = & \sum_{i=1}^{m} c_{io} x_i + \sum_{l=1}^{L} (1 - \delta_l) cp_{lo} z_l \\
\text{s.t.}\quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, r = 1, \ldots, s, \\
& \sum_{i=1}^{n} \lambda_j z_{lj} \leq z_{lo} + M\delta_l, l = 1, \ldots, L, \\
& -\sum_{j=1}^{n} \gamma_j z_{lj} \leq -z_{lo} + M(1 - \delta_l), \qquad l = 1, \ldots, L, \\
& \delta_l \in \{0, 1\}, \lambda_j \geq 0, \qquad l = 1, \ldots, L,\ j = 1, \ldots, n.
\end{aligned} \tag{8}$$

In model (8), c_{io} is the price of input i and cp_{lo} is the price of flexible index l for the DMUo under evaluation. x_i and z_i are variables corresponding to input index and flexible index, that at the optimality give the amount of input i and input l to be used by DMUo for produce the current outputs at minimum cost of objective function. Model (8) is a non-linear programming problem together with non-negative variables and $0, 1$ variables. Also, in the third set of constraints of Model (8), *i.e.*, $\sum_{j=1}^{n} \lambda_j z_{lj} \leq z_{lo} + M\delta_l,\ l = 1, \ldots, L$, must be revised so that the variable z_l should be placed instead of the value of the index specified by z_{lo}. The cost minimum modified model is presented as follows:

$$
\begin{aligned}
\min\ \ & c_o(y_o, x_o, c_{io}, cp_{lo}) = \sum_{i=1}^{m} c_{io}x_i + \sum_{l=1}^{L} cp_{lo}w_l \\
\text{s.t.}\ \ & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_i, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \quad r = 1,\dots,s, \\
& \sum_{j=1}^{n} \lambda_j z_{lj} \ge z_{lo} - M\gamma_l, \qquad l = 1,\dots,L, \\
& \sum_{j=1}^{n} \mu_{lj} z_{lj} \le w_l + M\beta_l, \qquad l = 1,\dots,L, \\
& \beta_l + \gamma_l = 1, \qquad l = 1,\dots,L, \\
& \beta_l, \gamma_l \in \{0,1\}, w_l \ge 0, \qquad l = 1,\dots,L, \\
& x_i \ge 0, \lambda_j \ge 0, \mu_{lj} \ge 0, \qquad l = 1,\dots,L, j = 1,\dots,n.
\end{aligned} \tag{9}
$$

where $w_l = \gamma_l z_l$, $\mu_{lj} = \gamma_l \lambda_j$, $\beta_l = \gamma_l \delta_l$ and M is a large number. In the optimal solution of model (9), if for flexible index, say, k, $\gamma_k = 1$, then $\beta_k = 0$, and the constraint $\sum_{j=1}^{n} \lambda_j z_{kj} \ge z_{ko} - M$ is redundant and the constraint $\sum_{j=1}^{n} \mu_{kj} z_{kj} \le w_k$ is active as an input constraint and the contribution of this constraint in the objective function is equal to $cp_{ko}w_k$. Cost efficiency is then obtained as the ratio of minimum cost with current prices (*i.e.*, the optimal solution to model (9)) to the current cost at DMUo, as follows:

$$
CE_o = \frac{\sum_{i=1}^{m} c_{io}x_i^* + \sum_{i=1}^{L} cp_{lo}w_L^*}{\sum_{i=1}^{m} c_{io}x_{io} + \sum_{i=1}^{L} cp_{lo}\gamma_l z_{lo}}. \tag{10}
$$

Theorem 3.1. *Model (9) is always feasible.*

Proof. Model (9) is feasible, because, with select the variables model (9) $\lambda_o = 1, \lambda_j = 0, j \ne o,\ j = 1,\dots,n$, $\beta_l = 0, \gamma_l = 1, w_l = 0,\ l = 1,\dots,L,\ \mu_{ij} = 0,\ l = 1,\dots,L,\ j = 1,\dots,n,\ x_i = x_{io},\ i = 1,\dots,m$, we obtain a feasible solution. This means that this solution satisfy in the constraints set model (9) and in other words, the solution space is not empty. □

Theorem 3.2. *Model (9) has the finite objective function in optimality.*

Proof. Suppose that $(\lambda^*, x^*, w^*, \mu^*, \beta^*, \gamma^*)$ is the optimal solution in evaluating DMUo . Let us the constant price of input vector is c and the price of flexible index vector is cp, then we have the value objective function $c_o^* = cx^* + cpw^*$. □

3.2. The cost Malmquist productivity index

In this section, we first compute the cost technical efficiency and the cost technology changes in presence of the flexible data at the time period t and $t + 1$ .Then in continue, we present the cost Malmquist index. Our proposed model for evaluate the cost technical efficiency DMUo at the time t and $t+1$ is given as follows:

$$
\begin{aligned}
\min\ & c_o^{\tau}(y_o^{\tau}, x_o^{\tau}, c_{io}^{\tau}, cp_{lo}^{\tau}) = \sum_{i=1}^{m} c_{io}^{\tau} x_i + \sum_{l=1}^{L} cp_{lo}^{\tau} w_l \\
\text{s.t.}\ & \sum_{j=1}^{n} \lambda_j x_{ij}^{\tau} \le x_i, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{\tau} \ge y_{ro}, \quad r = 1, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j z_{lj}^{\tau} \ge z_{lo}^{\tau} - M\gamma_l, \qquad l = 1, \ldots, L, \\
& \sum_{j=1}^{n} \mu_{lj} z_{lj}^{\tau} \le w_l + M\beta_l, \qquad l = 1, \ldots, L, \\
& \beta_l + \gamma_l = 1, \qquad l = 1, \ldots, L, \\
& \beta_l, \gamma_l \in \{0, 1\}, w_l \ge 0, \qquad l = 1, \ldots, L, \\
& x_i \ge 0, \lambda_j \ge 0, \mu_{lj} \ge 0, \qquad l = 1, \ldots, L, j = 1, \ldots, n.
\end{aligned}
\tag{11}
$$

where $\tau \in \{t, t+1\}$ and the variables in the above model are $\beta_l, \gamma_l, w_l, \lambda_j, \mu_{lj}, x_i$ and $cp_l = (cp_1, \ldots, cp_L)$ is the prices flexible vector related to the flexible index w_l in the objective function of the model (11). This model will be implemented n times in order to find the minimum cost of efficiency of the DMUo under evaluation in the presence of flexible data. The model (11) is solved into two periods and gives the cost efficiency as $c_o^t(y_o^t, x_o^t, c_{io}^t, cp_{lo}^t)$ and $c_o^{t+1}(y_o^{t+1}, x_o^{t+1}, c_{io}^{t+1}, cp_{lo}^{t+1}))$ for under-evaluated DMUo . Of course, it should also be noted that the production potential set should be considered the same for all units, because in such flexible models, some of the data play the role of input and some others play the role of output, and we want to be prevented of different production possibility [31,33].

The Malmquist index requires calculating efficiencies in a single period and a combined period for technological efficiencies. The single period has expressed in the model (11) and the combined periods are presented in two models (12) and (13) as follows:

$$
\begin{aligned}
\min\ & c_o^{t+1}(y_o^{t}, x_o^{t}, c_{io}^{t}, cp_{lo}^{t}) = \sum_{i=1}^{m} c_{io}^{t} x_i + \sum_{l=1}^{L} cp_{lo}^{t} w_l \\
\text{s.t.}\ & \sum_{j=1}^{n} \lambda_j x_{ij}^{t+1} \le x_i, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{t+1} \ge y_{ro}^{t}, \quad r = 1, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j z_{lj}^{t+1} \ge z_{lo}^{t} - M\gamma_l, \qquad l = 1, \ldots, L, \\
& \sum_{j=1}^{n} \mu_{lj} z_{lj}^{t+1} \le w_l + M\beta_l, \qquad l = 1, \ldots, L, \\
& \beta_l + \gamma_l = 1, \qquad l = 1, \ldots, L, \\
& \beta_l, \gamma_l \in \{0, 1\}, w_l \ge 0, \qquad l = 1, \ldots, L, \\
& x_i \ge 0, \lambda_j \ge 0, \mu_{lj} \ge 0, \qquad l = 1, \ldots, L, j = 1, \ldots, n.
\end{aligned}
\tag{12}
$$

and,

$$
\begin{aligned}
\min \ \ & c_o^t(y_o^{t+1}, x_o^{t+1}, c_{io}^{t+1}, cp_{lo}^{t+1}) = \sum_{i=1}^{m} c_{io}^{t+1} x_i + \sum_{l=1}^{L} cp_{lo}^{t+1} w_l \\
\text{s.t.} \ \ & \sum_{j=1}^{n} \lambda_j x_{ij}^t \leq x_i, \quad i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^t \geq y_{ro}^{t+1}, \quad r = 1, \dots, s, \\
& \sum_{j=1}^{n} \lambda_j z_{lj}^t \geq z_{lo}^{t+1} - M\gamma_l, \qquad l = 1, \dots, L, \\
& \sum_{j=1}^{n} \mu_{lj} z_{lj}^t \leq w_l + M\beta_l, \qquad l = 1, \dots, L, \\
& \beta_l + \gamma_l = 1, \qquad l = 1, \dots, L, \\
& \beta_l, \gamma_l \in \{0, 1\}, w_l \geq 0, \qquad l = 1, \dots, L, \\
& x_i \geq 0, \lambda_j \geq 0, \mu_{lj} \geq 0, \qquad l = 1, \dots, L, j = 1, \dots, n.
\end{aligned} \tag{13}
$$

Therefore, we obtain the cost Malmquist productivity index with flexible data (CMPIF) as follows for DMUo:

$$
CMPIF_o = \left[\frac{CE_o^t(y_o^{t+1}, x_o^{t+1}, c_{io}^{t+1}, cp_{lo}^{t+1})}{CE_o^t(y_o^t, x_o^t, c_{io}^t, cp_{lo}^t)} \times \frac{CE_o^{t+1}(y_o^{t+1}, x_o^{t+1}, c_{io}^{t+1}, cp_{lo}^{t+1})}{CE_o^{t+1}(y_o^t, x_o^t, c_{io}^t, cp_{lo}^t)} \right]^{\frac{1}{2}}. \tag{14}
$$

We can rewrite the CMPIF as the multiple of cost technological efficiency change (CTEC) and cost frontier shift (CFS):

$$
CMPIF_o = CTEC_o \times CFS_o \tag{15}
$$

where,

$$
\begin{aligned}
CTEC_o &= \frac{CE_o^t(y_o^{t+1}, x_o^{t+1}, c_{io}^{t+1}, cp_{lo}^{t+1})}{CE_o^t(y_o^t, x_o^t, c_{io}^t, cp_{lo}^t)} \\
CFS_o &= \left[\frac{CE_o^t(y_o^{t+1}, x_o^{t+1}, c_{io}^{t+1}, cp_{lo}^{t+1})}{CE_o^{t+1}(y_o^{t+1}, x_o^{t+1}, c_{io}^{t+1}, cp_{lo}^{t+1})} \times \frac{CE_o^t(y_o^t, x_o^t, c_{io}^t, cp_{lo}^t)}{CE_o^{t+1}(y_o^t, x_o^t, c_{io}^t, cp_{lo}^t)} \right]^{\frac{1}{2}}.
\end{aligned} \tag{16}
$$

From equation (15), four problems must be solved in order to compute each Malmquist index. It is obvious that for evaluating DMUo, it is not necessary that a part of DMUj ($j = 1, \dots, n$) lies on which the technology production frontier. Thus, the cost efficiency score CFSo may exceed one, that is, DMUo is more cost efficient than the best benchmark standard. This may be especially the case when the performance of observation over a frontier of a different period is evaluated. The solution to this problem requires that technologies be restricted to constant return to scale. This assumption is well eligible for our purposes. However, to value CMPIF> 1, indicates progress, CMPIF< 1, regress and CMPIF$= 1$, indicates that performance stayed constant in a system.

3.3. Two examples for illustration flexible index

In this section, we present two examples on a set of data with 10 DMUs. In the first case, DMUs have two inputs index and two outputs index and in the second case, the same data have one input index, two flexible indices, and one output index. The purpose of these two examples is to compare the application of the model to identify the type of flexible index.

TABLE 1. The set of 10 DMUs with two inputs and two outputs.

Indices	DMU1	DMU2	DMU3	DMU4	DMU5	DMU6	DMU7	DMU8	DMU9	DMU10
I1	20	19	25	27	22	55	33	31	30	50
I2	15	13	16	16	15	25	23	20	24	26
O1	10	15	16	18	64	23	22	15	19	25
O2	40	50	55	32	66	22	12	80	10	10

TABLE 2. The set of 10 DMUs with one input and one output and two flexible indices.

Indices	DMU1	DMU2	DMU3	DMU4	DMU5	DMU6	DMU7	DMU8	DMU9	DMU10
I1	20	19	25	27	22	55	33	31	30	50
Flexible1	15	13	16	16	15	25	23	20	24	26
O1	10	15	16	18	64	23	22	15	19	25
Flexible 2	40	50	55	32	66	22	12	80	10	10

State 1:

Consider the set of 10 DMUs each consumes two inputs in order to produce two outputs according to Table 1. The second to fifth columns of Table 3 shows the results of model (1) and relation (2) for minimum cost and cost efficiency evaluation when the data consists of two inputs with the price vector $(4, 2)$ and two outputs according to Table 1.

State 2:

Consider the set of the same 10 DMUs in Table 1, with one input index, two flexible indices, and one output index in Table 2.

The sixth to tenth columns show the obtained results to the minimize the cost and cost efficiency of the model (9) and relation (10) when the data include one input with price 4, two flexible indices with the price vector $(6, 5)$ and one output. The results obtained from 10 DMUs have been described in two cases, without the presence of the flexible index and with the presence of the two flexible indices in Table 3. In the first case, the data include two inputs and two outputs, so, the results of the evaluation of the units using model (1) and relation (2) are given in the second to fifth columns. The second and third columns show the optimal amount of minimum inputs x_1 and x_2, the fourth column expresses the minimum cost ω and the fifth column shows the cost efficiency CE for all units. In the second case, the data have one input, two flexible indices, and one output, so the results of their cost efficiency evaluation with the model (9) and relation (10) are presented in the sixth to tenth columns of Table 3. The sixth column shows the optimal input x_1 of the units, the seventh and eighth columns give values of two flexible indices w_1 and w_2 . The seventh and eighth columns of Table 3, show that both flexible indices are inputs indices in determining the minimum cost of the first unit, while for the second to fifth units the flexible indices have the role of outputs indices.

Also, for the sixth to tenth units, the first flexible index plays the role of input, but the second flexible index appears as output. Finally, the minimum cost C_o for units has given in the ninth column and their cost efficiency CE has also shown in the tenth column. By comparing the fifth and tenth columns of Table 3, it explains that cost efficiency is non-decreasing for all DMUs except the first DMU because both of its flexible indices are input.

Table 3. Cost efficiency in two cases without flexible indices and with flexible indices.

	Cost efficiency without flexible indices				Cost efficiency with two flexible indices				
DMUs	$x1$	$x2$	ω	CE	$x1$	$w1$	$w2$	Co	CE
DMU1	13.33	9.09	71.52	0.65	3.44	2.34	10.31	79.38	0.21
DMU2	16.67	11.36	89.39	0.88	18.51	0.00	0.00	74.05	0.97
DMU3	18.33	12.50	98.33	0.75	22.28	0.00	0.00	89.13	0.89
DMU4	10.67	7.27	57.21	0.41	21.06	0.00	0.00	84.26	0.78
DMU5	22.00	15.00	118.00	1.00	22.00	0.00	0.00	88.00	1.00
DMU6	7.91	5.39	42.41	0.16	7.91	5.39	0.00	63.97	0.17
DMU7	7.56	5.16	40.56	0.23	7.56	5.16	0.00	61.19	0.23
DMU8	26.67	18.18	143.03	0.87	28.72	0.00	0.00	114.87	0.93
DMU9	6.53	4.45	35.03	0.20	6.53	4.45	0.00	52.84	0.20
DMU10	8.59	5.86	46.09	0.18	8.59	5.86	0.00	69.53	0.20

4. An application in the Tehran banking industry

In this section, we describe a real application example from the article on achieving the products of bank branches (see Paradi, Zhou, 2013) [24] contain two inputs $(X1, X2)$ and three outputs $(Y1, Y2, Y3)$. We suppose all bank branches have two inputs $(X1, X2)$, one flexible (Y1) and two outputs $(Y2, Y3)$, which given in Table 4.

Inputs: human resource and location index of branches.
Flexible: deposits.
Outputs: facilities, and services.

The input of human resources has related to the employees of the branch and the input of the place is related to the physical location of the branch. The flexible index of branch deposits includes all types of methods of collecting cash by that branch. It is an input because it shows economic growth; also it can be as an output, because workforce, advertising, and money have been spent to obtain it. The output of the facility includes all the funds that have been paid by the branch in the form of various types of facilities. Also, the output of services is an indicator that includes all kinds of fee services in the form of card issuance, types of guarantees and opening of visual and long-term documentary credit, foreign exchange transaction fee and funds transfer fee by providing weighted coefficients by the branch to its customers.

The prices of 2 inputs and 1 flexible are also given as the vector $(2, 4, 6)$, which is constant for two time periods 2017 and 2018. In order to determine the cost Malmquist productivity index, the results obtained of the model (11) for the single period at the time t and $t+1$, and also two models (12) and (13) for the combined period have denoted in Table 5, respectively.

In Table 5, column 2 (CE_{11}), and column 5 (CE_{22}), show the cost efficiency values using model (11) in two periods 2017 and 2018, respectively. The cost efficiency technology has given in column 3 (CE_{12}) and column 4 (CE_{21}). (CE_{12}) points out the cost efficiency technology for DMUs in 2017 relative to efficiency frontier at the period 2018 and conversely, (CE_{21}) shows the cost efficiency technology for units in 2017 relative to the efficiency frontier at the period 2018. Columns 6 and 7 of Table 5, show two indices cost technological efficiency change (CTEC) and cost frontier shift (CFS), respectively. The units 15, 16,17,19,20,21,24,25,26,30,31,34,35 and 37 in both indices CTEC and CFS, have values greater than the unity amount.

Table 5, indicated the cost Malmquist productivity index with flexible (CMPIF), such that we find out CMPIF>1 for the 18 units, which have the progress in performance and also the regress in performance for the remaining units. Among them the DMU25 has the most CMPIF. As it can be seen in Table 5, there is not any

TABLE 4. Average values of inputs and outputs and price indices from 2017 to 2018.

DMU	$Y1$	$Y2$	$Y3$	$X1$	$X2$
1	609.85	811.50	550.50	2952.49	1031.00
2	998.95	982.45	832.05	4459.63	1140.00
3	340.80	980.90	639.50	2135.65	696.20
4	1441.00	2914.50	894.40	6565.92	1090.00
5	441.90	1284.50	739.40	2722.38	707.40
6	1129.00	1420.50	800.25	4551.81	1097.00
7	3149.00	781.70	3315.50	8026.07	1150.00
8	1035.00	823.80	771.90	4357.19	1032.00
9	1154.00	1313.00	659.80	4021.79	1070.00
10	828.70	1244.00	720.15	4562.83	1022.00
11	1414.00	2358.00	1000.00	6907.83	1061.00
12	2454.00	10896.50	2178.00	11601.31	674.80
13	1124.05	615.70	1021.60	3977.59	1008.00
14	1001.90	1623.50	762.65	3754.29	1068.00
15	1015.00	1007.10	542.50	3866.61	1048.00
16	909.20	600.15	1396.50	3786.60	1010.00
17	1205.00	1933.00	814.95	5621.49	1092.00
18	1508.50	2364.50	1099.00	6647.73	1089.00
19	1810.50	797.40	2235.50	7265.68	1062.00
20	880.60	1207.00	720.20	4729.05	988.00
21	895.95	1018.90	637.90	2948.40	1006.00
22	1380.50	1416.50	965.45	5228.99	1029.00
23	1015.60	1197.50	625.40	2961.06	1023.00
24	820.25	841.20	847.00	3215.65	980.00
25	716.40	718.45	712.20	2742.78	938.70
26	1483.50	659.35	611.75	2537.32	968.60
27	1099.15	338.90	1305.50	4131.15	951.20
28	1062.20	1015.90	601.30	2968.56	927.90
29	1103.00	383.25	861.35	3440.38	919.80
30	711.35	1003.45	814.05	3079.24	924.40
31	987.25	987.10	906.05	4698.42	894.00
32	518.15	825.85	544.90	2791.18	918.40
33	984.55	438.80	442.85	2716.58	941.80
34	766.90	554.10	346.40	2610.33	925.50
35	340.40	448.90	287.85	2092.02	937.70
36	530.85	427.50	295.95	2195.39	852.00
37	890.95	670.90	533.80	3110.52	819.10
38	241.80	543.40	362.40	1958.45	813.20
39	626.50	236.85	173.95	2159.22	813.20
40	198.20	204.55	372.15	1628.37	813.20
G-Mean	**889.2**	**904.8**	**722.1**	**3673.5**	**955.7**

unit CMPIF = 1, so they have stability in two periods. Also, 4 units of the 40 branches have CMPIF less than 0.5.

The results obtained from our method (CMPIF) and cost Malmquist in Tavallaaee *et al.* (CMT) [29], has shown in the eighth and ninth columns of Table 5, respectively. A simple comparison between them states a significant difference (Fig. 1). Because of in our method (CMPIF), the flexible deposit index has been accepted as an input index in the evaluation of the cost Malmquist, while in the Tavallaaee *et al.* (CMT) method, it

TABLE 5. The cost Malmquist productivity index with flexible data (CMPIF) and CMT.

DMUs	$CE11$	$CE12$	$CE21$	CE22	CTEC	CFS	CMPIF	CMT
1	0.7265	0.1652	0.1970	0.2032	0.2797	2.0651	0.57753	1.1515
2	0.6924	0.2296	0.1948	0.2017	0.2914	1.7068	0.49729	0.9698
3	0.6878	0.2893	0.3605	0.3710	0.5394	1.5198	0.81974	1.1620
4	0.5956	0.2031	0.2222	0.2216	0.3721	1.7147	0.63801	1.1290
5	0.7846	0.3252	0.3648	0.3747	0.4776	1.5326	0.73194	1.2682
6	0.4550	0.1902	0.2248	0.2281	0.5012	1.5358	0.76971	1.2099
7	0.4620	0.4373	0.4739	0.4481	0.9700	1.0570	1.02528	0.8593
8	0.3118	0.1902	0.2216	0.2101	0.6739	1.3149	0.88611	1.2330
9	0.3321	0.1393	0.2258	0.2144	0.6455	1.5846	1.02288	1.0566
10	0.3466	0.1581	0.2817	0.2674	0.7714	1.5196	1.17217	1.2069
11	0.5683	0.2320	0.2308	0.2194	0.3860	1.6054	0.61970	1.2484
12	0.4517	0.4851	0.4595	0.4798	1.0623	0.9443	1.00318	1.2512
13	0.3216	0.3069	0.2190	0.2096	0.6519	1.0463	0.68205	0.7238
14	0.2211	0.2195	0.2100	0.2073	0.9378	1.0100	0.94718	1.1679
15	0.1522	0.1500	0.1562	0.1536	1.0091	1.0158	1.02507	1.1783
16	0.3456	0.3300	0.4543	0.4309	1.2467	1.0508	1.31002	0.9707
17	0.1833	0.1839	0.2314	0.2287	1.2475	1.0043	1.25286	1.2364
18	0.2553	0.2537	0.2369	0.2339	0.9162	1.0095	0.92492	1.3199
19	0.3646	0.3472	0.5131	0.4860	1.3331	1.0528	1.40349	1.1987
20	0.2071	0.2042	0.2138	0.2084	1.0062	1.0200	1.02632	1.2446
21	0.1625	0.1601	0.2437	0.2370	1.4583	1.0215	1.48973	1.2041
22	0.2652	0.2601	0.1931	0.1875	0.7068	1.0249	0.72437	1.1623
23	0.1855	0.1828	0.1846	0.1818	0.9799	1.0151	0.99472	1.1826
24	0.2541	0.2442	0.2799	0.2712	1.0675	1.0361	1.10603	0.9926
25	0.1819	0.1762	0.3288	0.3163	1.7386	1.0360	1.80116	1.2177
26	0.1322	0.1282	0.1793	0.1728	1.3071	1.0344	1.35215	1.0719
27	0.4542	0.4294	0.2638	0.2501	0.5508	1.0560	0.58160	0.9508
28	0.2610	0.2566	0.1095	0.1073	0.4111	1.0188	0.41883	1.0115
29	0.2911	0.2761	0.1883	0.1801	0.6188	1.0499	0.64968	1.2117
30	0.2359	0.2312	0.3233	0.3109	1.3180	1.0300	1.35754	1.0870
31	0.2380	0.2317	0.2925	0.2813	1.1821	1.0334	1.22159	1.0740
32	0.2128	0.2100	0.1945	0.1882	0.8842	1.0234	0.90495	1.0751
33	0.1623	0.1567	0.1011	0.0976	0.6014	1.0358	0.62292	0.9881
34	0.0933	0.0926	0.1371	0.1328	1.4227	1.0198	1.45089	1.2015
35	0.0856	0.0845	0.1756	0.1705	1.9912	1.0218	2.03451	1.2610
36	0.1549	0.1500	0.0804	0.0796	0.5138	1.0216	0.52486	1.0551
37	0.1555	0.1522	0.2075	0.1980	1.2735	1.0348	1.31775	1.1059
38	0.2684	0.2614	0.0894	0.0883	0.3291	1.0196	0.33557	1.1176
39	0.0722	0.0703	0.0638	0.0621	0.8599	1.0272	0.88330	1.2579
40	0.2738	0.2612	0.0717	0.2619	0.2619	1.0456	0.27385	0.5837

has been considered as an output from the beginning. This meaningful difference, which shows the effect of the presence of flexible index in the computation of the cost Malmquist, is graphically described in Figure 1.

However, according to Table 5, calculating CMT with the presence a flexible index (CMPIF) indicates that 18 decision-making units have made progress. On the other hands, CMT values without a flexible index show that 32 decision-making units have progressed, and both of them gave the progress for 15 decision-making units.

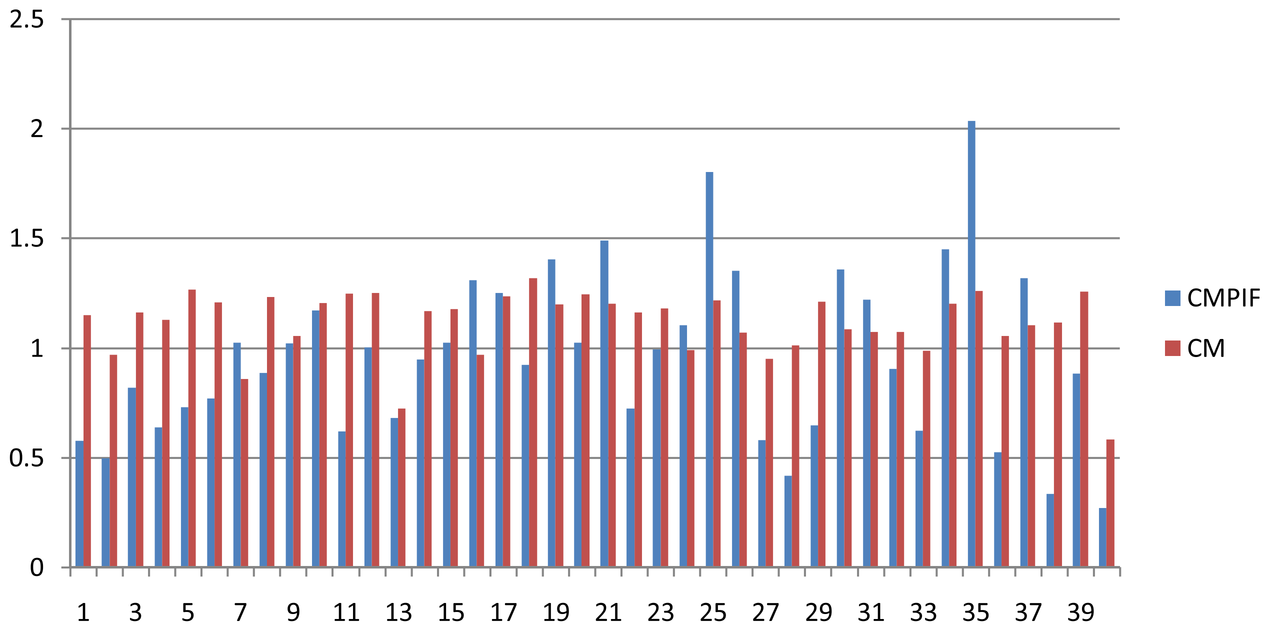

Figure 1. The cost Malmquist productivity with CMPIF and CMT.

5. Conclusions

Increased competition in banking have forced bank managers to choose the appropriate scale for banking services such that, they always evaluated their banking performance until they adopted the necessary strategies in order to improve it. Therefore, in some cases evaluations, we have dealt with variables that cannot be determined in the kind of their input or output. Therefore, it is necessary to use models in DEA to determine the input or output orientation. These types of variables are called flexible variables. Some research has been done in this field. In this article, we presented a cost efficiency model for solving the problem of complexity of variables in the nonlinear model objective function. Also, we obtained the cost Malmquist productivity index in presence of a flexible index in order to determine the progress and the regress of DMUs. Our goals in this type of proposed model have been to achieve the minimum cost for each DMU, which is also one of the DMs' goals. Therefore, we provided a suitable solution to advance the goals of DM in order to increase productivity and reduce costs for many production units, industries, etc. Also, in this paper, we showed the effect of flexible indices on the productivity of 40 industrial banks. The existence of such results can be helped managers to present better decisions in the future. In future work, one uses the other variables, such as fuzzy or random in order to obtain cost Malmquist productivity index.

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