RAIRO-Oper. Res. 56 (2022) 3955–3971
https://doi.org/10.1051/ro/2022192

RAIRO Operations Research
www.rairo-ro.org

A CLASS OF NEW SEARCH DIRECTIONS FOR FULL-NT STEP FEASIBLE INTERIOR POINT METHOD IN SEMIDEFINITE OPTIMIZATION

Loubna Guerra*

Abstract. In this paper, based on Darvay *et al.*'s strategy for linear optimization (LO) (Z. Darvay and P.R. Takács, *Optim. Lett.* **12** (2018) 1099–1116.), we extend Kheirfam *et al.*'s feasible primal-dual path-following interior point algorithm for LO (B. Kheirfam and A. Nasrollahi, *Asian-Eur. J. Math.* **1** (2020) 2050014.) to semidefinite optimization (SDO) problems in order to define a class of new search directions. The algorithm uses only full Nesterov-Todd (NT) step at each iteration to find an ϵ-approximated solution to SDO. Polynomial complexity of the proposed algorithm is established which is as good as the LO analogue. Finally, we present some numerical results to prove the efficiency of the proposed algorithm.

Mathematics Subject Classification. 90C51, 90C25, 90C22.

Received April 30, 2022. Accepted October 26, 2022.

1. Introduction

Semidefinite optimization (SDO) problems are convex optimization problems, including linear optimization (LO), which minimize a linear function with the matrix variable over the intersection of an affine set and the cone of positive semidefinite matrices. SDO problems have a lot of significant applications in continuous and combinatorial optimization (see, *e.g.*, [3, 21]).

In the last decade, SDO has become a very active research area in mathematical programming because of the extension of the most algorithms for LO to the SDO case. Several primal-dual interior-point methods (IPMs) suggested for LO have been successfully extended to SDO [7, 11, 14, 18], convex quadratic semidefinite optimization (CQSDO) [2, 10] and other optimization problems [9, 19, 20, 22] due to their polynomial complexity and practical efficiency. The first primal-dual feasible IPM with a full-Newton step for LO was proposed by Roos *et al.* [16]. Later on, De Klerk [7], Achache and Guerra [2] extended Roos *et al.*'s algorithm for LO to SDO and CQSDO by using the full Nesterov-Todd (NT) direction as a search direction, respectively. Finding the search directions plays a crucial role in IPMs. In 2003, Darvay [4] introduced a new strategy for defining search directions for LO problems. The strategy is based on an algebraic equivalent transformation (AET) of the standard centering equations of the central path $\psi(\frac{xz}{\mu}) = \psi(e)$ where $\psi(t) = \sqrt{t}$. Achache [1], Wang and Bai [18, 19], extended Darvay's algorithm for LO to convex quadratic optimization (CQO), SDO and second-order

Keywords. Semidefinite optimization, interior point method, full Nesterov-Todd step, new search directions, algebraic equivalent transformation, polynomial complexity.

Département de Recherche Opérationnelle, Faculté des Mathématiques et d'Informatique, Université Mohamed El Bachir El Ibrahimi, Bordj Bou Arréridj, El Anasser 34030, Algérie.
*Corresponding author: loubna.guerra@univ-bba.dz, lguerra.maths@yahoo.fr

cone optimization (SOCO), respectively. In 2016, Darvay *et al.* [6] developed a new full-Newton step feasible IPM for LO based on a new reformulation of the standard centering equations of the central path with $\psi(t) = t - \sqrt{t}$. Kheirfam [11], generalized this method for SDO and derived the currently best-known iteration bound for SDO problems. In 2018, Darvay and Takács [5] designed a feasible primal-dual interior point algorithm for LO. Their algorithm is based on a new reformulation of the nonlinear equations of the central path $\psi(\frac{xz}{\mu}) = \psi((\frac{xz}{\mu})^{\frac{1}{2}})$ where $\psi(t) = t^2$, for $t > \frac{1}{\sqrt{2}}$. They established that the iteration bound of it is $O\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$. Recently, Kheirfam *et al.* [12] extended this study to case $\psi(t) = t^p$ with $t > \frac{1}{\sqrt[p]{2}}$ and $p \geq 2$ to determine a class of the search directions in LO and proved that the suggested approach has the same complexity bound obtained by Darvay *et al.* [5].

Motivated by the mentioned works, we propose a new feasible primal-dual path-following interior point algorithm for SDO based on a new transformation to define a class of new search directions. We adopt the basic analysis used in [12] for the SDO case. The iteration bound for the algorithm with the small-update method is as good as the bound for the LO case [5,12]. Furthermore, our analysis is relatively simple and straightforward to the LO analogue.

The outline of the paper is as follows. In Section 2, the SDO problem and the central path are presented. In Section 3, we extend Darvay's new technique for LO to SDO and derive a class of new search directions for SDO problems based on the AET with $\psi(t) = t^p$ for $t > \frac{1}{\sqrt[p]{2}}$ and $p \geq 2$. In Section 4, we present a new primal-dual interior point algorithm for SDO. In Section 5, the polynomial complexity result is established where we give the detailed proofs of it. Some numerical results are provided in Section 6. Finally, a conclusion is stated in Section 7.

The following notations are used throughout the paper.$\mathbb{R}^n$ denotes the space of vectors with n components. $\mathbb{S}^n$ denotes the space of real symmetric matrices of order n and $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$) denotes the cone of $n \times n$ symmetric positive definite (positive semidefinite) matrices. Furthermore, $X \succeq 0$ ($X \succ 0$) means that $X \in \mathbb{S}^n_+$ ($X \in \mathbb{S}^n_{++}$). For any matrix A, $\lambda_i(A)$ denote the i^{th} eigenvalues of A with $\lambda_{\min}(A)$ the smallest one and $\det A$ denotes its determinant whereas $Tr(A) = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i$ denotes its trace where a_{ii} is the diagonal elements of A, $\|.\|_F$ denote the Frobenius norm and the symbol $A \bullet B$ denotes the trace inner-product in $\mathbb{S}^n$defined by $A \bullet B = \mathbf{Tr}(AB) = \sum_{i,j=1}^n A_{ij}B_{ij}$. The symmetric positive definite square root of any symmetric positive definite matrix X is denoted by $X^{1/2}$. For $f(x)$, $g(x) : \mathbb{R}^n_+ \to \mathbb{R}^n_{++}$, $f(x) = O(g(x))$ if $f(x) \leq kg(x)$ for some positive constant k. Finally, the notation $A \sim B \Leftrightarrow A = SBS^{-1}$ for some invertible matrix S, means the similarity between the two matrices $A, B \in \mathbb{R}^{n\times n}$, and the identity matrix is denoted by I.

2. The central path

The standard primal form of semidefinite optimization (SDO) problems is as follows

$$(\mathcal{P}) \quad \min_X \{C \bullet X \mid \; A_i \bullet X = b_i, i = 1, \ldots, m, \; X \succeq 0\},$$

and its Lagrange dual problem

$$(\mathcal{D}) \quad \max_{(y,\, Z)} \{b^\top y \mid \; \sum\nolimits_{i=1}^m y_i A_i + Z = C, \; Z \succeq 0, \, y \in \mathbb{R}^m\},$$

where C, $A_i \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$.

Throughout the paper, we make the following assumptions on $(\mathcal{P})$ and $(\mathcal{D})$.

- **Independence condition.** The matrices $A_i, i = 1, \ldots, m$ are linearly independent.
- **Interior point condition (IPC).** There exists a triple (X^0, y^0, Z^0) such that:

$$A_i \bullet X^0 = b_i, \; i = 1, \ldots, m, \sum_{i=1}^m y_i^0 A_i + Z^0 = C, \; X^0 \succ 0, Z^0 \; \succ 0.$$

If the IPC holds, it is well known that finding an optimal solutions of $(\mathcal{P})$ and $(\mathcal{D})$ is equivalent to solving the following system:

$$\begin{cases} A_i \bullet X & = b_i, i = 1, \ldots, m, X \succeq 0, \\ \sum\limits_{i=1}^{m} y_i A_i + Z & = C,\ y \in \mathbb{R}^m, Z \succeq 0, \\ XZ & = 0. \end{cases} \tag{1}$$

The basic idea of primal-dual IPMs is to replace the third equation $XZ = 0$ in the system (1), the so-called *complementarity condition* for $(\mathcal{P})$ and $(\mathcal{D})$, by the parameterized equation $XZ = \mu I$ $(\mu > 0)$. Thus we consider

$$\begin{cases} A_i \bullet X & = b_i, i = 1, \ldots, m, X \succ 0, \\ \sum\limits_{i=1}^{m} y_i A_i + Z & = C, Z \succ 0, \\ XZ & = \mu I. \end{cases} \tag{2}$$

Since the IPC holds and the A_i are linearly independent, the parameterized system (2), has a unique solution $(X(\mu), y(\mu), Z(\mu))$ for any $\mu > 0$ [13,15]. The set of all such solutions defines the *central-path* of $(\mathcal{P})$ and $(\mathcal{D})$. If $\mu \to 0$, then the limit of the *central-path* exists and since the limit satisfies the complementarity condition, the limit yields a primal-dual optimal solutions for $(\mathcal{P})$ and $(\mathcal{D})$ [8].

3. A class of new search directions based on Darvay et al.'s technique

In [5], Darvay and Takács proposed a new reformulation to obtain a new search directions for LO by replacing the standard centering equation $xz = \mu e$ with $\psi\left(\frac{xz}{\mu}\right) = \psi\left(\sqrt{\frac{xz}{\mu}}\right)$, where $\psi(.)$ is the continuously differentiable vector function induced by function $\psi(t)$ on (κ, ∞) such that $2t\psi^{'}(t^2) - \psi^{'}(t) > 0$, for all $t > \kappa$ $(0 < \kappa < 1)$. Inspired by [5], we replace the standard centering equation $XZ = \mu I$ by $\psi\left(\frac{XZ}{\mu}\right) = \psi\left(\left(\frac{XZ}{\mu}\right)^{\frac{1}{2}}\right)$, then the system (2) can be written as:

$$\begin{cases} A_i \bullet X & = b_i, i = 1, \ldots, m, X \succ 0, \\ \sum\limits_{i=1}^{m} y_i A_i + Z & = C, Z \succ 0, \\ \psi\left(\frac{XZ}{\mu}\right) & = \psi\left(\left(\frac{XZ}{\mu}\right)^{\frac{1}{2}}\right). \end{cases} \tag{3}$$

Applying Newton method's on system (3), we obtain the following system for the search directions $\Delta X, \Delta y$ and ΔZ

$$\begin{cases} A_i \bullet \Delta X & = 0, i = 1, \ldots, m, \\ \sum\limits_{i=1}^{m} \Delta y_i A_i + \Delta Z & = 0, \\ \psi\left(\frac{XZ}{\mu} + \frac{X\Delta Z + \Delta XZ + \Delta X \Delta Z}{\mu}\right) & = \psi\left(\left(\frac{XZ}{\mu} + \frac{X\Delta Z + \Delta XZ + \Delta X \Delta Z}{\mu}\right)^{\frac{1}{2}}\right). \end{cases} \tag{4}$$

Applying Lemma 2.5 in [18], the third equation of the last system can be written as

$$\psi\left(\frac{XZ}{\mu}\right) + \psi^{'}\left(\frac{XZ}{\mu}\right)\left(\frac{X\Delta Z + \Delta XZ}{\mu}\right) - \psi\left(\left(\frac{XZ}{\mu}\right)^{\frac{1}{2}}\right) - \frac{1}{2}\left(\frac{XZ}{\mu}\right)^{\frac{-1}{2}}\psi^{'}\left(\left(\frac{XZ}{\mu}\right)^{\frac{1}{2}}\right)\left(\frac{X\Delta Z + \Delta XZ}{\mu}\right) = 0,$$

which equivalent to

$$\Delta XZ + X\Delta Z = \mu[\psi^{'}\left(\frac{XZ}{\mu}\right) - \frac{1}{2}\left(\frac{XZ}{\mu}\right)^{\frac{-1}{2}}\psi^{'}\left(\left(\frac{XZ}{\mu}\right)^{\frac{1}{2}}\right)]^{-1}[\psi\left(\left(\frac{XZ}{\mu}\right)^{\frac{1}{2}}\right) - \psi\left(\frac{XZ}{\mu}\right)].$$

Then we consider the following system

$$\begin{cases} A_i \bullet \Delta X & = 0, i = 1, \ldots, m, \\ \sum\limits_{i=1}^{m} \Delta y_i A_i + \Delta Z & = 0, \\ \Delta X + X\Delta Z Z^{-1} & = \mu[\psi'\left(\frac{XZ}{\mu}\right) - \frac{1}{2}\left(\frac{XZ}{\mu}\right)^{\frac{-1}{2}} \psi'\left(\left(\frac{XZ}{\mu}\right)^{\frac{1}{2}}\right)]^{-1}\left[\psi\left(\left(\frac{XZ}{\mu}\right)^{\frac{1}{2}}\right) - \psi\left(\frac{XZ}{\mu}\right)\right] Z^{-1}. \end{cases} \tag{5}$$

to obtain search directions (ΔX, Δy, ΔZ). It is obvious that ΔZ is symmetric due to the second equation in (5) but ΔX may be not symmetric. Many researchers have proposed several methods for symmetrizing the third equation in (5) such that the resulting new system has a unique symmetric solution.
In this paper, we use the Nesterov-Todd symmetrization scheme [2,7,10,11,17–19,21], which defines the so-called NT-direction. Let us define the matrix

$$P = X^{\frac{1}{2}}\left(X^{\frac{1}{2}}ZX^{\frac{1}{2}}\right)^{-\frac{1}{2}}X^{\frac{1}{2}} = Z^{-\frac{1}{2}}\left(Z^{\frac{1}{2}}XZ^{\frac{1}{2}}\right)^{\frac{1}{2}}Z^{-\frac{1}{2}}.$$

We replace the term $X\Delta Z Z^{-1}$ in the third equation of (5) by $P\Delta Z P^T$. Then the system (5) becomes

$$\begin{cases} A_i \bullet \Delta X & = 0, i = 1, \ldots, m, \\ \sum\limits_{i=1}^{m} \Delta y_i A_i + \Delta Z & = 0, \\ \Delta X + P\Delta Z P^T & = \mu[\psi'\left(\frac{XZ}{\mu}\right) - \frac{1}{2}\left(\frac{XZ}{\mu}\right)^{\frac{-1}{2}} \psi'\left(\left(\frac{XZ}{\mu}\right)^{\frac{1}{2}}\right)]^{-1}\left[\psi\left(\frac{XZ}{\mu}\right)^{\frac{1}{2}} - \psi\left(\frac{XZ}{\mu}\right)\right] Z^{-1}. \end{cases} \tag{6}$$

Furthermore, we define $D = P^{\frac{1}{2}}$, where $P^{\frac{1}{2}}$ denotes the symmetric square roote of P. The matrix D can be used to scale X and Z to the same matrix V as follows

$$V := \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DZD. \tag{7}$$

Note that both matrices D and V are symmetric and positive definite. It is easy to verify from (7) that $V^2 = \frac{1}{\mu} D^{-1} XZD$ *i.e.*, $V^2 \sim \frac{1}{\mu} XZ$. In addition, the scaling directions D_X and D_Z are:

$$D_X \ : \ = \frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}, \ \ D_Z := \frac{1}{\sqrt{\mu}} D\Delta Z D, \tag{8}$$
$$\bar{A}_i = \frac{1}{\sqrt{\mu}} DA_iD, \ i = 1, \ldots, m.$$

Then it follows from (6) that the scaled NT search directions $(D_X, \Delta y, D_Z)$ are defined by the following system

$$\begin{cases} \bar{A}_i \bullet D_X & = 0, i = 1, \ldots, m, \\ \sum\limits_{i=1}^{m} (\Delta y)_i \bar{A}_i + D_Z & = 0, \\ D_X + D_Z & = P_V. \end{cases} \tag{9}$$

Since the A_i are linearly independent so the $\bar{A}_i$, then the system (9) has a unique solution $D_X, \Delta y$, and D_Z with D_X and D_Z are symmetric matrices, and

$$P_V = \left[2V\left(D\psi'\left(V^2\right)D^{-1}\right) - \left(D\psi'(V)D^{-1}\right)\right]^{-1}\left[2\left(D\psi(V)D^{-1}\right) - 2\left(D\psi(V^2)D^{-1}\right)\right].$$

Darvay *et al.* [5] developed a full Newton step primal-dual path-following IPM for LO. They also established that their approach solves the LO problem in polynomial time and has $O(\sqrt{n}L)$-iteration complexity bound.

Their analysis is based on the function $\psi(t) = t^2$ case, and Kheirfam [12] extended this study to the case $\psi(t) = t^p$ for $t > \frac{1}{\sqrt[p]{2}}$ and $p \geq 2$ to derive a class of new search directions for LO problems.

In this paper, we extend the proposed strategy in [5] with $\psi(t) = t^p$ for $t > \frac{1}{\sqrt[p]{2}}$ and $p \geq 2$ to the SDO case, this yields

$$P_V = \frac{2}{p}(V - V^{p+1})(2V^p - I)^{-1}, \text{ with } \lambda_{\min}(V) \geq \frac{1}{\sqrt[p]{2}} \text{ and } p \geq 2. \tag{10}$$

We begin by recalling the following technical lemma.

Lemma 3.1. *Let P_V be defined by (10). Then*

$$I - \frac{P_V^2}{4} \preceq V^2 + VP_V \preceq I + ((2p-1)^2 - 1)\frac{P_V^2}{4}.$$

Proof. From (10), we have

$$\begin{aligned} V^2 + VP_V &= V^2 + \frac{2}{p}(V^2 - V^{p+2})(2V^p - I)^{-1}, \\ &= V^2(2V^p - I)(2V^p - I)^{-1} + \frac{2}{p}(V^2 - V^{p+2})(2V^p - I)^{-1}, \\ &= \frac{1}{p}\left[2pV^{p+2} - pV^2 + 2V^2 - 2V^{p+2}\right](2V^p - I)^{-1}, \\ &= \frac{1}{p}\left[(2pV^p - pI) + (2p-2)V^{p+2} - (p-2)V^2 - 2pV^p + pI\right](2V^p - I)^{-1}, \\ &= I + \frac{1}{p}\left[(2p-2)V^{p+2} - (p-2)V^2 - 2pV^p + pI\right](2V^p - I)^{-1}. \end{aligned}$$

Due to spectral theorem for symmetric matrices [18], since $V \in \mathbb{S}^n_+$, we obtain

$$V = Q^{-1}diag(\lambda_1(V), \ldots, \lambda_n(V))Q,$$

where Q is any orthonormal matrix ($Q^T = Q^{-1}$) that diagonalizes V. Then

$$V^2 + VP_V - I = Q^{-1}diag(\varphi(\lambda_1(V)), \ldots, \varphi(\lambda_n(V)))Q, \tag{11}$$

where $\varphi(\lambda_i(V)) = \frac{(2p-2)\lambda_i^{p+2}(V) - (p-2)\lambda_i^2(V) - 2p\lambda_i^p(V) + p}{p(2\lambda_i^p(V) - 1)}$, for $i = 1, \ldots, n$.

Let us consider $f(t) = (2p-2)t^{p+2} - (p-2)t^2 - 2pt^p + p$, for all $t > \frac{1}{\sqrt[p]{2}}$ and $p \geq 2$. One easily verify that $f'(1) = 0$ and $f''(1) = 6p^2 - 4p > 0$. Hence, $f(t) \geq f(1) = 0$ for all $t \in \mathbb{R}$. From this, together with $2t^p - 1 > 0$ for $t > \frac{1}{\sqrt[p]{2}}$ and (11), it follows that $0 \preceq V^2 + VP_V - I$. In addition, since $\frac{P_V^2}{4} \succ 0$, we obtain for $\lambda_{\min}(V) \geq \frac{1}{\sqrt[p]{2}}$ and $p \geq 2$, $I - \frac{P_V^2}{4} \preceq V^2 + VP_V$.

To prove the second inequality, we consider the following function

$$
\begin{aligned}
g(t) &= \frac{(2p-2)t^{p+2}-(p-2)t^2-2pt^p+p}{p(2t^p-1)}, \text{for all } t > \frac{1}{\sqrt[p]{2}} \text{ and } p \geq 2, \\
&= \frac{(1-t)^2((2p-2)t^p+2(2p-2)t^{p-1}+2(2p-3)t^{p-2}+...+2pt+p)}{p(2t^p-1)} \\
&\quad \times \frac{p(2t^p-1)(t+t^2+t^3+...+t^p)^2}{p(2t^p-1)(t+t^2+t^3+...+t^p)^2}, \\
&= \left(\frac{t-t^{p+1}}{p(2t^p-1)}\right)^2 \frac{p(2t^p-1)((2p-2)t^p+2(2p-2)t^{p-1}+2(2p-3)t^{p-2}+...+2pt+p)}{(t+t^2+t^3+...+t^p)^2}, \\
&\leq \left(\frac{t-t^{p+1}}{p(2t^p-1)}\right)^2 \frac{2p(2p-2)t^p(t^p+2t^{p-1}+3t^{p-2}+...+1)}{(t+t^2+t^3+...+t^p)^2}, \\
&= 2p(2p-2)\left(\frac{t-t^{p+1}}{p(2t^p-1)}\right)^2 \frac{(t^{2p}+2t^{2p-1}+3t^{2p-2}+...+t^p)}{(t+t^2+t^3+...+t^p)^2}, \\
&\leq 2p(2p-2)\left(\frac{t-t^{p+1}}{p(2t^p-1)}\right)^2 = ((2p-1)^2-1)\left(\frac{t-t^{p+1}}{p(2t^p-1)}\right)^2.
\end{aligned}
$$

Substituting this bound into (11), we obtain

$$
\begin{aligned}
V^2+VP_V-I &\preceq ((2p-1)^2-1)Q^{-1}diag\left(\left(\frac{\lambda_1(V)-\lambda_1^{p+1}(V)}{p(2\lambda_1^p(V)-1)}\right)^2,\dots,\left(\frac{\lambda_n(V)-\lambda_n^{p+1}(V)}{p(2\lambda_n^p(V)-1)}\right)^2\right)Q, \\
&= ((2p-1)^2-1)\frac{P_V^2}{4},
\end{aligned}
$$

where the last equality follows from the fact that $\frac{P_V^2}{4} \succ 0$. This completes the proof. □

For the analysis of the algorithm, we define a norm-based proximity measure $\delta(X,Z;\mu)$ as follows:

$$\delta := \delta(X,Z;\mu) = \frac{1}{2}\|P_V\|_F = \frac{1}{p}\left\|(V-V^{p+1})(2V^p-I)^{-1}\right\|_F. \tag{12}$$

One can easily check that

$$\delta(X,Z;\mu) = 0 \Leftrightarrow V = I \Leftrightarrow XZ = \mu I.$$

Hence, δ is a distance which measure the closeness of primal-dual points (X,y,Z) to the central path.

4. The algorithm

A primal-dual path following interior point algorithm based on new search directions for SDO is given in Figure 1.

5. Analysis of the algorithm

In this section, we will show that the proposed algorithm can solve the SDO in polynomial time.
To simplify the analysis of the Algorithm 4, we introduce $Q_V = D_X - D_Z$. Furthermore, from the third equation of (9), it follows that

$$D_X = \frac{P_V+Q_V}{2},\ D_Z = \frac{P_V-Q_V}{2},\ D_{XZ} = \frac{P_V^2-Q_V^2}{4} = \frac{D_XD_Z+D_ZD_X}{2}. \tag{13}$$

Input:
An accuracy parameter $\epsilon > 0$;
A fixed barrier update parameter $0 < \theta < 1$ (default $\theta = \frac{1}{2p^4\sqrt{n}}$);
A threshold parameter $0 < \tau < 1$ (default $\tau = \frac{1}{2((p-1)^2+p)}$);
A strictly feasible point (X^0, y^0, Z^0) and $\mu^0 = \frac{X^0 \bullet Z^0}{n}$ such that $\lambda_{\min}\left(\sqrt{\frac{X^0Z^0}{\mu^0}}\right) > \frac{1}{\sqrt[p]{2}}$, $\delta(X^0, Z^0; \mu^0) \leq \tau$;
begin
$X := X^0;\ y := y^0;\ Z := Z^0;\ \mu = \mu^0;$
While $X \bullet Z > \epsilon$ **do**
begin
solve the system (9) and use (8) to obtain $(\Delta X, \Delta y, \Delta Z)$
update $X_+ := X + \Delta X$, $y_+ := y + \Delta y$, $Z_+ := Z + \Delta Z$,
$\mu_+ := (1-\theta)\mu$;
end
end

FIGURE 1. Algorithm 4.

Due to the orthogonality of the matrices D_X and D_Z we obtain

$$\|P_V\|_F = \|Q_V\|_F = 2\delta(X, Z; \mu). \tag{14}$$

We start first by stating the following technical lemmas, that will be used later. Let $0 \leq \alpha \leq 1$.

Lemma 5.1. *[7] Let $X(\alpha) := X + \alpha\Delta X$, $Z(\alpha) := Z + \alpha\Delta Z$. If one has*

$$\det(X(\alpha)Z(\alpha)) > 0, \forall 0 \leq \alpha \leq \bar{\alpha}$$

then $X(\bar{\alpha}) \succ 0$ and $Z(\bar{\alpha}) \succ 0$.

Lemma 5.2. *[7] Let $Q \in \mathbb{R}^{n\times n}$, be symmetric ($Q = Q^\top$) and $M \in \mathbb{R}^{n\times n}$ be skew-symmetric ($M = -M^\top$). One has $\det(Q + M) > 0$ if $Q \succ 0$. Moreover, if $\lambda_i(Q + M) \in \mathbb{R}$ for $i = 1, ..., n$, then*

$$0 < \lambda_{\min}(Q) \leq \lambda_{\min}(Q + M) \leq \lambda_{\max}(Q + M) \leq \lambda_{\max}(Q).$$

The next lemma shows the strict feasibility of a full NT-step under the condition $\delta < 1$.

Lemma 5.3. *Let $\delta < 1$ and assume that $\lambda_{\min}(V) > \frac{1}{\sqrt[p]{2}}$, then the full NT-step is strictly feasible.*

Proof. Let $X(\alpha) = (X + \alpha\Delta X)$, $Z(\alpha) = (Z + \alpha\Delta Z)$ with $0 \leq \alpha \leq 1$. From (7) and (8), it follows that

$$\begin{aligned} X(\alpha)Z(\alpha) &= XZ + \alpha(\Delta XZ + X\Delta Z) + \alpha^2\Delta X\Delta Z, \\ &= \mu D\left[V^2 + \alpha(D_XV + VD_Z) + \alpha^2 D_XD_Z\right]D^{-1}, \\ &\sim \mu\left[V^2 + \alpha(D_XV + VD_Z) + \alpha^2 D_XD_Z\right], \end{aligned}$$

and which can be written as

$$X(\alpha)Z(\alpha) \sim B(\alpha) + M(\alpha) \tag{15}$$

where

$$M(\alpha) = \mu\alpha[D_XV - VD_X + \frac{1}{2}\alpha(D_XD_Z - D_ZD_X)],$$

is skew-symmetric. Then by Lemma 5.2, it follows that $\det(X(\alpha)Z(\alpha)) > 0$, if the matrix

$$\begin{aligned} B(\alpha) &= \mu\left[V^2 + \alpha V(D_Z + D_X) + \alpha^2 D_{XZ}\right], \\ &= \mu\left[V^2 + \alpha V P_V + \alpha^2 D_{XZ}\right], \end{aligned}$$

is positive definite. Now, in view of (13) and Lemma 3.1, $B(\alpha)$ can be written as:

$$\begin{aligned} B(\alpha) &= \mu\left[V^2 + \alpha V P_V + \alpha^2 \frac{P_V^2 - Q_V^2}{4}\right], \\ &= \mu\left[(1-\alpha)V^2 + \alpha(V^2 + V P_V) + \alpha^2 \frac{P_V^2 - Q_V^2}{4}\right], \\ &\succeq \left[(1-\alpha)V^2 + \alpha\left(I - \frac{P_V^2}{4}\right) + \alpha^2 \frac{P_V^2 - Q_V^2}{4}\right], \\ &= (1-\alpha)V^2 + \alpha\left(I - (1-\alpha)\frac{P_V^2}{4} - \alpha\frac{Q_V^2}{4}\right), \end{aligned}$$

Then, $B(\alpha) \succ 0$ if $\alpha \leq 1$ and

$$\left\|(1-\alpha)\frac{P_V^2}{4} - \alpha\frac{Q_V^2}{4}\right\|_F \leq (1-\alpha)\left\|\frac{P_V^2}{4}\right\|_F + \alpha\left\|\frac{Q_V^2}{4}\right\|_F \leq (1-\alpha)\frac{\|P_V\|_F^2}{4} + \alpha\frac{\|Q_V\|_F^2}{4} = \delta^2 < 1,$$

where the last equality is due to (14). In addition, since $X(0) \succ 0$ and $Z(0) \succ 0$, Lemma 5.1, implies that $X(1) \succ 0$ and $Z(1) \succ 0$. This completes the proof. □

For simplicity, we may write $X_+ = X(1)$ and $Z_+ = Z(1)$, then

$$\mu V_+^2 \sim X_+ Z_+.$$

The following lemma gives a lower bound for the smallest eigenvalue of V_+^2, denoted by $\lambda_{\min}(V_+^2)$, after a full NT-step.

Lemma 5.4. *Let $\delta < 1$ then we have*

$$\lambda_{\min}(V_+^2) \geq (1-\delta^2),$$

where $\lambda_{\min}(V_+^2)$ is the smallest eigenvalue of V_+^2.

Proof. From (15), in Lemma 5.3, letting $\alpha = 1$, we get

$$\mu V_+^2 \sim X_+ Z_+ \sim B(1) + M(1) \sim \mu(V^2 + V P_V) + \mu\frac{P_V^2 - Q_V^2}{4} + M(1).$$

It follows that

$$\lambda_{\min}(V_+^2) = \lambda_{\min}(V^2 + V P_V + \frac{P_V^2}{4} - \frac{Q_V^2}{4} + \frac{1}{\mu}M(1)).$$

Using Lemmas 3.1, 5.2 and the skew-symmetric of $M(1)$, it implies that:

$$\lambda_{\min}(V_+^2) \geq \lambda_{\min}(I - \frac{Q_V^2}{4}) \geq 1 - \lambda_{\max}\left(\frac{Q_V^2}{4}\right) \geq 1 - \left\|\frac{Q_V^2}{4}\right\|_F \geq 1 - \frac{\|Q_V\|_F^2}{4} = 1 - \delta^2.$$

Where the last equality is follows from (14). This completes the proof. □

The next result shows the quadratic convergence of the proximity.

Lemma 5.5. *If* $\delta := \delta(X, Z; \mu) < \sqrt{1 - \frac{1}{\sqrt[p]{4}}}$ *and assume that* $\lambda_{\min}(V) > \frac{1}{\sqrt[p]{2}}$. *Then* $\lambda_{\min}(V_+) > \frac{1}{\sqrt[p]{2}}$ *and*

$$\delta_+ := \delta(X_+, Z_+; \mu) \leq \frac{(1-\delta^2)^{\frac{p}{2}} + (1-\delta^2)^{\frac{p-1}{2}} + \dots + (1-\delta^2)^{\frac{1}{2}}}{p(2(1-\delta^2)^{\frac{p}{2}} - 1)(1 + (1-\delta^2)^{\frac{1}{2}})}((2p-1)^2 + 1)\delta^2.$$

Proof. Let $\delta < \sqrt{1 - \frac{1}{\sqrt[p]{4}}}$ and $\lambda_{\min}(V) > \frac{1}{\sqrt[p]{2}}$. Then by Lemma 5.4, we deduce that

$$\lambda_{\min}(V_+) \geq \sqrt{1-\delta^2} > \sqrt{1 - \left(1 - \frac{1}{\sqrt[p]{4}}\right)} = \sqrt{\frac{1}{4^{\frac{1}{p}}}} = \frac{1}{\sqrt[p]{2}},$$

which implies that $\lambda_{\min}(V_+) > \frac{1}{\sqrt[p]{2}}$. Since

$$V_+ - V_+^{p+1} = (V_+^P + V_+^{p-1} + \dots + V_+)(I - V_+),$$

we have

$$\begin{aligned}
p\delta(X_+, Z_+; \mu) &= \left\|(V_+ - V_+^{p+1})(2V_+^p - I)^{-1}\right\|_F, \\
&= \left\|(V_+ - V_+^{p+1})(I + V_+)(I + V_+)^{-1}(2V_+^p - I)^{-1}\right\|_F, \\
&= \left\|(V_+^P + V_+^{p-1} + \dots + V_+)(I + V_+)^{-1}(2V_+^p - I)^{-1}(I - V_+^2)\right\|_F, \\
&= \left(\sum_{i=1}^n \left(\frac{(\lambda_i(V_+^P) + \lambda_i(V_+^{p-1}) + \dots + \lambda_i(V_+))(1 - \lambda_i(V_+^2))}{(1 + \lambda_i(V_+))(2\lambda_i(V_+^p) - 1)}\right)^2\right)^{\frac{1}{2}}, \\
&\leq \max_{i=1}^n \left(\frac{(\lambda_i(V_+^P) + \lambda_i(V_+^{p-1}) + \dots + \lambda_i(V_+))}{(1 + \lambda_i(V_+))(2\lambda_i(V_+^p) - 1)}\right)\left(\sum_{i=1}^n (1 - \lambda_i(V_+^2))^2\right)^{\frac{1}{2}}, \\
&= \frac{(\lambda_{\min}(V_+^P) + \lambda_{\min}(V_+^{p-1}) + \dots + \lambda_{\min}(V_+))}{(1 + \lambda_{\min}(V_+))(2\lambda_{\min}(V_+^p) - 1)}\left\|I - V_+^2\right\|_F,
\end{aligned}$$

where the last equality follows from the fact that the function $f(t) = \frac{t^P + t^{p-1} + \dots + t}{(2t^p - 1)(1+t)}$ is decreasing on $t > \frac{1}{\sqrt[p]{2}}$. By using Lemma 5.4, we obtain

$$\delta(X_+, Z_+; \mu) \leq \frac{(1-\delta^2)^{\frac{p}{2}} + (1-\delta^2)^{\frac{p-1}{2}} + \dots + (1-\delta^2)^{\frac{1}{2}}}{p(2(1-\delta^2)^{\frac{p}{2}} - 1)(1 + (1-\delta^2)^{\frac{1}{2}})}\left\|I - V_+^2\right\|_F. \tag{16}$$

Now, we prove that

$$\left\|I - V_+^2\right\|_F \leq ((2p-1)^2 + 1)\delta^2.$$

Since $X(1)Z(1) \sim \mu V_+^2$ then by Lemmas 5.3, 3.1 and (13), we deduce that

$$\begin{aligned}
V_+^2 &\sim V^2 + VP_V + D_{XZ} + \frac{1}{\mu}M(1), \\
&\preceq I + ((2p-1)^2 - 1)\frac{P_V^2}{4} + \frac{P_V^2}{4} - \frac{Q_V^2}{4} + \frac{1}{\mu}M(1), \\
&= I + (2p-1)^2\frac{P_V^2}{4} - \frac{Q_V^2}{4} + \frac{1}{\mu}M(1),
\end{aligned}$$

consequently

$$V_+^2 - I \preceq (2p-1)^2 \frac{P_V^2}{4} - \frac{Q_V^2}{4} + \frac{1}{\mu} M(1).$$

By using the Frobenius norm, we obtain

$$\begin{aligned}
\left\| I - V_+^2 \right\|_F &\leq \left\| -(2p-1)^2 \frac{P_V^2}{4} + \frac{Q_V^2}{4} - \frac{1}{\mu} M(1) \right\|_F, \\
&\leq \left\| -(2p-1)^2 \frac{P_V^2}{4} \right\|_F + \left\| \frac{Q_V^2}{4} - \frac{1}{\mu} M(1) \right\|_F, \\
&\leq (2p-1)^2 \left\| \frac{P_V^2}{4} \right\|_F + \left\| \frac{Q_V^2}{4} \right\|_F, \\
&\leq (2p-1)^2 \frac{\|P_V\|_F^2}{4} + \frac{\|Q_V\|_F^2}{4}, \\
&= ((2p-1)^2 + 1)\delta^2. \qquad (17)
\end{aligned}$$

Where the last equality is due to (14) and the third inequality becomes from the following inequality

$$\begin{aligned}
\left\| \frac{Q_V^2}{4} - \frac{1}{\mu} M(1) \right\|_F^2 &= \operatorname{tr} \left[\left(\frac{Q_V^2}{4} - \frac{1}{\mu} M(1) \right) \left(\frac{Q_V^2}{4} - \frac{1}{\mu} M(1) \right)^T \right], \\
&= \operatorname{tr} \left[\left(\frac{Q_V^2}{4} \right)^2 + \frac{Q_V^2}{4} \frac{1}{\mu} M^T(1) + \frac{1}{\mu} M^T(1) \frac{Q_V^2}{4} - \frac{1}{\mu^2} M(1) M^T(1) \right], \\
&\leq \operatorname{tr} \left[\left(\frac{Q_V^2}{4} \right)^2 \right] = \left\| \frac{Q_V^2}{4} \right\|_F^2.
\end{aligned}$$

Here, the inequality is due to $\frac{Q_V^2}{4} \frac{1}{\mu} M^T(1) + \frac{1}{\mu} M^T(1) \frac{Q_V^2}{4}$ is skew-symmetric and $M(1)M^T(1)$ is positive semidefinite. Substituting (16) into (17) yields the result. This completes the proof. □

The next lemma, shows the influence of a full NT-step on the duality gap.

Lemma 5.6. *After a full NT-step, we have*

$$X_+ \bullet Z_+ \leq \mu(n + ((2p-1)^2 - 1)\delta^2). \qquad (18)$$

Proof. From (15) in the proof of Lemma 5.3, we have

$$X_+ \bullet Z_+ = \mu \operatorname{Tr}(V_+^2) = \mu \operatorname{Tr}(V^2 + V P_V + D_{XZ} + \frac{1}{\mu} M(1)),$$

by skew-symmetry of $M(1)$ and Lemma 3.1, it implies

$$\begin{aligned}
X_+ \bullet Z_+ &= \mu \operatorname{Tr}(V^2 + V P_V + D_{XZ}), \\
&\leq \mu \operatorname{Tr}(I + ((2p-1)^2 - 1) \frac{P_V^2}{4} + D_{XZ}), \\
&= \mu(n + ((2p-1)^2 - 1) \operatorname{Tr} \left(\frac{P_V^2}{4} \right) + \operatorname{Tr}(D_{XZ})), \\
&= \mu(n + ((2p-1)^2 - 1)\delta^2 + D_X \bullet D_Z).
\end{aligned}$$

From the orthogonality of the matrices D_X and D_Z, we get

$$X_+ \bullet Z_+ \leq \mu(n + ((2p-1)^2 - 1)\delta^2).$$

This completes the proof. □

In the next lemma, we investigate the effect of a full NT-step on the proximity after an update of the parameter μ.

Lemma 5.7. *Let* $\delta := \delta(X, Z; \mu) < \frac{1}{2((p-1)^2+p)}$, $\lambda_{\min}(V) > \frac{1}{\sqrt[p]{2}}$ *and* $\mu_+ = (1-\theta)\mu$ *with* $0 < \theta < 1$. *Then* $\lambda_{\min}\left(\hat{V}_+\right) > \frac{1}{\sqrt[p]{2}}$ *where* $\hat{V}_+^2 := \frac{V_+^2}{1-\theta} \sim \frac{X_+Z_+}{\mu_+}$ *and*

$$\delta(X_+, Z_+; \mu_+) \leq \frac{\beta^p + \eta\beta^{p-1} + ... + \eta^{p-1}\beta}{p\eta(2\beta^p - \eta^p)(\eta+\beta)}(\theta\sqrt{n} + ((2p-1)^2+1)\delta^2),$$

with $\beta := \sqrt{1-\delta^2}$ *and* $\eta := \sqrt{1-\theta}$.
In addition, if $\theta = \frac{1}{2p^4\sqrt{n}}$, *then* $\delta(X_+, Z_+; \mu_+) < \frac{1}{2((p-1)^2+p)}$.

Proof. From Lemma 5.4 and the hypothesis of this Lemma, it follows that $\lambda_{\min}(V_+) > \frac{1}{\sqrt[p]{2}}$. Hence, $\hat{V}_+ = \frac{V_+}{\sqrt{1-\theta}}$ and since $\frac{1}{\sqrt{1-\theta}} > 1$ for all $0 < \theta < 1$, then we get

$$\lambda_{\min}\left(\hat{V}_+\right) > \frac{1}{\sqrt{1-\theta}}\lambda_{\min}(V_+) > \frac{1}{\sqrt[p]{2}}.$$

Now, to simplify the notation we write $\eta := \sqrt{1-\theta}$ and $\beta = \sqrt{1-\delta^2}$. In this way, $\hat{V}_+ = \frac{1}{\eta}V_+$. Hence, we have

$$\begin{aligned}
p\delta(X_+, Z_+; \mu_+) &= \left\|(\hat{V}_+ - \hat{V}_+^{p+1})(2\hat{V}_+^p - I)^{-1}\right\|_F,\\
&= \left\|(\hat{V}_+ - \hat{V}_+^{p+1})(I+\hat{V}_+)(I+\hat{V}_+)^{-1}(2\hat{V}_+^p - I)^{-1}\right\|_F,\\
&= \left\|(\hat{V}_+^P + \hat{V}_+^{p-1} + ... + \hat{V}_+)(I+\hat{V}_+)^{-1}(2\hat{V}_+^p - I)^{-1}(I - \hat{V}_+^2)\right\|_F,\\
&= \left(\sum_{i=1}^n \left(\frac{(\lambda_i(\hat{V}_+^P) + \lambda_i(\hat{V}_+^{p-1}) + ... + \lambda_i(\hat{V}_+))(1-\lambda_i(\hat{V}_+^2))}{(2\lambda_i(\hat{V}_+^p) - 1)(1+\lambda_i(\hat{V}_+))}\right)^2\right)^{\frac{1}{2}},\\
&\leq \max_{i=1}^n \left(\frac{(\lambda_i(\hat{V}_+^P) + \lambda_i(\hat{V}_+^{p-1}) + ... + \lambda_i(\hat{V}_+))}{(2\lambda_i(\hat{V}_+^p) - 1)(1+\lambda_i(\hat{V}_+))}\right)\left(\sum_{i=1}^n (1-\lambda_i(\hat{V}_+^2))^2\right)^{\frac{1}{2}},\\
&= \frac{(\lambda_{\min}(\hat{V}_+^P) + \lambda_{\min}(\hat{V}_+^{p-1}) + ... + \lambda_{\min}(\hat{V}_+))}{(2\lambda_{\min}(\hat{V}_+^p) - 1)(1+\lambda_{\min}(\hat{V}_+))}\left\|I - \hat{V}_+^2\right\|_F,
\end{aligned}$$

where the last equality follows from the fact that the function $f(t) = \frac{t^P + t^{p-1} + ... + t}{(2t^p-1)(1+t)}$ is decreasing on $t > \frac{1}{\sqrt[p]{2}}$. Now, by using Lemma 5.4, one has

$$\lambda_{\min}(\hat{V}_+) = \frac{1}{\eta}\lambda_{\min}(V_+) \geq \frac{1}{\eta}\sqrt{1-\delta^2} = \frac{\beta}{\eta},$$

which implies that $f(t) \leq f(\frac{\beta}{\eta})$. Hence, we get

$$\delta(X_+, Z_+; \mu_+) \leq \frac{1}{p\eta}\frac{\beta^p + \eta\beta^{p-1} + ... + \eta^{p-1}\beta}{(2\beta^p - \eta^p)(\eta+\beta)}\left\|\eta^2 I - V_+^2\right\|_F. \tag{19}$$

On the other hand, we have

$$\begin{aligned}
\left\|\eta^2 I - V_+^2\right\|_F &= \left\|(1-\theta)I - V_+^2\right\|_F,\\
&\leq \theta\left\|I\right\|_F + \left\|(I - V_+^2)\right\|_F,\\
&\leq \theta\sqrt{n} + ((2p-1)^2+1)\delta^2. \qquad (20)
\end{aligned}$$

We now combine the results in (19) and (20), and this yields the desired result. To prove the second part, let us consider the function $u(\eta) = \frac{\beta^p+\eta\beta^{p-1}+...+\eta^{p-1}\beta}{p(2\beta^p-\eta^p)(\eta+\beta)}$, for $0 < \eta < 1$, $\beta > \frac{1}{\sqrt[p]{2}}$ and $p \geq 2$, which is increasing with respect to η. So, $u(\eta) < u(1)$. Combining this result with the upper bound of $\delta(X_+, Z_+; \mu_+)$ yields

$$\delta(X_+, Z_+; \mu_+) \leq \frac{\beta^p + \beta^{p-1} + ... + \beta}{p\eta(2\beta^p - 1)(1+\beta)}(\theta\sqrt{n} + ((2p-1)^2+1)\delta^2), \tag{21}$$

Now, let us define $z(\beta) = \frac{\beta^p+\beta^{p-1}+...+\beta}{p(2\beta^p-1)(1+\beta)}$, for $\beta > \frac{1}{\sqrt[p]{2}}$, which is decreasing with respect to β. From this along with $\beta = \sqrt{1-\delta^2} > \sqrt{1 - \frac{1}{4((p-1)^2+p)^2}} := \gamma(p)$ and the fact that $\gamma(p)$ increasing for $p \geq 2$, we get

$$z(\beta) < z(\gamma(p)) < z(\gamma(\infty)) = z(1) = \frac{1}{2}.$$

Substituting this bound into (21), and using $\theta = \frac{1}{2p^4\sqrt{n}}$ and $\delta < \frac{1}{2((p-1)^2+p)}$, gives

$$\begin{aligned}\delta(X_+, Z_+; \mu_+) &\leq \frac{1}{2\sqrt{1-\theta}}(\theta\sqrt{n} + ((2p-1)^2+1)\delta^2),\\ &\leq \left(\frac{1}{2p^4} + \frac{(2p-1)^2+1}{4((p-1)^2+p)^2}\right)\frac{p^2}{\sqrt{2(2p^4-1)}},\\ &= h(p)\frac{1}{2((p-1)^2+p)}.\end{aligned}$$

where $h(p) = \left(\frac{((p-1)^2+p)}{p^4} + \frac{(2p-1)^2+1}{2((p-1)^2+p)}\right)\frac{p^2}{\sqrt{2(2p^4-1)}}$, for $p \geq 2$, which is increasing with respect to p and $\lim_{p\to+\infty} h(p) = 1$. Consequently,

$$\delta(X_+, Z_+; \mu_+) \leq \frac{1}{2((p-1)^2+p)}.$$

This completes the proof. □

From Lemma 5.7, we deduce that Algorithm 4 is well defined. The next lemma gives an upper bound for the number of iterations produced by Algorithm 4.

Lemma 5.8. *Let X^{k+1} and Z^{k+1} be the $(k+1)-th$ iterate generated by Algorithm 4, with $\mu := \mu_k$. Then*

$$X^{k+1} \bullet Z^{k+1} \leq \epsilon,$$

if

$$k \geq \left\lceil \frac{1}{\theta}\log\left(\frac{\mu_0(n + \frac{(2p-1)^2-1}{4((p-1)^2+p)^2})}{\epsilon}\right)\right\rceil.$$

Proof. In view of (18) in Lemma 5.6, it follows that:

$$\begin{aligned}X^{k+1} \bullet Z^{k+1} &\leq \mu_k(n + ((2p-1)^2-1)\delta^2),\\ &\leq \mu_k\left(n + \frac{(2p-1)^2-1}{4((p-1)^2+p)^2}\right),\end{aligned}$$

with

$$\mu_k = (1-\theta)\mu_{k-1} = (1-\theta)^k\mu_0.$$

Hence, we have

$$X^{k+1} \bullet Z^{k+1} \leq (1-\theta)^k \mu_0 \left(n + \frac{(2p-1)^2 - 1}{4((p-1)^2+p)^2} \right).$$

Thus the inequality $X^{k+1} \bullet Z^{k+1} \leq \epsilon$ holds if

$$(1-\theta)^k \mu_0 \left(n + \frac{(2p-1)^2 - 1}{4((p-1)^2+p)^2} \right) \leq \epsilon.$$

Now, taking logarithms, we may write

$$k \log(1-\theta) \leq \log \epsilon - \log \left(\mu_0 \left(n + \tfrac{(2p-1)^2 - 1}{4((p-1)^2+p)^2} \right) \right),$$

and since $-\log(1-\theta) \geq \theta$ for $0 < \theta < 1$, then the inequality is satisfied if

$$k \geq \frac{1}{\theta} \log \left(\frac{\mu_0 \left(n + \tfrac{(2p-1)^2 - 1}{4((p-1)^2+p)^2} \right)}{\epsilon} \right).$$

This completes the proof. □

For $\theta = \frac{1}{2p^4\sqrt{n}}$, we obtain the following theorem.

Theorem 5.9. *Let $\theta = \frac{1}{2p^4\sqrt{n}}$. Then Algorithm 4 requires at most*

$$O\left(p^4 \sqrt{n} \log \left(\frac{n}{\epsilon} \right) \right)$$

iterations, if $X^0 = Z^0 = I$. This choice of initial point can be done by the embedding technique [7].

Proof. It is a straightforward from Lemma 5.8. This completes the proof. □

6. Numerical results

In order to compare the efficiency of the algorithm with the existing methods and to show the influence of the parameter p on the number of iterations produced by the algorithm, we present some numerical results under Matlab 8.1 where the implementation is done on a computer with an Intel core 2.3 GHz processor and 4 GB RAM, for solving some semidefinite optimization problems.

Note that the value of the parameter p may be very large, which leads to a very small value of the parameter θ, see Theorem 5.9. This motivated us to make some changes in the implementation of the proposed algorithm. The initial primal dual point (X^0, y^0, Z^0) with $\mu^0 = \frac{X^0 \bullet Z^0}{n}$ is chosen such that the pair is strictly feasible, the proximity $\delta(X^0, Z^0; \mu^0) \leq \tau$ and the smallest eigenvalue of the matrix V is greater than a positive constant. At each iteration, the value of the parameter μ was calculated as $\mu_+ = \sigma \frac{\min\{(X_+ Z_+)_{ii} \,:\, 1 \leq i \leq n\}}{lb}$, where $0 < \sigma < 1$ and lb is a given lower bound, which in this case is $0.5^{\frac{2}{p}}$, $p \geq 2$ and $\sigma = 0.2$. The technique for determining the value of the parameter μ_+ ensures that $\lambda_{\min}(V) \geq \frac{1}{\sqrt[p]{2}}$ with $p \geq 2$, which is signifiant in our case for the used search direction. Moreover, to guarantee that the iterates remain interior, we use the following strategy: we compute at each iteration a maximum step size $\alpha_{\max}$ such that $X + \xi \alpha_{\max} \Delta X \succ 0$ and $Z + \xi \alpha_{\max} \Delta Z \succ 0$ with $\alpha_{\max} = \min(\alpha_X, \alpha_Z)$ and $\xi \in (0,1)$, where α_X and α_Z are the primal and the dual feasible step size given by $\alpha_X = \min_{i=1}^{n} \alpha_X^{'}[i]$, with

$$\alpha_X^{'}[i] = \begin{cases} \frac{-1}{\lambda_i\left(X^{\frac{-1}{2}} \Delta X X^{\frac{-1}{2}}\right)} & , \text{ if } \lambda_i\left(X^{\frac{-1}{2}} \Delta X X^{\frac{-1}{2}}\right) < 0 \\ 1 & , \text{ if } \lambda_i\left(X^{\frac{-1}{2}} \Delta X X^{\frac{-1}{2}}\right) \geq 0, \end{cases}$$

and $\alpha_Z = \min_{i=1}^{n} \alpha_Z^{'}[i]$, with

$$\alpha_Z^{'}[i] = \begin{cases} \frac{-1}{\lambda_i\left(Z^{\frac{-1}{2}}\Delta Z Z^{\frac{-1}{2}}\right)} & \text{, if } \lambda_i\left(Z^{\frac{-1}{2}}\Delta Z Z^{\frac{-1}{2}}\right) < 0 \\ 1 & \text{, if } \lambda_i\left(Z^{\frac{-1}{2}}\Delta Z Z^{\frac{-1}{2}}\right) \geq 0. \end{cases}$$

To ensure the strict feasibility of the new iterates we used a factor $\xi = 0.95$. In our computational study, we compared our algorithm where $\psi(t) = t^p$, $p \geq 2$ with the variant of interior point algorithms that use the following AET for solving SDO problems: $\psi(t) = t$, $\psi(t) = \sqrt{t}$ and $\psi(t) = t - \sqrt{t}$ (see [7, 11, 18], respectively) where the value of lb is $\frac{1}{4}$. In all cases, the accuracy parameter had a value $\epsilon = 10^{-5}$. Here, we use the following notations: "iter" means the number of iterations performed by the algorithm in order to get an approximate optimal solution. "CPU" denotes the time (in seconds) necessary to get an approximate optimal solution for SDO.

Problem 6.1. *We consider the SDO problem in [10], where* $m = 3$, $n = 5$, $b = [-2, 2, -2]^\top$,

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & 1 & -1 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ -2 & 1 & -2 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 2 & 2 & -1 & -1 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 3 & 3 & -3 & 1 & 1 \\ 3 & 5 & 3 & 1 & 2 \\ -3 & 3 & -1 & 1 & 2 \\ 1 & 1 & 1 & -3 & -1 \\ 1 & 2 & 2 & -1 & -1 \end{bmatrix}.$$

We take $X^0 = Z^0 = I$ *and* $y^0 = [1, 1, 1]^\top$ *as a feasible starting point. An exact optimal solution for Problem 6.1 is given by*

$$X^* = \begin{bmatrix} 0.0914 & -0.0718 & 0.0169 & 0.0649 & -0.1583 \\ -0.0718 & 0.0724 & -0.0183 & -0.0602 & 0.1676 \\ 0.0169 & -0.0183 & 0.0103 & -0.0084 & -0.0772 \\ 0.0649 & -0.0602 & -0.0084 & 0.1481 & 0.0056 \\ -0.1583 & 0.1676 & -0.0772 & 0.0056 & 0.6022 \end{bmatrix},$$

$$Z^* = \begin{bmatrix} 1.4338 & 0.5754 & -0.0295 & -0.4043 & 0.2169 \\ 0.5754 & 1.0965 & 0.3401 & 0.2169 & -0.1120 \\ -0.0295 & 0.3401 & 1.1874 & 0.2169 & 0.0478 \\ -0.4043 & 0.2169 & 0.2169 & 0.2831 & -0.1415 \\ 0.2169 & -0.1120 & 0.0478 & -0.1415 & 0.0957 \end{bmatrix}$$

and $y^* = \begin{bmatrix} 0.8585 & 1.0937 & 0.7831 \end{bmatrix}^\top$ *, the optimal value of both problems is equal to* -1.0957. *We summarize the obtained numerical results in Table 1 where the parameter* p *used in the implementation is as follows:*

$$p \in \{2,\ 3,\ 5,\ 7,\ 15,\ 20,\ 30\}.$$

Problem 6.2. *We consider the SDO problem, where* $n = 2m$, $b[i] = 2$, $i = 1, \ldots, m$ *and*

$$C[i,j] = \begin{cases} -1 & \text{if } i = j \text{ and } i \leq m, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_i[j,k] = \begin{cases} 1 & \text{if } j = k = i \text{ or } j = k = i + m, \\ 0 & \text{otherwise,} \end{cases}$$

TABLE 1. Number of iterations for Problem 6.1.

$\psi(t)$	t^p							$t-\sqrt{t}$	$\sqrt{t}$	t
	p									
	2	3	5	7	15	20	30			
iter	31	40	65	93	203	272	410	67	61	64
CPU	0.1571	0.1587	0.1789	0.2503	0.2841	0.3422	0.5793	0.2444	0.2388	0.2336

TABLE 2. Number of iterations for Problem 6.2.

$(m,n)\backslash\psi(t)$		t^p					$t-\sqrt{t}$	$\sqrt{t}$	t
		p							
		2	3	5	7	10			
$(5,10)$	iter	33	43	69	98	141	71	64	68
	CPU	0.2163	0.1010	0.1482	0.1877	0.2658	0.1314	0.1649	0.1382
$(10,20)$	iter	35	45	73	102	148	75	67	71
	CPU	0.1339	0.1377	0.2022	0.2668	0.3992	0.2070	0.1801	0.1884
$(50,100)$	iter	39	50	80	113	164	83	74	79
	CPU	1.9779	2.5945	3.4456	3.6638	6.0864	3.5441	3.3478	2.4177
$(100,200)$	iter	40	52	84	118	171	86	77	82
	CPU	6.0358	6.2007	9.5844	9.1948	28.0350	12.8844	10.1662	6.8034
$(200,400)$	iter	42	54	87	123	178	89	80	85
	CPU	21.9279	28.2211	47.4171	68.2180	137.2114	61.2352	51.1357	58.8938

we consider the following starting points:

$$X^0[i,j] = \begin{cases} 2-\gamma & \textit{if} \quad i=j=1,\ldots,m, \\ \gamma & \textit{if} \quad i=j=m+1,\ldots,n, \\ 0 & \textit{otherwise}, \end{cases}$$

$$Z^0[i,j] = \begin{cases} \frac{1}{\gamma}-1 & \textit{if} \quad i=j=1,\ldots,m, \\ \frac{1}{\gamma} & \textit{if} \quad i=j=m+1,\ldots,n, \\ 0 & \textit{otherwise}, \end{cases}$$

and $y^0[i] = -\frac{1}{\gamma}, \quad i=1,\ldots,m$, *with* $\gamma = (4-\sqrt{8})/2$. *An exact optimal solution for Problem 6.2 is given by:*

$$X^*[i,j] = \begin{cases} 2 & \textit{if} \quad i=j=1,\ldots,m, \\ 0 & \textit{othewise}, \end{cases}$$

$$Z^*[i,j] = \begin{cases} 1 & \textit{if} \quad i=j=m+1,\ldots,n, \\ 0 & \textit{othewise}, \end{cases}$$

and $y^*[i] = -1, \quad i=1,\ldots,m$. *The optimal value of both problems is equal to* $-n$. *We summarize the obtained numerical results in Table 2 where the parameter* p *used in the implementation is as follows:*

$$p \in \{2,\ 3\ ,\ 5,\ 7,\ 10\}.$$

Comment. Across the numerical results obtained by the algorithm the minimal number of iterations is achieved by the AET $\psi(t) = t^p$ with $p = 2$ for different size (m,n).

7. Conclusion

In this work we have extended a primal-dual path-following interior-point method for LO to SDO problems with full NT-step. Based on the new Darvay's technique [5], we used the function $\psi(t) = t^p$ with $p \geq 2$ in order to determine a class of new search directions. The associated short-step algorithm deserves the best well-known polynomial complexity, which is the same iteration bound as in the LO case. Moreover, the resulting analysis is relatively simple and straightforward to the LO analogue in [12]. We also presented some numerical results to show the efficiency of the proposed method.

Acknowledgements. The author is very grateful and would like to thank the Editor-in-Chief and the anonymous referees for their useful suggestions which significantly improved the presentation of this paper.

References

[1] M. Achache, A new primal-dual path-following method for convex quadratic programming. *Comput. Appl. Math.* **25** (2006) 97–110.

[2] M. Achache and L. Guerra, A full Nesterov-Todd-step feasible primal-dual interior point algorithm for convex quadratic semi-definite optimization. *Appl. Math. Comput.* **231** (2014) 581–590.

[3] F. Alizadeh, Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM J. Optim.* **5** (1995) 13–51.

[4] Z. Darvay, New interior-point algorithms in linear optimization. *Adv. Model. Optim.* **5** (2003) 51–92.

[5] Z. Darvay and P.R. Takács, New method for determining search directions for interior-point algorithms in linear optimization. *Optim. Lett.* **12** (2018) 1099–1116.

[6] Zs. Darvay, I.M. Papp and P.R. Takács, Complexity analysis of a full-Newton step interior-point method for linear optimization. *Period. Math. Hung.* **73** (2016) 27–42.

[7] E. de Klerk, Aspects of Semidefinite Programming: Interior Point Algorithms and Selected Applications. Kluwer Academic Publishers, Dordrecht, The Netherlands (2002).

[8] M. Halicka, E. De Klerk and C. Roos, On the convergence of the central path in semidefinite optimization. *SIAM J. Optim.* **12** (2002) 1090–1099.

[9] G. Gu, M. Zangiabadi and C. Roos, Full Nesterov-Todd step infeasible interior- point method for symmetric optimization. *Eur. J. Oper. Res.* **214** (2011) 473–484.

[10] L. Guerra and M. Achache, A parametric kernel function generating the best known iteration bound for large-update methods for CQSDO. *Stat. Optim. Inf. Comput.* **8** (2020) 876–889.

[11] B. Kheirfam, New complexity analysis of a full Nesterov–Todd step interior-point method for semidefinite optimization. *Asian-Eur. J. Math.* **3** (2017) 1750070.

[12] B. Kheirfam and A. Nasrollahi, An extension for identifying search directions for interior-point methods in linear optimization. *Asian-Eur. J. Math.* **1** (2020) 2050014.

[13] M. Kojima, S. Shindoh and S. Hara, Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices. *SIAM J. Optim.* **7** (1997) 86–125.

[14] H. Mansouri and C. Roos, A new full Newton-step $O(n)$ infeasible interior-point algorithm for semidefinite optimization. *Numer. Algorithms* **52** (2009) 225–255.

[15] Y.E. Nesterov and A.S. Nemirovskii, Interior-Point Polynomial Algorithms in Convex Programming. SIAM, Philadelphia, PA (1994).

[16] C. Roos, T. Terlaky and J.Ph. Vial, Theory and algorithms for linear optimization. An interior point approach, John-Wiley, Sons, Chichester, UK (1997).

[17] M.J. Todd, K.C. Toh and R.H. Tütüncü, On the Nesterov-Todd direction in semidefinite programming. *SIAM J. Optim.* **8** (1998) 769–796.

[18] G.Q. Wang and Y.Q. Bai, A new primal-dual path-following interior point algorithm for semidefinite programming. *J. Math. Anal. Appl.* **353** (2009) 339–349.

[19] G.Q. Wang and Y.Q. Bai, A primal-dual path-following interior-point algorithm for second-order cone optimization with full Nesterov-Todd step. *Appl. Math. Comput.* **215** (2009) 1047–1061.

[20] G.Q. Wang and Y.Q. Bai, A new full Nesterov-Todd step primal-dual path-following interior-point algorithm for symmetric optimization. *J. Optim. Theory Appl.* **154** (2012) 966–985.

[21] H. Wolkowicz, R. Saigal and L. Vandenberghe, Handbook of Semidefinite Programming, Theory, Algorithms, and Applications. Kluwer Academic Publishers, Dordrecht, The Netherlands (2000).

[22] M. Zangiabadi, G. Gu and C. Roos, Full Nesterov-Todd step primal-dual interior- point mathods for second-order cone optimization. *J. Optim. Theory Appl.* **158** (2013) 816–858.