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# SUN TOUGHNESS AND PATH-FACTOR UNIFORM GRAPHS

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**Abstract.** A path-factor is a spanning subgraph $F$ of $G$ such that each component of $F$ is a path of order at least two. Let $k$ be an integer with $k \geq 2$. A $P_{\geq k}$-factor is a spanning subgraph of $G$ whose components are paths of order at least $k$. A graph $G$ is called a $P_{\geq k}$-factor covered graph if for any edge $e$ of $G$, $G$ admits a $P_{\geq k}$-factor covering $e$. A graph $G$ is called a $P_{\geq k}$-factor uniform graph if for any two distinct edges $e_1$ and $e_2$ of $G$, $G$ has a $P_{\geq k}$-factor covering $e_1$ and excluding $e_2$. In this article, we claim that (i) a 4-edge-connected graph $G$ is a $P_{\geq 3}$-factor uniform graph if its sun toughness $s(G) \geq 1$; (ii) a 4-connected graph $G$ is a $P_{\geq 3}$-factor uniform graph if its sun toughness $s(G) > \frac{4}{5}$.

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## 1. Introduction

All graphs discussed here are finite, undirected and loopless, and have no multiple edges. Let $G$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. An edge joining vertices $u$ and $v$ is denoted by $uv$. For a vertex $v$ of $G$, the degree of $v$ in $G$ is denoted by $d_G(v)$. For $X \subseteq V(G)$ and $E' \subseteq E(G)$, we denote by $G - X$ the subgraph derived from $G$ by removing vertices in $X$ together with the edges incident to vertices in $X$, and by $G - E'$ the subgraph obtained from $G$ by deleting all edges in $E'$. A set $X \subseteq V(G)$ is called an independent set of $G$ if no two vertices in $X$ are adjacent to each other. We use $\kappa(G)$, $\lambda(G)$ and $\omega(G)$ to denote the vertex connectivity, the edge connectivity and the number of connected components of $G$, respectively. We denote the path and the complete graph with $n$ vertices by $P_n$ and $K_n$, respectively. Let $G_1$ and $G_2$ be two graphs. Then the union of $G_1$ and $G_2$ is denoted by $G_1 \cup G_2$, and the join of $G_1$ and $G_2$ is denoted by $G_1 \vee G_2$.

We first introduce two parameters for a graph, namely, the binding number and the isolated toughness. The binding number of $G$ is defined by Woodall [9] as

$$bind(G) = min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

The isolated toughness of $G$ is defined by Yang *et al.* [10] as

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$$I(G) = \min\left\{\frac{|X|}{i(G-X)} : X \subseteq V(G), i(G-X) \geq 2\right\}$$

if $G$ is not a complete graph; otherwise, $I(G) = +\infty$.

The relationships between binding number, isolated toughness and graph factors can be found in [2, 6, 17, 24, 25]. Many other results on graph factors can be discovered in [7, 11, 14–16, 21].

A path-factor is a spanning subgraph $F$ of $G$ such that each component of $F$ is a path of order at least two. Let $k$ be an integer with $k \geq 2$. A $P_{\geq k}$-factor is a spanning subgraph of $G$ whose components are paths of order at least $k$. In order to characterize a graph possessing a $P_{\geq 3}$-factor, Kaneko [3] introduced the concept of a sun. A spanning subgraph $F$ of $G$ is called a 1-factor if $d_F(v) = 1$ for all $v \in V(G)$. A graph $H$ is called a factor-critical graph if $H - v$ admits a 1-factor for every $v \in V(H)$. Let $H$ be a factor-critical graph with vertex set $V(H) = \{v_1, v_2, \cdots, v_n\}$. By adding new vertices $\{u_1, u_2, \cdots, u_n\}$ together with new edges $\{v_i u_i : 1 \leq i \leq n\}$ to $H$, we acquire a new graph, which is called a sun. According to Kaneko, $K_1$ and $K_2$ are also suns. Usually, $K_1$ and $K_2$ are called small suns and the others are called big suns (with at least six vertices). The number of sun components of $G$ is denoted by $sun(G)$.

Kano *et al.* [5] acquired a sufficient condition for the existence of a $P_{\geq 3}$-factor in a graph. Wang and Zhang [8] gave a result on the existence of a $P_{\geq 3}$-factor in a graph. Zhou *et al.* [13, 20] derived some results on $P_{\geq 3}$-factors in graphs. Kaneko [3] provided a necessary and sufficient condition for a graph possessing a $P_{\geq 3}$-factor. Kano *et al.* [4] posed its shorter proof.

**Theorem 1.** *([3, 4]). A graph $G$ possesses a $P_{\geq 3}$-factor if and only if*

$$sun(G - X) \leq 2|X|$$

*for any vertex subset $X$ of $G$.*

A graph $G$ is called a $P_{\geq k}$-factor covered graph if for any edge $e$ of $G$, $G$ admits a $P_{\geq k}$-factor covering $e$, which was first defined by Zhang and Zhou [12]. Furthermore, they [12] presented a characterization for a graph to be a $P_{\geq 3}$-factor covered graph, which is shown in the following.

**Theorem 2.** *([12]). A connected graph $G$ is a $P_{\geq 3}$-factor covered graph if and only if*

$$sun(G - X) \leq 2|X| - \varepsilon(X)$$

*for any vertex subset $X$ of $G$, where $\varepsilon(X)$ is defined by*

$$\varepsilon(X) = \begin{cases} 2, \textit{if } X \textit{ is not an independent set;} \\ 1, \textit{if } X \textit{ is a nonempty independent set and } G - X \textit{ has} \\ \quad \textit{a non} - \textit{sun component;} \\ 0, \textit{otherwise.} \end{cases}$$

A graph $G$ is called a $P_{\geq k}$-factor uniform graph if for any two distinct edges $e_1$ and $e_2$ of $G$, $G$ has a $P_{\geq k}$-factor covering $e_1$ and excluding $e_2$, which was first defined by Zhou and Sun [18]. Furthermore, they [18] derived a binding number condition for a graph to be a $P_{\geq 3}$-factor uniform graph. Gao and Wang [1] improved Zhou and Sun's result on $P_{\geq 3}$-factor uniform graphs. Zhou and Bian [19] showed two sufficient conditions for the existence of a $P_{\geq 3}$-factor uniform graph. Zhou *et al.* [22] provided an isolated toughness condition for a graph to possess a $P_{\geq 3}$-factor uniform graph. The sun toughness of a graph $G$ is denoted by $s(G)$ and defined by

$$s(G) = \min\left\{\frac{|X|}{sun(G-X)} : X \subseteq V(G), sun(G-X) \geq 2\right\}$$

if $G$ is not a complete graph; otherwise, $s(G) = +\infty$. Zhou *et al.* [23] presented a sun toughness condition for a graph to be a $P_{\geq 3}$-factor uniform graph, which is stated as follows.

**Theorem 3.** *([23]). Let $G$ be a 3-edge-connected graph. Then $G$ is a $P_{\geq 3}$-factor uniform graph if its sun toughness $s(G) > 1$.*

The purpose of this paper is to weaken the sun toughness condition in Theorem 3 by assuming that $G$ is 4-edge-connected or $G$ is 4-connected.

**Theorem 4.** *Let $G$ be a 4-edge-connected graph. Then $G$ is a $P_{\geq 3}$-factor uniform graph if its sun toughness $s(G) \geq 1$.*

**Theorem 5.** *Let $G$ be a 4-connected graph. Then $G$ is a $P_{\geq 3}$-factor uniform graph if its sun toughness $s(G) > \frac{4}{5}$.*

## 2. The proofs of main theorems

*Proof of Theorem 4.* If $G$ is a complete graph, then $G$ is obviously a $P_{\geq 3}$-factor uniform graph. Hence, we may assume that $G$ is not a complete graph. Since $G$ is 4-edge-connected, we admit $|V(G)| \geq 5$.

We proceed to verify Theorem 4 by contradiction. Suppose that there exists an edge $e = uv$ in $G$ such that $G' = G - e$ is not a $P_{\geq 3}$-factor covered graph. Then by Theorem 2, we have

$$sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \tag{1}$$

for some vertex subset $X$ of $G'$.

**Claim 1.** $|X| = 3$.

*Proof.* If $|X| = 0$, then $\varepsilon(X) = 0$. According to (1), we obtain $sun(G') \geq 1$. On the other hand, since $G'$ is connected, we admit $sun(G') \leq \omega(G') = 1$. Hence, $sun(G') = \omega(G') = 1$, which implies that $G'$ is a sun. Note that $|V(G')| = |V(G)| \geq 5$. Therefore, $G'$ is a big sun with at least six vertices. Clearly, $G'$ has at least three vertices with degree 1, and so $G$ admits at least one vertex with degree 1, which contradicts that $G$ is 4-edge-connected.

If $1 \leq |X| \leq 2$, then it follows from (1) and $\varepsilon(X) \leq |X|$ that $sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq |X| + 1 \geq 2$, which implies that $G' - X$ has at least two sun components. If $G' - X$ admits a sun component $K_1 = \{w\}$, then $d_{G'-X}(w) = 0$. Thus, $d_G(w) \leq d_{G'}(w) + 1 \leq d_{G'-X}(w) + |X| + 1 = |X| + 1 = 2$, which contradicts our assumption that $G$ is 4-edge-connected. If $G' - X$ does not admit a sun component $K_1$, then $G' - X$ has at least four vertices with degree 1, and so $G$ has at least two vertices with degree 1, which contradicts that $G$ is 4-edge-connected.

If $|X| \geq 4$, then $\varepsilon(X) \leq 2$. By (1), we admit $sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1 \geq 7$, and so $sun(G - X) \geq sun(G' - X) - 2 \geq 2|X| - 3 \geq 5$. Combining this with $s(G) \geq 1$, we obtain

$$1 \leq s(G) \leq \frac{|X|}{sun(G - X)} \leq \frac{|X|}{2|X| - 3} = \frac{1}{2} + \frac{3}{4|X| - 6} \leq \frac{1}{2} + \frac{3}{4 \times 4 - 6} = \frac{4}{5},$$

which is a contradiction. This completes the proof of Claim 1. □

Note that $sun(G - X) \geq sun(G' - X) - 2$. The following proof will be divided into two cases.

**Case 1.** $sun(G - X) \geq sun(G' - X) - 1$.

It follows from (1), $\varepsilon(X) \leq 2$, $s(G) \geq 1$ and Claim 1 that

$$1 \leq s(G) \leq \frac{|X|}{sun(G-X)} \leq \frac{|X|}{sun(G'-X)-1} \leq \frac{|X|}{2|X|-\varepsilon(X)} \leq \frac{|X|}{2|X|-2} = \frac{3}{2\times 3-2} = \frac{3}{4},$$

a contradiction.

**Case 2.** $sun(G-X) = sun(G'-X)-2$.

In this case, we may let $e = uv$ join two sun components $H_1$ and $H_2$ of $G'-X$, where $u \in V(H_1)$ and $v \in V(H_2)$. Note that $sun(G-X) = sun(G'-X)-2$. Hence, $H_1 \neq K_1$ or $H_2 \neq K_1$ (otherwise, $sun(G-X) = sun(G'-X)-1$). Without loss of generality, we may assume $H_1 \neq K_1$.

**Subcase 2.1.** $H_1 = K_2$.

Obviously, $sun(G-X-u) = sun(G-X-u-e) = sun(G'-X-u) = sun(G'-X)$. According to (1), $\varepsilon(X) \leq 2$, $s(G) \geq 1$ and Claim 1, we have

$$1 \leq s(G) \leq \frac{|X \cup \{u\}|}{sun(G-X-u)} = \frac{|X|+1}{sun(G'-X)} \leq \frac{|X|+1}{2|X|-\varepsilon(X)+1} \leq \frac{|X|+1}{2|X|-1} = \frac{4}{5},$$

a contradiction.

**Subcase 2.2.** $H_1$ is a big sun component.

We write $R_1$ for the factor-critical graph of $H_1$, and $\exists w \in V(R_1)$ with $uw \in E(H_1)$. Then $sun(G-X-u-V(R_1)\setminus\{w\}) = sun(G'-X-u-V(R_1)\setminus\{w\}) = sun(G'-X)-1+|V(R_1)|$. In view of (1), $\varepsilon(X) \leq 2$, $s(G) \geq 1$ and Claim 1, we derive

$$\begin{aligned} 1 \leq s(G) &\leq \frac{|X \cup \{u\} \cup (V(R_1)\setminus\{w\})|}{sun(G-X-u-V(R_1)\setminus\{w\})} \\ &= \frac{|X|+|V(R_1)|}{sun(G'-X)-1+|V(R_1)|} \leq \frac{|X|+|V(R_1)|}{2|X|-\varepsilon(X)+|V(R_1)|} \\ &\leq \frac{|X|+|V(R_1)|}{2|X|-2+|V(R_1)|} = \frac{3+|V(R_1)|}{4+|V(R_1)|} < 1, \end{aligned}$$

which is a contradiction. The proof of Theorem 4 is complete. □

*Proof of Theorem 5.* If $G$ is a complete graph, then $G$ is clearly a $P_{\geq 3}$-factor uniform graph. Hence, we may assume that $G$ is not a complete graph. Since $G$ is 4-connected, we admit $|V(G)| \geq 6$.

We proceed to demonstrate Theorem 5 by contradiction. Suppose that there exists an edge $e = uv$ in $G$ such that $G' = G-e$ is not a $P_{\geq 3}$-factor covered graph. Then it follows from Theorem 2 that

$$sun(G'-X) \geq 2|X| - \varepsilon(X) + 1 \tag{2}$$

for some vertex subset $X$ of $G'$.

**Claim 2.** $|X| = 3$.

*Proof.* The proof of Claim 2 is similar to that of Claim 1 in Theorem 4. □

In view of (2), $\varepsilon(X) \leq 2$ and Claim 2, we obtain

$$sun(G'-X) \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1 = 5. \tag{3}$$

Note that $sun(G-X) \geq sun(G'-X)-2$. Combining this with (3), we admit

$$\omega(G-X) \geq sun(G-X) \geq sun(G'-X) - 2 \geq 3. \tag{4}$$

On the other hand, since $G$ is 4-connected, it follows from Claim 2 that $\omega(G-X) = 1$, which contradicts (4). We completes the proof of Theorem 5. □

## 3. Remarks

**Remark 1.** Let $H_1, H_2, H_3, H_4$ and $H_5$ be five big suns. We write $R_1$ for the factor-critical graph of $H_1$, and $R_2$ for the factor-critical graph of $H_2$. Let $u \in V(H_1) \setminus V(R_1)$, $w \in V(R_1)$, $v \in V(H_2) \setminus V(R_2)$ and $uw \in E(H_1)$. We denote by $H$ the graph with vertex set $V(H) = V(H_1) \cup V(H_2)$ and edge set $E(H) = E(H_1) \cup E(H_2) \cup \{e\}$, where $e = uw$.

To show that the bound of $s(G)$ in Theorem 4 is sharp, we construct a graph $G = K_3 \vee (H \cup H_3 \cup H_4 \cup H_5)$. Obviously, $G$ is 4-edge-connected, and $s(G) = \frac{|V(K_3)\cup\{u\}\cup(V(R_1)\setminus\{w\})|}{sun(G-(V(K_3)\cup\{u\}\cup(V(R_1)\setminus\{w\})))} = \frac{3+|V(R_1)|}{4+|V(R_1)|} \to 1$ $(|V(R_1)| \to \infty)$. Let $X = V(K_3)$ and $G' = G - e$. Then $\varepsilon(X) = 2$ and

$$sun(G' - X) = 5 > 4 = 2|X| - \varepsilon(X).$$

In view of Theorem 2, $G'$ is not a $P_{\geq 3}$-factor covered graph, and so $G$ is not a $P_{\geq 3}$-factor uniform graph.

**Remark 2.** Now, we show that 4-edge-connected in Theorem 4 cannot be replaced by 3-edge-connected. We construct a graph $G = K_2 \vee (H \cup P_3)$, where $P_3 = v_0v_1v_2$ and $H$ $(\neq K_1)$ is a sun. It is obvious that $G$ is 3-edge-connected and $s(G) = \frac{|V(K_2)\cup\{v_1\}|}{sun(G-(V(K_2)\cup\{v_1\}))} = \frac{3}{3} = 1$. Let $G' = G - e$ for $e \in E(P_3)$ and $X = V(K_2)$. Then $\varepsilon(X) = 2$, and so

$$sun(G' - X) = 3 > 2 = 2|X| - \varepsilon(X).$$

In terms of Theorem 2, $G'$ is not a $P_{\geq 3}$-factor covered graph, and so $G$ is not a $P_{\geq 3}$-factor uniform graph.

**Remark 3.** In what follows, we show that $s(G) > \frac{4}{5}$ and 4-connected in Theorem 5 cannot be replaced by $s(G) \geq \frac{4}{5}$ and 3-connected. We construct a graph $G = K_3 \vee (3K_2 \cup P_3)$, where $P_3 = v_0v_1v_2$. We easily see that $G$ is 3-connected and $s(G) = \frac{|V(K_3)\cup\{v_1\}|}{sun(G-(V(K_3)\cup\{v_1\}))} = \frac{4}{5}$. Let $G' = G - e$ for $e \in E(P_3)$ and $X = V(K_3)$. Then $\varepsilon(X) = 2$, and so

$$sun(G' - X) = 5 > 4 = 2|X| - \varepsilon(X).$$

According to Theorem 2, $G'$ is not a $P_{\geq 3}$-factor covered graph, and so $G$ is not a $P_{\geq 3}$-factor uniform graph.

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