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HERMITE–HADAMARD TYPE INEQUALITY FOR (E, F)-CONVEX FUNCTIONS AND GEODESIC (E, F)-CONVEX FUNCTIONS

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Abstract. The main aim of the present paper is to introduce geodesic (E, F)-convex sets and geodesic (E, F)-functions on a Riemannian manifold. Furthermore, some basic properties of these mappings are investigated. Moreover, the Hadamard-type inequalities for (E, F)-convex functions are proven.

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1. Introduction

Convex optimization has an increasing impact on many areas of mathematics, practical applications, and applied sciences. The idea of convexity has been developed and generalized in numerous directions due to its uses and significance, see [1, 10, 19, 20]. E-convexity of sets and functions was introduced in 1999 [22].

Many other researchers are studied further, improved, generalized, and extended E-conexity such as E-convex hull, E-convex cone, E-affine sets, semi semi E-convex For more results on E-convexity see *e.g.*, [1,3,9,10,17,20]. Also, E-convex sets and functions are extended to another class called (E, F)-convex sets and (E, F)-convex functions [5, 6].

The geodesic convexity was introduced in [11, 21]. Moreover, geodesic E-convex sets and geodesic E-convex functions were introduced on Riemannian manifolds in [4].

2. Notations and preliminaries

In this section, some definitions and known results of convex, E-convex and (E, F)-functions in real numbers sets are presented. Also, geodesic convex, geodesic E-convex functions and some results about Riemannian manifolds, which will be used throughout the paper, are given.

Definition 2.1. Let $U \subseteq \mathbb{R}$ be an interval, then $f : U \longrightarrow \mathbb{R}$ is called convex if

$$f(t\omega_1 + (1-t)\omega_2) \leq tf(\omega_1) + (1-t)f(\omega_2), \quad \forall \omega_1, \omega_2 \in U, \quad t \in [0,1]. \tag{2.1}$$

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Definition 2.2. A function $E : [\omega_1, \omega_2] \longrightarrow [\omega_1, \omega_2]$ where $[\omega_1, \omega_2] \subseteq \mathbb{R}$. A function $f : [\omega_1, \omega_2] \longrightarrow \mathbb{R}$ is called an E-convex function is

$$f(tE(\mu_1) + (1-t)E(\mu_2)) \le tf(E(\mu_1)) + (1-t)f(E(\mu_2)), \quad \forall \mu_1, \mu_2 \in [\omega_1, \omega_2], t \in [0,1],$$

for more results on this kind of function, see [14, 22].

Definition 2.3 ([5])**.** U is called (E,F)-convex set if

$$tE(\omega_1) + (1-t)F(\omega_2) \in U, \quad \forall \omega_1, \omega_2 \in U, t \in [0,1].$$

Definition 2.4. A function f is called (E,F)-convex function if U is (E,F)-convex set and

$$f(tE(\omega_1) + (1-t)F(\omega_2)) \le tf(E(\omega_1)) + (1-t)f(F(\omega_2)),$$

$\forall \omega_1, \omega_2 \in U$ and $t \in [0,1]$.

If we replace the space $\mathbb{R}^n$ by a Riemannian manifold N. Assume that (N, f) is a complete m-dimensional Riemannian manifold with Riemannian connection $\bigtriangledown$. Given a piecewise C^1 path $\gamma : [\omega_1, \omega_2] \longrightarrow N$ joining χ_1 to χ_2, that is, $\gamma(\omega_1) = \chi_2$ and $\gamma(\omega_2) = \chi_1$, the length of γ is defined by

$$L(\gamma) = \int_{u_1}^{u_2} \|\acute{\gamma}(\lambda)\|_{\gamma(\lambda)} \, \mathrm{d}\lambda.$$

For any two points $\chi_1, \chi_2 \in N$, we define

$$d(\chi_1, \chi_2) = \inf\{L(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ path joining } \chi_1 \text{ to } \chi_2\}.$$

Then d is a metric which induces the original topology on N.

Every Riemannian manifold there is a unique determined Riemannian connection, called a Levi-Civita connection, denoted by $\bigtriangledown_{A_1} A_2$, for any vector fields $A_1, A_2 \in N$. Also, a smooth path γ is a geodesic if and only if its tangent vector is a parallel vector field along the path γ, *i.e.*, γ satisfies the equation $\bigtriangledown_{\gamma'} \gamma' = 0$. Any path γ joining ω_1 and ω_2 in N such that $L(\gamma) = d(\omega_1, \omega_2)$ is a geodesic and is called a minimal geodesic. Finally, let N as a C^∞ complete n-dimensional Riemannian manifold with metric g and Levi-Civita connection $\bigtriangledown$. Moreover, considering that the points $\omega_1, \omega_2 \in N$ and $\gamma \colon [0,1] \longrightarrow N$ is a geodesic joining ω_1, ω_2, *i.e.*, $\gamma_{\omega_1,\omega_2}(0) = \omega_2$ and $\gamma_{\omega_1,\omega_2}(1) = \omega_1$.

Definition 2.5 ([21])**.** A set U is totally convex if U contains every geodesic $\gamma_{\omega_1,\omega_2}$ of N whose end points ω_1 and ω_2 are in U.

Definition 2.6 ([21])**.** A subset $U \subseteq N$ is called totally convex if and only if U contains every geodesic $\gamma_{\omega_1,\omega_2}$ of N whose endpoints ω_1 and ω_2 are in U.

Definition 2.7 ([21])**.** A function $f : U \subset N \longrightarrow \mathbb{R}$ is called geodesic convex if and only if for all geodesic arcs $\gamma_{\omega_1,\omega_2}$, then

$$f(\gamma_{\omega_1,\omega_2}(t)) \le tf(\omega_1) + (1-t)f(\omega_2)$$

for each $\omega_1, \omega_2 \in U$ and $t \in [0,1]$.

The notion of a geodesic E-convex function on a complete Riemannian manifold has been discussed in [4, 8, 13, 14, 16].

Definition 2.8 ([4])**.** A set $U \subset N$ is geodesic E-convex where $E : N \longrightarrow N$, iff there exists a unique geodesic $\gamma_{E(\omega_1),E(\omega_2)}(t)$ of length $d(\omega_1, \omega_2)$ which belong to U for every $\omega_1, \omega_2 \in U$ and $t \in [0,1]$.

Definition 2.9 ([4])**.** A function $f : U \longrightarrow \mathbb{R}$ is called geodesic E-convex if U is geodesic E-convex set and

$$f(\gamma_{E(\omega_1),E(\omega_2)}) \le tf(E(\omega_1)) + (1-t)f(E(\omega_2)), \quad \forall \omega_1, \omega_2 \in U, t \in [0,1].$$

The next section is devoted to the study of some properties of (E,F)-convex functions like Hermite–Hadamard-type inequalities. In Section 4, the concepts of geodesic (E,F)-convex set and geodesic (E,F)-convex function on N are introduced. Also, some properties of the geodesic (E,F)-convex function are given.

3. Some properties of (E, F)-convex functions

The Hadamard-type inequality for E-convex given in [15] is as follows:

Theorem 3.1. *Assume that $E : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous increasing function and assume that $\omega_1, \omega_2 \in J$ with $\omega_1 < \omega_2$. Assume that $f : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is an E-convex function on $[\omega_1, \omega_2]$, then*

$$f\left(\frac{E(\omega_1) + E(\omega_2)}{2}\right) \leq \frac{1}{E(\omega_2) - E(\omega_1)} \int_{E(\omega_1)}^{E(\omega_2)} f(E(t))\,\mathrm{d}E(t) \leq \frac{f(E(\omega_1)) + f(E(\omega_2))}{2}.$$

Publications [2,7,12,18,23] are recommended for readers interested in generalizations of the Hadamard-type inequality.

Now, we present the Hermite–Hadamard-type inequalities for (E, F)-convex as follows:

Theorem 3.2. *Assume that $E, F : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ are continuous increasing functions and assume that $\omega_1, \omega_2 \in J$ with $\omega_1 < \omega_2$. Assume that $f : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is an (E, F)-convex function on $[\omega_1, \omega_2]$, then*

$$f\left(\frac{E(\omega_1) + F(\omega_2)}{2}\right) \leq \frac{1}{F(\omega_2) - E(\omega_1)} \int_{E(\omega_1)}^{F(\omega_2)} f(x)\,\mathrm{d}x \leq \frac{f(E(\omega_1)) + f(F(\omega_2))}{2}. \tag{3.1}$$

Proof. Since f is (E, F)-convex function, then

$$f(tE(\omega_1) + (1-t)F(\omega_2)) \leq tf(E(\omega_1)) + (1-t)f(F(\omega_2)), \quad \forall \omega_1, \omega_2 \in U, t \in [0, 1]. \tag{3.2}$$

Put $t = \frac{1}{2}$, then

$$\begin{aligned} f\left(\frac{E(\omega_1) + F(\omega_2)}{2}\right) &= f\left(\frac{tE(\omega_1) + (1-t)F(\omega_2)}{2} + \frac{(1-t)E(\omega_1) + tF(\omega_2)}{2}\right) \\ &\leq \frac{1}{2}[f(tE(\omega_1) + (1-t)F(\omega_2)) + f((1-t)E(\omega_1) + tF(\omega_2))]. \end{aligned} \tag{3.3}$$

Integrating both sides of (3.3) with respect to t over $(0, 1)$, it follows that

$$f\left(\frac{E(\omega_1) + F(\omega_2)}{2}\right) \leq \frac{1}{2}\left[\int_0^1 f(tE(\omega_1) + (1-t)F(\omega_2))\,\mathrm{d}t + \int_0^1 f((1-t)E(\omega_1) + tF(\omega_2))\,\mathrm{d}t\right].$$

In the first integral, we put $x = tE(\omega_1) + (1-t)F(\omega_2)$ and in the second integral we also put $x = (1-t)E(\omega_1) + tF(\omega_2)$, then

$$f\left(\frac{E(\omega_1) + F(\omega_2)}{2}\right) \leq \frac{1}{F(\omega_2) - E(\omega_1)} \int_{E(\omega_1)}^{F(\omega_2)} f(x)\,\mathrm{d}x. \tag{3.4}$$

Now, we prove the second inequality of (3.1) by integrating both sides of the inequality (3.2) with respect to t over $(0, 1)$, then we obtain

$$\int_0^1 f(tE(\omega_1) + (1-t)F(\omega_2))\,\mathrm{d}t \leq \frac{1}{2}[f(E(\omega_1)) + f(F(\omega_2))].$$

Let $x = tE(\omega_1) + (1-t)F(\omega_2)$, then

$$\frac{1}{F(\omega_2) - E(\omega_1)} \int_{E(\omega_1)}^{F(\omega_2)} f(x)\,\mathrm{d}x \leq \frac{1}{2}[f(E(\omega_1)) + f(F(\omega_2))]. \tag{3.5}$$

From inequalities (3.4) and (3.5), we get the result. □

Theorem 3.3. *Assume that $f : U \longrightarrow \mathbb{R}$ is (E,F)-convex function on U, then the following inequality holds:*

$$\frac{1}{F(\omega_2)+E(\omega_1)}\int_{E(\omega_1)}^{F(\omega_2)} f(x)f(E(\omega_1)+F(\omega_2)-x)\,\mathrm{d}x \leq \frac{1}{6}\big[f^2(E(\omega_1))+f^2(F(\omega_2))\big] + \frac{2}{3}f(E(\omega_1))f(F(\omega_2)).$$

Proof. Since f is (E,F)-convex function, then

$$f(tE(\omega_1)+(1-t)F(\omega_2)) \leq tf(E(\omega_1))+(1-t)f(F(\omega_2)), \quad \forall \omega_1,\omega_2 \in U, t\in[0,1] \tag{3.6}$$
$$f((1-t)E(\omega_1)+tF(\omega_2)) \leq (1-t)f(E(\omega_1))+tf(F(\omega_2)), \quad \forall \omega_1,\omega_2 \in U, t\in[0,1]. \tag{3.7}$$

Multiplying both sides of (3.6) by (3.7), we have

$$\begin{aligned} f(tE(\omega_1)&+(1-t)F(\omega_2))f((1-t)E(\omega_1)+tF(\omega_2)) \\ &\leq t^2 f(E(\omega_1))f(F(\omega_2))+(1-t)^2 f(E(\omega_1))f(F(\omega_2)) \\ &\quad + t(1-t)f^2(E(\omega_1))+t(1-t)f^2(F(\omega_2)) \\ &= t(1-t)\big[f^2(E(\omega_1))+f^2(F(\omega_2))\big]+\big(t^2+(1-t)^2\big)f(E(\omega_1))f(F(\omega_2)). \end{aligned} \tag{3.8}$$

Integration inequality (3.8) with respect to t over $(0,1)$, then

$$\int_0^1 f(tE(\omega_1)+(1-t)F(\omega_2))f((1-t)E(\omega_1)+tF(\omega_2))\,\mathrm{d}t \leq \frac{1}{6}\big[f^2(E(\omega_1))+f^2(F(\omega_2))\big] + \frac{2}{3}f(E(\omega_1))f(F(\omega_2)). \tag{3.9}$$

We get the result if we put $x = tE(\omega_1)+(1-t)F(\omega_2)$. □

Theorem 3.4. *Assume that $f_1 : U \longrightarrow \mathbb{R}$ and $f_1 : U \longrightarrow \mathbb{R}$ are (E,F)-convex functions, then the following inequality holds:*

$$\frac{3}{F(\omega_2)-E(\omega_1)}\int_{E(\omega_1)}^{F(\omega_2)} f_1(x)f_2(x)\,\mathrm{d}x \leq f_1(E(\omega_1))f_2(E(\omega_1))+f_1(F(\omega_2))f_2(F(\omega_2)) + \frac{1}{2}[f_1(E(\omega_1))f_2(F(\omega_2))+f_1(F(\omega_2))f_2(E(\omega_1))].$$

Proof. Since f_1 and f_2 are (E,F)-convex functions, then

$$f_1(tE(\omega_1)+(1-t)F(\omega_2)) \leq tf_1(E(\omega_1))+(1-t)f_1(F(\omega_2)), \quad \forall \omega_1,\omega_2 \in U, t\in[0,1] \tag{3.10}$$
$$f_2((1-t)E(\omega_1)+tF(\omega_2)) \leq (1-t)f_2(E(\omega_1))+tf_2(F(\omega_2)), \quad \forall \omega_1,\omega_2 \in U, t\in[0,1]. \tag{3.11}$$

Multiplying both sides of (3.10) by (3.11), we have

$$\begin{aligned} f_1(tE(\omega_1)&+(1-t)F(\omega_2))f_2((1-t)E(\omega_1)+tF(\omega_2)) \\ &\leq t^2 f_1(E(\omega_1))f_2(F(\omega_2))+(1-t)^2 f_1(E(\omega_2))f_2(F(\omega_2)) \\ &\quad + t(1-t)[f(E(\omega_1))f_2(F(\omega_2))+f_1(F(\omega_2))f_2(E(\omega_1))]. \end{aligned} \tag{3.12}$$

Integration inequality (3.12) with respect to t over $(0,1)$, then

$$\int_0^1 f_1(tE(\omega_1)+(1-t)F(\omega_2))f_2((1-t)E(\omega_1)+tF(\omega_2))\,\mathrm{d}t \leq \frac{1}{3}[f_1(E(\omega_1))f_2(E(\omega_1))+f_1(F(\omega_2))f_2(F(\omega_2))] + \frac{1}{6}[f_1(E(\omega_1))f_2(F(\omega_2))+f_1(F(\omega_2))f_2(E(\omega_1))].$$

If we put $x = tE(\omega_1)+(1-t)F(\omega_2)$, we get the result. □

4. Geodesic(E, F)-convex functions

In this section, we introduce a new geodesic convexity on a Riemannian manifold that is called a geodesic (E, F)-convex function and study some of its properties.

Definition 4.1. Assume that $E, F : N \longrightarrow N$ are two mappings. A subset U of N is called geodesic (E, F)-convex iff there exists a unique geodesic $\gamma_{E(\omega_1),F(\omega_2)}(t)$ of length $d(\omega_1, \omega_2)$, which belongs to U, $\forall \omega_1, \omega_2 \in U$ and $t \in [0, 1]$.

Proposition 4.2. *If a set U is geodesic (E, F)-convex set. Then, $E(U) \subseteq U$ and $F(U) \subseteq U$.*

Proof. Since U is geodesic (E, F)-convex set, then $\gamma_{E(\omega_1),F(\omega_2)}(t) \in U, \forall \omega_1, \omega_2 \in U$ and $t \in [0, 1]$. When $t = 1$, then we have $E(\omega_1) \in \omega_1$, *i.e.*, $E(U) \subseteq U$. Also, when $t = 0$, then we have $F(\omega_2) \in U$, *i.e.*, $F(U) \subseteq U$. □

Proposition 4.3. *If $E(U) \cup F(U)$ is convex and $E(U) \cup F(U) \subseteq U$, then U is geodesic (E, F)-convex.*

Proof. Let $\omega_1, \omega_2 \in U$, then $E(\omega_1), F(\omega_1) \in E(U) \cup F(U)$. Since $E(U) \cup F(U)$ are convex, then $\gamma_{E(\omega_1),F(\omega_2)}(t) \in E(U) \cup F(U) \subseteq U, \forall t \in [0, 1]$, that means U is geodesic (E, F)-convex. □

Example 4.4. Assume that U is given as in Figure 1, E is a mapping from U to white cat and F is a mapping from U to black cat. Then U is neither geodesic E-convex nor geodesic F-convex, since there is $\omega_1, \omega_2 \in U$ where $\gamma_{E(\omega_1),E(\omega_2)}(t) \notin U$, also $\gamma_{F(\omega_1),F(\omega_2)}(t) \notin U$, on the other hand $\gamma_{E(\omega_1),f(\omega_2)}(t) \in U, \forall \omega_1, \omega_2 \in U$ which gives that U is geodesic (E, F)-convex.

Theorem 4.5. *If $(U_i)_{i\in I}$ is an arbitrary collection of geodesic (E, F)-convex subsets of N with respect to $E : N \longrightarrow N$ and $F : N \longrightarrow N$, then their intersection $\cap_{i\in I} U_i$ is a geodesic (E, F)-convex subset of N.*

Proof. Assume that $(U_i)_{i\in I}$ is a collection of geodesic (E, F)-convex. If $\cap_{i\in I} U_i = \phi$, then the result is obvious. Now, let $\omega_1, \omega_2 \in \cap_{i\in I} U_i$, then $\omega_1, \omega_2 \in U_i, \forall i$. Hence, $\gamma_{E(\omega_1),F(\omega_2)}(t) \in U_i, \forall i, t \in [0, 1]$, which implies that $\gamma_{E(\omega_1),F(\omega_2)}(t) \in \cap_{i\in I} U_i, t \in [0, 1]$. □

Remark 4.6. The above theorem is not true in general for the union of geodesic (E, F)-convex sunsets of N.

Lemma 4.7. *Assume that $U \subseteq N$ is geodesic (E_1, F_1)-convex and geodesic (E_1, F_1)-convex set. Then U is geodesic $(E_1 \circ E_2, F_1 \circ F_2)$-convex set.*

Proof. Consider U is geodesic (E_1, F_1)-convex and geodesic (E_1, F_1)-convex subset of N, and $\omega_1, \omega_2 \in U$. Assume, on the contrary, that there is $t \in [0, 1]$ such that $\gamma_{(E_1\circ E_2)(\omega_1),(F_1\circ F_2)(\omega_2)}(t) \notin U$. Put $\rho_1 = E_2(\omega_1), \rho_2 = F_2(\omega_2)$, then by Proposition 4.2, we have $\rho_1, \rho_2 \in U$, that is $\gamma_{E_1(\rho_1),F_1(\rho_2)}(t) \in U$ which is contradicts the assumption. Hence, U is geodesic $(E_1 \circ E_2, F_1 \circ F_2)$-convex set. □

Definition 4.8. Assume that $U \times \mathbb{R} \subseteq N \times \mathbb{R}$, $E, F : N \longrightarrow N$ and $E^*, F^* : \mathbb{R} \longrightarrow \mathbb{R}$. The set $U \times \mathbb{R}$ is called geodesic $(E, F) \times (E^*, F^*)$-convex, if

$$\big(\gamma_{E(\omega_1),F(\tau_1)}(t), tE^*(\omega_2) + (1 - t)F^*(\tau_2)\big) \in U \times \mathbb{R}$$

$\forall (\omega_1, \omega_2), (\tau_1, \tau_2) \in U \times \mathbb{R}$ and $t \in [0, 1]$.

A characterization between geodesic (E, F)-convex of $U \subseteq N$ and $U \times \mathbb{R}$ is given in the next proposition.

Proposition 4.9. *A is geodesic (E, F)-convex iff $U \times \mathbb{R}$ is geodesic $(E, F) \times (E^*, F^*)$-convex*

FIGURE 1. U is a geodesic (E, F)-convex set.

Proof. For all $\omega_1, \tau_1 \in U, \omega_2, \tau_2 \in \mathbb{R}$ and $t \in [0,1]$, we have $\gamma_{E(\omega_1),F(\tau_1)}(t) \in U$ and $tE^*(\omega_2) + (1-t)F^*(\tau_2) \in \mathbb{R}$. Hence,

$$\left(\gamma_{E(\omega_1),F(\tau_1)}(t), tE^*(\omega_2) + (1-t)F^*(\tau_2)\right) \in U \times \mathbb{R},$$

then $U \times \mathbb{R}$ is geodesic $(E, F) \times (E^*, F^*)$-convex. By using the same method, we can obtain other direction. □

The following definition is generalized from the definition of (E, F)-convex function which is called a geodesic (E, F)-convex function on a geodesic (E, F)-convex sumset of a Riemannian manifold.

Definition 4.10. Let $U \subseteq N$ be a geodesic (E, F)-convex set. A real-valued function $f : U \longrightarrow \mathbb{R}$ is called a geodesic (E, F)-convex function iff

$$f\left(\gamma_{E(\omega_1),F(\omega_2)}(t)\right) \leq tf(E(\omega_1)) + (1-t)f(F(\omega_2)), \forall \omega_1, \omega_2 \in U, t \in [0,1]. \tag{4.1}$$

If the inequality above is strict $\forall \omega_1, \omega_2 \in U, E(\omega_1) \neq F(\omega_2)$ for all $t \in [0,1]$, then f is called strictly geodesic (E, F)-convex.

The following remark shows that some special cases of the geodesic (E, F)-convex function.

Remark 4.11. (1) If N is 1-dimension Eucledian space, then f is called (E, F)-convex function [5].
(2) If $E = F$, then f is called geodesic E-convex function [4].
(3) If $E = F = I$, then f is called geodesic convex function [21].

Example 4.12. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$f(x) = \begin{cases} 2, & \text{if } a \in [0,2], \\ 1, & \text{if otherwise.} \end{cases}$$

Let $E, F : \mathbb{R} \longrightarrow \mathbb{R}$ be given as $E(\omega_1) = 0$ and $F(\omega_1) = \frac{1}{2}$. Assume the geodesic γ is defined as

$$\gamma_{E(\omega_1),F(\omega_2)}(t) = \begin{cases} \omega_2 + t(\omega_1 - \omega_2), & \text{if } ab \geq 0, \\ \omega_2 + t(\omega_2 - \omega_1), & \text{if } ab < 0, \end{cases}$$

where $t \in (0,1)$. Then,

$$f(\gamma_{E(\omega_1),F(\omega_2)}) \leq f(F(\omega_2) + t(E(\omega_1) - F(\omega_2))),$$

$\forall \omega_1, \omega_2 \in \mathbb{R}$, hence f is geodesic (E, F)-convex function.

Next, some properties of geodesic (E, F)-convex functions are given which remain $U \subseteq N$ is geodesic (E, F)-convex set unless we mention otherwise.

Theorem 4.13. *If $f_i : N \longrightarrow \mathbb{R}, i = 1, 2, \ldots, m$ are geodesic (E, F)-convex functions. Then, $f = \sum_{i=1}^{m} \eta_i f_i$ is geodesic (E, F)-convex function on U, $\forall \eta_i \in \mathbb{R}, \eta_i \geq 0, i = 1, 2, \ldots, m$.*

Proof. Since f_i are geodesic (E,F)-convex functions for all i, then

$$f_i\Big(\gamma_{E_{(\omega_1)},F(\omega_2)}\Big) \leq tf_i(E(\omega_1)) + (1-t)f_i(F(\omega_2)), \quad \forall i, t \in [0,1],$$

then

$$\eta_i f_i\Big(\gamma_{E_{(\omega_1)},F(\omega_2)}\Big) \leq t\eta_i f_i(E(\omega_1)) + (1-t)\eta_i f_i(F(\omega_2))$$

or

$$\sum_{i=1}^{n} \eta_i f_i\Big(\gamma_{E_{(\omega_1)},F(\omega_2)}\Big) \leq t\sum_{i=1}^{n} \eta_i f_i(E(\omega_1)) + (1-t)\sum_{i=1}^{n} \eta_i f_i(F(\omega_2)).$$

That is the result. □

Proposition 4.14. *Assume that $f_i : U \longrightarrow \mathbb{R}, \forall i \in I$ is a family of above-bounded and geodesic (E,F)-convex function on U. Then the function $f : U \longrightarrow \mathbb{R}$ which is defined as $f(\omega_1) = \sup_{i\in I} f_i(\omega_1)$ is also geodesic (E,F)-convex function on U.*

Proof. For all $\omega_1, \omega_2 \in U$ and $t \in [0,1]$, we have

$$\begin{aligned} f\big(\gamma_{E(\omega_1),F(\omega_2)}(t)\big) &= \sup_i f_i\big(\gamma_{E(\omega_1),F(\omega_2)}(t)\big) \\ &\leq \sup_i (tf_i(E(\omega_1)) + (1-t)f_i(F(\omega_2))) \\ &= t\sup_i f_i(E(\omega_1)) + (1-t)\sup_i f_i(F(\omega_2)), \\ &= tf(E(\omega_1)) + (1-t)f(F(\omega_2)). \end{aligned}$$

Hence, f is a geodesic (E,F)-convex function. □

Proposition 4.15. *Assume that f is geodesic (E,F)-convex function on U and $H : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing convex function, then Hof is a geodesic (E,F)-convex function on U.*

Proof. From the assumption

$$f\Big(\gamma_{E_{(\omega_1)},F(\omega_2)}\Big) \leq tf(E(\omega_1)) + (1-t)f(F(\omega_2)), \quad \forall t \in [0,1].$$

Now,

$$(Hof)\Big(\gamma_{E_{(\omega_1)},F(\omega_2)}\Big) \leq Ho(tf(E(\omega_1)) + (1-t)f(F(\omega_2))), \quad \forall t \in [0,1].$$

Since H is non-decreasing convex, then

$$(Hof)\Big(\gamma_{E_{(\omega_1)},F(\omega_2)}\Big) \leq t(Hof)(E(\omega_1)) + (1-t)(Hof)(F(\omega_2)),$$

that means Hof is geodesic (E,F)-convex function on U. □

Theorem 4.16. *If $f : U \longrightarrow \mathbb{R}$ is a geodesic (E,F)-convex function on U, then the level set $G_\mu = \{\omega : \omega \in U, f(\omega) \leq \mu\}$ is geodesic (E,F)-convex for each $\mu \in \mathbb{R}$.*

Proof. Since f is geodesic (E,F)-convex function on U, for all $\omega_1, \omega_2 \in U$, we have $E(\omega_1), F(\omega_2) \in U$

$$f\Big(\gamma_{E_{(\omega_1)},F(\omega_2)}\Big) \leq tf(E(\omega_1)) + (1-t)f(F(\omega_2)) \leq t\mu + (1-t)\mu = \mu,$$

this implies that $\gamma_{E(\omega_1),F(\omega_2)} \subseteq G_\mu$ and G_μ is geodesic (E,F)-convex set. □

Corollary 4.17. *Assume that $f_i : U \longrightarrow \mathbb{R}$ are geodesic (E,F)-convex functions on U, then the set $G = \{\omega : \omega \in U, f_i(\omega) \leq 0, \forall i\}$ is geodesic (E,F)-convex.*

The proof of this corollary is directly from Proposition 4.2 and Theorem 4.16.

5. Conclusions

In this work, geodesic (E,F)-convex sets and geodesic (E,F)-functions on Riemannian manifold are introduced. Some properties of this type of convexity are established.

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