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PRICING RAINBOW OPTION FOR UNCERTAIN FINANCIAL MARKET

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Abstract. Rainbow option refers to the option whose payoff depends on at least two underlying risky assets, which is justifiably one of the most significant tool to hedge risk brought by the uncertainty from financial market. Hence, option pricing problem is always an issue with great attention. In this paper, we assume that the multiple dynamic stock prices obey uncertain differential equations without sharing dividends in the framework of uncertainty theory. Then we discuss the rainbow option pricing problem for multiple stocks in a financial market with uncertain information, give the concepts and derive pricing formulas for five scenarios including maximum call, minimum call, maximum put, minimum put, and put on 2 and call on 1. Moreover, some corresponding examples are respectively taken to illustrate the pricing formulas in five cases.

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1. Introduction

Rainbow option is a financial instrument exposed to at least two sources of uncertainty brought by different underlying assets and its value also comes from the price of these risky assets. In real financial markets, there are always several underlying assets existing simultaneously, therefore rainbow option is an issue deserving to be explored in depth. In fact, the theory of this option can trace back to 1970s. For example, Margrabe [16] evaluated the option relative to one asset being exchanged for another one, which is justifiably regarded as the most eminent early option pricing paper and opens up a new epoch of option pricing for multiple assets. As an extension, Stulz [18] delivered and priced "best of assets or cash" option that purchase or sell the maximum or minimum of the risky assets at a strike price at due date. While, it was not until 1991 that the concept of rainbow option was proposed by Rubinstein [17].

To the best of our knowledge, current study with regard to rainbow option is under the framework of probability theory, where the dynamic change of asset price is characterized by stochastic process and assumed to subject to a stochastic differential equation. However, uncertain process, another tool used to describe dynamic uncertainty associated with human beings' belief degree, was proposed by Liu [10]. Following that, Liu [11] put forward Liu process, a special type of uncertain process, which is the cornerstone of constructing uncertain calculus and uncertain differential equation. By using uncertain differential equation, the investigation of uncertain fiance entered a period of rapid development. For example, Chen and Gao [2] took the lead in presenting

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uncertain interest model. Then the zero-coupon bond model was studied by Jiao and Yao [8], and the interest rate ceiling/floor models model are proposed and discussed by Zhang *et al.* [23]. Additionally,uncertain currency model was established by Liu *et al.* [14]. Additionally, uncertain currency model was established by Liu *et al.* [14]. Wang and Ning [21] made some improvements referring to Liu-Chen-Ralescu model, and presented an uncertain currency model with floating interest rates. Then Gao *et al.* [7] studied American barrier option for uncertain currency model.

With the presentation of uncertain asset model, option pricing issue was paid close attention which is indubitably the core problem in financial theory. Liu [11] presented uncertain stock model where the stock price was modeled by uncertain differential equation, and proved European option pricing formulas for an uncertain stock model including put on and call on. Then Chen [1] respectively proved European option pricing formulas and American option pricing formulas for an uncertain stock model including put on and call on. Then focusing on exotic option, Asian option pricing in uncertain financial market was first studied by Sun and Chen [19]. Yang [22] and Gao *et al.* [4] respectively studied Asian barrier option and American barrier option for the uncertain stock model. Gao *et al.* [5] studied Geometric Asian barrier option and proved its pricing formulas for the uncertain stock model. Next a new stock was presented by Lu *et al.* [15] and its European option pricing problem was explored. And the pricing problem of European barrier option was investigated by Tian *et al.* [20] for the uncertain mean-reverting stock model.

Up to present, the existing literature under uncertainty theory most studied the option including only one asset. For tackling with the risk carried by the dynamic change of assets price, option is an important derivative in financial market. While, in reality, many assets usually coexist in an market at the same time. Hence Gao *et al.* [3] studied multi-asset option pricing in an uncertain financial market with jump risk. And Gao *et al.* [6] discussed Asian rainbow option for the uncertain multi-stock model. But the European rainbow option pricing problem has not been studied. Admittedly, we will introduce European rainbow option into the uncertain financial market composed of multiple stocks. Furthermore, option pricing formulas will be proved under five scenarios concluding maximum call, minimum call, maximum put, minimum put, and put on 2 and call on 1. The remainder of this paper is assigned as follows. Section 2 is planed to introduce some necessary concepts and propertied. Section 3 expresses settings, symbols and presents a model used in next sections. Then we discuss the rainbow option pricing problem including pricing formulas and their properties under maximum call, minimum call, maximum put, minimum put, and put on 2 and call on 1 scenarios in Section 4. Finally, brief summary and expectation in future are give in Section 5.

2. Preliminaries

In this part, lots of basic definitions and key properties are introduced relative to the uncertain variable, the uncertain process, and the uncertain differential equation.

Assumed that Γ is a set that is nonempty, and assumed that $\mathcal{L}$ is a σ-algebra on Γ, where any element Λ taken from $\mathcal{L}$ is regarded as an event. The we assign a number $\mathcal{M}\{\Lambda\}$ to interpret the degree of human belief that the event Λ may occur. For coping with belief degrees rationally, three axioms were proposed in [9] which are:

Axiom 1. (Normality Axiom) For the universal set Γ, we have $\mathcal{M}\{\Gamma\} = 1$;

Axiom 2. (Duality Axiom) For each event Λ, we have $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$;

Axiom 3. (Subadditivity Axiom) The following equality

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$$

holds for any denumerable sequence of events $\Lambda_1, \Lambda_2, \cdots$.

Definition 2.1. (Liu [9]) If set function $\mathcal{M}$ satisfies normality, duality and subadditivity axioms, then it is called to be an uncertain measure.

We call $(\Gamma, \mathcal{L}, \mathcal{M})$ an uncertainty space. Then Liu [11] introduce Product axiom

Axiom 4. (Product Axiom) Providing that $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ $(k = 1, 2, \cdots)$ are uncertainty spaces. Then $\mathcal{M}$ is an uncertain measure if it satisfies

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrary events respectively chosen from $\mathcal{L}_k$ $(k = 1, 2, \cdots)$.

Definition 2.2. (Liu [9]) If ξ is measurable from $(\Gamma, \mathcal{L}, \mathcal{M})$ to a real number set, i.e., the following set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event for each real Borel set B, then ξ is called an uncertain variable.

Definition 2.3. (Liu [9]) The uncertainty distribution of uncertain variable ξ is defined as

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any real number x.

An uncertainty distribution $\Phi(x)$ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$. Inverse uncertainty distribution plays an important role in the operations of independent uncertain variables. In the following, the concept of inverse uncertainty distribution will be presented.

Definition 2.4. (Liu [12]) Supposed that there is an uncertain variable ξ owing regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

Definition 2.5. (Liu [11]) An uncertain variable sequence of $\xi_1, \xi_2, \cdots, \xi_n$ is regarded to be independent mutually if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

is tenable for any real Borel sets $B_1, B_2, \cdots, B_n$.

Definition 2.6. (Liu [9]) Assumed that ξ is an uncertain variable. Then ξ owns an expected value defined by the following formula

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \le x\}\mathrm{d}x$$

under condition that one of the two integrals is finite at least.

Theorem 2.7. *(Liu [9]) Supposed that ξ is an uncertain variable owning regular uncertainty distribution Φ, and its expected value exists. Then its expected value is*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

Definition 2.8. (Liu [10]) Assumed that $(\Gamma, \mathcal{L}, \mathcal{M})$ is the uncertain space, and assumed that set T is totally ordered. If function X_t is measurable from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to a real number set, i.e., for arbitrary given t, $\{X_t \in B\}$ is an event for any real Borel set B, then we say X_t an uncertain process.

TABLE 1. Notations.

Symbol	Meaning
S_{it}	Spot price of the stock i, $i = 1, 2, \cdots, n$
X_t	Spot price of the bond
K_i	Strike price of the rainbow option w.r.t. stock i, $i = 1, 2, \cdots, n$
τ	Expiry date of the option
r	Riskless interest rate
μ_i	Log-drifts (revenue rate) of the stock i, $i = 1, 2, \cdots, n$
σ_i	Log-diffusions (revenue variance) of the stock i, $i = 1, 2, \cdots, n$
C_{it}	Independent Liu processes, $i = 1, 2, \cdots, n$

Definition 2.9. (Liu [11]) The uncertain process C_t is called to be a Liu process, supposed that
(i) its initial value is 0, i.e., $C_0 = 0$, and all of its sample paths are Lipschitz continuous almost,
(ii) the increments of C_t is independent and stationary,
(iii) each increment $C_{s+t} - C_s$ obeys normal uncertainty distribution whose expected value is 0 and variance is t^2 such that its uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re.$$

Theorem 2.10. *(Liu [13]) For a sequence of independent uncertain processes $X_{1t}, X_{2t}, \cdots, X_{nt}$ whose regular uncertainty distributions are respectively $\Phi_{1t}, \Phi_{2t}, \cdots, \Phi_{nt}$, if there is a continuous function $f(x_1, x_2, \cdots, x_n)$ being strictly increasing in regard to $x_1, x_2, \cdots, x_m$ and strictly decreasing in regard to $x_{m+1}, x_{m+2}, \cdots, x_n$, then*

$$X_t = f(X_{1t}, X_{2t}, \cdots, X_{nt})$$

possesses the following inverse uncertainty distribution which is

$$\Phi_t^{-1}(\alpha) = f(\Phi_{1t}^{-1}(\alpha), \cdots, \Phi_{mt}^{-1}(\alpha), \Phi_{m+1,t}^{-1}(1-\alpha) \cdots, \Phi_{nt}^{-1}(1-\alpha)).$$

Definition 2.11. (Liu [10]) Supposed there is a Liu process C_t and continuous functions f and g. Then for any given initial value X_0,

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is said to be an uncertain differential equation with X_0.

3. NOTATION, SETTING AND MODEL

This section is employed to introduce multiple uncertain stock model including some notations and settings. Before giving the model, we define some variables as shown in Table 1.

The system for dynamic uncertain financial market is

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}S_{1t} = \mu_1 S_{1t}\mathrm{d}t + \sigma_1 S_{1t}\mathrm{d}C_{1t} \\ \mathrm{d}S_{2t} = \mu_2 S_{2t}\mathrm{d}t + \sigma_2 S_{2t}\mathrm{d}C_{2t} \\ \qquad\vdots \\ \mathrm{d}S_{nt} = \mu_n S_{nt}\mathrm{d}t + \sigma_n S_{nt}\mathrm{d}C_{nt}. \end{cases} \tag{1}$$

From Model (1), we can obtain the analytic solutions of the prices of bond X_t and stock $i,\ i = 1, 2, \cdots, n$ which are

$$\begin{cases} X_t = X_0 \exp(rt) \\ S_{1t} = S_{10} \exp(\mu_1 t + \sigma_1 C_{1t}) \\ S_{2t} = S_{20} \exp(\mu_2 t + \sigma_2 C_{2t}) \\ \qquad\quad \vdots \\ S_{nt} = S_{n0} \exp(\mu_n t + \sigma_n C_{nt}), \end{cases}$$

where the inverse uncertainty distributions of $S_{it},\ i = 1, 2, \cdots, n$ are

$$\begin{cases} \Phi_{1t}^{-1}(\alpha) = S_{10} \exp\left(\mu_1 t + \frac{\sigma_1 t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right) \\ \Phi_{2t}^{-1}(\alpha) = S_{20} \exp\left(\mu_2 t + \frac{\sigma_2 t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right) \\ \qquad\quad \vdots \\ \Phi_{nt}^{-1}(\alpha) = S_{n0} \exp\left(\mu_n t + \frac{\sigma_n t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right). \end{cases}$$

4. Main results

In this section, we pioneer a new type of option which is rainbow option for hedging risk brought by at least two stocks existing simultaneously in an uncertain financial market. It can be considered as the option whose payoff depends on more than one underlying risky asset and each asset is supposed to be as a color of the rainbow. And rainbow option is usually said call or put on the best or worst of n underlying assets, which is going to be investigated in five scenarios including maximum call, minimum call, maximum put, minimum put, and put on 2 and call on 1.

4.1. Maximum call scenario

In this section, the scenario of maximum call option is discussed. Considering n underlying assets in an uncertain market with upward trend, in order to hedge the risk of excessive premium, a contract is provided for the investor (she) to purchase the best-performing stock $S_{i\tau}$ at corresponding strike price K_i at expiry date τ where every stock has its own strike price, while this is just a right instead of obligation. Here, best-performing stock means that its spot price minus the strike price being largest, that is, its revenue is best. Hence, the investor should pay for acquiring the right to the bank (he). However, how much should the contract price be reasonable? Next we will discuss this problem according to pair price principle.

Supposed that $f^c_{\max}$ denotes the price of maximum call option. Thus the investor should pay $f^c_{\max}$ at time 0 for holding the opportunity to purchase the best-performing stock at strike price K_i at expiry date τ with revenue $\max\big(S_{1\tau} - K_1, S_{2\tau} - K_2, \cdots, S_{n\tau} - K_n\big)$, if the spot price of the best-performing stock is more than K_i. Otherwise, she will abandon this right. Then the present income of the investor is $\exp(-r\tau)\big[\max\big((S_{1\tau} - K_1)^+, (S_{2\tau} - K_2)^+, \cdots, (S_{n\tau} - K_n)^+\big)\big]$. Thus, at time zero, the investor has the net revenue which is

$$\exp(-r\tau) \max\left[(S_{1\tau} - K_1)^+, (S_{2\tau} - K_2)^+, \cdots, (S_{n\tau} - K_n)^+\right] - f^c_{\max}.$$

And focusing from the point of bank, he will receive $f^c_{\max}$ for selling the contract at time zero. If the spot price of best-performing stock is more than K, then the bank should give the payment whose value is $\max\big(S_{1\tau}-K_1, S_{2\tau}-K_2, \cdots, S_{n\tau}-K_n\big)$. Otherwise, he has no loss. That is, the present income of the bank is $-\exp(-r\tau)\big[\max\big((S_{1\tau}-K_1)^+, (S_{2\tau} - K_2)^+, \cdots, (S_{n\tau} - K_n)^+\big)\big]$. Naturally, at time zero, the investor has the net revenue which is

$$f^c_{\max} - \exp(-r\tau)\big[\max\big((S_{1\tau} - K_1)^+, (S_{2\tau} - K_2)^+, \cdots, (S_{n\tau} - K_n)^+\big)\big].$$

According to fair price principle that the expected net revenue of the investor should equal to that of the bank, so we can derive the following relationship

$$\exp(-r\tau)E\Big[\max_{1\le i\le n}(S_{i\tau}-K_i)^+\Big]-f^c_{\max}=f^c_{\max}-\exp(-r\tau)E\Big[\max_{1\le i\le n}(S_{i\tau}-K_i)^+\Big]$$

which indicates

$$f^c_{\max}=\exp(-r\tau)E\Big[\max_{1\le i\le n}(S_{i\tau}-K_i)^+\Big].$$

That is, the price of maximum call option is the expected present value of revenue of the best stock.

Definition 4.1. Assumed that the expiry date is τ and the strike price is K in a maximum call option. Then the price of the maximum call option is

$$f^c_{\max}=\exp(-r\tau)E\Big[\max_{1\le i\le n}(S_{i\tau}-K_i)^+\Big].$$

Theorem 4.2. *Considering the same conditions in Definition 4.1, the price of the maximum call option of Model (1) is*

$$f^c_{\max}=\exp(-r\tau)\int_0^1\max_{1\le i\le n}\left[S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K_i\right]^+\mathrm{d}\alpha.$$

Proof. Due to $C_{\mathrm{it}}, i=1,2\cdots,n$ being independent, we may assert that $S_{1t},S_{2t},\cdots,S_{nt}$ are independent uncertain processes for any $t, 0\le t\le\tau$. Noting that $\max\big[(x_1-K_1)^+,(x_2-K_2)^+,\cdots,(x_n-K_n)^+\big]$ is strictly increasing regarded to $x_1,x_2,\cdots,x_n$, so we obtain that $\max\big[(S_{1\tau}-K_1)^+,(S_{2\tau}-K_2)^+,\cdots,(S_{n\tau})-K_n)^+\big]$ has an inverse uncertainty distribution

$$\begin{aligned}\Psi^{-1}_\tau(\alpha)&=\max\Big[\big(\Phi^{-1}_{1\tau}(\alpha)-K_1\big)^+,\big(\Phi^{-1}_{2\tau}(\alpha)-K_2\big)^+,\cdots,\big(\Phi^{-1}_{n\tau}(\alpha)-K_n\big)^+\Big]\\&=\max_{1\le i\le n}\left[S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K_i\right]^+\end{aligned}\qquad(2)$$

from Theorem 2.10. Consider that $\exp(-r\tau)$ is just a real function and has no relation to uncertainty. It follows from equation (2) and Theorem 2.7 that $\exp(-r\tau)\max\big[(S_{1\tau}-K_1)^+,(S_{2\tau}-K_2)^+,\cdots,(S_{n\tau})-K_n)^+\big]$ has the following expected value, that is,

$$f^c_{\max}=\exp(-r\tau)\int_0^1\max_{1\le i\le n}\left[S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K_i\right]^+\mathrm{d}\alpha.$$

Thus the proof is accomplished. □

If $K_1=K_2=\cdots=K_n=K$, then what is the formula for option pricing? In this situation, we obtain

$$\begin{aligned}&\max\big[(S_{1\tau}-K_1)^+,(S_{2\tau}-K_2)^+,\cdots,(S_{n\tau}-K_n)^+\big]\\&=\max\big[S_{1\tau}-K_1,S_{2\tau}-K_2,\cdots,S_{n\tau}-K_n,0\big]\\&=\max\big[\max(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})-K,0\big]\\&=\big[\max(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})-K\big]^+\end{aligned}$$

whose inverse uncertainty distribution is

$$\begin{aligned}\Psi^{-1}_\tau(\alpha)&=\big[\max\big(\Phi^{-1}_{1\tau}(\alpha),\Phi^{-1}_{2\tau}(\alpha),\cdots,\Phi^{-1}_{n\tau}(\alpha)\big)-K\big]^+\\&=\left[\max_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)-K\right]^+\end{aligned}\qquad(3)$$

from Theorem 2.10. Thus we give the following theorem where the pricing formula is giving under the assumption of $K_1=K_2=\cdots=K_n=K$.

Theorem 4.3. *Considering the same conditions in Definition 4.1, the price of the maximum call option of Model (1) with $K_1 = K_2 = \cdots = K_n = K$ is*

$$f^c_{\max} = \exp(-r\tau) \int_0^1 \left[\max_{1\le i\le n} \left(S_{i0} \exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right)\right) - K \right]^+ \mathrm{d}\alpha.$$

Proof. According to Theorem 4.2 and equation (3), the result can be derived directly. □

Remark 4.4. We usually regard such maximum call option as the extension of the standard call option for a single asset.

Theorem 4.5. *The rainbow option $f^c_{\max}$ of maximum call scenario for Model (1) has the following properties:*

1) $f^c_{\max}$ is respectively increasing with respect to S_{i0}, μ_i and σ_i;

2) $f^c_{\max}$ is decreasing with respect to K.

Proof. 1) The function

$$g = \left[\max_{1\le i\le n} \left(S_{i0} \exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right)\right) - K \right]^+$$

is increasing concerning S_{i0} if other parameters is fixed. Hence, $f^c_{\max}$ is increasing with respect to S_{i0}. The proof of variables μ_i and σ_i is similar to that of S_{i0}.

2) The function

$$g = \left[\max_{1\le i\le n} \left(S_{i0} \exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right)\right) - K \right]^+$$

is decreasing concerning K if other parameters is fixed. Hence, $f^c_{\max}$ is increasing with respect to K.

□

Example 4.6. Considering Model (1), let $n = 3$ and set $\tau = 10, r = 0.03, K = 16, \mu_1 = 0.04, \mu_2 = 0.05, \mu_3 = 0.07, \sigma_1 = 0.05, \sigma_2 = 0.06, \sigma_3 = 0.09, S_{10} = 10, S_{20} = 13$ and $S_{30} = 15$. Then we find that the maximum call option price $f^c_{\max} = 0.2028$.

4.2. Minimum call scenario

In this section, the scenario of minimum call is studied. Considering n underlying assets in the uncertain market possessing upward trend, in order to hedge risk of undue premium,

a contract is supplied for the investor to buy the minimum stock $S_{i\tau}$ at corresponding strike price K_i at expiry date τ where every stock has its own strike price, while this is just a right instead of obligation. Here, minimum stock means that its spot price minus the strike price being smallest. Hence, the investor should pay for the right to the bank. However, how much should the contract price be appropriate? Next we will probe this problem according to the pair price principle.

Assumed that $f^c_{\min}$ represent the price of minimum call option. Thus the investor should pay $f^c_{\min}$ at time 0 for holding the opportunity to buy the minimum stock $S_{i\tau}$ at strike price K_i at expiry date τ with income $\min\big(S_{1\tau} - K_1, S_{2\tau} - K_2, \cdots, S_{n\tau} - K_n\big)$, if the spot price of the minimum stock is more than K. On the contrary, she will give up this opportunity. Thus the present income of the investor is $\exp(-r\tau)\big[\min\big((S_{1\tau} - K_1)^+, (S_{2\tau} - K_2)^+, \cdots, (S_{n\tau} - K_n)^+\big)\big]$. Thus, at time zero, the investor has the net revenue which is

$$\exp(-r\tau) \min\left[(S_{1\tau} - K_1)^+, (S_{2\tau} - K_2)^+, \cdots, (S_{n\tau} - K_n)^+\right] - f^c_{\min}.$$

And from the standpoint of bank, he will receive $f^c_{\min}$ for selling the contract at time zero. If the spot price of best stock is more than K, then the bank should give the payment whose value is $\min\big(S_{1\tau}-K_1, S_{2\tau}-K_2,\cdots,S_{n\tau}-K_n\big)$. Contrarily, he does not lose anything. In other words, the present income of the bank is $-\min\big[(S_{1\tau}-K_1)^+,(S_{2\tau}-K_2)^+,\cdots,(S_{n\tau}-K_n)^+\big]$. Naturally, at time zero, the bank has the net revenue which is

$$f^c_{\min}-\exp(-r\tau)\min\big[(S_{1\tau}-K_1)^+,(S_{2\tau}-K_2)^+,\cdots,(S_{n\tau}-K_n)^+\big].$$

According to fair price principle that the expected net revenue of the investor should equal to the one of the bank, we can derive the following relationship

$$\exp(-r\tau)E\Big[\min_{1\le i\le n}(S_{i\tau}-K_i)^+\Big]-f^c_{\max}=f^c_{\max}-\exp(-r\tau)E\Big[\min_{1\le i\le n}(S_{i\tau}-K_i)^+\Big]$$

which implies

$$f^c_{\min}=\exp(-r\tau)E\Big[\min_{1\le i\le n}(S_{i\tau}-K_i)^+\Big].$$

That is, the price of minimum call option is the expected present value of revenue of best stock.

Definition 4.7. Assumed that the expiry date is τ and the strike price is K in a minimum call option. Then the price of the minimum call option is

$$f^c_{\min}=\exp(-r\tau)E\Big[\min_{1\le i\le n}(S_{i\tau}-K_i)^+\Big].$$

Theorem 4.8. *Considering the same conditions in Definition 4.7, the price of the minimum call option of Model (1) is*

$$f^c_{\min}=\exp(-r\tau)\int_0^1\min_{1\le i\le n}\left[S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K_i\right]^+\mathrm{d}\alpha.$$

Proof. Due to $C_{it}, i=1,2\cdots,n$ being independent, we may claim that $S_{1t},S_{nt},\cdots,S_{nt}$ are independent uncertain processes for any $t, 0\le t\le\tau$. Noting that $\min\big[(x_1-K_1)^+,(x_2-K_2)^+,\cdots,(x_n-K_n)^+\big]$ is strictly humdrum increasing regarded to $x_1,x_2,\cdots,x_n$, so we obtain that $\min\big[(S_{1\tau}-K_1)^+,(S_{2\tau}-K_2)^+,\cdots,(S_{n\tau})-K_n)^+\big]$ has an inverse uncertainty distribution

$$\begin{aligned}\Psi_\tau^{-1}(\alpha)&=\min\Big[\big(\Phi_{1\tau}^{-1}(\alpha)-K_1\big)^+,\big(\Phi_{2\tau}^{-1}(\alpha)-K_2\big)^+,\cdots,\big(\Phi_{n\tau}^{-1}(\alpha)-K_N\big)^+\Big]\\&=\min_{1\le i\le n}\left[S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K_i\right]^+\end{aligned}\tag{4}$$

from Theorem 2.10. Consider that $\exp(-r\tau)$ is just a real function and has no relation to uncertainty. It follows from equation (4) and Theorem 2.7 that $\exp(-r\tau)\min\big[(S_{1\tau}-K_1)^+,(S_{2\tau}-K_2)^+,\cdots,(S_{n\tau})-K_n)^+\big]$ has the following expected value, that is,

$$f^c_{\min}=\exp(-r\tau)\int_0^1\min_{1\le i\le n}\left[S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K_i\right]^+\mathrm{d}\alpha.$$

Thus the proof is finished. □

If $K_1=K_2=\cdots=K_n=K$, then what is the formula for option pricing? In this situation, we obtain

$$\begin{aligned}&\min\big[(S_{1\tau}-K_1)^+,(S_{2\tau}-K_2)^+,\cdots,(S_{n\tau}-K_n)^+\big]\\&=\min\big[S_{1\tau}-K_1,S_{2\tau}-K_2,\cdots,S_{n\tau}-K_n,0\big]\\&=\min\big[\max(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})-K,0\big]\\&=\big[\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})-K\big]^+\end{aligned}$$

whose inverse uncertainty distribution is

$$\begin{aligned}\Psi_\tau^{-1}(\alpha) &= \left[\min\left(\Phi_{1\tau}^{-1}(\alpha), \Phi_{2\tau}^{-1}(\alpha), \cdots, \Phi_{n\tau}^{-1}(\alpha)\right) - K\right]^+ \\ &= \left[\min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right) - K\right]^+ \end{aligned} \tag{5}$$

from Theorem 2.10. Thus we give the following theorem where the pricing formula is giving under the assumption of $K_1 = K_2 = \cdots = K_n = K$.

Theorem 4.9. *Considering the same conditions in Definition 4.1, the price of the maximum call option of Model (1) is*

$$f_{\max}^c = \exp(-r\tau)\int_0^1 \left[\max_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right) - K\right]^+ \mathrm{d}\alpha,$$

if $K_1 = K_2 = \cdots = K_n = K$.

Proof. According to Theorem 4.8 and equation (3), the result can be derived directly. □

Remark 4.10. We usually regard such minimum call option as the extension of the standard call option for a single asset.

Theorem 4.11. *The rainbow option $f_{\min}^c$ of minimum call scenario for Model (1) has the following properties:*

1) $f_{\min}^c$ is respectively increasing with respect to S_{i0}, μ_i and σ_i;

2) $f_{\min}^c$ is decreasing with respect to K.

Proof. 1) The function

$$g = \left[\min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right) - K\right]^+$$

is increasing concerning S_{i0} if other parameters is fixed. Hence, $f_{\min}^c$ is increasing with respect to S_{i0}. The proof of variables μ_i and σ_i is similar to that of S_{i0}.

2) The function

$$g = \left[\min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right) - K\right]^+$$

is decreasing concerning K if other parameters is fixed. Hence, $f_{\min}^c$ is increasing with respect to K.

□

Example 4.12. Considering Model (1), let $n = 3$ and set $\tau = 10, r = 0.03, K = 16, \mu_1 = 0.04, \mu_2 = 0.05, \mu_3 = 0.07, \sigma_1 = 0.05, \sigma_2 = 0.06, \sigma_3 = 0.09, S_{10} = 10, S_{20} = 13$ and $S_{30} = 15$. Then we find that the minimum call option price $f_{\min}^c = 0.1248$.

4.3. Maximum put scenario

In this section, we are going to inquire some useful result under the scenario of maximum put. Considering n underlying assets in the uncertain market which appear down trend, in order to hedge the risk of downtick, a contract is put forward for the investor to sell the best-performing stock $S_{i\tau}$ at corresponding strike price K_i at expiry date τ where every stock has its own strike price, while this is just a right rather than an obligation. Here, best-performing stock means that the strike price minus its spot price being largest, that is, revenue of investor is best. Hence, the investor should pay for the right to the bank. But, how can we price a rational value to satisfy both the investor and the bank? In the following, we will resolve this problem according to pair price principle.

Supposed that $f^d_{\max}$ denotes the price of maximum put option. Thus the investor should pay $f^d_{\max}$ at time 0 for holding the opportunity to sell the best-performing stock $S_{i\tau}$ at strike price K_i at expiry date τ with revenue $\max\big(K_1 - S_{1\tau}, K_2 - S_{2\tau}, \cdots, K_n - S_{n\tau}\big)$, if the spot price of the best-performing stock $S_{i\tau}$ is less than K_i. Otherwise, she plans to give up the right. Hence the present income of the investor is $\exp(-r\tau)\max\big[(K_1 - S_{1\tau})^+, (K_2 - S_{2\tau})^+, \cdots, (K_n - S_{n\tau})^+\big]$. Thus, at time zero, the investor has the net revenue which is

$$\exp(-r\tau)\max\big[(K_1 - S_{1\tau})^+, (K_2 - S_{2\tau})^+, \cdots, (K_n - S_{n\tau})^+\big] - f^d_{\max}.$$

And focusing from the point of bank, he will receive $f^d_{\max}$ for selling the contract at time zero. If the spot price of best stock is more than K, then the bank should give the payment whose value is $\max\big(K_1 - S_{1\tau}, K_2 - S_{2\tau}, \cdots, K_n - S_{n\tau}\big)$. Inversely, he has no loss. The two cases imply that the present income of the bank is $-\exp(-r\tau)\max\big[(K_1 - S_{1\tau})^+, (K_2 - S_{2\tau})^+, \cdots, (K_n - S_{n\tau})^+\big]$. Naturally, at time zero, the investor has the net revenue which is

$$f^d_{\max} - \exp(-r\tau)\max\big[(K_1 - S_{1\tau})^+, (K_2 - S_{2\tau})^+, \cdots, (K_n - S_{n\tau})^+\big].$$

According to fair price principle that the expected net revenue of the investor should equal to the one of the bank, we can derive the following relationship

$$\exp(-r\tau)E\Big[\max_{1\le i\le n}(K_i - S_{i\tau})^+\Big] - f^c_{\max} = f^c_{\max} - \exp(-r\tau)E\Big[\max_{1\le i\le n}(K_i - S_{i\tau})^+\Big]$$

which indicates

$$f^d_{\max} = \exp(-r\tau)E\Big[\max_{1\le i\le n}(K_i - S_{i\tau})^+\Big].$$

That is, the price of maximum put option is the expected present value of revenue of best stock.

Definition 4.13. Assumed that the expiry date is τ and the strike price is K_i with respect to S_{it} in a maximum put option. Then the price of the maximum put option is

$$f^d_{\max} = \exp(-r\tau)\Big[\max_{1\le i\le n}(K_i - S_{i\tau})^+\Big].$$

Theorem 4.14. *Considering the same conditions in Definition 4.13, the price of the maximum put option of Model (1) is*

$$f^d_{\max} = \exp(-r\tau)\int_0^1 \max_{1\le i\le n}\left[K_i - S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+ \mathrm{d}\alpha.$$

Proof. Thanks to $C_{it}, i = 1, 2\cdots, n$ being independent, we may make sure that the uncertain processes $S_{1t}, S_{nt}, \cdots, S_{nt}$ are independent for any $t, 0 \le t \le \tau$. Noting that $\max\big[(K_1 - x_1)^+, (K_2 - x_2)^+, \cdots, (K_n - x_n)^+\big]$ is strictly humdrum decreasing regarded to $x_1, x_2, \cdots, x_n$, so we obtain that $\max\big[(K_1 - S_{1\tau})^+, (K_2 - S_{2\tau})^+, \cdots, (K_n - S_{n\tau})^+\big]$ has an inverse uncertainty distribution

$$\begin{aligned}\Psi_\tau^{-1}(\alpha) &= \max\Big[\big(K_1 - \Phi_{1\tau}^{-1}(1-\alpha)\big)^+, \big(K_2 - \Phi_{2\tau}^{-1}(1-\alpha)\big)^+, \cdots, \big(K_n - \Phi_{n\tau}^{-1}(1-\alpha)\big)^+\Big] \\ &= \max_{1\le i\le n}\left[K_i - S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]^+ \qquad (6)\end{aligned}$$

from Theorem 2.10. Consider that $\exp(-r\tau)$ is just a real function and has no relation to uncertainty. It follows from equation (6) and Theorem 2.7 that $\exp(-r\tau)\max\left[(K_1-S_{1\tau})^+,(K_2-S_{2\tau})^+,\cdots,(K_n-S_{n\tau})^+\right]$ has the following expected value, that is,

$$f^d_{\max}=\exp(-r\tau)\int_0^1\max_{1\le i\le n}\left[K_i-S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]^+\mathrm{d}\alpha.$$

By using variable substitution $y=1-\alpha$, the following equality is derived

$$f^d_{\max}=\exp(-r\tau)\int_0^1\max_{1\le i\le n}\left[K_i-S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+\mathrm{d}\alpha.$$

Thus the proof is done. □

If $K_1=K_2=\cdots=K_n=K$, then what is the formula for option pricing? In this situation, we obtain

$$\begin{aligned}&\max\left[(K_1-S_{1\tau})^+,(K_2-S_{2\tau})^+,\cdots,(K_n-S_{n\tau})^+\right]\\&=\max\left[K_1-S_{1\tau})^+,K_2-S_{2\tau})^+,\cdots,K_n-S_{n\tau},0\right]\\&=\max\left[K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau}),0\right]\\&=\left[K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})\right]^+\end{aligned}$$

whose inverse uncertainty distribution is

$$\begin{aligned}\Psi_\tau^{-1}(\alpha)&=\left[K-\min\left(\Phi_{1\tau}^{-1}(1-\alpha),\Phi_{2\tau}^{-1}(1-\alpha),\cdots,\Phi_{n\tau}^{-1}(1-\alpha)\right)\right]^+\\&=\left[K-\min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right)\right]^+\end{aligned}\tag{7}$$

from Theorem 2.10. Thus we give the following theorem where the pricing formula is giving under the assumption of $K_1=K_2=\cdots=K_n=K$.

Theorem 4.15. *Considering the same conditions in Definition 4.13, the price of the call on max option of Model (1) with $K_1=K_2=\cdots=K_n=K$ is*

$$f^d_{\max}=\exp(-r\tau)\int_0^1\left[K-\min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\right]^+\mathrm{d}\alpha.$$

Proof. According to Theorem 4.14 and equation (7), the result can be derived directly. □

Remark 4.16. We usually regard such maximum put option as the extension of the standard put option for an individual asset.

Theorem 4.17. *The rainbow option $f^d_{\max}$ of maximum put scenario for Model (1) has the following properties:*

1) $f^d_{\max}$ is respectively decreasing with respect to S_{i0},μ_i and σ_i;

2) $f^d_{\max}$ is increasing with respect to K.

Proof. 1) The function

$$g=\left[K-\min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\right]^+$$

is decreasing concerning S_{i0} if other parameters is fixed. Hence, $f^d_{\max}$ is increasing with respect to S_{i0}. The proof of variables μ_i and σ_i is similar to that of S_{i0}.

2) The function

$$g=\left[K-\min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\right]^+$$

is increasing concerning K if other parameters is fixed. Hence, $f^d_{\max}$ is increasing with respect to K.

□

Example 4.18. Considering Model (1), let $n=3$ and set $\tau=20, r=0.03, K=18, \mu_1=-0.04, \mu_2=-0.05, \mu_3=-0.07, \sigma_1=0.01, \sigma_2=0.01, \sigma_3=0.01, S_{10}=20, S_{20}=21$ and $S_{30}=19$. Then we obtain $f^d_{\max}=0.028$.

4.4. Minimum put scenario

This section is applied to studying the scenario of minimum put. Considering n underlying assets in the uncertain market which appear down trend, in order to hedge the risk of downtick, a contract is provided for the investor to sell the minimum stock at strike price K at expiry date τ, while this is just a right instead of obligation. Hence, the investor should pay for the right to the bank. However, how can we make an appropriate price satisfy the bank and investor? Next we will resolve this problem according to the pair price principle.

Assumed that $f^d_{\min}$ represents the price of minimum put option. Thus the investor should pay $f^d_{\min}$ at time 0 for holding the opportunity to sell the minimum stock at strike price K at expiry date τ, if the spot price of the minimum stock is less than K. If not, she can discard the right. Then the present income of the investor is $\exp(-r\tau)\big[K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})\big]^+$. Thus, at time zero, the investor has the net revenue which is

$$\exp(-r\tau)\big[K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})\big]^+-f^d_{\min}.$$

And from the point of bank, he will receive $f^d_{\min}$ for selling the contract at time zero. If the spot price of minimum stock is less than K, then the bank should give the payment whose value is $K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})$. If not, it has no influence on the bank. That is, the present income of the bank is $-\big[K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})\big]^+$. Naturally, at time zero, the investor has the net revenue which is

$$f^d_{\min}-\exp(-r\tau)\big[K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})\big]^+.$$

According to fair price principle that the expected net revenue of the investor should equal to the one of the bank, we can derive the following relationship

$$\exp(-r\tau)E\Big[\big(K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})\big)^+\Big]-f^d_{\min}=f^d_{\min}-\exp(-r\tau)E\Big[\big(K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})\big)^+\Big],$$

which indicates

$$f^d_{\min}=\exp(-r\tau)E\Big[\big(K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})\big)^+\Big].$$

That is, the price of minimum put option is the expected present value of revenue of minimum stock.

Definition 4.19. Assumed that the expiry date is τ and the strike price is K in a minimum put option. Then the price of the minimum put option is

$$f^d_{\min}=\exp(-r\tau)E\Big[\big(K-\min(S_{1\tau},S_{2\tau},\cdots,S_{n\tau})\big)^+\Big].$$

Theorem 4.20. *Considering the same conditions in Definition 4.19, the price of the maximum call option of Model (1) is*

$$f^d_{\min}=\exp(-r\tau)\int_0^1\left[K-\min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau+\frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\right]^+\mathrm{d}\alpha.$$

Proof. Because of $C_{it}, i = 1, 2 \cdots, n$ being independent, we may assert that uncertain processes $S_{1t}, S_{2t}, \cdots, S_{nt}$ should be independent for any $t, 0 \le t \le \tau$. Noting that $\big[K - \min(x_1, x_2, \cdots, x_n)\big]^+$ is strictly humdrum decreasing regarded to $x_1, x_2, \cdots, x_n$, so we obtain that $\big[K - \min(S_{1\tau}, S_{2\tau}, \cdots, S_{n\tau})\big]^+$ has an inverse uncertainty distribution

$$\begin{aligned}\Psi_\tau^{-1}(\alpha) &= \left[K - \min\left(\Phi_{1\tau}^{-1}(1-\alpha), \Phi_{2\tau}^{-1}(1-\alpha), \cdots, \Phi_{n\tau}^{-1}(1-\alpha)\right)\right]^+ \\ &= \left[K - \min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right)\right]^+ \end{aligned} \tag{8}$$

from Theorem 2.10. Consider that $\exp(-r\tau)$ is just a real function and has no relation to uncertainty. It follows from equation (8) and Theorem 2.7 that $\exp(-r\tau)\big[\min(S_{1\tau}, S_{2\tau}, \cdots, S_{n\tau}) - K\big]^+$ has the following expected value, that is,

$$f_{\min}^d = \exp(-r\tau)\int_0^1 \left[K - \min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right)\right]^+ \mathrm{d}\alpha.$$

By using variable substitution $y = 1 - \alpha$, the following equality is derived

$$f_{\min}^d = \exp(-r\tau)\int_0^1 \left[K - \min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\right]^+ \mathrm{d}\alpha.$$

Thus the proof is fulfilled. □

Theorem 4.21. *The rainbow option $f_{\min}^d$ of maximum put scenario for Model (1) has the following properties:*

1) $f_{\min}^d$ is respectively decreasing with respect to S_{i0}, μ_i and σ_i;

2) $f_{\min}^d$ is increasing with respect to K.

Proof. 1) The function

$$g = \left[K - \min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\right]^+$$

is decreasing concerning S_{i0} if other parameters is fixed. Hence, $f_{\min}^d$ is increasing with respect to S_{i0}. The proof of variables μ_i and σ_i is similar to that of S_{i0}.

2) The function

$$g = \left[K - \min_{1\le i\le n}\left(S_{i0}\exp\left(\mu_i\tau + \frac{\sigma_i\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)\right]^+$$

is increasing concerning K if other parameters is fixed. Hence, $f_{\min}^d$ is increasing with respect to K.

□

Example 4.22. Considering Model (1), let $n = 3$ and set $\tau = 20, r = 0.03, K = 18, \mu_1 = -0.04, \mu_2 = -0.05, \mu_3 = -0.07, \sigma_1 = 0.01, \sigma_2 = 0.01, \sigma_3 = 0.01, S_{10} = 20, S_{20} = 21$ and $S_{30} = 19$. Then we have $f_{\min}^d = 0.0046$.

4.5. Put on 2 and call on 1 scenario

We employ this section to study the colorred put on 2 and call on 1 option, which is actually to evaluate the option to exchange one stock for the other at maturity date τ. Considering two underlying asset in an uncertain market, one has an upward trend and the other has downward trend. Hence, a type of option is provided for the investor, where they can buy the asset 1 at the price of asset 2. This conceptually likes a call option on asset 1, in which the exercise price is uncertain, dynamic and is the price of asset 2 in reality. However, this is just a right rather than an obligation. Hence, the investor should pay something for obtaining the opportunity. However, how much should the contract price be reasonable? Next we will discuss this issue according to the pair price principle.

Supposed that f denotes the price of put on 2 and call on 1 option. Thus the investor should pay f at time 0 for holding the opportunity to buy the stock 1 at strike price stock 2 at due date τ if the spot price of the stock 1 is more than that of stock 2. On the contrary, she could give up the right. Following that, the present income of the investor is $\exp(-r\tau)\big(S_{1\tau}-S_{2\tau}\big)^+$. Naturally, at time zero, the investor has the net revenue which is

$$\exp(-r\tau)\big(S_{1\tau}-S_{2\tau}\big)^+ - f.$$

And for the bank, he will receive f for selling the contract at time zero. If the spot price of stock 1 is more than stock 2, then the bank should give the payment whose value is $S_{1\tau}-S_{2\tau}$. If not, he has no any loss. In other words, the present income of the bank is $-\big(S_{1\tau}-S_{2\tau}\big)^+$. Naturally, at time zero, the investor has the net revenue which is

$$f-\exp(-r\tau)\big(S_{1\tau}-S_{2\tau}\big)^+.$$

According to fair price principle that the expected net revenue of the investor should equal to the one of the bank, therefore we can derive the following relationship

$$\exp(-r\tau)E\Big[\big(S_{1\tau}-S_{2\tau}\big)^+\Big]-f=f-\exp(-r\tau)E\Big[\big(S_{1\tau}-S_{2\tau}\big)^+\Big]$$

which indicates

$$f=\exp(-r\tau)E\Big[\big(S_{1\tau}-S_{2\tau}\big)^+\Big].$$

That is, the price of put on 2 call on 1 option is the expected present value of revenue of stock 1.

Definition 4.23. Assumed that the expiry date is τ and the strike price is K in a put on 2 and call on 1 option. Then the price of the put on 2 and call on 1 option is

$$f=\exp(-r\tau)E\Big[\big(S_{1\tau}-S_{2\tau}\big)^+\Big].$$

Theorem 4.24. *Considering the same conditions in Definition 4.23, the price of the put on 2 and call on 1 option of Model (1) is*

$$f=\exp(-r\tau)\int_0^1\left[S_{10}\exp\left(\mu_1\tau+\frac{\sigma_1\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-S_{20}\exp\left(\mu_2\tau+\frac{\sigma_2\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+\mathrm{d}\alpha$$

where $i=2$.

Proof. Noting that $C_{it}, i=1,2$ being independent, we may conclude that S_{1t}, S_{2t} are independent uncertain processes for any $t, 0\le t\le\tau$. Due to $\big(x_1-x_2\big)^+$ being strictly humdrum increasing regarded to x_1 and strictly humdrum decreasing regarded to x_2, so we obtain that $\big(S_{1\tau}-S_{2\tau}\big)^+$ has an inverse uncertainty distribution

$$\begin{aligned}\Psi_\tau^{-1}(\alpha)&=\big[\Phi_{1\tau}^{-1}(\alpha)-\Phi_{2\tau}^{-1}(1-\alpha)\big]^+\\&=\left[S_{i0}\exp\left(\mu_1\tau+\frac{\sigma_1\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-S_{i0}\exp\left(\mu_2\tau+\frac{\sigma_2\tau\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]^+\end{aligned}\qquad(9)$$

from Theorem 2.10. Consider that $\exp(-r\tau)$ is just a real function and has no relation to uncertainty. It follows from equation (9) and Theorem 2.7 that $\exp(-r\tau)\big(S_{1\tau}-S_{2\tau}\big)^+$ has the following expected value, that is,

$$f=\exp(-r\tau)\int_0^1\left[S_{10}\exp\left(\mu_1\tau+\frac{\sigma_1\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-S_{20}\exp\left(\mu_2\tau+\frac{\sigma_2\tau\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]^+\mathrm{d}\alpha.$$

For one case, if $S_{1\tau}>S_{2\tau}$, then by using variable substitution $y=1-\alpha$, the following equality is derived

$$\begin{aligned}f&=\exp(-r\tau)\int_0^1\left[S_{10}\exp\left(\mu_1\tau+\frac{\sigma_1\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-S_{20}\exp\left(\mu_2\tau+\frac{\sigma_2 t\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]\mathrm{d}\alpha\\
&=\exp(-r\tau)\int_0^1 S_{10}\exp\left(\mu_1\tau+\frac{\sigma_1\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}\alpha-\exp(-r\tau)\int_0^1 S_{20}\exp\left(\mu_2 t+\frac{\sigma_2\tau\sqrt{3}}{\pi}\mathrm{d}\alpha\ln\frac{1-\alpha}{\alpha}\right)\mathrm{d}\alpha\\
&=\exp(-r\tau)\int_0^1 S_{10}\exp\left(\mu_1\tau+\frac{\sigma_1\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}\alpha-\exp(-r\tau)\int_0^1 S_{20}\exp\left(\mu_2\tau+\frac{\sigma_2\tau\sqrt{3}}{\pi}\mathrm{d}\alpha\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}\alpha\\
&=\exp(-r\tau)\int_0^1\left[S_{10}\exp\left(\mu_1\tau+\frac{\sigma_1\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-S_{20}\exp\left(\mu_2\tau+\frac{\sigma_2\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]\mathrm{d}\alpha.\end{aligned}$$

For the other case, if $S_{1\tau}\le S_{2\tau}$, then $f=0$. Thus we derive

$$f=\exp(-r\tau)\int_0^1\left[S_{10}\exp\left(\mu_1\tau+\frac{\sigma_1\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-S_{20}\exp\left(\mu_2\tau+\frac{\sigma_2\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+\mathrm{d}\alpha.$$

Thus the proof is completed. □

Theorem 4.25. *The rainbow option f of maximum put scenario for Model (1) where $i=2$ has the following properties:*

1) f is respectively increasing with respect to S_{10},μ_1 and σ_1;

2) f is decreasing with respect to S_{20},μ_2 and σ_2.

Proof. 1) The function

$$g=\left[S_{10}\exp\left(\mu_1\tau+\frac{\sigma_1\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-S_{20}\exp\left(\mu_2\tau+\frac{\sigma_2\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+$$

is decreasing concerning S_{10} if other parameters is fixed. Hence, f is increasing with respect to S_{10}. The proof of variables μ_1 and σ_1 is similar to that of S_{10}.

2) The function

$$g=\left[S_{10}\exp\left(\mu_1\tau+\frac{\sigma_1\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-S_{20}\exp\left(\mu_2\tau+\frac{\sigma_2\tau\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+$$

is decreasing concerning S_{20} if other parameters is fixed. Hence, f is decreasing with respect to S_{20}. The proof of variables μ_2 and σ_2 is similar to that of S_{20}.

□

Example 4.26. Considering Model (1), let $n=3$ and set $\tau=10, r=0.03, K=18, \mu_1=0.05, \mu_2=0.03, \sigma_1=0.03, \sigma_2=0.025, S_{10}=13$, and $S_{20}=10$. Then we have $f=0.0148$.

5. Conclusions

In this paper, we assume that the dynamic stock prices obey uncertain differential equation without sharing dividends. Based on the multiple uncertain stock model, we studied the rainbow option pricing issue for multiple stocks in a financial market with uncertain information, presented the concepts of corresponding option and derived pricing formulas for five scenarios including maximum call, minimum call, maximum put, minimum put, and put on 2 and call on 1. In the future, some extensions can be considered. For example, we can introduce the rainbow option to uncertain interest market and uncertain currency market under the frame of uncertainty theory, respectively. In addition, we can also design rainbow option into other types such as American option type and Asian option type. Furthermore, the rainbow option can also be combined with barrier option. Then these new combined options can be used into stock market, interest market and currency market.

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