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PATH-FACTOR CRITICAL COVERED GRAPHS AND PATH-FACTOR UNIFORM GRAPHS

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Abstract. A path-factor in a graph G is a spanning subgraph F of G such that every component of F is a path. Let d and n be two nonnegative integers with $d \geq 2$. A $P_{\geq d}$-factor of G is its spanning subgraph each of whose components is a path with at least d vertices. A graph G is called a $P_{\geq d}$-factor covered graph if for any $e \in E(G)$, G admits a $P_{\geq d}$-factor containing e. A graph G is called a $(P_{\geq d}, n)$-factor critical covered graph if for any $N \subseteq V(G)$ with $|N| = n$, the graph $G - N$ is a $P_{\geq d}$-factor covered graph. A graph G is called a $P_{\geq d}$-factor uniform graph if for any $e \in E(G)$, the graph $G - e$ is a $P_{\geq d}$-factor covered graph. In this paper, we verify the following two results: (i) An $(n+1)$-connected graph G of order at least $n+3$ is a $(P_{\geq 3}, n)$-factor critical covered graph if G satisfies $\delta(G) > \frac{\alpha(G)+2n+3}{2}$; (ii) Every regular graph G with degree $r \geq 2$ is a $P_{\geq 3}$-factor uniform graph.

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1. Introduction

In this work, the graphs considered are finite, undirected and simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is the number $|V(G)|$ of its vertices. The degree of $x \in V(G)$ is denoted by $d_G(x)$. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and write $G - X = G[V(G) \setminus X]$. A set $X \subseteq V(G)$ is called an independent set of G if $G[X]$ does not possess edges. Let $\alpha(G)$, $\delta(G)$, $\omega(G)$ and $i(G)$ the independence number, the minimum degree, the number of connected components and the number of isolated vertices in G, respectively. The binding number of G, denoted by $\mathrm{bind}(G)$, is defined by

$$\mathrm{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

The isolated toughness of G, denoted by $I(G)$, was defined by

$$I(G) = \min\left\{\frac{|X|}{i(G-X)} : X \subseteq V(G), i(G-X) \geq 2\right\}$$

Keywords. Graph, minimum degree, independence number, $P_{\geq 3}$-factor, $(P_{\geq 3}, n)$-factor critical covered graph, $P_{\geq 3}$-factor uniform graph.

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if G is not a complete graph; otherwise, $I(G) = +\infty$. The sun toughness of G is denoted by $s(G)$ and defined by

$$s(G) = \min\left\{\frac{|X|}{\text{sun}(G-X)} : X \subseteq V(G),\ \text{sun}(G-X) \geq 2\right\}$$

if G is not a complete graph; and $s(G) = +\infty$ if G is a complete graph. The path and the complete graph of order d are denoted by P_d and K_d, respectively. Let G_1 and G_2 be two graphs. We use $G_1 \vee G_2$ and $G_1 \cup G_2$ to denote the join and the union of G_1 and G_2, respectively.

A path-factor in a graph G is a spanning subgraph F of G such that every component of F is a path. Let $d \geq 2$ and $n \geq 0$ be two integers. A $P_{\geq d}$-factor of a graph G is its spanning subgraph each of whose components is a path with at least d vertices. A graph G is called a $P_{\geq d}$-factor covered graph if for any $e \in E(G)$, G has a $P_{\geq d}$-factor covering e. A graph G is called a $(P_{\geq d}, n)$-factor critical covered graph if for any $N \subseteq V(G)$ with $|N| = n$, the graph $G - N$ is a $P_{\geq d}$-factor covered graph. A graph G is called a $P_{\geq d}$-factor uniform graph if for any $e \in E(G)$, the graph $G - e$ is a $P_{\geq d}$-factor covered graph.

We may simulate real-world networks by graphs. The vertices of the graph correspond to the nodes of the network, and the edges of the graph represent the links between the nodes in the network. Henceforth, we replace "network" by the term "graph". Thus we may utilize some graphic parameters to characterize the vulnerability and robustness of the network, for instance, independence number, toughness, binding number and minimum degree, and so on. In data transmission networks, the data transmission between two sites of a network goes through a path between two corresponding vertices of a corresponding graph. Hence, the availability of data transmission in the network is equivalent to the existence of path-factor in the corresponding graph which is generated by the network. When some nodes are damaged and a special channel is assigned, the possibility of data transmission in a data transmission network is equal to the existence of path-factor critical covered graph. When a special channel is damaged and another special channel is assigned, the possibility of data transmission in a data transmission network is equivalent to the existence of a path-factor uniform graph. The study on the existence of path-factors, path-factor critical covered graphs or path-factor uniform graphs under specific network structures can help scientists to design and create networks with high data transmission rates. In this work, we investigate the existence of path-factor critical covered graphs and path-factor uniform graphs which play an important role in studying data transmissions of data transmission networks.

Asratian and Casselgren [1], Egawa *et al.* [4] investigated the existence of path-factors in graphs. Kelmans [8] derived some results on path-factors in claw-free graphs. Matsubara *et al.* [11] presented degree sum conditions for bipartite graphs to admit path-factors. Kano *et al.* [7] established a relationship between the number of isolated vertices and $P_{\geq 3}$-factors in graphs. Wang [12] verified that a bipartite graph G contains a $P_{\geq 3}$-factor if and only if $i(G - X - M) \leq 2|X| + |M|$ for every $X \subseteq V(G)$ and independent $M \subseteq E(G)$. Kano *et al.* [6] claimed that every connected cubic bipartite graph of order at least 8 has a $P_{\geq 8}$-factor. Zhou *et al.* [27] obtained some results on $P_{\geq 3}$-factors in graphs with given properties. Dai *et al.* [3] gave some sufficient conditions for graphs to be $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs. Zhou [21] established a relationship between $P_{\geq 3}$-factor covered graphs and neighborhoods of independent sets. Kouider and Ouatiki [9], Bekkai [2], Zhou [22], Zhou *et al.* [30], Yuan and Hao [16] posed some relationships between independence number and graph factors. Zhou *et al.* [28], Zhou [18–20], Wang and Zhang [13, 14], Zhou and Liu [24] established some relationships between minimum degree (or degree) and graph factors.

A matching M in G is a subset of $E(G)$ in which no two edges admit a vertex in common. A matching M of G with $V(M) = V(G)$ is called a perfect matching. A graph H is called a factor-critical graph if $H - x$ contains a perfect matching for any $x \in V(H)$. Let H be a factor-critical graph with $V(H) = \{x_1, x_2, \cdots, x_n\}$. To characterize those graphs with a $P_{\geq 3}$-factor, Kaneko [5] introduced the concept of a sun. A graph R is called a sun if $R = K_1$, $R = K_2$ or R is the corona of H with at least three vertices, that is, R is derived from H by adding n new vertices $y_1, y_2, \cdots, y_n$ together with n new edges $x_1y_1, x_2y_2, \cdots, x_ny_n$ to H. Obviously, $d_R(y_i) = 1$ for all i, $1 \leq i \leq n$. A sun of order at least 6 is called a big sun. A sun component of G is a component isomorphic to a sun in G. Let $\text{sun}(G)$ denote the number of sun components in G.

Kaneko [5] provided a criterion for a graph with a $P_{\geq 3}$-factor, which is the following theorem.

Theorem 1.1 ([5]). *A graph G contains a $P_{\geq 3}$-factor if and only if*

$$\mathrm{sun}(G-X) \leq 2|X|$$

for all $X \subseteq V(G)$.

Later *et al.* [17] extend Theorem 1.1, and obtained a characterization for a $P_{\geq 3}$-factor covered graph which is stated as follows.

Theorem 1.2 ([17]). *A connected graph G is a $P_{\geq 3}$-factor covered graph if and only if*

$$\mathrm{sun}(G-X) \leq 2|X| - \varepsilon(X)$$

for any $X \subseteq V(G)$, where $\varepsilon(X)$ is defined by

$$\varepsilon(X) = \begin{cases} 2, & \text{if } X \text{ is not an independent set;} \\ 1, & \text{if } X \text{ is a nonempty independent set and } G-X \text{ has} \\ & \text{a non-sun component;} \\ 0, & \text{otherwise.} \end{cases}$$

A graph G is called a $(P_{\geq d}, n)$-factor critical covered graph if for any $N \subseteq V(G)$ with $|N| = n$, the graph $G - N$ is a $P_{\geq d}$-factor covered graph. Zhou *et al.* [29] showed a sun toughness condition for a graph to be a $(P_{\geq 3}, n)$-factor critical covered graph. Zhou *et al.* [31] demonstrated two results on the existence of $(P_{\geq 3}, n)$-factor critical covered graphs depending on toughness and isolated toughness. Wang and Zhang [15] improved the previous isolated toughness condition to guarantee the existence of $(P_{\geq 3}, n)$-factor critical covered graphs. The following results on $(P_{\geq d}, n)$-factor critical covered graphs are known.

Theorem 1.3 ([29]). *An $(n+r+1)$-connected graph G is a $(P_{\geq 3}, n)$-factor critical covered graph if its sun toughness $s(G) > \frac{n+r+1}{2r+1}$, where $n \geq 0$ and $r \geq 1$ are integers.*

Theorem 1.4 ([15]). *Let n and λ be two nonnegative integers. Then an $(n+\lambda+2)$-connected graph G is a $(P_{\geq 3}, n)$-factor-critical covered graph if its isolated toughness $I(G) > \frac{n+3\lambda+5}{2\lambda+3}$.*

A graph G is called a $P_{\geq d}$-factor uniform graph if for any $e \in E(G)$, the graph $G - e$ is a $P_{\geq d}$-factor covered graph. Zhou and Sun [25] showed binding number conditions for a graph to be $P_{\geq 2}$-factor and $P_{\geq 3}$-factor uniform graphs, respectively. Liu [10] obtained an improved binding number condition for a graph to be a $P_{\geq 3}$-factor uniform graph. Zhou and Bian [23] derived two results on the existence of $P_{\geq 3}$-factor uniform graphs. Zhou *et al.* [26] derived isolated toughness conditions for graphs to be $P_{\geq 2}$-factor and $P_{\geq 3}$-factor uniform graphs, respectively. The following results on $P_{\geq 3}$-factor uniform graphs are known.

Theorem 1.5 ([10]). *A 3-connected graph G is a $P_{\geq 3}$-factor uniform graph if* $\mathrm{bind}(G) > \frac{10}{7}$.

Theorem 1.6 ([26]). *Let r be a nonnegative integer. An $(r+3)$-edge-connected graph G is a $P_{\geq 3}$-factor uniform graph if its isolated toughness $I(G) > \frac{3r+6}{2r+3}$.*

It is natural and interesting to present some new graphic parameter conditions to guarantee that a graph is a $(P_{\geq 3}, n)$-factor critical covered graph or a $P_{\geq 3}$-factor uniform graph. In this work, we continue to study $P_{\geq 3}$-factor uniform graphs and $P_{\geq 3}$-factor uniform graphs, and pose a new sufficient condition for the existence of $(P_{\geq 3}, n)$-factor critical covered graphs and $P_{\geq 3}$-factor uniform graphs, respectively.

Theorem 1.7. *Let $n \geq 0$ be an integer, and let G be an $(n+1)$-connected graph of order at least $n+3$. If G satisfies*

$$\delta(G) > \frac{\alpha(G) + 2n + 3}{2},$$

then G is a $(P_{\geq 3}, n)$-factor critical covered graph.

Theorem 1.8. *Every regular graph G with degree $r \geq 2$ is a $P_{\geq 3}$-factor uniform graph.*

This work is organized as follows. In Section 2, we give the proof and sharpness of Theorem 1.7. In Section 2, we give the proof of Theorem 1.8.

2. The proof of Theorem 1.7

Proof of Theorem 1.7. Let $G' = G - N$ for any $N \subseteq V(G)$ with $|N| = n$. To prove Theorem 1.7, we only need to verify that G' is a $P_{\geq 3}$-factor covered graph. Suppose, to the contrary, that G' is not a $P_{\geq 3}$-factor covered graph. Then it follows from Theorem 1.2 that

$$\text{sun}(G' - X) \geq 2|X| - \varepsilon(X) + 1 \tag{2.1}$$

for some $X \subseteq V(G')$.

Claim 1. $|X| \neq 0$.

Proof. Assume that $|X| = 0$. Note that G is an $(n+1)$-connected graph. Then G' is a connected graph, and so $\omega(G') = 1$. Using (2.1) and $\varepsilon(X) = 0$, we deduce

$$1 = \omega(G') \geq \text{sun}(G') = \text{sun}(G' - X) \geq 2|X| - \varepsilon(X) + 1 = 1,$$

which implies that G' is a sun. Since $|V(G)| \geq n + 3$, we have $|V(G')| \geq 3$. Combining these with the concept of a big sun, we know that G' is a big sun. Let H be the factor-critical graph of G'. Then $V(G') \setminus V(H)$ is an independent set of G', and so

$$\alpha(G) \geq \alpha(G') \geq |V(G') \setminus V(H)| = \frac{|V(G')|}{2}. \tag{2.2}$$

Note that $d_{G'}(x) = 1$ for any $x \in V(G') \setminus V(H)$. We select $t \in V(G') \setminus V(H)$. Then we have

$$\delta(G) \leq d_G(t) \leq d_{G-N}(t) + |N| = d_{G'}(t) + n = n + 1. \tag{2.3}$$

By virtue of (2.2), (2.3) and $\delta(G) > \frac{\alpha(G)+2n+3}{2}$, we derive

$$n + 1 \geq \delta(G) > \frac{\alpha(G) + 2n + 3}{2} \geq \frac{\frac{|V(G')|}{2} + 2n + 3}{2} = \frac{|V(G')| + 4n + 6}{4},$$

which implies that $|V(G')| < 0$, which contradicts that $|V(G')| \geq 3$. Claim 1 is verified. □

According to (2.1), Claim 1 and $\varepsilon(X) \leq 2$, we deduce

$$\text{sun}(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1 \geq 1. \tag{2.4}$$

By virtue of (2.4) and the definition of sun component, there exists a vertex t in $G' - X$ such that $d_{G'-X}(t) \leq 1$. Thus, we infer

$$\delta(G) \leq d_G(t) \leq d_{G-N}(t) + |N| = d_{G'}(t) + n \leq d_{G'-X}(t) + |X| + n \leq |X| + n + 1. \tag{2.5}$$

Set $\text{sun}(G' - X) = r$. Let $R_1, R_2, \cdots, R_r$ be r sun components in $G' - X$. Select $t_i \in V(R_i)$, $1 \leq i \leq r$. Then $\{t_1, t_2, \cdots, t_r\}$ is an independent set of $G' - X$, and so $\alpha(G' - X) \geq \text{sun}(G' - X) = r$. Thus, we have

$$\alpha(G) \geq \alpha(G') \geq \alpha(G' - X) \geq \text{sun}(G' - X) = r. \tag{2.6}$$

It follows from (2.4) and (2.6) that

$$\alpha(G) \geq \text{sun}(G' - X) \geq 2|X| - 1. \tag{2.7}$$

In terms of (2.7) and $\delta(G) > \frac{\alpha(G)+2n+3}{2}$, we derive

$$\delta(G) > \frac{\alpha(G) + 2n + 3}{2} \geq \frac{2|X| - 1 + 2n + 3}{2} = |X| + n + 1,$$

which contradicts (2.5). This completes the proof of Theorem 1.7. □

Remark 2.1. Next, we show that the condition

$$\delta(G) > \frac{\alpha(G) + 2n + 3}{2}$$

in Theorem 1.7 is sharp, namely, it cannot be replaced by

$$\delta(G) \geq \frac{\alpha(G) + 2n + 3}{2}.$$

Let n and r be two nonnegative integers with $r \geq 1$. In what follows, we construct a graph $G = K_{n+r+1} \vee ((2r+1)K_2)$. It is clear that $\delta(G) = n + r + 2$, $\alpha(G) = 2r + 1$ and G is an $(n + r + 1)$-connected graph. Thus, we easily infer

$$\delta(G) = n + r + 2 = \frac{\alpha(G) + 2n + 3}{2}.$$

For any $N \subseteq V(K_{n+r+1})$ with $|N| = n$, we write $G' = G - N = K_{r+1} \vee ((2r+1)K_2)$. We select $X = V(K_{r+1}) \subseteq V(G')$, and X is not an independent set of G'. Then we see $|X| = r + 1$, $\mathrm{sun}(G' - X) = 2r + 1$ and $\varepsilon(X) = 2$. Thus, we deduce

$$\mathrm{sun}(G' - X) = 2r + 1 > 2r = 2(r + 1) - 2 = 2|X| - \varepsilon(X).$$

Combining this with Theorem 1.2, G' is not a $P_{\geq 3}$-factor covered graph, and so G is not a $(P_{\geq 3}, n)$-factor critical covered graph.

Set $n = 0$ in Theorem 1.7. Then we easily obtain the following corollary.

Corollary 2.2. *Let G be a connected graph of order at least 3. If G satisfies*

$$\delta(G) > \frac{\alpha(G) + 3}{2},$$

then G is a $P_{\geq 3}$-factor covered graph.

3. The proof of Theorem 1.8

Proof of Theorem 1.8. Without loss of generality, we may assume that G is a connected graph. Otherwise, we consider every connected component of G.

Let $G' = G - e$ for any $e = uv \in E(G)$. To prove Theorem 1.8, we only need to verify that G' is a $P_{\geq 3}$-factor covered graph. Suppose, to the contrary, that G' is not a $P_{\geq 3}$-factor covered graph. According to Theorem 1.2 there exists a vertex subset X of G' such that

$$\mathrm{sun}(G' - X) = a + b + c \geq 2|X| - \varepsilon(X) + 1, \tag{3.1}$$

where a is the number of sun components R of $G' - X$ satisfying $R = K_1$, b is the number of sun components R of $G' - X$ satisfying $R = K_2$ and c is the number of big sun components R of $G' - X$.

For any $x \in V(aK_1)$, the degree of x in aK_1 is 0. For any $x \in V(bK_2)$, the degree of x in bK_2 is 1. For each big sun component R, R admits at least three vertices of degree exactly one. Note that G is a regular graph with degree $r \geq 2$ and $G' = G - e$. We easily see that the following inequality holds:

$$ar + 2b(r - 1) + 3c(r - 1) - 2 \leq r|X|. \tag{3.2}$$

Case 1. $|X| = 0$.

Obviously, $\varepsilon(X) = 0$. Using (3.1), we deduce

$$\mathrm{sun}(G') \geq 1.$$

Note that G is connected. Hence, we have

$$\mathrm{sun}(G') \leq \omega(G') \leq \omega(G) + 1 = 2.$$

Thus, we derive $1 \leq \mathrm{sun}(G') \leq \omega(G') \leq 2$.

Subcase 1.1. $\omega(G') = 1$.

In this subcase, we have $\mathrm{sun}(G') = 1$, and so G' is a sun. Note that $r \geq 2$. Hence, $|V(G')| = |V(G)| \geq 3$, and G' is a big sun. Thus, there exist at least three vertices in G' with degree 1, and so G admits at least one vertex with degree 1, which contradicts that G is a regular graph with degree $r \geq 2$.

Subcase 1.2. $\omega(G') = 2$.

In this subcase, G' has a sun component R and another component Q. If $R = K_1$, then $G = K_1 \cup Q \cup \{e\}$. Thus, we know $\delta(G) = 1$. If $R = K_2$, then $G = K_2 \cup Q \cup \{e\}$. Hence, $\delta(G) = 1$. If R is a big sun, then R has at least three vertices with degree 1. Thus $G = R \cup Q \cup \{e\}$ and $\delta(G) = 1$. In conclusion, we always have $\delta(G) = 1$ in Subcase 1.2. Recall that G is a regular graph with degree $r \geq 2$. Then $1 = \delta(G) \geq 2$, which is a contradiction.

Case 2. $|X| = 1$.

Subcase 2.1. $G' - X$ does not admit a non-sun component.

Clearly, $\varepsilon(X) = 0$. Then using (3.1) and $|X| = 1$, we infer

$$a + b + c \geq 2|X| - \varepsilon(X) + 1 = 3. \tag{3.3}$$

It follows from (3.2), (3.3), $|X| = 1$ and $r \geq 2$ that

$$\begin{aligned} 0 &\geq ar + 2b(r-1) + 3c(r-1) - 2 - r|X| \\ &= ar + 2b(r-1) + 3c(r-1) - 2 - r \\ &= (a + 2b + 3c - 1)r - 2b - 3c - 2 \\ &\geq 2(a + 2b + 3c - 1) - 2b - 3c - 2 \\ &= 2a + 2b + 3c - 4 \geq 2(a + b + c) - 4 \geq 2, \end{aligned}$$

which is a contradiction.

Subcase 2.2. $G' - X$ admits a non-sun component.

Obviously, $\varepsilon(X) = 1$. In terms of (3.1) and $|X| = 1$, we get

$$a + b + c \geq 2|X| - \varepsilon(X) + 1 = 2. \tag{3.4}$$

Note that G is a regular graph with degree $r \geq 2$, $G' = G - e$ and $G' - X$ admits a non-sun component. Thus, we deduce

$$ar + 2b(r-1) + 3c(r-1) + 1 - 2 \leq r|X|. \tag{3.5}$$

It follows from (3.4), (3.5), $|X| = 1$ and $r \geq 2$ that

$$\begin{aligned} 0 &\geq ar + 2b(r-1) + 3c(r-1) - 1 - r|X| \\ &= ar + 2b(r-1) + 3c(r-1) - 1 - r \\ &= (a + 2b + 3c - 1)r - 2b - 3c - 1 \\ &\geq 2(a + 2b + 3c - 1) - 2b - 3c - 1 \\ &= 2a + 2b + 3c - 3 \geq 2(a + b + c) - 3 \geq 1, \end{aligned}$$

which is a contradiction.

Case 3. $|X| = 2$.

Subcase 3.1. X is an independent set of G'.

Obviously, $\varepsilon(X) \leq 1$. By means of (3.1) and $|X| = 2$, we derive

$$a + b + c \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| = 4. \tag{3.6}$$

By virtue of (3.2), (3.6), $|X| = 2$ and $r \geq 2$ that

$$\begin{aligned} 0 &\geq ar + 2b(r-1) + 3c(r-1) - 2 - r|X| \\ &= ar + 2b(r-1) + 3c(r-1) - 2 - 2r \\ &= (a + 2b + 3c - 2)r - 2b - 3c - 2 \\ &\geq 2(a + 2b + 3c - 2) - 2b - 3c - 2 \\ &= 2a + 2b + 3c - 6 \\ &\geq 2(a + b + c) - 6, \end{aligned}$$

which implies that $a + b + c \leq 3$, which contradicts (3.6).

Subcase 3.2. X is not an independent set of G'.

In this subcase, $\varepsilon(X) = 2$. In terms of (3.1) and $|X| = 2$, we have

$$a + b + c \geq 2|X| - \varepsilon(X) + 1 = 2|X| - 1 = 3. \tag{3.7}$$

For any $x \in X$, $d_{G[X]}(x) = 1$. Note that G is a regular graph with degree $r \geq 2$. Thus, we admit the following inequality:

$$ar + 2b(r-1) + 3c(r-1) - 2 \leq (r-1)|X|. \tag{3.8}$$

According to (3.7), (3.8), $|X| = 2$ and $r \geq 2$ that

$$\begin{aligned} 0 &\geq ar + 2b(r-1) + 3c(r-1) - 2 - (r-1)|X| \\ &= ar + 2b(r-1) + 3c(r-1) - 2 - 2(r-1) \\ &= (a + 2b + 3c - 2)r - 2b - 3c \\ &\geq 2(a + 2b + 3c - 2) - 2b - 3c \\ &= 2a + 2b + 3c - 4 \\ &\geq 2(a + b + c) - 4, \end{aligned}$$

which implies that $a + b + c \leq 2$, which contradicts (3.7).

Case 4. $|X| \geq 3$.

From (3.1) and $\varepsilon(X) \leq 2$, we infer

$$a + b + c \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1. \tag{3.9}$$

It follows from (3.2), (3.9), $r \geq 2$ and $|X| \geq 3$ that

$$\begin{aligned} 0 &\geq ar + 2b(r-1) + 3c(r-1) - 2 - r|X| \\ &= (a + 2b + 3c - |X|)r - 2b - 3c - 2 \\ &\geq 2(a + 2b + 3c - |X|) - 2b - 3c - 2 \\ &= 2a + 2b + 3c - 2|X| - 2 \\ &\geq 2(a + b + c) - 2|X| - 2 \\ &\geq 2(2|X| - 1) - 2|X| - 2 \\ &= 2|X| - 4, \end{aligned}$$

which implies that $|X| \leq 2$, which contradicts $|X| \geq 3$. This completes the proof of Theorem 1.8.

$\square$

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References

[1] A. Asratian and C. Casselgren, On path factors of $(3, 4)$-biregular bigraphs. *Graphs Combin.* **24** (2008) 405–411.
[2] S. Bekkai, Minimum degree, independence number and pseudo $[2, b]$-factors in graphs. *Discrete Appl. Math.* **162** (2014) 108–114.
[3] G. Dai, Z. Zhang, Y. Hang and X. Zhang, Some degree conditions for $P_{\geq k}$-factor covered graphs. *RAIRO-Oper. Res.* **55** (2021) 2907–2913.
[4] Y. Egawa, M. Furuya and K. Ozeki, Sufficient conditions for the existence of a path-factor which are related to odd components. *J. Graph Theory* **89** (2018) 327–340.
[5] A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two. *J. Combin. Theory Ser. B* **88** (2003) 195–218.
[6] M. Kano, C. Lee and K. Suzuki, Path and cycle factors of cubic bipartite graphs. *Discuss. Math. Graph Theory* **28** (2008) 551–556.
[7] M. Kano, H. Lu and Q. Yu, Component factors with large components in graphs. *Appl. Math. Lett.* **23** (2010) 385–389.
[8] A. Kelmans, Packing 3-vertex paths in claw-free graphs and related topics. *Discrete Appl. Math.* **159** (2011) 112–127.
[9] M. Kouider and S. Ouatiki, Sufficient condition for the existence of an even $[a, b]$-factor in graph. *Graphs Combin.* **29** (2013) 1051–1057.
[10] H. Liu, Binding number for path-factor uniform graphs. *Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci.* **23** (2022) 25–32.
[11] R. Matsubara, H. Matsumura, M. Tsugaki and T. Yamashita, Degree sum conditions for path-factors with specified end vertices in bipartite graphs. *Discrete Math.* **340** (2017) 87–95.
[12] H. Wang, Path factors of bipartite graphs. *J. Graph Theory* **18** (1994) 161–167.
[13] S. Wang and W. Zhang, Research on fractional critical covered graphs. *Probl. Inf. Transm.* **56** (2020) 270–277.
[14] S. Wang and W. Zhang, On k-orthogonal factorizations in networks. *RAIRO-Oper. Res.* **55** (2021) 969–977.
[15] S. Wang and W. Zhang, Isolated toughness for path factors in networks. *RAIRO-Oper. Res.* **56** (2022) 2613–2619.
[16] Y. Yuan and R. Hao, Independence number, connectivity and all fractional (a, b, k)-critical graphs. *Discuss. Math. Graph Theory* **39** (2019) 183–190.
[17] H. Zhang and S. Zhou, Characterizations for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs. *Discrete Math.* **309** (2009) 2067–2076.
[18] S. Zhou, A result on fractional (a, b, k)-critical covered graphs. *Acta Math. Appl. Sin. Engl. Ser.* **37** (2021) 657–664.
[19] S. Zhou, A neighborhood union condition for fractional (a, b, k)-critical covered graphs. *Discrete Appl. Math.* **323** (2022) 343–348.
[20] S. Zhou, A note of generalization of fractional ID-factor-critical graphs. *Fund. Inform.* **187** (2022) 61–69.
[21] S. Zhou, Path factors and neighborhoods of independent sets in graphs. *Acta Math. Appl. Sin. Engl. Ser.* (2022). DOI: 10.1007/s10255-022-1096-2.
[22] S. Zhou, Remarks on restricted fractional (g, f)-factors in graphs. *Discrete Appl. Math.* (2022). DOI: 10.1016/j.dam.2022.07.020.
[23] S. Zhou and Q. Bian, The existence of path-factor uniform graphs with large connectivity. *RAIRO-Oper. Res.* **56** (2022) 2919–2927.
[24] S. Zhou and H. Liu, Discussions on orthogonal factorizations in digraphs. *Acta Math. Appl. Sin. Engl. Ser.* **38** (2022) 417–425.
[25] S. Zhou and Z. Sun, Binding number conditions for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor uniform graphs. *Discrete Math.* **343** (2020) 111715.
[26] S. Zhou, Z. Sun and Q. Bian, Isolated toughness and path-factor uniform graphs (II). *Indian J. Pure Appl. Math.* (2022). DOI: 10.1007/s13226-022-00286-x.
[27] S. Zhou, Q. Bian and Q. Pan, Path factors in subgraphs. *Discrete Appl. Math.* **319** (2022) 183–191.
[28] S. Zhou, H. Liu and Y. Xu, A note on fractional ID-$[a, b]$-factor-critical covered graphs. *Discrete Appl. Math.* **319** (2022) 511–516.
[29] S. Zhou, J. Wu and Q. Bian, On path-factor critical deleted (or covered) graphs. *Aequationes Math.* **96** (2022) 795–802.
[30] S. Zhou, J. Wu and H. Liu, Independence number and connectivity for fractional (a, b, k)-critical covered graphs. *RAIRO-Oper. Res.* **56** (2022) 2535–2542.

[31] S. Zhou, J. Wu and Y. Xu, Toughness, isolated toughness and path factors in graphs. *Bull. Aust. Math. Soc.* **106** (2022) 195–202.