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SINGLE MACHINE SLK/DIF DUE WINDOW ASSIGNMENT PROBLEM WITH GENERAL JOB-DEPENDENT POSITIONAL EFFECT AND OUTSOURCING OPERATIONS

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Abstract. This paper considers the SLK/DIF due window assignment problems on a single machine with a general job-dependent positional effect, where the actual processing time of a job is subject to a job-dependent and positional effect. Production schedulers may decide to outsource/reject some jobs by paying the corresponding costs. Based on whether to accept the tardiness jobs, we consider two different objectives under slack and different due window assignment, respectively. The first objective function is to minimize the weighted sum of earliness, tardiness, due window starting time, due window size, and outsourcing costs, while the second objective function is to minimize the cost function that includes earliness, due window starting time, due window size and outsourcing costs. We study the structural properties of two due window assignment methods and develop polynomial-time solution algorithms for the considered problems. A numerical example proves the advantage of the outsourcing decision and the distinction between two different objectives.

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1. Introduction

Due-date quotation (due window assignment) and production scheduling are the most common and significant problems in modern manufacturing and supply chain management and have received considerable attention from both practitioners and researchers. Integrated due windows assignment and production scheduling (IDWPS) refers to the two types of decisions: sales planning decisions (due windows assignment) and job sequencing decisions (production scheduling). Because of the practical significance of due window assignment, there is increasing attention has been paid to the research of IDWPS. A recent survey [8], contains about 60 references on various models consisting of different due-window cases, different objective functions, etc. However, this survey focuses only on the case where the manufacturer processes all jobs and does not mention due window assignment problems under outsourcing operations considerations. In this paper, we focus on due windows assignment and the single machine scheduling problems with general job-dependent positional effects and outsourcing options.

Keywords. Scheduling, Single machine, Due window assignment, Job-dependent positional effect, outsourcing.

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Today it is often expected that the job(s) are finished in a certain time interval (due window) rather than at a single point in time (due date) in the manufacturing industry. Scheduling problems with due window assignment are a generalization of the classical due date assignment [11]. In the last twenty years, various due date assignment models have been widely studied and mainly include common due date (CON), slack due date (SLK), and different due date (DIF) in the literature. Two comprehensive surveys on various due date assignment models and due date assignment under special conditions on job processing are given by Gordon *et al.* [4,5]. Similarly, there are also three due window assignment models (CON/SLK/DIF due window) in the literature [8]. However, compared to the research of three due date assignment models, most studies focus on the common due window assignment model, which assumes that all jobs are assigned a same due window; the readers may refer to Janiak *et al.* [8].

Due window assignment with a common flow allowance (called slack due window) is first considered in Janiak *et al.* [7]. Under SLK due window assignment model, each job has a job-dependent due window and the due window size is equal for all jobs. For each job, the starting time of due window is equal to its actual processing time plus the job-independent parameter, *i.e.*, $d_j^1 = p_j + q_1$; the completion time of its due window is equal to its actual processing time plus the job-independent parameter, *i.e.*, $d_j^2 = p_j + q_2$. The due window size is $D = q_2 - q_1$, where q_1, q_2 are the decision variables. Mosheiov and Oron [17] studied a single machine scheduling problem of determining the size, starting time of due windows, and a processing schedule to minimize the total costs contained the earliness, tardiness, due window size, and starting time of due window. They introduce an efficient polynomial-time algorithm to solve the problem. Following Mosheiov and Oron [17], Mor and Mosheiov [15] study a single machine scheduling problem with a maintenance activity under SLK due window assignment. They consider several versions of maintenance activity: constant maintenance duration and variable maintenance duration which is dependent on its starting time or position. Ultimately, they prove that all the problems based on all the combinations of these maintenance activity versions can be solved in polynomial time. Yang *et al.* [25] consider the SLK due window assignment problem with learning effect and deteriorating jobs and point out the incorrectness of the solution methods in Wang *et al.* [22]. They present their corrections and provide corrected results. Recent studies in this vein contain Ji *et al.* [9] for group scheduling , Mor and Mosheiov [16] for proportionate flow shop, Yin *et al.* [27] and Li *et al.* [14] which integrate with controllable processing time, Zhao and Tang [30] and Yue and Wan [29] which integrate with deterioration effect, etc.

Compared with CON/SLK due window model, there is little research on scheduling problems with DIF due window assignment although scheduling problems with multiple due window assignment have been studied. For example, Yang *et al.* [24] study a single machine scheduling with multiple due windows and controllable processing times. Li and Zhao [12] and Ji *et al.* [10] studied group scheduling with multiple due windows. As far as we know, the related work of considering DIF due-window assignment is Yue and Wan [29]. They consider the single machine scheduling problems with deteriorating jobs (*i.e.*, the job's actual processing time is dependent on its starting processing time) and SLK/DIF due window assignment. They study the optimality properties for all problems and develop polynomial-time algorithms for solving these problems. In this paper, we consider the single machine scheduling problems with general job-dependent positional effect and outsourcing for SLK/DIF due window assignment, in which the job's actual processing time is dependent on the job itself and its position in a schedule.

To quote satisfactory due dates or due windows, production schedulers may outsource a subset of jobs. Moreover, outsourcing can also reduce the operational costs and the effect of aging effect which often occurs in manufacturing workshops. With the development of cloud manufacturing, outsourcing has become available at the operational level. The use of outsourcing options brings a profound impact on due windows assignment and job scheduling. In the scheduling context, outsourcing is tackled as rejection or order acceptance. And the main aim is to determine whether a job is homemade or not. Scheduling problems with rejection or order acceptance can see in the survey paper [19,20]. Recent studies in this vein contain Zhong and Ou [32], Wang and Truong [21], and so on. Considering the outsourcing operations and varying processing times of jobs, Gordon and Strusevich [3] study a single machine scheduling problem with positionally dependent processing times and due date assignment in which all acceptance jobs are not tardy. They present polynomial-time dynamic programming algorithms for CON/SLK due date assignment methods. Then, Li *et al.* [13] consider the single machine scheduling problem with deteriorating jobs and due date assignment. Yin *et al.* (2014) consider a single-machine scheduling problem with due-date assignment and deteriorating maintenance activities. For the case of accepting tardy jobs, Iranpoor *et al.* [6] explore the novel extensive problem of assigning due-dates to selected jobs and the rate-modifying maintenance. Zhao *et al.* [31] investigate the single machine scheduling problems with position-dependent processing times for the three popular due date assignment. Moreover, Zhao and Tang [30] consider a single machine scheduling with both deterioration and positional effects and SLK due-window assignment. Gerstl *et al.* [2] study the minmax scheduling problem with acceptable lead-times and DIF due dates.

Despite many researchers have studied due date assignment and CON due-window assignment for the scheduling problems with outsourcing, integrated SLK/DIF due window assignment and scheduling problems with outsourcing have not been studied yet. To model a real production system, this paper focuses on SLK/DIF due window assignment in the framework of scheduling with general job-dependent positional effect and outsourcing. We investigate ways to minimize the total costs through a combination of due window assignment, production scheduling, and job outsourcing. According to whether the accepted jobs are allowed to delay, we study two different objective functions. Our main contributions in this paper are (1) we extend the model of due-date assignment methods under outsourcing operations to the new model of SLK/DIF due-window assignment methods under outsourcing operations, (2) we extend the application of outsourcing to the scheduling with general job-dependent positional effect and due-windows assignment, (3) we investigate how to set due windows and determine a processing schedule for the different objective functions.

As a practical application of the scheduling problem under study, we consider the production and delivery of rotten perishable goods in the manufacturing industry. Assume that these goods(jobs) belong to different customers, so the delivery time(due window) to be prescribed for a job has to be negotiated between the manufacturer and customer. Many real-life contracts between suppliers and customers allow some flexibility to the supplier, reflected by such a time window. When a job is finished after the time window, it will incur a tardiness penalty which relies on how late the finished jobs are delivered. When a job is finished before the time window, it will incur an earliness penalty which depends on how early the jobs are produced. Due to the time limit requirements for the rotten perishable goods and the influence of deterioration/learning effect in processing, it is very important and difficult to quote satisfactory due windows and obtain a reasonable job production sequencing. For this purpose, the manufacturer decides to outsource some jobs by paying the corresponding costs. Based on these reasons, the above-described situation can be modeled single machine SLK/DIF due window assignment problems with general job-dependent positional effect and outsourcing operations.

The rest of the paper is organized as follows. The following section gives the problem statements formally. In Section 3, we consider the first objective for SLK/DIF due window assignment and present the corresponding algorithm. In Section 4, we consider the second objective for SLK/DIF due window assignment and present the corresponding algorithm. In Section 5, a numerical example is given to prove the advantage of outsourcing decisions and the distinction between two different objectives. The last section concludes the paper and suggests topics for future research.

2. Problem formulation

Consider a manufacturer production system that consists of n independent jobs, $N = \{J_1, J_2, \ldots, J_n\}$, that are to be processed without preemption on a single machine at time zero. The machine can handle at most one job at a time. Each job J_j has a normal processing time p_j. The actual processing time of the job J_j, if scheduled in position r, is defined as $p^A_{jr} = p_j g_j(r)$, $j, r = 1, 2, \ldots, n$, where $g_j(r)$ is a function that specifies a job-dependent positional effect.

The job is either accepted and processed on the machine or is outsourced. When a job J_j is outsourced, a job-dependent cost e_j will occur. Each job J_j has a unique due window $[d^1_j, d^2_j]$, where $d^1_j \geq 0$ and $d^2_j (d^2_j \geq d^1_j)$ are the due window starting time and completion time of the job J_j respectively. The due window size for job

J_j is defined as $D_j = d_j^2 - d_j^1$. When a common flow allowance is assumed, the due window starting time for job J_j is defined as its actual processing time plus a job-independent parameter q_1, *i.e.*, $d_j^1 = p_{jr}^A + q_1$. Similarly, the due window completion time for job J_j is defined as its actual processing time plus a job-independent parameter q_2, *i.e.*, $d_j^2 = p_{jr}^A + q_2 (q_2 \geq q_1)$. We refer to the case above as SLK due window in short. When each job can be assigned a different due window starting time d_j^1 and completion time d_j^2 with no restrictions, we refer to the case as DIF due window in short.

We split the jobs of set N into two subsets denoted by O and R, where O is the set of accepted jobs and R is the set of outsourced jobs. Let h denotes the number of jobs to be accepted and processed by the manufacturer. In this paper, our objective is to obtain an outsourcing scheme and determine a processing schedule π, the due window starting time d_j^1, and size D_j to minimize the following two objective functions:

$$Z\big(\pi, d_j^1, D_j\big) = \sum_{j \in O} \big(\alpha E_j + \beta T_j + \gamma d_j^1 + \eta D_j\big) + \sum_{j \in R} e_j \tag{1}$$

$$Z\big(\pi, d_j^1, D_j\big) = \sum_{j=1}^{n} \big(\alpha E_j + e_j U_j + \gamma d_j^1 + \eta D_j\big) \tag{2}$$

where $E_j = \max\{0, d_j^1 - C_j\}$ is the earliness of job J_j; $T_j = \max\{0, C_j - d_j^2\}$ is the tardiness for job J_j; U_j is the tardiness indicator variable for job J_j, *i.e.*, $U_j = 1$ if $C_j > d_j^2$ and $U_j = 0$ if $C_j \leq d_j^2$; α, β, γ and η are non-negative parameters representing the cost of one unit of earliness, tardiness, (delaying) the due window starting time, (increasing) the due window size.

For the second objective function, the manufacturer just accepts the early jobs and outsources all the tardy jobs. However, under this circumstance, the manufacturer needs to pay the outsourcing cost along with the due window assignment cost for tardiness jobs. It is quite common in real life, especially when the manufacturer must fulfill its delivery commitment to improving the quality of service or the customer is very important and the delay penalty is very high.

In this paper, we consider the SLK and DIF due window assignment methods for each objective function. Using the traditional notation for describing scheduling problems, we can denote the two sub-problems with the objective function in equation (1) under SLK/DIF due window models as:

$$1|p_j g_j(r), \mathrm{SLK}| \sum_{j \in O} \big(\alpha E_j + \beta T_j + \gamma d_j^1 + \eta D\big) + \sum_{j \in R} e_j \tag{3}$$

$$1|p_j g_j(r), \mathrm{DIF}| \sum_{j \in O} \big(\alpha E_j + \beta T_j + \gamma d_j^1 + \eta D_j\big) + \sum_{j \in R} e_j. \tag{4}$$

And denote the two sub-problems with the objective function in equation (2) under SLK/DIF due window models as:

$$1|p_j g_j(r), \mathrm{SLK}| \sum_{j=1}^{n} \big(\alpha E_j + e_j U_j + \gamma d_j^1 + \eta D\big) \tag{5}$$

$$1|p_j g_j(r), \mathrm{DIF}| \sum_{j=1}^{n} \big(\alpha E_j + e_j U_j + \gamma d_j^1 + \eta D_j\big). \tag{6}$$

In the following, we first consider the total earliness and tardiness with due window assignment (ETDW) under the SLK/DIF due window assignment models, then analyze the total earliness and the number of tardy jobs with due window assignment (ENTDW) under the SLK/DIF due window assignment models.

3. The ETDW problem

Under SLK due window assignment model, the manufacturer needs to determine an outsourcing scheme and a schedule, a due window starting time and a common due window size for the accepted jobs to minimize the total costs.

3.1. The problem $1|p_jg_j(r), \text{SLK}|\sum_{j\in O}(\alpha E_j+\beta T_j+\gamma d_j^1+\eta D)+\sum_{j\in R}e_j$

In this subsection, we show that the problem can be solved in $O(n^4)$ time. First, we consider the special case of this problem, *i.e.*, $O{=}N$. For ease of presentation, we refer to the special case as problem ***P1***: $1|p_jg_j(r), \text{SLK}|\sum_{j=1}^{n}(\alpha E_j+\beta T_j+\gamma d_j^1+\eta D)$. Before developing the result, some structural properties of an optimal schedule are first provided.

Lemma 1. *All the jobs are processed consecutively without idle time from time 0.*

Proof. It is straightforward. See Mosheiov and Oron [17]. □

Lemma 2. *Prior to a certain position in a schedule, all the jobs are early, i.e.,* $C_j \le d_j^1$ *implies* $C_{j-1}\le d_{j-1}^1$ *for all* j*; Starting from a certain position in a schedule, all the jobs are tardy, i.e.,* $C_j\ge d_j^2$ *implies* $C_{j+1}\ge d_{j+1}^2$ *for all* j.

Proof. For a given schedule $\pi=[J_{[1]},J_{[2]},\ldots,J_{[n]}]$, if $C_{[j]}\le d_{[j]}^1$, it follows that $C_{[j-1]}+g_{[j]}(j)p_{[j]}\le g_{[j]}(j)p_{[j]}+q_1\Rightarrow C_{[j-1]}\le q_1$, *i.e.*, $C_{[j-2]}\le q_1$. So we can get $C_{[j-2]}+g_{[j-1]}(j-1)p_{[j-1]}\le q_1+g_{[j-1]}(j-1)p_{[j-1]}$. Thus, we have $C_{[j-1]}\le d_{[j-1]}^1$. Similarly, if $C_{[j]}\ge d_{[j]}^2$, it follows that $C_{[j-1]}+g_{[j]}(j)p_{[j]}\ge g_{[j]}(j)p_{[j]}+q_2\Rightarrow C_{[j-1]}\ge q_2$. So we can get $C_{[j]}\ge q_2$. Therefore, $C_{[j]}+g_{[j+1]}(j+1)p_{[j+1]}\ge q_{[2]}+g_{[j+1]}(j+1)p_{[j+1]}$, *i.e.*, $C_{[j+1]}\ge d_{[j+1]}^2$. □

Lemma 3. *For a given schedule* π*, there exist the optimal values of* q_1 *and* q_2 *which are equal to the completion time of one job separately, i.e.,* $q_1=C_{[k]}$ *and* $q_2=C_{[l]}$.

Proof. Suppose that there exists a schedule π starting at time zero and containing jobs in the kth and lth positions such that $C_{[k]}<q_1<C_{[k+1]}$ and $C_{[l]}<q_2<C_{[l+1]}$. Let $\Delta_1=q_1-C_{[k]}$ and $\Delta_2=q_2-C_{[l]}$, we can get $0<\Delta_1<p_{[k+1]}g_{[k+1]}(k+1)$ and $0<\Delta_2<p_{[l+1]}g_{[l+1]}(l+1)$.

Based on the above assumption, it is easy to verify that the job $J_{[k+1]}$ is earliness and the job $J_{[l+2]}$ is tardiness. Therefore, the minimal total cost as a function of Δ_1 and Δ_2 given by

$$Z(\Delta_1,\Delta_2)=\alpha\sum_{r=1}^{k+1}E_{[r]}+\beta\sum_{r=l+2}^{n}T_{[r]}+\gamma\sum_{r=1}^{n}d_{[r]}^1+\eta\sum_{r=1}^{n}(q_2-q_1).$$

We treat each cost component separately:

$$\begin{aligned}\alpha\sum_{r=1}^{k+1}\left(d_{[r]}^1-C_{[r]}\right)&=\alpha\Big[\Big(\Big(p_{[1]}^A+q_1\Big)-C_{[1]}\Big)+\Big(\Big(p_{[2]}^A+q_1\Big)-C_{[2]}\Big)+\cdots+\Big(\Big(p_{[k+1]}^A+q_1\Big)-C_{[k+1]}\Big)\Big]\\&=\alpha\Big[\Big(\Big(p_{[1]}^A+C_{[k]}\Big)-C_{[1]}+\Delta_1\Big)+\Big(\Big(p_{[2]}^A+C_{[k]}\Big)-C_{[2]}+\Delta_1\Big)+\cdots\\&\quad+\Big(\Big(p_{[k+1]}^A+C_{[k]}\Big)-C_{[k+1]}+\Delta_1\Big)\Big]\\&=\alpha(k+1)\Delta_1+\alpha\sum_{r=1}^{k}rp_{[r]}^A;\end{aligned}$$

$$\beta\sum_{r=l+2}^{n}\left(C_{[r]}-d_{[r]}^2\right)=\beta\Big[\Big(C_{[l+2]}-\Big(p_{[l+2]}^A+q_2\Big)\Big)+\cdots+\Big(C_{[n]}-\Big(p_{[n]}^A+q_2\Big)\Big)\Big]$$

$$= \beta\Big[\Big(C_{[l+2]} - \Big(p^A_{[l+2]} + C_{[l]} + \Delta_2\Big)\Big) + \cdots + \Big(C_{[n]} - \Big(p^A_{[n]} + C_{[l]} + \Delta_2\Big)\Big)\Big]$$

$$= -\beta(n-l+1)\Delta_2 + \beta\sum_{r=l+2}^{n}(n-r)p^A_{[r]};$$

$$\gamma\sum_{r=1}^{n} d^1_{[r]} = \gamma\sum_{r=1}^{n}\Big(p^A_{[r]} + q_1\Big) = \gamma\Big[\Big(p^A_{[1]} + C_{[k]} + \Delta_1\Big) + \Big(p^A_{[2]} + C_{[k]} + \Delta_1\Big) + \cdots + \Big(p^A_{[n]} + C_{[k]} + \Delta_1\Big)\Big]$$

$$= n\gamma\Delta_1 + \gamma\left(\sum_{r=1}^{n} p^A_{[r]} + nC_{[k]}\right);$$

$$\eta\sum_{r=1}^{n}(q_2 - q_1) = n\eta\big(C_{[l]} + \Delta_2 - C_{[k]} - \Delta_1\big) = n\eta(\Delta_2 - \Delta_1) + n\eta\sum_{r=k+1}^{l} p^A_{[r]}.$$

Therefore, we can express the total cost as: $Z(\Delta_1, \Delta_2) = A\Delta_1 + B\Delta_2 + C$, where $A = [\alpha(k+1) + n\gamma - n\eta]$, $B = [n\eta - \beta(n-l-1)]$,

$$C = \alpha\sum_{r=1}^{k} r p^A_{[r]} + \beta\sum_{r=l+2}^{n}(n-r)p^A_{[r]} + \gamma\left(\sum_{r=1}^{k}(n+1)p^A_{[r]} + \sum_{r=k+1}^{n} p^A_{[r]}\right) + n\eta\sum_{r=k+1}^{l} p^A_{[r]}.$$

Clearly, A, B and C are constants and independent of Δ_1 and Δ_2. Following Yang and Yang [23], we conclude that one schedule which has a lower cost can be obtained by either decreasing Δ_1 and/or Δ_2 to zero, or by increasing them to $p_{[k+1]}g_{[k+1]}(k+1)$ and $p_{[l+1]}g_{[l+1]}(l+1)$ respectively. That is to say, there exists an optimal schedule in which q_1 and q_2 coincide with the completion times of some jobs. □

Lemma 4. *For any specified schedule, there exists the optimal* SLK *due window in which the indexes* $k = \max\left(\left\lfloor \frac{n(\eta-\gamma)}{\alpha}\right\rfloor, 0\right)$ *and* $l = \max\left(\left\lfloor \frac{n(\beta-\eta)}{\beta}\right\rfloor, 0\right)$.

Proof. See in Chen *et al.* [1]. □

Note that the values of k and l are independent of the job processing times and the job sequence. Thus, for any given schedule π, $q_1 = C_{[k]} = \sum_{i=1}^{k} p^A_{[i]}$ and $q_2 = C_{[l]} = \sum_{i=1}^{l} p^A_{[i]}$. Since $d^1_{[k]} = p^A_{[k]} + q_1 = p^A_{[k]} + C_{[k]} > C_{[k]}$ and $d^1_{[k+1]} = p^A_{[k+1]} + q_1 = p^A_{[k+1]} + C_{[k]} = C_{[k+1]}$, we can get jobs in position $1, 2, \ldots, k$ are the early jobs. Similarly, because $d^2_{[l]} = p^A_{[l]} + q_2 = p^A_{[l]} + C_{[l]} > C_{[l]}$ and $d^2_{[l+1]} = p^A_{[l+1]} + q_2 = p^A_{[l+1]} + C_{[l]} = C_{[l+1]}$, jobs in position $l+2, \ldots, n$ are the tardy jobs. According to descriptions above, we have the following results:

$$E_{[j]} = d^1_{[j]} - C_{[j]} = p^A_{[j]} + C_{[k]} - C_{[j]} = \sum_{i=j}^{k} p^A_{[i]}, \qquad \text{for } j = 1, 2, \ldots, k,$$

$$T_{[l]} = 0,$$

$$T_{[l+1]} = \max\Big\{0, C_{[l+1]} - d^2_{[l+1]}\Big\} = 0,$$

$$T_{[j]} = C_{[j]} - d^2_{[j]} = C_{[j]} - \Big(p^A_{[j]} + C_{[l]}\Big) = \sum_{i=l+1}^{j-1} p^A_{[i]}, \qquad \text{for } j = l+2, \ldots, n,$$

$$d^1_{[j]} = p^A_{[j]} + q_1 = p^A_{[j]} + \sum_{i=1}^{k} p^A_{[i]}, \qquad \text{for } j = 1, 2, \ldots, n,$$

$$d^2_{[j]} = p^A_{[j]} + q_2 = p^A_{[j]} + \sum_{i=1}^{l} p^A_{[i]}, \qquad \text{for } j = 1, 2, \ldots, n,$$

$$D = d^2_{[j]} - d^1_{[j]} = \sum_{i=k+1}^{l} p^A_{[i]}.$$

Substituting $E_{[j]}, T_{[j]}, d^1_{[j]}, D$ into the objective function of problem ***P1***, we can get a new expression of the objection function as follows:

$$Z = \sum_{j=1}^{k} (\alpha j + (n+1)\gamma) p^A_{[j]} + \sum_{j=k+1}^{l} (n\eta + \gamma) p^A_{[j]} + \sum_{j=l+1}^{n} (\beta(n-j) + \gamma) p^A_{[j]}.$$

By substituting for the $p^A_{[j]}$ we get the total cost of

$$Z = \sum_{j=1}^{k} (\alpha j + (n+1)\gamma) p_{[j]} g_{[j]}(j) + \sum_{j=k+1}^{l} (n\eta + \gamma) p_{[j]} g_{[j]}(j) + \sum_{j=l+1}^{n} (\beta(n-j) + \gamma) p_{[j]} g_{[j]}(j). \tag{7}$$

Based on the properties proved above, we introduce in the following procedure for finding the optimal solution. Define x_{jr} as a 0/1 variable such that $x_{jr} = 1$ if job J_j is scheduled in the rth position to be processed on the machine and $x_{jr} = 0$ otherwise. Then the problem ***P1*** can be formulated as the following assignment problem:

$$\begin{aligned} \text{Minimize } & \sum_{j=1}^{n} \sum_{r=1}^{n} \omega_{jr} x_{jr} \\ \text{Subject to } & \sum_{r=1}^{n} x_{jr} = 1, \qquad j = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_{jr} = 1, \qquad r = 1, 2, \ldots, n \\ & x_{jr} = \{0, 1\}, \qquad j, r = 1, 2, \ldots, n, \end{aligned}$$

where

$$\omega_{jr} = \begin{cases} [\alpha r + (n+1)\gamma] p_j g_j(r), & \text{for } r = 1, 2, \ldots, k, \\ [n\eta + \gamma] p_j g_j(r), & \text{for } r = k+1, \ldots, l, \\ [\beta(n-r) + \gamma] p_j g_j(r), & \text{for } r = l+1, \ldots, n. \end{cases} \tag{8}$$

In the following, we consider the case of the problem with $h (1 \le h \le n)$ accepted jobs. As mentioned earlier, manufacturers do not need to pay attention to the production scheduling of outsourced jobs, thus we can get a solution that the h accepted jobs are scheduled in the first h positions and the $n - h$ rejected jobs are scheduled in the remaining positions in any order. Therefore, for a given h, we can formulate the model (3) as:

$$\begin{aligned} \textbf{\textit{P2(h)}} \text{ Minimize } & \sum_{j=1}^{n} \sum_{r=1}^{n} \theta_{jr} x_{jr} \\ \text{Subject to } & \sum_{r=1}^{n} x_{jr} = 1, \qquad j = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_{jr} = 1, \qquad r = 1, 2, \ldots, n \\ & x_{jr} = \{0, 1\}, \qquad j, r = 1, 2, \ldots, n, \end{aligned}$$

where

$$\theta_{jr} = \begin{cases} [\alpha r + (h+1)\gamma]p_j g_j(r), & \text{for } r = 1, 2, \ldots, k, \\ [h\eta + \gamma]p_j g_j(r), & \text{for } r = k+1, \ldots, l, \\ [\beta(h-r) + \gamma]p_j g_j(r), & \text{for } r = l+1, \ldots, h, \\ e_j, & \text{for } r = h+1, \ldots, n. \end{cases}$$

Now, based on Lemmas 1–4 and the above analysis, we present the following solution algorithm for the problem $1|p_j g_j(r), \mathrm{SLK}|\sum_{j\in O}(\alpha E_j + \beta T_j + \gamma d_j^1 + \eta D) + \sum_{j\in R} e_j$.

Algorithm 1.

Step 1. For $h = 0$, calculate $F(0) = \sum_{j=1}^{n} e_j$;
Step 2. For $h = 1, 2, \ldots, n$:
Step 2.1. Calculate the indices k and l according to Lemma 4;
Step 2.2. Solve the assignment problem *P2(h)* to obtain the corresponding schedule and the total cost $F(h)$;
Step 3. Choose the minimum cost $F^* = \min\{F(h),\ h = 0, 1, 2, \ldots, n\}$ and get the set of outsourced jobs and the corresponding job processing schedule;
Step 4. For the resulting job schedule, calculate q_1, q_2 and the due-window size D. Stop.

Theorem 1. *The problem* $1|p_j g_j(r), \mathrm{SLK}|\sum_{j\in O}(\alpha E_j + \beta T_j + \gamma d_j^1 + \eta D) + \sum_{j\in R} e_j$ *can be solved in* $O(n^4)$ *time.*

Proof. The correctness of Algorithm 1 follows from Lemmas 1 to 4. For initialization of Step 1, it can be finished in constant time. Once the value h is given, we can take $O(n^3)$ time to solve the associated assignment problem in Step 2. Since there are at most n cases, thus Step 2 totally takes at most $O(n^4)$ time. Finally, Steps 3 and 4 are to calculate the optimal due window assignment, which takes at most $O(n)$ time. Therefore, the overall computational complexity of Algorithm 1 is $O(n^4)$. □

3.2. The problem $1|p_j g_j(r), \mathrm{DIF}|\sum_{j\in O}(\alpha E_j + \beta T_j + \gamma d_j^1 + \eta D_j) + \sum_{j\in R} e_j$

Under the DIF due window model, the manufacturer needs to determine an outsourcing scheme and a processing schedule, a due window starting time, and a size for each accepted job to minimize the total costs. According to the notations of E_j, T_j, d_j^1 and D_j, the objective function of problem (4) can be written as

$$Z' = \sum_{j=1}^{h}\left[\alpha \max\Big(0, d_{[j]}^1 - C_{[j]}\Big) + \beta \max\Big(0, C_{[j]} - d_{[j]}^2\Big) + \gamma d_{[j]}^1 + \eta\Big(d_{[j]}^2 - d_{[j]}^1\Big)\right] + \sum_{j\in R} e_j. \tag{9}$$

Define $Z = \sum_{j=1}^{h}[\alpha\max(0, d_{[j]}^1 - C_{[j]}) + \beta\max(0, C_{[j]} - d_{[j]}^2) + \gamma d_{[j]}^1 + \eta(d_{[j]}^2 - d_{[j]}^1)]$. Obviously, for a given h and the schedule $\pi = [J_{[1]}, J_{[2]}, \ldots, J_{[h]}]$, the due window assignment problem has a separable objective function. Therefore, for a given h and schedule π, we can determine the optimal due window starting time and due window size for the job J_j by minimizing the following objective:

$$Z_j = \alpha\max\big(0, d_j^1 - C_j\big) + \beta\max\big(0, C_j - d_j^2\big) + \gamma d_j^1 + \eta\big(d_j^2 - d_j^1\big). \tag{10}$$

Lemma 5. *For a given set of accepted jobs and schedule* π*, the optimal due window starting time* d_j^1 *and the optimal due window completion time* d_j^2 *for job* J_j *is no greater than its completion time* C_j*, i.e.,* $d_j^1 \le d_j^2 \le C_j$.

Proof. If $d_j^1 \le C_j < d_j^2$, according to equation (10), the total cost for job J_j is $Z_j = \gamma d_j^1 + \eta(d_j^2 - d_j^1)$. When shifting the due-window completion time d_j^2 of job J_j to the left to make it equal to its completion time C_j, the total cost for job J_j (denoted $\tilde{Z}_j$) is given by $\tilde{Z}_j = \gamma d_j^1 + \eta(C_j - d_j^1)$. It is easy to see that $\tilde{Z}_j < Z_j$.

If $C_j < d_j^1 \leq d_j^2$, according to equation (10), the total cost for job J_j is $Z_j = \alpha(d_j^1 - C_j) + \gamma d_j^1 + \eta(d_j^2 - d_j^1)$. When shifting the due-window starting time d_j^1 and completion time d_j^2 of job J_j to the left to make them equal to its completion time C_j, the total cost for job J_j (denoted $\tilde{Z}_j$) is given by $\tilde{Z}_j = \gamma C_j$. It is easy to see that $\tilde{Z}_j < Z_j$. □

Lemma 6. *For a given accepted job set and schedule π, the optimal due window starting time d_j^1 and the optimal due window completion time d_j^2 for job J_j can be assigned according to the following rule:*

(1) *If $\gamma \geq \eta$ and $\eta \geq \beta$, then set $d_j^1 = d_j^2 = 0$;*

(2) *If $\gamma \geq \eta$ and $\eta < \beta$, then set $d_j^1 = 0$ and $d_j^2 = C_j$;*

(3) *If $\gamma < \eta$ and $\gamma \geq \beta$, then set $d_j^1 = d_j^2 = 0$;*

(4) *If $\gamma < \eta$ and $\gamma < \beta$, then set $d_j^1 = d_j^2 = C_j$.*

Proof. According to Lemma 5, for job J_j, we have $d_j^1 \leq d_j^2 \leq C_j$. Therefore, equation (10) can be simplified as $Z_j = \beta(C_j - d_j^2) + \gamma d_j^1 + \eta(d_j^2 - d_j^1) = \beta C_j + (\gamma - \eta)d_j^1 + (\eta - \beta)d_j^2$. From the above equation, if $\gamma \geq \eta$, we set $d_j^1 = 0$. Then we only need to determine d_j^2 by minimizing $\beta C_j + (\eta - \beta)d_j^2$. If $\eta \geq \beta$, set $d_j^2 = 0$; otherwise, set $d_j^2 = C_j$.

If $\gamma < \eta$, we set $d_j^1 = d_j^2$. The following thing is to determine d_j^2 by minimizing $\beta C_j + (\gamma - \beta)d_j^2$. Clearly, if $\gamma \geq \beta$, set $d_j^2 = 0$; otherwise, set $d_j^2 = C_j$. □

From Lemmas 5 and 6, for a given schedule of accepted jobs, the optimal due window assignment can be determined easily.

Lemma 7. *For a given accepted job set, the optimal schedule can be obtained in $O(n^3)$ time.*

Proof. With Lemmas 5 and 6, the objective function in equation (9) of model (4) can be transformed into the following three cases: (1) $\beta \sum_{j=1}^n C_j$; (2) $\eta \sum_{j=1}^n C_j$; (3) $\gamma \sum_{j=1}^n C_j$. For all three cases, it is easy to verify that $1|p_j g_j(r)| \sum_{j \in O} C_j$ can be formulated as an assignment problem respectively, which can be solved in $O(n^3)$ time. □

For a given h, it is easy to determine the optimal processing schedule and the due windows by Lemmas 5–7. The remaining problem is how to obtain the outsourcing scheme. Based on the above analysis, model (4) can be transformed into problem $\boldsymbol{P}3$:$1|p_j g_j(r)|\xi \sum_{j \in O} C_j + \sum_{j \in R} e_j$, where $\xi \in \{\beta, \gamma, \eta\}$. And the problem $1|p_j g_j(r)|\xi \sum_{j \in O} C_j + \sum_{j \in R} e_j$ can be solved in $O(n^4)$ time by formulating an assignment problem [18].

In the following we present the following solution algorithm for model (4).

Algorithm 2.

Step 1. For $h = 0$, calculate $F(0) = \sum_{j=1}^n e_j$ and determine the value of ξ;
Step 2. For $h = 1, 2, \ldots, n$, solve the assignment problem and calculate the corresponding objective value $F(h)$;
Step 3. Choose the minimum cost $F^* = \min\{F(h),\ h = 0, 1, 2, \ldots, n\}$ and get the set of outsourced jobs and the corresponding job processing schedule;
Step 4. For the resulting job schedule, calculate the optimal due window starting time and completion time for each job according to Lemma 6;
Step 5. Calculate the optimal due window size $D_j = d_j^2 - d_j^1$ and the objective function value according to equation (9). Stop.

Theorem 2. *The problem $1|p_j g_j(r), \mathrm{DIF}| \sum_{j \in O} (\alpha E_j + \beta T_j + \gamma d_j^1 + \eta D_j) + \sum_{j \in R} e_j$ can be solved in $O(n^4)$ time.*

Proof. The correctness of Algorithm 2 follows from Lemmas 5 to 7. In Algorithm 2, Step 1 can be finished in constant time. Step 2 needs to solve n linear assignment problems which each one can be solved in $O(n^3)$ time. Steps 3–5 is to determine the optimal due window assignment for accepted jobs and calculate the objective function value, which takes at most $O(n)$ time. Therefore, the overall computational complexity of Algorithm 2 is $O(n^4)$. □

4. The ENTDW problem

Similarly, we begin by considering the SLK due window model where the manufacturer needs to determine an outsourcing scheme and a production schedule for the accepted jobs, a due window starting time, and a common due window size for all jobs to minimize the total costs for earliness, number of tardy jobs, due window starting time and common due window size.

4.1. The problem $1|p_jg_j(r), \mathrm{SLK}|\sum_{j=1}^{n}(\alpha E_j + e_jU_j + \gamma d_j^1 + \eta D)$

For the fixed job processing times, the following results are given by Yin *et al.* (2014).

Lemma 8. *For problem* $1|\mathrm{SLK}|\sum_{j=1}^{n}(\alpha E_j + e_jU_j + \gamma d_j^1 + \eta D)$*, an optimal schedule exists that satisfies the following properties:*

(1) *All the jobs are processed consecutively without idle time from time 0;*
(2) *Starting from a certain position in the schedule, all the jobs are tardy, i.e.,* $C_j \geq d_j^2$ *implies* $C_{j+1} \geq d_{j+1}^2$ *for all* j*;*
(3) *Prior to a certain position in the schedule, all the jobs are early, i.e.,* $C_j \leq d_j^1$ *implies* $C_{j-1} \leq d_{j-1}^1$ *for all* j*;*
(4) *The optimal* q_1 *value equals* $C_{[k^*]}$ *and the optimal* q_2 *value equals* $C_{[l^*]}$ *with* $k^* \leq l^* \leq n$*, where* $k^* = \max\left\{\left\lfloor \frac{n(\eta-\gamma)}{\alpha}\right\rfloor, 0\right\}$.

The above Lemma 8 continues to hold for problem $1|p_jg_j(r), \mathrm{SLK}|\sum_{j=1}^{n}(\alpha E_j + e_jU_j + \gamma d_j^1 + \eta D)$ and be easily verified similarly to the proof of Lemma 2. Thus, for any given schedule π and the given value l, $q_1, q_2, d_{[j]}^1, d_{[j]}^2, D$ and $E_{[j]}$ can be calculated by the corresponding equations in Section 3. Hence, we get a new expression of the objective function as follows:

$$Z = \sum_{j=1}^{k}(\alpha j + (n+1)\gamma)p_{[j]}^A + \sum_{j=k+1}^{l}(n\eta + \gamma)p_{[j]}^A + \sum_{j=l+1}^{n}\gamma p_{[j]}^A + \sum_{j=l+2}^{n} e_{[j]}. \tag{11}$$

Now, define $\omega_{[j]}(l)$ as follows:

$$\omega_{[j]}(l) = \begin{cases} \alpha r + (n+1)\gamma, & \text{for } j = 1, 2, \dots, k, \\ n\eta + \gamma, & \text{for } j = k+1, \dots, l, \\ \gamma, & \text{for } j = l+1, \dots, n. \end{cases}$$

Clearly, $\omega_{[j]}(l), j = 1, 2, \dots, n$, denotes the positional penalty for a given l.

Theorem 3. *For problem* $1|p_jg_j(r), \mathrm{SLK}|\sum_{j=1}^{n}(\alpha E_j + e_jU_j + \gamma d_j^1 + \eta D)$ *with a given* l *value, the optimal schedule can be determined by solving a linear assignment problem in* $O(n^3)$ *time.*

Proof. For $1 \leq j, r \leq n$, let

$$\omega_{jr}(l) = \begin{cases} [\alpha r + (n+1)\gamma]p_jg_j(r), & \text{for } r = 1, 2, \dots, k, \\ [n\eta + \gamma]p_jg_j(r), & \text{for } r = k+1, \dots, l, \\ \gamma p_jg_j(r), & \text{for } r = l+1, \dots, n. \end{cases}$$

If we define the value $c_{jr}(l)$ by

$$c_{jr}(l) = \begin{cases} \omega_{jr}(l), & r = 1, 2, \ldots, l+1, \\ \omega_{jr}(l) + e_j, & r = l+2, \ldots, n, \end{cases} \tag{12}$$

then $c_{jr}(l)$ represents the minimum possible cost resulting from assigning job J_j to position r in the schedule under the assumption that q_2 equals the completion time of the l-th job in the schedule. Define x_{jr} as a 0/1 variable such that $x_{jr} = 1$ if job J_j is assigned to position r and $x_{jr} = 0$ otherwise. Then for a given l, model (5) can be formulated as the following assignment problem.

$$P3(l) \text{ Minimize } Z(l) = \sum_{j=1}^{n}\sum_{r=1}^{n} c_{jr}(l)x_{jr}$$

$$\text{Subject to } \sum_{r=1}^{n} x_{jr} = 1, \qquad j = 1, 2, \ldots, n,$$

$$\sum_{j=1}^{n} x_{jr} = 1, \qquad r = 1, 2, \ldots, n,$$

$$x_{jr} = \{0, 1\}, \qquad j, r = 1, 2, \ldots, n.$$

It is well known that the linear assignment problem can be solved in $O(n^3)$ time. □

Because the l value is unknown, we have to enumerate the possible l values and solve the corresponding series of assignment problems. Based on the above analysis, we present a solution algorithm for problem $1|p_jg_j(r), \mathrm{SLK}|\sum_{j=1}^{n}(\alpha E_j + e_jU_j + \gamma d_j^1 + \eta D)$.

Algorithm 3.

Step 1. Calculate the indices k^* according to Lemma 8;
Step 2. For $k^* \le l \le n$ do:
  Step 2.1. Calculate the $c_{jr}(l)$ values according to (12);
  Step 2.2. Solve the assignment problem $P3(l)$ to obtain the corresponding job schedule and the total cost $Z(l)$;
Step 3. Choose the minimum cost $Z^* = \min\{Z(l), l = k, \ldots, n\}$ and get the set of outsourced jobs and the corresponding job processing schedule;
Step 4. For the resulting job schedule, calculate the optimal values of q_1, q_2 and the due-window size D. Stop.

Theorem 4. *The problem* $1|p_jg_j(r), \mathrm{SLK}|\sum_{j=1}^{n}(\alpha E_j + e_jU_j + \gamma d_j^1 + \eta D)$ *can be solved in* $O(n^4)$ *time.*

Proof. The correctness of Algorithm 3 follows from Lemma 8 and Theorem 3. Each iteration of Step 2 takes $O(n^3)$ steps, which is the complexity of solving the linear assignment problem for a given l value. Because we perform at most $O(n)$ iterations of Step 2, the overall complexity of this step is $O(n^4)$. In addition, Steps 1, 3, and 4 can be finished in constant time. Therefore, the overall complexity of Algorithm 3 is $O(n^4)$. □

4.2. The problem $1|p_jg_j(r), \mathrm{DIF}|\sum_{j=1}^{n}(\alpha E_j + e_jU_j + \gamma d_j^1 + \eta D_j)$

In an optimal solution, the starting/completing time of due windows for tardiness jobs(outsourcing jobs) is equal to 0 and all tardiness jobs are scheduled behind the accepted jobs. Similarly, with notations E_j, U_j, d_j^1 and d_j^2 defined in Section 2, the objective function of model (6) can be written as

$$Z = \sum_{j=1}^{n}\left[\alpha \max\left(0, d_{[j]}^1 - C_{[j]}\right) + e_{[j]}U_{[j]} + \gamma d_{[j]}^1 + \eta\left(d_{[j]}^2 - d_{[j]}^1\right)\right]. \tag{13}$$

Obviously, for a given schedule $\pi = [J_{[1]}, J_{[2]}, \dots, J_{[n]}]$, model (6) has a separable objective function. Therefore, for a given schedule π, we can determine the optimal due window starting time and due window size for job J_j by minimizing the following objective:

$$Z_j = \alpha \max\big(0, d_j^1 - C_j\big) + e_j U_j + \gamma d_j^1 + \eta\big(d_j^2 - d_j^1\big). \tag{14}$$

Lemma 9. *For a given schedule π, the optimal due window starting time d_j^1 and the optimal due window completion time d_j^2 for job J_j are no greater than its completion time C_j, i.e., $d_j^1 \le d_j^2 \le C_j$.*

Proof. This lemma can be proved similarly as that for Lemma 5. □

Lemma 10. *For a given schedule π, the optimal due window starting time d_j^1 and the optimal due window completion time d_j^2 for job J_j can be assigned according to the following rule:*

(1) *If $\gamma \ge \eta$ and $e_j \ge \eta C_j$, then set $d_j^1 = 0$ and $d_j^2 = C_j$;*

(2) *If $\gamma \ge \eta$ and $e_j < \eta C_j$, then set $d_j^1 = d_j^2 = 0$;*

(3) *If $\gamma < \eta$ and $e_j \ge \gamma C_j$, then set $d_j^1 = d_j^2 = C_j$;*

(4) *If $\gamma < \eta$ and $e_j < \gamma C_j$, then set $d_j^1 = d_j^2 = 0$.*

Proof. According to Lemma 9, for job J_j, we have $d_j^1 \le d_j^2 \le C_j$. Thus equation (14) can be simplified as $Z_j = e_j U_j + (\gamma - \eta) d_j^1 + \eta d_j^2$.

① If $\gamma \ge \eta$, set $d_j^1 = 0$, then the remaining thing is to determine d_j^2 by minimizing $e_j U_j + \eta d_j^2$. To minimize $e_j U_j + \eta d_j^2$, we consider two cases. If $d_j^2 = C_j$, the total cost is ηC_j; if $d_j^2 < C_j$, the total cost is $e_j + \eta d_j^2$, which is increasing in d_j^2, thus $d_j^2 = 0$. Comparing the two cases above, we have that if $e_j < \eta C_j$, set $d_j^2 = 0$; if $e_j \ge \eta C_j$, $d_j^2 = C_j$.

② If $\gamma < \eta$, $d_j^1 = d_j^2$, then the remaining thing is to determine d_j^2 by minimizing $e_j U_j + \gamma d_j^2$. Similarly, if $d_j^2 = C_j$, the total cost is γC_j; if $d_j^2 < C_j$, the total cost is $e_j + \gamma d_j^2$, which is increasing in d_j^2, thus $d_j^2 = 0$. Comparing the two cases, we have that if $e_j \ge \gamma C_j$, set $d_j^2 = C_j$; if $e_j < \gamma C_j$, set $d_j^2 = 0$. □

Lemma 11. *The optimal schedule can be obtained in $O(n^4)$ time.*

Proof. With Lemmas 9 and 10, the objective function in equation (13) of the model (6) can be transformed into the following two cases: $\eta \sum_{j\in O} C_j + \sum_{j\in R} e_j$ or $\gamma \sum_{j\in O} C_j + \sum_{j\in R} e_j$. For the two cases, it is easy to verify that all the two cases can be formulated as a corresponding assignment problem respectively, which can be solved in $O(n^4)$ time. □

Based on Lemmas 9–11, we present a solution algorithm to solve model (6).

Algorithm 4.

Step 1. For $h = 0$, calculate $F(0) = \sum_{j=1}^n e_j$;

Step 2. For $h = 1, 2, \dots, n$, solve the assignment problem and calculate the corresponding objective value $F(h)$;

Step 3. Choose the minimum cost $F^* = \min\{F(h),\ h = 0, 1, 2, \dots, n\}$ and get the set of outsourced jobs and the corresponding job processing schedule;

Step 4. For the resulting job schedule, calculate the optimal due window starting time and completion time for each job according to Lemma 10;

Step 5. Calculate the optimal due window size $D_j = d_j^2 - d_j^1$ and the objective function value according to equation (13). Stop.

Theorem 5. *The problem $1|p_j g_j(r), \mathrm{DIF}| \sum_{j=1}^n (\alpha E_j + e_j U_j + \gamma d_j^1 + \eta D_j)$ can be solved in $O(n^4)$ time.*

Proof. The correctness of Algorithm 4 follows from Lemmas 9 to 11. In Algorithm 4, Step 1 can be finished in constant time. For getting the set of outsourced jobs and the job schedule, Step 2 needs to solve n linear assignment problems which each one can be solved in $O(n^3)$ time. Steps 3–5 is to determine the optimal due window assignment for all jobs and calculate the objective function value, which takes at most $O(n)$ time. Therefore, the overall computational complexity of Algorithm 4 is $O(n^4)$. □

TABLE 1. Parameters table of jobs.

Jobs	J1	J2	J3	J4	J5	J6	J7	J8
p_j	3	7	3	8	8	10	2	4
a_j	0.4	0.1	0.3	0.4	0.3	0.2	0.2	0.1
e_j	65	100	50	40	85	60	100	80

TABLE 2. Comparison of various methods ($a_j > 0$).

Cases	Schedule	Outsourcing jobs	Objective
Model (3)	$(1, 8, 7, 2, 5)$	$(3, 4, 6)$	305.48
Model (4)	$(1, 3, 7, 8, 2, 5, 6)$	(4)	180.81
Model (5)	$(1, 3, 8, 7, 2)$	$(4, 5, 6)$	431.94
Model (6)	$(1, 3, 7, 8, 2, 5, 6)$	(4)	180.81

TABLE 3. Comparison of various methods ($a_j < 0$).

Cases	Schedule	Outsourcing jobs	Objective
Model (3)	$(8, 7, 1, 3, 5, 2, 4)$	(6)	237.52
Model (4)	$(7, 3, 1, 8, 4, 5, 2, 6)$	(0)	111.01
Model (5)	$(2, 8, 7, 3, 1, 5)$	$(4, 6)$	324.96
Model (6)	$(7, 3, 1, 8, 4, 5, 2, 6)$	(0)	111.01

5. Numerical examples

A commonly used scenario for the application of the problems studied in this paper is the manufacturing industry. In this section, by using the production of the rotten perishable goods as an example, experiments are performed and the computation results obtained are used to verify the effectiveness of the algorithms. Moreover, the examples will show the effect of the different due window assignment method on the objective function.

The manufacturer has the single-machine production equipment and receives a set of jobs to be processed. The specific functional form of the general job-dependent positional effect is denoted by $p_{jr}^A = p_j r^{a_j}$. Other parameters related to the jobs are shown in Table 1. In this study, the weights are $\alpha = 2$, $\beta = 5$, $\gamma = 1, \eta = 2$.

For the SLK and DIF due window assignment methods, we can respectively get the solution by the Algorithms 1–4 (see Tab. 2). The algorithms' running steps involve the case that the manufacturer does not outsource any jobs. Therefore, Table 2 no longer lists this result of no outsourcing separately.

In order to verify the influence of other function forms on the studied problems, we calculate the objective function values when learning effects occur. Based on the above example, the relevant results obtained are shown in Table 3.

As seen from the above results through Tables 2 and 3, the benefit of outsourcing can be achieved by comparing our solution subject to the schedule without outsourcing. The due window assignment methods have various effects on the objective function value, and DIF due-window assignment has fewer costs than SLK due-window assignment. Moreover, under the same due window assignment method, outsourcing all the tardy jobs will generate more costs than accepting and processing some tardy jobs. The manager should choose the appropriate due window assignment method according to the importance of the current customer, the production capacity, and so on. The solutions of the manuscript designed provide an effective basis for the manufacturer to negotiate the due windows with the customer.

Table 4. Test results for randomly generated cases.

Instances	Model (3)		Model (4)		Model (5)		Model (6)	
	OBJ	NJ	OBJ	NJ	OBJ	NJ	OBJ	NJ
$\alpha = 2$	6694	17	5962	31	28052	51	5962	31
$\alpha = 3$	6722	17	5962	31	16814	34	5962	31
$\alpha = 4$	6734	16	5962	31	12951	26	5962	31
$\alpha = 5$	6740	16	5962	31	11074	21	5962	31
$\alpha = 6$	6745	16	5962	31	9942	17	5962	31
$\alpha = 7$	6748	16	5962	31	9526	15	5962	31
$\alpha = 8$	6751	16	5962	31	9187	13	5962	31
$\alpha = 9$	6752	16	5962	31	9091	12	5962	31
$\alpha = 10$	6753	16	5962	31	8985	11	5962	31

Notes. OBJ – Objective function value; NJ – The number of jobs accepted for processing.

To further explore the effect of different parameters on due window assignment models, we randomly generated a larger numerical example. The number of jobs to be processed is 100. The processing times of the jobs were randomly generated between 1 and 10. The outsourcing costs were generated between 50 and 100. The deterioration rate a_j associated with the job was generated between 0.01 and 0.1. In the case, the storage of the rotten perishable goods may be at risk of spoilage. Therefore, we focus on the effect of the parameter α on the objective function and outsourcing decision. The relevant results obtained are shown in Table 4.

As can be seen in Table 4, compared with the other three models, the operation cost corresponding to Model (5) is the highest, which reflects that the manufacturer will pay more operation costs when the jobs do not allow delay. In addition, it is not difficult to find out from the second and sixth columns that the total cost of SLK due window assignment is higher than that of DIF due window assignment. According to Lemmas 7 and 11, the objective functions of Models (4) and (6) are not related to α; therefore, their objective function values do not change.

The application of outsourcing strategies at the scheduling provides a very valuable opportunity for manufacturers to control operational costs and improve customer satisfaction. Some managerial insights from the study are summarized as follows.

The study suggests that when outsourcing is an option for the jobs to be processed, the manufacturers should actively adopt this strategy. The results of numerical experiments also reveal that the introduction of the outsourcing decision leads to fewer operating costs, regardless of the due window assignment method adopted. The outsourcing strategy greatly improves the flexibility of production scheduling and provides a new solution to cope with the impacts caused by aging effects, etc.

Furthermore, Different due window assignment methods have a significant impact on the effectiveness of the scheduling scheme. A separate due window for each job will yield better results than a common due window size for all jobs. When DIF due window assignment method is used, none of the jobs will be early. When delays are not allowed to occur, the manufacturer will have a greater operating costs. Therefore, the manufacturer should choose the appropriate due window assignment method according to the importance of the current customer, the status of the workshop, and so on.

In addition, the weight setting in the objective function affects the difficulty of problem solving. The size of the weight directly affects the determination of due windows, especially for DIF due window assignment models. It can be seen that the sales department of the manufacturers should communicate more with customers and strengthen the cooperation with the production department, so as to make the complex multiple centralized decision-making problems clearer.

6. Conclusions

Due window assignment and outsourcing have been the most important topics in supply chain management. In this paper, we study the due window assignment methods for scheduling on a single machine with general job-dependent positional effect and outsourcing, where the due windows are assignable according to the SLK/DIF due window assignment methods. Depending on whether the tardiness jobs are outsourced, two objectives under each due window assignment method were investigated. The first one is to minimize a cost function that includes earliness, tardiness, due window starting time, due window size, and outsourcing costs, while the second is to minimize the total costs for earliness, the number of tardy jobs, due window starting time and due window size. The optimality properties of SLK/DIF due windows are analyzed and the polynomial-time solution algorithm is provided for each problem.

For future research, extending the problems to the multi-machine setting is challenging and valuable since the manufacturer operates several parallel production facilities in practice. Moreover, studying the limited outsourcing costs or the constrained capacity which frequently appear in the reality is also worthy of investigation.

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