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# OPTIMALITY CONDITIONS FOR MPECS IN TERMS OF DIRECTIONAL UPPER CONVEXIFACTORS

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**Abstract.** In this paper, we investigate necessary and sufficient optimality conditions for mathematical programs with equilibrium constraints. For this goal, we introduce an appropriate type of MPEC regularity condition and a stationary concept given in terms of directional upper convexificators and directional upper semi-regular convexificators. The appearing functions are not necessarily smooth/locally Lipschitz/convex/continuous, and the continuity directions' sets are not assumed to be compact or convex. Finally, notions of directional pseudoconvexity and directional quasiconvexity are used to establish sufficient optimality conditions for MPECs.

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## 1. Introduction

In this paper, we investigate the following mathematical program with equilibrium constraints:

$$(MPEC) : \begin{cases} \text{Minimize } f(x) \\ \text{s.t.} \begin{cases} g(x) \leq 0,\ h(x) = 0, \\ G(x) \geq 0,\ H(x) \geq 0,\ G(x)^\top H(x) = 0, \end{cases} \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R},\ g : \mathbb{R}^n \to \mathbb{R}^m,\ h : \mathbb{R}^n \to \mathbb{R}^p,\ G : \mathbb{R}^n \to \mathbb{R}^l$ and $H : \mathbb{R}^n \to \mathbb{R}^l$ are lower semicontinuous functions; $n,\ m,\ p,\ l \in \mathbb{N}$.

Such a problem has been discussed by several authors at various levels of generality [1, 7–10, 20, 28]. In [8], Flegel and Kanzow presented a straightforward and elementary approach to first-order optimality conditions for the MPECs and showed that Fritz-John approach leads to a new optimality condition under a Mangasarian-Fromovitz-type assumption. In [9], the authors introduced a new Abadie-type constraint qualification for the MPECs and showed it to be weaker than any of the existing ones. In [1], Ardali *et al.* defined nonsmooth stationary conditions based on the convexificators and showed that generalized strong stationary is the first-order optimality condition under a generalized standard Abadie constraint qualification.

The notion of convexificator can be seen as a generalization of the idea of subdifferential. For a locally Lipschitz function, most known subdifferentials are convexificators and these subdifferentials may contain the convex hull

*Keywords.* Directional upper convexificators, directional upper semi-regular convexificators, regularity conditions, mathematical programs with equilibrium constraints, optimality conditions.

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of a convexificator [16]. Noting that convexificators admitted by discontinuous functions may be unbounded and because the boundedness of convexificators is of crucial importance in many well-known results, Dempe and Pilecka [3] developed the concept of directional convexificators. They were able to create a convexificator for a given lower semicontinuous function using directional convexificators, presuming convexity and closedness of the set of continuity directions (see [3], Cor. 2 and Prop. 1). Notice that directional convexificators are closed sets which can be bounded and/or strictly included in convexificators (see Example 2.10). Using this new tool, Gadhi [11] established mean value conditions in terms of directional convexificators and formulate variational inequalities of Stampacchia and Minty type in terms of directional convexificators; he used these variational inequalities as a tool to find out necessary and sufficient conditions for a point to be an optimal solution of an inherent optimization problem. In [14], Gadhi *et al.* gave optimality conditions for a set valued optimization problem using support functions of set valued mappings.

Motivated by the above work of Dempe and Pilecka [3], we investigate necessary and sufficient optimality conditions for (MPEC) where data functions are not necessarily smooth/locally Lipschitz/convex/continuous. Because the directional upper (semi-regular) convexificator of such a data function can be bounded while the upper (semi-regular) convexificator is not, our results may be applicable in situations where other results imposing local Lipschitzity or continuity are not (see Example 3.11). To achieve our goal, we introduce an alternative stationarity concept and a generalized Abadie-type regularity condition using directional upper (semi-regular) convexificator; and, assuming the feasible set is locally star-shaped, we show that alternative stationarity is in fact a first-order necessary optimality condition for MPECs. Unlike Dempe and Pilecka [3] and Gadhi *et al.* [14], we do not assume that the sets of all continuity directions are convex or compact. Under some directional generalized convexities, we establish sufficient optimality conditions for $(MPEC)$. Notice that directional upper semi-regular convexificators are not necessarily upper semi-regular convexificators; moreover, they may not even be directional upper regular convexificators (see Example 2.11).

The outline of the paper is as follows: Section 2 describes the preliminary and basic definitions; Sections 3 and 4 establish the main results; and Section 5 provides a conclusion.

## 2. Preliminaries

Throughout this section, let $\mathbb{R}^n$ be the usual $n$-dimensional Euclidean space. Given a nonempty subset $S$ of $\mathbb{R}^n$, the closure, convex hull, and convex cone (including the origin) generated by $S$ are denoted respectively by $cl\ S$, $conv\ S$ and $pos\ S$. The negative polar cone of $S$ is defined by

$$S^- := \{v \in \mathbb{R}^n \mid \langle x, v\rangle \leq 0,\ \forall x \in S\}.$$

Let $x \in cl\ S$. The cone of feasible directions of $S$ at $x$, the cone of weak feasible directions of $S$ at $x$, and the contingent cone of $S$ at $x$ are given by

$$\mathcal{F}(S,x) = \{v \in \mathbb{R}^n : \exists \delta > 0,\ \forall \alpha \in (0,\delta) \text{ such that } x + \alpha v \in S\},$$

$$W(S,x) = \{v \in \mathbb{R}^n : \exists t_n \to 0^+ \text{ such that } x + t_n v \in S,\ \forall n\}$$

and

$$T(S,x) = \{v \in \mathbb{R}^n : \exists t_n \to 0^+,\ \exists v_n \to v \text{ such that } x + t_n v_n \in S,\ \forall n\}.$$

Notice that, for all $x \in cl\ S$, we have

$$\mathcal{F}(S,x) \subseteq W(S,x) \subseteq T(S,x). \tag{1}$$

The regular (Fréchet) normal cone $N_S(x)$ of $S$ at $x \in S$, following Definition 6.3 of [27], is defined by

$$N_S(x) = \left\{ v \in \mathbb{R}^n : \limsup_{y \longmapsto x,\ y \in S,\ y \neq x} \frac{\langle v, y - x\rangle}{\|y - x\|} \leq 0 \right\}.$$

Observe that $N_S(x) = T(S,x)^-$, see Theorem 6.28a of [27]. On the one hand, $\mathcal{F}(S,x)$ is not necessarily convex or closed. On the other hand, $T(S,x)$ is closed but not necessarily convex. When $S$ is convex, $T(S,\overline{x})$ is also convex and $\mathcal{F}(S,x)$ merges with $W(S,x)$, and we have $\mathcal{F}(S,x) = W(S,x)$, $T(S,x) = cl\ \mathcal{F}(S,x)$ and

$$N_S(x) = \{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leq 0,\ \forall y \in S\}.$$

**Definition 2.1.** [6] A nonempty set $S \subseteq \mathbb{R}^n$ is said to be locally star-shaped at $\overline{x} \in S$, if there exists some scalar $a(\overline{x}, x) \in (0,1]$, corresponding to $\overline{x}$ and each $x \in S$, such that

$$\overline{x} + \lambda(x - \overline{x}) \in S,\ \text{for all}\ \lambda \in (0, a(\overline{x}, x)).$$

If $a(\overline{x}, x) = 1$ for each $x \in S$, then $S$ is said to be star-shaped at $\overline{x}$.

Open sets and convex sets, for instance, are locally star-shaped at each of their elements, whereas cones are locally star-shaped at their origin. If $S$ is closed and is locally star-shaped at each $\overline{x} \in S$, then $S$ is convex [21]. However, there exist locally star-shaped sets (at some $\overline{x}$) that are neither star-shaped nor locally convex (at $\overline{x}$). For example,

$$S = \mathbb{R}^2 \setminus \left\{(x,y) \in \mathbb{R}^2 : x^2 = y \text{ and } x \neq 0\right\}$$

is locally star-shaped at $\overline{x} = (0,0)$ and is neither star-shaped nor locally convex at $\overline{x}$ [18].

The following result is due to Kabgani and Soleimani-damaneh; for more details see Theorem 3.1 of [18].

**Proposition 2.2.** *[18] Assume that $\Omega$ is locally star-shaped at $\overline{x} \in \Omega$. Then*

$$T(\Omega, \overline{x}) = cl\ \mathcal{F}(\Omega, \overline{x}).$$

**Remark 2.3.** In case $\Omega$ is locally star-shaped at $\overline{x} \in \Omega$, according to Proposition 2.2 together with inclusions (1), we have

$$(\Omega, \overline{x}) = cl\ W(\Omega, \overline{x}).$$

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function and let $x \in \mathbb{R}^n$ such that $f(x)$ is finite. The expressions

$$f^-(x,d) = \liminf_{t \longmapsto 0^+} \frac{f(x+td) - f(x)}{t} \text{ and } f^+(x,d) = \limsup_{t \longmapsto 0^+} \frac{f(x+td) - f(x)}{t}$$

signify, respectively, the lower and upper Dini directional derivatives of $f$ at $x$ in the direction $d$. When $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, both of the above derivatives exist finitely.

**Definition 2.4.** [5] The function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have an upper convexifactor $\emptyset \neq \partial^u f(x) \subseteq \mathbb{R}^n$ at $x$ if $\partial^u f(x)$ is closed and, for each $d \in \mathbb{R}^n$,

$$f^-(x,d) \leq \sup_{x^* \in \partial^u f(x)} \langle x^*, d\rangle.$$

The function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have an upper semi-regular convexifactor $\emptyset \neq \partial^{us} f(x) \subseteq \mathbb{R}^n$ at $x$ if $\partial^{us} f(x)$ is closed and, for each $d \in \mathbb{R}^n$,

$$f^+(x,d) \leq \sup_{x^* \in \partial^{us} f(x)} \langle x^*, d\rangle.$$

**Remark 2.5.** The class of functions that admit an upper (semi-regular) convexifactor is extensive. Observe that Gâteaux differentiable functions and regular functions in the sense of Clarke [2] are members of this class. Clarke subdifferentials of locally Lipschitz functions and tangential subdifferentials of tangentially convex functions [23] are both upper (semi-regular) convexifactors.

**Remark 2.6.** It is worth noting that the upper convexificator for a given function is not always unique. In certain instances, it is possible to find an upper convexificator that is smaller than the most well-known subdifferentials, such as those of Clarke [2], Michel-Penot [25], and Mordukhovich [24]. Demyanov and Jeyakumar's concept of minimal upper convexificator [4] appears promising for this purpose. In [16], Jeyakumar and Luc presented conditions for unique minimal upper convexificators in terms of the set of extreme points [16].

We shall need the following definition.

**Definition 2.7.** [3] Consider $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. A vector $d \in \mathbb{R}^n$ is a continuity direction of $f$ at the point $x \in \mathbb{R}^n$ if for all sequences $\{t_k\} \subset \mathbb{R}$ with $\{t_k\} \searrow 0$ we have

$$\lim_{k \longmapsto \infty} f(x + t_k d) \ = \ f(x).$$

The set of all continuity directions of $f$ at $x$ is denoted by $\mathcal{D}$ or $\mathcal{D}_f(x)$.

Note that the Fréchet normal cone to $\mathcal{D}$ at $\overline{d} = 0_n$ is given by $N_{\mathcal{D}}(0_n) = T(\mathcal{D}, 0_n)^-$.

**Remark 2.8.** The set $\mathcal{D}$ is a non-empty cone (it always contains $0_n$) which is not necessarily closed or convex. When $\mathcal{D}$ is convex, $T(\mathcal{D}, 0_n)$ is also convex, and thus $N_{\mathcal{D}}(0_n) = \mathcal{D}^-$.

Dempe and Pilecka introduced directional convexificators using the set of continuity directions. For more details, see Definition 3 of [3].

**Definition 2.9.** [3] Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function.

- $f$ admits a directional upper convexificator $\emptyset \neq \partial^u_{\mathcal{D}} f(x)$ at $x \in \mathbb{R}^n$ if the set $\partial^u_{\mathcal{D}} f(x)$ is closed and for each $d \in \mathcal{D}$ we have:
$$f^-(x, d) \leq \sup_{x^* \in \partial^u_{\mathcal{D}} f(x)} \langle x^*, d \rangle.$$
- $f$ admits a directional upper semi-regular convexificator $\emptyset \neq \partial^{us}_{\mathcal{D}} f(x)$ at $x \in \mathbb{R}^n$ if the set $\partial^{us}_{\mathcal{D}} f(x)$ is closed and for each $d \in \mathcal{D}$ we have:
$$f^+(x, d) \leq \sup_{x^* \in \partial^{us}_{\mathcal{D}} f(x)} \langle x^*, d \rangle. \tag{2}$$

In the case where $f$ is continuous at $x \in \mathbb{R}^n$, we have $\mathcal{D} = \mathcal{D}_f(x) = \mathbb{R}^n$ and the directional upper convexificator (resp. directional upper semi-regular convexificator) coincides with the upper convexificator (resp. upper semi-regular convexificator). If inequality (2) holds as equality for every $d \in \mathcal{D}$, then $\partial^{us}_{\mathcal{D}} f(x)$ is known as a directional upper regular convexificator of $f$ at $x$; for more details see Definition 3 of [3]. The following example shows that a directional upper convexificator is not necessarily an upper convexificator.

**Example 2.10.** Consider the function

$$\forall x = (x_1, x_2) \in \mathbb{R}^2 : f(x) = \begin{cases} 2x_2 - 1 & \text{if} \quad x_1 = 0,\ x_2 > 0, \\ -3x_1 - 1 & \text{if} \quad x_1 < 0,\ x_2 = 0, \\ |x_1| - |x_2| - 2 & \text{elsewhere} \end{cases}$$

at the point $\overline{x} = (0, 0)$.

- The set of all continuity directions
$$\mathcal{D} = \mathbb{R}^2 \setminus (\{0\} \times (0, +\infty) \cup (-\infty, 0) \times \{0\})$$
is neither closed nor convex. The normal cone to the set $\mathcal{D}$ equals $N_{\mathcal{D}}(0, 0) = \{(0, 0)\}$.

– The function $f$ admits $\partial_{\mathcal{D}}^{u} f(\overline{x}) = \{(1,-1),(-1,1)\}$ as a directional upper convexificator at $\overline{x}$. Notice that this directional upper convexificator is not an upper convexificator of $f$ at $\overline{x}$. Indeed, for $\overline{d} = (0,1)$, we have

$$+\infty = f^{-}\left(\overline{x},\overline{d}\right) > 1 = \sup_{x^* \in \partial_{\mathcal{D}}^{u} f(\overline{x})} \langle x^*, \overline{d}\rangle.$$

Observe that both $\partial_{\mathcal{D}}^{u} f(\overline{x})$ and $\partial_{\mathcal{D}}^{u} f(\overline{x}) + N_{\mathcal{D}}(0,0)$ are compact sets.

Example 2.11 shows that a directional upper semi-regular convexificator is not necessarily an upper semi-regular convexificator; further, it may not even be a directional upper regular convexificator.

**Example 2.11.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1,x_2) = \begin{cases} 0 & \text{if} \quad x_1 \geq 0, \\ x_2^2 + 1 & \text{if} \quad x_1 < 0. \end{cases}$$

– The set of all continuity directions of $f$ at $\overline{x} = (0,0)$ is $\mathcal{D} = \mathbb{R}^+ \times \mathbb{R}$.
– $\partial_{\mathcal{D}}^{us} f(\overline{x}) = \{(1,0)\}$ is a directional upper semi-regular convexificator at $\overline{x}$. Indeed, $\partial_{\mathcal{D}}^{us} f(\overline{x})$ is closed and for each $d = (d_1, d_2) \in \mathcal{D}$, we have

$$f^{+}(\overline{x}, d) = 0 \leq d_1 = \sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\overline{x})} \langle x^*, d\rangle.$$

- $\partial_{\mathcal{D}}^{us} f(\overline{x})$ is not an upper semi-regular convexificator of $f$ at $\overline{x}$. Indeed, for $\widetilde{d} = (-1,0) \in \mathbb{R}^2$, we have

$$f^{+}\left(\overline{x}, \widetilde{d}\right) = +\infty > -1 = \sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\overline{x})} \langle x^*, \widetilde{d}\rangle.$$

- $\partial_{\mathcal{D}}^{us} f(\overline{x})$ is not a directional upper regular convexificator of $f$ at $\overline{x}$. Indeed, for $\overline{d} = (1,0) \in \mathcal{D}$, we have

$$f^{+}\left(\overline{x}, \overline{d}\right) = 0 \neq 1 = \sup_{x^* \in \partial_{\mathcal{D}}^{us} f(\overline{x})} \langle x^*, \overline{d}\rangle.$$

The following lemma is of interest for our investigations.

**Lemma 2.12.** *[13] Let $\mathcal{B}$ a nonempty, convex and compact set and $\mathcal{A}$ be a convex cone. If*

$$\sup_{v \in \mathcal{B}} \langle v, d\rangle \geq 0, \ \textit{for all } d \in \mathcal{A}^-$$

*then* $0 \in \mathcal{B} + cl\mathcal{A}$.

## 3. Necessary optimality conditions

Let $\Omega$ be the feasible set of $(MPEC)$ defined by

$$\Omega := \left\{ x \in \mathbb{R}^n : g(x) \leq 0, \ h(x) = 0, \ G(x) \geq 0, \ H(x) \geq 0, \ G(x)^t H(x) = 0 \right\}.$$

Let $\overline{x} \in \Omega$ and let

$$I := \{1, \ \ldots, \ m\}, \ J := \{1, \ \ldots, \ p\}, \ \mathcal{L} := \{1, \ \ldots, \ l\} \text{ and } I(\overline{x}) := \{i \in I : g_i(\overline{x}) = 0\}.$$

Consider the sets

$$A := \{i \in \mathcal{L} : G_i(\overline{x}) = 0, \ H_i(\overline{x}) > 0\}, \ B := \{i \in \mathcal{L} : G_i(\overline{x}) = 0, \ H_i(\overline{x}) = 0\},$$

and

$$D := \{i \in \mathcal{L} : G_i(\overline{x}) > 0,\ H_i(\overline{x}) = 0\}.$$

The set $B$ is known as the degenerate set. If it is empty, the vector $\overline{x}$ is said to fulfill strict complementarity [28] and we have $\mathcal{L} = A \cup D$. Throughout this section, we assume that $B$ is a nonempty set. A partition of $B$ is of the form $(B_1, B_2)$ where $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$. We denote the set of all partitions of $B$ by $P(B)$. Now, we recall a nonlinear program $MPEC(B_1, B_2)$ as given by Ye [28], with respect to a partition $(B_1, B_2)$ of $B$, given by

$$MPEC(B_1, B_2) : \begin{cases} \text{Minimize } f(x) \\ \text{s.t.} \begin{cases} g(x) \leq 0,\ h(x) = 0, \\ G_i(x) \geq 0,\ i \in B_1,\ H_j(x) \geq 0,\ j \in B_2, \\ G_i(x) = 0,\ i \in A \cup B_2,\ H_j(x) = 0,\ j \in D \cup B_1. \end{cases} \end{cases}$$

Notice that $\overline{x} \in \Omega$ is a local optimal solution of $MPEC$ if and only if it is a local optimal solution of $MPEC(B_1, B_2)$ for all $(B_1, B_2) \in P(B)$.

For the rest of the paper, we will make use of the following assumptions.

**Assumption 3.1.** *The function $f$ admits a compact directional upper semi-regular convexificator $\partial^{us}_{\mathcal{D}} f(\overline{x})$ at $\overline{x} \in \Omega$.*

**Assumption 3.2.** *The functions $g_i, i \in I(\overline{x})$, $h_j,\ j \in J$, $G_s,\ s \in A \cup B_2$, and $H_\tau,\ \tau \in D \cup B_1$, admit directional upper convexificators $\partial^u_{\mathcal{D}} g_i(\overline{x}), i \in I(\overline{x})$, $\partial^u_{\mathcal{D}} h_j(\overline{x}), j \in J$, $\partial^u_{\mathcal{D}} G_s(\overline{x}), s \in A \cup B_2$, and $\partial^u_{\mathcal{D}} H_\tau(\overline{x}), \tau \in D \cup B_1$.*

**Assumption 3.3.** *The functions $(-h_j), j \in J$, $(-G_s), s \in A \cup B$, and $(-H_\tau), \tau \in D \cup B$, admit directional upper convexificators $\partial^u_{\mathcal{D}}(-h_j)(\overline{x}), j \in J$, $\partial^u_{\mathcal{D}}(-G_s)(\overline{x}), s \in A \cup B$, and $\partial^u_{\mathcal{D}}(-H_\tau)(\overline{x}), \tau \in D \cup B$.*

Here, $\mathcal{D}$ is the set of all continuity directions of the functions $f,\ g_i,\ i \in I(\overline{x}), h_j,\ (-h_j), j \in J,\ G_s,\ (-G_s), s \in A \cup B$, and $H_\tau,\ (-H_\tau), \tau \in D \cup B$.

Now, assuming that all of the constraint functions have directional upper convexificators at $\overline{x}$, we introduce the following notations:

$$T_{\mathcal{D}}(\Omega, \overline{x}) := T(\Omega, \overline{x}) \cap \mathcal{D},\ W_{\mathcal{D}}(\Omega, \overline{x}) := W(\Omega, \overline{x}) \cap \mathcal{D} \text{ and } \Xi(\overline{x}) := \Gamma(\overline{x}) \cup N_{\mathcal{D}}(0_n),$$

where

$$\begin{aligned} \Gamma(\overline{x}) : = & \left( \bigcup_{i \in I(\overline{x})} conv\ \partial^u_{\mathcal{D}} g_i(\overline{x}) \right) \cup \left( \bigcup_{i \in J} conv\ \partial^u_{\mathcal{D}} h_i(\overline{x}) \right) \cup \left( \bigcup_{i \in J} conv\ \partial^u_{\mathcal{D}}(-h_i)(\overline{x}) \right) \\ & \cup \left( \bigcup_{i \in A \cup B_2} (conv\ \partial^u_{\mathcal{D}} G_i(\overline{x}) \cup conv\ \partial^u_{\mathcal{D}}(-G_i)(\overline{x})) \right) \cup \left( \bigcup_{i \in D \cup B_1} (conv\ \partial^u_{\mathcal{D}} H_i(\overline{x}) \cup conv\ \partial^u_{\mathcal{D}}(-H_i)(\overline{x})) \right) \\ & \cup \left( \bigcup_{i \in B_1} conv\ \partial^u_{\mathcal{D}}(-G_i)(\overline{x}) \right) \cup \left( \bigcup_{i \in B_2} conv\ \partial^u_{\mathcal{D}}(-H_i)(\overline{x}) \right). \end{aligned}$$

Using the aforementioned notations and the concept of a directional upper convexificator, we can now state our Abadie regularity condition.

**Definition 3.4.** Suppose that Assumptions 3.2 and 3.3 hold for some $(B_1, B_2) \in P(B)$. We say that the Abadie regularity condition $\partial_{\mathcal{D}} - ARC(B_1, B_2)$ holds at $\overline{x} \in \Omega$ if

$$\{0_n\} \neq \Xi(\overline{x})^- \subseteq T_{\mathcal{D}}(\Omega, \overline{x}).$$

**Remark 3.5.** The preceding regularity condition extends a number ones addressed in the literature. Indeed, if all the constraint functions are continuous, $\mathcal{D} = \mathbb{R}^n$ and $\partial_{\mathcal{D}} - ARC\,(B_1, B_2)$ reduces to the generalized MPEC Abadie constraint qualification given by Ardali *et al.* in Definition 3.2 of [1]. If in addition $h \equiv 0,\ G \equiv 0$, and $H \equiv 0$, it merges with the Abadie constraint qualification $(ACQ)$ presented by Li and Zhang in [22].

In the following definition, we introduce a generalized alternatively stationarity concept in terms of directional upper convexifactors. For continuous functions, Definition 3.6 merges with Definition 4.3 of [1] and Definition 4.1 of [12] since in this case $N_{\mathcal{D}}\,(0_n) = \{0_n\}$ and $\mathcal{D} = \mathbb{R}^n$.

**Definition 3.6.** A feasible point $\overline{x}$ of $MPEC$ is said to be a generalized alternatively stationary point if there exists a vector $\left(\lambda^g, \lambda^h, \mu^h, \lambda^G, \lambda^H, \mu^G, \mu^H\right) \in \mathbb{R}^m \times \mathbb{R}^{2p} \times \mathbb{R}^{2l} \times \mathbb{R}^{2l}$ such that

$$0 \in \begin{bmatrix} conv\ \partial^{us}_{\mathcal{D}} f\,(\overline{x}) + \sum_{i=1}^{m} \lambda_i^g\ conv\ \partial^{u}_{\mathcal{D}} g_i\,(\overline{x}) \\ + \sum_{i\in J} \mu_i^h\ conv\ \partial^{u}_{\mathcal{D}} h_i\,(\overline{x}) + \sum_{i\in J} \lambda_i^h\ conv\ \partial^{u}_{\mathcal{D}} \left(-h_i\right)(\overline{x}) \\ + \sum_{i=1}^{l} \lambda_i^G\ conv\ \partial^{u}_{\mathcal{D}} \left(-G_i\right)(\overline{x}) + \sum_{i=1}^{l} \lambda_i^H\ conv\ \partial^{u}_{\mathcal{D}} \left(-H_i\right)(\overline{x}) \\ + \sum_{i=1}^{l} \mu_i^G\ conv\ \partial^{u}_{\mathcal{D}} G_i\,(\overline{x}) + \sum_{i=1}^{l} \mu_i^H\ conv\ \partial^{u}_{\mathcal{D}} H_i\,(\overline{x}) + N_{\mathcal{D}}\,(0_n)\,. \end{bmatrix} \tag{3}$$

with

$$\lambda_i^g\ g_i\,(\overline{x}) = 0,\ \forall i \in I \tag{4}$$

and

$$\begin{cases} \mu_i^G = 0 \text{ or } \mu_i^H = 0,\ \forall i \in B, \\ \lambda_i^G = 0,\ \mu_i^G = 0,\ \forall i \in D, \\ \lambda_i^H = 0,\ \mu_i^H = 0,\ \forall i \in A, \\ \lambda_i^G,\ \lambda_i^H,\ \mu_i^G,\ \mu_i^H \geq 0,\ \forall i \in \mathcal{L}, \\ \lambda_i^g \geq 0,\ \forall i \in I,\ \text{ and } \lambda_i^h \geq 0,\ \mu_i^h \geq 0,\ \forall i \in J. \end{cases} \tag{5}$$

**Remark 3.7.** Observe that if all function s are differentiable and the upper convexificator is replaced by the upper regular convexifactor in the preceding stationary notion, then this concept reduces to the A-stationary condition given by Flegel and Kanzow in [10] and by Flegel in [7].

**Proposition 3.8.** *Let $\overline{x}$ be a local optimal solution of $MPEC$ where Assumption 3.1 holds. Then,*

$$\sup_{\eta \in \partial^{us}_{\mathcal{D}} f(\overline{x})} \langle \eta, v\rangle \geq 0,\ \forall v \in cl\ W_{\mathcal{D}}(\Omega, \overline{x}). \tag{6}$$

*Proof.* Let $v \in cl\ W_{\mathcal{D}}(\Omega, \overline{x})$ be arbitrary. Then, there exist $v_s \in W_{\mathcal{D}}(\Omega, \overline{x})$ such that $v_s \to v$ as $s \to \infty$. Consequently, $v_s \in W(\Omega, \overline{x}) \cap \mathcal{D}$ and thus we can find a sequence $t_s^q \to 0^+$ such that $\overline{x} + t_s^q v_s \in \Omega,\ \forall q \in \mathbb{N}$. For $q$ large enough, since $\overline{x}$ is a local optimal solution of $f$ over $\Omega$, we have $f\,(\overline{x} + t_s^q v_s) \geq f\,(\overline{x})$. Then,

$$\frac{f\,(\overline{x} + t_s^q v_s) - (\overline{x})}{t_s^q} \geq 0, \quad \text{for sufficiently large } q.$$

Thus,

$$f_d^+\,(\overline{x}, v_s) = \limsup_{q} \frac{f\,(\overline{x} + t_s^q v_s) - f\,(\overline{x})}{t_s^q} \geq 0. \tag{7}$$

Using the upper semi-regularity of $\partial^{us}_{\mathcal{D}} f\,(\overline{x})$ at $\overline{x}$, since $v_s \in \mathcal{D}$, we get

$$\sup_{\eta \in \partial^{us}_{\mathcal{D}} f(\overline{x})} \langle \eta, v_s\rangle \geq 0.$$

Since $\partial_{\mathcal{D}}^{us} f(\overline{x})$ is compact and taking the limit as $s \to \infty$, we obtain

$$\sup_{\eta \in \partial_{\mathcal{D}}^{us} f(\overline{x})} \langle \eta, v \rangle \geq 0.$$

Because $v$ is arbitrarily chosen in $cl\ W_{\mathcal{D}}(\Omega, \overline{x})$, we can deduce the desired inquality (6). □

**Theorem 3.9.** *Let $\overline{x}$ be a local optimal solution of $MPEC$. Suppose that $\mathcal{D} \neq \{0_n\}$, that $\Omega$ is locally star-shaped at $\overline{x}$ and that Assumption 3.1 holds. If, in addition, Assumptions 3.2 and 3.3 are true for a partition $(B_1, B_2)$ of $B$ such that $\partial_{\mathcal{D}} - ARC\,(B_1, B_2)$ holds at $\overline{x}$ and $pos\ \Xi(\overline{x})$ is closed, then $\overline{x}$ is a generalized alternatively stationary point.*

*Proof.* Let $\overline{x}$ be a local optimal solution of $MPEC$. By Proposition 3.8, we have

$$\sup_{\eta \in \partial_{\mathcal{D}}^{us} f(\overline{x})} \langle \eta, v \rangle \geq 0,\ \forall v \in cl\ W_{\mathcal{D}}(\Omega, \overline{x}).$$

Consequently,

$$\sup_{\eta \in conv \partial_{\mathcal{D}}^{us} f(\overline{x})} \langle \eta, v \rangle \geq 0,\ \text{for all } v \in cl\ W_{\mathcal{D}}(\Omega, \overline{x}).$$

Since $\Omega$ is locally star-shaped at $\overline{x}$, we have $T_{\mathcal{D}}(\Omega, \overline{x}) = cl\ W_{\mathcal{D}}(\Omega, \overline{x})$, and thus

$$\sup_{\eta \in conv \partial_{\mathcal{D}}^{us} f(\overline{x})} \langle \eta, v \rangle \geq 0,\ \text{for all } v \in T_{\mathcal{D}}(\Omega, \overline{x}).$$

– Since $\partial_{\mathcal{D}} - ARC\,(B_1, B_2)$ holds at $\overline{x}$, we have

$$\sup_{\eta \in conv \partial_{\mathcal{D}}^{us} f(\overline{x})} \langle \eta, v \rangle \geq 0,\ \text{for all } v \in \Xi(\overline{x})^{-}.$$

Since $\Xi(\overline{x}) \subseteq pos\ \Xi(\overline{x})$, we get

$$\sup_{\eta \in conv \partial_{\mathcal{D}}^{us} f(\overline{x})} \langle \eta, v \rangle \geq 0,\ \text{for all } v \in (pos\ \Xi(\overline{x}))^{-}.$$

– Since $\partial_{\mathcal{D}}^{us} f(\overline{x})$ is compact, $conv\ \partial_{\mathcal{D}}^{us} f(\overline{x})$ is also a compact set (see [15], Thm. 1.4.3). By Lemma 2.12, we get

$$0 \in conv\ \partial_{\mathcal{D}}^{us} f(\overline{x}) + cl\ pos\ \Xi(\overline{x}).$$

- Since $pos\ \Xi(\overline{x})$ is closed, we obtain

$$0 \in conv\ \partial_{\mathcal{D}}^{us} f(\overline{x}) + pos\ \Gamma(\overline{x}) + pos\ N_{\mathcal{D}}(0_n).$$

Since $N_{\mathcal{D}}(0_n)$ is a convex cone, we get $pos\ N_{\mathcal{D}}(0_n) = N_{\mathcal{D}}(0_n)$. Then, there exist scalars $\lambda_i^g \geq 0,\ i \in I(\overline{x}), \mu_i^h \geq 0,\ \lambda_i^h \geq 0,\ i \in J,\ \mu_i^G \geq 0,\ i \in A \cup B_2,\ \lambda_i^G \geq 0,\ i \in A \cup B,\ \mu_i^H \geq 0,\ i \in D \cup B_1$, and $\lambda_i^H \geq 0,\ i \in D \cup B$, such that

$$0 \in \left[ \begin{array}{c} conv\ \partial_{\mathcal{D}}^{us} f(\overline{x}) + \sum\limits_{i \in I(\overline{x})} \lambda_i^g\ conv\ \partial_{\mathcal{D}}^{u} g_i(\overline{x}) \\ + \sum\limits_{i \in J} \mu_i^h\ conv\ \partial_{\mathcal{D}}^{u} h_i(\overline{x}) + \sum\limits_{i \in J} \lambda_i^h\ conv\ \partial_{\mathcal{D}}^{u}(-h_i)(\overline{x}) \\ + \sum\limits_{i \in A \cup B_2} \mu_i^G\ conv\ \partial_{\mathcal{D}}^{u} G_i(\overline{x}) + \sum\limits_{i \in A \cup B} \lambda_i^G\ conv\ \partial_{\mathcal{D}}^{u}(-G_i)(\overline{x}) \\ + \sum\limits_{i \in D \cup B_1} \mu_i^H\ conv\ \partial_{\mathcal{D}}^{u} H_i(\overline{x}) + \sum\limits_{i \in D \cup B} \lambda_i^H\ conv\ \partial_{\mathcal{D}}^{u}(-H_i)(\overline{x}) + N_{\mathcal{D}}(0_n). \end{array} \right].$$

- Setting

$$\begin{cases} \mu_i^G = 0, \ \forall i \in D \cup B_1 \\ \mu_i^H = 0, \ \forall i \in A \cup B_2 \\ \lambda_i^G = 0, \ \forall i \in D \\ \lambda_i^H = 0, \ \forall i \in A \end{cases}$$

we obtain (3), (4) and (5). The proof is then finished.

□

**Remark 3.10.** The additional condition mentioned above, the closedness of *pos* $\Xi(\overline{x})$, has been previously used by several authors in the continuous case (see [1,17] and [19]). Observe that if *conv* $\Xi(\overline{x})$ is a polyhedral set containing the origin, then *pos* $\Xi(\overline{x})$ is a polyhedral convex cone Corollary 19.7.1 of [26] and, thus, closed. Notice that *pos* $\Xi(\overline{x}) =$ *pos conv* $\Xi(\overline{x})$.

The following example provides a case where Theorem 3.9 is applicable while both Theorem 4.4 of [20] and Theorem 4.5 of [1] are not. Observe that in Example 3.11, the objective function $f$ is not continuous; thus not locally Lipschitz and consequently Theorem 4.4 of [20] and Theorem 4.5 of [1] cannot be used with this last property imposed.

**Example 3.11.** Consider the following nonsmooth optimization problem:

$$(MPEC): \begin{cases} \text{Minimize } f(x_1, x_2) \\ \text{s.t.} \begin{cases} g(x_1, x_2) \leq 0, \ h(x_1, x_2) = 0, \\ G(x_1, x_2) \geq 0, \ H(x_1, x_2) \geq 0, \ G(x_1, x_2)^\top H(x_1, x_2) = 0, \end{cases} \end{cases}$$

where $g(x_1, x_2) = |x_2|, h(x_1, x_2) = 0, \ H(x_1, x_2) = x_2$

$$G(x_1, x_2) = \begin{cases} x_1 & \text{if} \quad x_2 \geq 0 \\ x_2 + 1 & \text{elsewhere.} \end{cases}$$

and

$$f(x_1, x_2) = \begin{cases} 2x_2 - 1 & \text{if} \quad x_1 = 0, \ x_2 > 0, \\ -3x_1 - 1 & \text{if} \quad x_1 < 0, \ x_2 = 0, \\ |x_1| - |x_2| - 2 & \text{elsewhere.} \end{cases}$$

– On the one hand, since

$$\mathcal{D}_f(\overline{x}) = \mathbb{R}^2 \setminus (\{0\} \times (0, +\infty) \cup (-\infty, 0) \times \{0\}), \mathcal{D}_g(\overline{x}, \overline{y}) = D_h(\overline{x}, \overline{y}) = D_H(\overline{x}, \overline{y}) = \mathbb{R} \times \mathbb{R}$$

and

$$\mathcal{D}_G(\overline{x}, \overline{y}) = \mathbb{R} \times \mathbb{R}^+$$

we have

$$\mathcal{D} = \left(\mathbb{R} \times \mathbb{R}^+\right) \setminus (\{0\} \times (0, +\infty) \cup (-\infty, 0) \times \{0\}).$$

Consequently,

$$N_{\mathcal{D}}(0_2) = \{0\} \times \mathbb{R}^-.$$

– On the other hand, $\overline{x} = (0, 0)$ is an optimal solution of $(MPEC)$. Moreover, $A = D = \emptyset, \ B = \{1\}, I = \{1\}, \ J = \{1\}, \Omega = \mathbb{R}^+ \times \{0\}, W(\Omega, \overline{x}) = \mathbb{R}^+ \times \{0\}$ and $W_{\mathcal{D}}(\Omega, \overline{x}) = \mathbb{R}^+ \times \{0\}$.
  - $\partial_{\mathcal{D}} - ARC(B_1, B_2)$ holds at $\overline{x}$.
    - ∘ $\partial_{\mathcal{D}}^{us} f(\overline{x}) = \{(1, -1), (-1, 1)\}$ is a compact directional upper semi-regular convexificator of $f$ at $\overline{x}$.
    - ∘ $\partial_{\mathcal{D}}^{u} g(\overline{x}) = \{(0, 1)\}, \partial_{\mathcal{D}}^{u} h(\overline{x}) = \{(0, 0)\}, \ \partial_{\mathcal{D}}^{u} G(\overline{x}) = \{(1, 0)\}$ and $\partial_{\mathcal{D}}^{u} H(\overline{x}) = \{(0, 1)\}$ are directional upper convexificators of $g, \ h, \ G$ and $H$ at $\overline{x}$.

- For $B_1 = \{1\}$ and $B_2 = \emptyset$, we have

$$\Gamma(\overline{x}) = \{(0,1),(0,1),(0,-1),(-1,0)\}.$$

Consequently,

$$\Xi(\overline{x}) = \{(0,1),(0,-1),(-1,0)\} \cup \left(\{0\} \times \mathbb{R}^-\right).$$

Then,

$$\Xi(\overline{x})^- = \mathbb{R}^+ \times \{0\}.$$

Since $T_{\mathcal{D}}(\Omega,\overline{x}) = \mathbb{R}^+ \times \{0\}$, we deduce that $\{(0,0)\} \neq \Xi(\overline{x})^- \subseteq T_{\mathcal{D}}(\Omega,\overline{x})$.

- *pos* $\Xi(\overline{x})$ is closed. Indeed,

$$pos\ \Xi(\overline{x}) = \mathbb{R}^- \times \mathbb{R}.$$

– By taking $\lambda^g = 2$, $\mu^G = \frac{2}{3}$, $\lambda^h = \mu^h = \mu^H = 0$, $\lambda^H = \frac{1}{3}$ and $\lambda^G = 1$, since $\left(0,-\frac{4}{3}\right) \in N_{\mathcal{D}}(0_2)$ and $\left(\frac{1}{3},-\frac{1}{3}\right) \in conv\ \partial^{us}_{\mathcal{D}} f(\overline{x})$, we get

$$0 \in \begin{bmatrix} conv\ \partial^{us}_{\mathcal{D}} f(\overline{x}) + \lambda^g\ conv\ \partial^{u}_{\mathcal{D}} g(\overline{x}) \\ +\mu^h\ conv\ \partial^{u}_{\mathcal{D}} h(\overline{x}) + \lambda^h\ conv\ \partial^{u}_{\mathcal{D}} (-h)(\overline{x}) \\ +\lambda^G\ conv\ \partial^{u}_{\mathcal{D}} (-G)(\overline{x}) + \lambda^H\ conv\ \partial^{u}_{\mathcal{D}} (-H)(\overline{x}) \\ +\mu^G\ conv\ \partial^{u}_{\mathcal{D}} G(\overline{x}) + \mu^H\ conv\ \partial^{u}_{\mathcal{D}} H(\overline{x}) + N_{\mathcal{D}}(0_2). \end{bmatrix}$$

**Remark 3.12.** It is clear that the smaller the directional upper (semi-regular) convexificator is, the more useful the optimality conditions using this directional upper (semi-regular) convexificator are. Notice that our findings are established independent of the directional upper (semi-regular) convexificators utilized. As a consequence, the results in this work are valid for each directional upper (semi-regular) convexificator.

## 4. Sufficient optimality conditions

In order to get sufficient optimality conditions, we need the following notions.

**Definition 4.1.** Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\overline{x} \in \mathbb{R}^n$. We assume that $f$ admits a directional upper (semi-regular) convexificator $\partial^u_{\mathcal{D}} f(\overline{x}) \subseteq \mathbb{R}^n$ at $\overline{x}$.

– $f$ is said to be $\partial^u_{\mathcal{D}}$-convex at $\overline{x}$ iff for all $x \in \mathbb{R}^n$:

$$\langle \xi, x-\overline{x}\rangle \leq f(x) - f(\overline{x}), \text{for all } \xi \in conv\ \partial^u_{\mathcal{D}} f(\overline{x}) + N_{\mathcal{D}}(0_n).$$

– $f$ is said to be $\partial^u_{\mathcal{D}}$-quasiconvex at $\overline{x}$ iff for all $x \in \mathbb{R}^n$:

$$f(x) - f(\overline{x}) \leq 0 \Rightarrow \langle \xi, x-\overline{x}\rangle \leq 0, \text{ for all } \xi \in conv\ \partial^u_{\mathcal{D}} f(\overline{x}) + N_{\mathcal{D}}(0_n).$$

– $f$ is said to be $\partial^u_{\mathcal{D}}$-pseudoconvex at $\overline{x}$ iff for all $x \in \mathbb{R}^n$:

$$f(x) - f(\overline{x}) < 0 \Rightarrow \langle \xi, x-\overline{x}\rangle < 0, \text{ for all } \xi \in conv\ \partial^u_{\mathcal{D}} f(\overline{x}) + N_{\mathcal{D}}(0_n).$$

– $f$ is said to be $\partial^u_{\mathcal{D}}$-quasilinear at $\overline{x}$ iff $f$ and $(-f)$ are both $\partial^u_{\mathcal{D}}$-quasiconvex at $\overline{x}$.

Let $\overline{x} \in \Omega$ be a feasible point satisfying the generalized alternatively stationary condition and let

$$\mathcal{S} := B^+_G \cup B^+_H \cup B^+ \cup A^+ \cup D^+$$

where

$$B^+_G := \left\{i \in B : \mu^G_i = 0 \text{ and } \mu^H_i > 0\right\}, B^+_H := \left\{i \in B : \mu^G_i > 0 \text{ and } \mu^H_i = 0\right\},$$

$$B^+ := \left\{i \in B : \mu^G_i > 0 \text{ and } \mu^H_i > 0\right\}, A^+ := \left\{i \in A : \mu^G_i > 0\right\} \text{ and } D^+ := \left\{i \in D : \mu^H_i > 0\right\}.$$

Here, $\mu^G$ and $\mu^H$ are the multipliers associated to the point $\overline{x}$ which satisfies the generalized alternatively stationary condition.

**Theorem 4.2.** *Let $\overline{x} \in \Omega$ be a feasible point for $(MPEC)$ where the generalized alternatively stationary condition holds. Assume $\mathcal{S}$ is empty, $f$ is $\partial^u_{\mathcal{D}}-$pseudoconvex at $\overline{x}$, $g_i,\ i \in I(\overline{x}), -G_i,\ i \in A \cup B,\ -H_i,\ i \in D \cup B$, are $\partial^u_{\mathcal{D}}$-quasiconvex at $\overline{x}$ and $h_i,\ i \in J$, is $\partial^u_{\mathcal{D}}$-quasilinear at $\overline{x}$. Then $\overline{x}$ is a global optimal solution of $(MPEC)$.*

*Proof.* Suppose that $\overline{x}$ is not a global optimal solution of $(MPEC)$. Then, there exists $x_0 \in \Omega$ such that $f(\overline{x}) > f(x_0)$. Since $f$ is $\partial^u_{\mathcal{D}}$-pseudoconvex at $\overline{x}$, we get

$$\langle \xi^*,\ x_0 - \overline{x} \rangle < 0,\ \text{for all } \xi^* \in conv\ \partial^u_{\mathcal{D}} f(\overline{x}) + N_{\mathcal{D}}(0_n). \tag{8}$$

Using (3), we can find $\xi \in conv\ \partial^{us}_{\mathcal{D}} f(\overline{x}), \zeta_i \in conv\ \partial^u_{\mathcal{D}} g_i(\overline{x}), \varsigma_i \in conv\ \partial^u_{\mathcal{D}} h_i(\overline{x}), \rho_i \in conv\ \partial^u_{\mathcal{D}}(-h_i)(\overline{x}), \gamma_i^* \in conv\ \partial^u_{\mathcal{D}}(-G_i)(\overline{x}), \gamma_i^{**} \in conv\ \partial^u_{\mathcal{D}} G_i(\overline{x}), \theta_i^* \in conv\ \partial^u_{\mathcal{D}}(-H_i)(\overline{x}), \theta_i^{**} \in conv\ \partial^u_{\mathcal{D}} H_i(\overline{x})$ and $\tau^* \in N_{\mathcal{D}}(0_n)$ such that

$$0 = \xi + \sum_{i=1}^{m} \lambda_i^g\ \zeta_i + \sum_{i \in J} \mu_i^h \varsigma_i + \sum_{i \in J} \lambda_i^h \rho_i + \sum_{i=1}^{l} \lambda_i^G \gamma_i^* + \sum_{i=1}^{l} \lambda_i^H \theta_i^* + \sum_{i=1}^{l} \mu_i^G\ \gamma_i^{**} + \sum_{i=1}^{l} \mu_i^H\ \theta_i^{**} + \tau^*.$$

Then,

$$\begin{aligned} 0 = \langle \xi,\ x_0 - \overline{x} \rangle &+ \sum_{i=1}^{m} \lambda_i^g\ \langle \zeta_i,\ x_0 - \overline{x} \rangle + \sum_{i \in J} \mu_i^h \langle \varsigma_i,\ x_0 - \overline{x} \rangle \\ &+ \sum_{i \in J} \lambda_i^h \langle \rho_i,\ x_0 - \overline{x} \rangle + \sum_{i=1}^{l} \lambda_i^G \langle \gamma_i^*,\ x_0 - \overline{x} \rangle + \sum_{i=1}^{l} \lambda_i^H \langle \theta_i^*,\ x_0 - \overline{x} \rangle \\ &+ \sum_{i=1}^{l} \mu_i^G\ \langle \gamma_i^{**},\ x_0 - \overline{x} \rangle + \sum_{i=1}^{l} \mu_i^H\ \langle \theta_i^{**},\ x_0 - \overline{x} \rangle + \langle \tau^*,\ x_0 - \overline{x} \rangle. \end{aligned}$$

– Observing that

$$\begin{cases} g_i(x_0) \le g(\overline{x}),\ i \in I(\overline{x}), \\ h_i(x_0) = h_i(\overline{x}) = 0,\ i \in J, \\ (-G_i)(x_0) \le (-G_i)(\overline{x}),\ i \in A \cup B, \\ (-H_i)(x_0) \le (-H_i)(\overline{x}),\ i \in D \cup B, \end{cases}$$

we get

$$\begin{cases} g_i(x_0) - g(\overline{x}) \le 0,\ i \in I(\overline{x}), \\ h_i(x_0) - h_i(\overline{x}) = 0,\ i \in J, \\ (-G_i)(x_0) - (-G_i)(\overline{x}) \le 0,\ i \in A \cup B, \\ (-H_i)(x_0) - (-H_i)(\overline{x}) \le 0,\ i \in D \cup B, \end{cases}$$

- By the $\partial^u_{\mathcal{D}}$-quasiconvexity of $g_i,\ i \in I(\overline{x}),\ -G_i,\ i \in A \cup B,\ -H_i,\ i \in D \cup B$, at $\overline{x}$, as $0 \in N_{\mathcal{D}}(0_n)$, we obtain

$$\begin{cases} \langle \zeta_i, x_0 - \overline{x} \rangle \le 0,\ \text{for all } i \in I(\overline{x}), \\ \langle \gamma_i^*, x_0 - \overline{x} \rangle \le 0,\ i \in A \cup B, \\ \langle \theta_i^*, x_0 - \overline{x} \rangle \le 0,\ i \in D \cup B. \end{cases}$$

  Then,

$$\left\langle \sum_{i \in I(\overline{x})} \zeta_i, x_0 - \overline{x} \right\rangle \le 0,\ \left\langle \sum_{i \in A \cup B} \lambda_i^G \gamma_i^*, x_0 - \overline{x} \right\rangle \le 0 \text{ and } \left\langle \sum_{i \in D \cup B} \lambda_i^H \theta_i^*, x_0 - \overline{x} \right\rangle \le 0.$$

  ∘ By (5), we have $\lambda_i^G = 0$ for all $i \in D$, and $\lambda_i^H = 0$, for all $i \in A$. Consequently,

$$\left\langle \sum_{i=1}^{l} \lambda_i^G \gamma_i^*, x_0 - \overline{x} \right\rangle \le 0 \tag{9}$$

and

$$\left\langle \sum_{i=1}^{l} \lambda_i^H \theta_i^*, x_0 - \overline{x} \right\rangle \leq 0. \tag{10}$$

- By (4), we have $\lambda_i^g = 0$ for all $i \notin I(\overline{x})$. Consequently,

$$\left\langle \sum_{i=1}^{m} \lambda_i^g \zeta_i, x_0 - \overline{x} \right\rangle \leq 0. \tag{11}$$

- By the $\partial_{\mathcal{D}}^u$-quasilinearity of $h_i,\ i \in J$, at $\overline{x}$, as $0 \in N_{\mathcal{D}}(0_n)$, we get

$$\left\langle \sum_{i \in J} \lambda_i^g \varsigma_i, x_0 - \overline{x} \right\rangle \leq 0, \text{ for all } i \in J \tag{12}$$

and

$$\left\langle \sum_{i \in J} \rho_i, x_0 - \overline{x} \right\rangle \leq 0, \text{ for all } i \in J. \tag{13}$$

- From the emptiness of $\mathcal{S}$, we deduce that $\mu_i^G = 0$ and $\mu_i^H = 0$, for all $i \in \mathcal{L}$. Then,

$$\sum_{i=1}^{l} \mu_i^G \ \langle \gamma_i^{**},\ x_0 - \overline{x} \rangle + \sum_{i=1}^{l} \mu_i^H \ \langle \theta_i^{**},\ x_0 - \overline{x} \rangle = 0. \tag{14}$$

– Summing (9) – (14), we obtain

$$\begin{aligned} 0 \geq & \left\langle \sum_{i=1}^{m} \lambda_i^g \zeta_i, x_0 - \overline{x} \right\rangle + \left\langle \sum_{i \in J} \mu_i^h \varsigma_i, x_0 - \overline{x} \right\rangle + \left\langle \sum_{i \in J} \lambda_i^h \varsigma_i \rho_i, x_0 - \overline{x} \right\rangle \\ & + \left\langle \sum_{i=1}^{l} \lambda_i^G \gamma_i^*, x_0 - \overline{x} \right\rangle + \left\langle \sum_{i=1}^{l} \lambda_i^H \theta_i^*, x_0 - \overline{x} \right\rangle \\ & + \sum_{i=1}^{l} \mu_i^G \ \langle \gamma_i^{**},\ x_0 - \overline{x} \rangle + \sum_{i=1}^{l} \mu_i^H \ \langle \theta_i^{**},\ x_0 - \overline{x} \rangle ; \end{aligned}$$

which implies

$$0 \leq \langle \xi + \tau,\ x_0 - \overline{x} \rangle . \tag{15}$$

Since $\xi \in conv\ \partial_{\mathcal{D}}^{us} f(\overline{x})$ and $\tau^* \in N_{\mathcal{D}}(0_n)$, Inequality (15) contradicts (8).

□

## 5. Conclusion

This work was about a nonsmooth mathematical program with equilibrium constraints ($MPEC$) in which the functions are not always locally Lipschitz or continuous. Using directional upper convexificators and directional upper semi-regular convexificators, we introduced an alternative stationarity concept. Under an appropriate Abadie regularity condition, given in terms of directional upper convexificators, we established that alternative stationarity is a first-order necessary optimality condition. Unlike Dempe and Pilecka (Journal of Global Optimization 61: 769–788, 2015), we reach our goal without resorting to convexificators; the reason is that we do not assume that the sets of all continuity directions are convex or closed. The obtained results are given in terms of directional upper convexificators and directional upper semi-regular convexificators. In order to get sufficient optimality conditions, we made use of $\partial_{\mathcal{D}}^u$-pseudoconvexity and $\partial_{\mathcal{D}}^u$-quasiconvexity on the functions.

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