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TOUGHNESS AND ISOLATED TOUGHNESS CONDITIONS FOR PATH-FACTOR CRITICAL COVERED GRAPHS

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Abstract. Given a graph G and an integer $k \geq 2$. A spanning subgraph H of G is called a $P_{\geq k}$-factor of G if every component of H is a path with at least k vertices. A graph G is said to be $P_{\geq k}$-factor covered if for any $e \in E(G)$, G admits a $P_{\geq k}$-factor including e. A graph G is called a $(P_{\geq k}, n)$-factor critical covered graph if $G - V'$ is $P_{\geq k}$-factor covered for any $V' \subseteq V(G)$ with $|V'| = n$. In this paper, we study the toughness and isolated toughness conditions for $(P_{\geq k}, n)$-factor critical covered graphs, where $k = 2,\ 3$. Let G be a $(n+1)$-connected graph. It is shown that (i) G is a $(P_{\geq 2}, n)$-factor critical covered graph if its toughness $\tau(G) > \frac{n+2}{3}$; (ii) G is a $(P_{\geq 2}, n)$-factor critical covered graph if its isolated toughness $I(G) > \frac{n+1}{2}$; (iii) G is a $(P_{\geq 3}, n)$-factor critical covered graph if $\tau(G) > \frac{n+2}{3}$ and $|V(G)| \geq n+3$; (iv) G is a $(P_{\geq 3}, n)$-factor critical covered graph if $I(G) > \frac{n+3}{2}$ and $|V(G)| \geq n+3$. Furthermore, we claim that these conditions are best possible in some sense.

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1. Introduction

All graphs considered here are finite and simple. We refer to [5] for the notation and terminologies not defined here. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Given a vertex $v \in V(G)$, let $d_G(v)$ denote the degree of v, that the number of edges incident to v in G. If $d_G(v) = 0$ for some vertex v in G, then v is said to be an isolated vertex in G. Let $i(G)$ be the number of isolated vertices of G. For any subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, and $G - S := G[V(G) \setminus S]$ is the resulting graph after deleting the vertices of S from G. The number of connected components of a graph G is denoted by $\omega(G)$. We write $\kappa(G)$ for the vertex connectivity of G.

Next, we introduce two parameters for a graph, namely, the toughness and the isolated toughness. The *toughness* of G was first introduced by Chvátal [6] as

$$\tau(G) = \min\left\{\frac{|X|}{\omega(G-X)} : X \subseteq V(G),\ \omega(G-X) \geq 2\right\},$$

if G is not complete; otherwise, $\tau(G) = +\infty$.

Keywords. Graph, Path-factor, Toughness, Isolated toughness, $(P_{\geq 2}, n)$-factor critical covered graph, $(P_{\geq 3}, n)$-factor critical covered graph.

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The *isolated toughness* of G was defined by Yang *et al.* [20] as

$$I(G)=\min\left\{\frac{|X|}{i(G-X)}: X\subseteq V(G),\ i(G-X)\geq 2\right\},$$

if G is not complete; otherwise, $I(G)=+\infty$.

A subgraph H of G is called a spanning subgraph of G if $V(H)=V(G)$ and $E(H)\subseteq E(G)$. For a family of connected graphs $\mathcal{F}$, a spanning subgraph H of a graph G is called an $\mathcal{F}$-factor of G if each component of H is isomorphic to some graph in $\mathcal{F}$. A spanning subgraph H of a graph G is called a $P_{\geq k}$-factor of G if every component of H is isomorphic to a path of order at least k, where $k\geq 2$ is an integer. For example, a $P_{\geq 3}$-factor means a graph factor in which every component is a path of order at least three.

Since Tutte proposed the well-known Tutte 1-factor theorem [17], there are many results on path-factor [3, 8, 10, 12, 14, 15, 18, 27] and path-factor covered graphs [7, 9, 19, 22–26, 28]. For more other results on graph factors, we refer the reader to the survey papers and books [1, 16, 21].

Akiyama *et al.* [2] provided a criterion for a graph having a $P_{\geq 2}$-factor as follows.

Theorem 1.1 (Akiyama *et al.* [2])**.** *A graph G has a $P_{\geq 2}$-factor if and only if $i(G-X)\leq 2|X|$ for all $X\subseteq V(G)$.*

In order to characterize a graph possessing a $P_{\geq 3}$-factor, Kaneko [11] put forward the concept of a sun as follows. A graph R is called factor-critical if $R-v$ has a 1-factor for each $v\in V(R)$. Let R be a factor-critical graph and $V(R)=\{v_1,v_2,\ldots,v_n\}$. By adding new vertices $\{u_1,u_2,\ldots,u_n\}$ together with new edges $\{v_iu_i: 1\leq i\leq n\}$ to R, the resulting graph is called a sun. Note that, according to Kaneko [11], we regard K_1 and K_2 also as a sun, respectively. Usually, the suns other than K_1 and K_2 are also called big suns. It is called a sun component of G if the component of G is isomorphic to a sun. We denote by $\mathrm{sun}(G)$ the number of sun components in G.

Kaneko [11] gave a characterization for a graph having a $P_{\geq 3}$-factor, for which Kano *et al.* [13] presented a simpler proof.

Theorem 1.2 (Kaneko [11])**.** *A graph G has a $P_{\geq 3}$-factor if and only if $\mathrm{sun}(G-X)\leq 2|X|$ for all $X\subseteq V(G)$.*

Zhang and Zhou [22] first defined a graph G to be $P_{\geq k}$-factor covered if G admits a $P_{\geq k}$-factor containing e for any $e\in E(G)$. In the same paper, they obtained a characterization for $P_{\geq 2}$-factor covered graphs and $P_{\geq 3}$-factor covered graphs, respectively.

Theorem 1.3 (Zhang and Zhou [22])**.** *Let G be a connected graph. Then G is a $P_{\geq 2}$-factor covered graph if and only if $i(G-X)\leq 2|X|-\varepsilon_1(X)$ for all $X\subseteq V(G)$, where $\varepsilon_1(X)$ is defined by*

$$\varepsilon_1(X)=\begin{cases}2 & \text{if } X\neq\emptyset \text{ and } X \text{ is not an independent set;}\\ 1 & \text{if } X \text{ is a nonempty independent set and there exists } a\\ & \text{nontrivial component of } G-X;\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 1.4 (Zhang and Zhou [22])**.** *Let G be a connected graph. Then G is a $P_{\geq 3}$-factor covered graph if and only if $\mathrm{sun}(G-X)\leq 2|X|-\varepsilon_2(X)$ for all $X\subseteq V(G)$, where $\varepsilon_2(X)$ is defined by*

$$\varepsilon_2(X)=\begin{cases}2 & \text{if } X\neq\emptyset \text{ and } X \text{ is not an independent set;}\\ 1 & \text{if } X \text{ is a nonempty independent set and there exists } a\\ & \text{non}-\text{sun component of } G-X;\\ 0 & \text{otherwise.}\end{cases}$$

A graph G is a $(P_{\geq k},n)$-factor critical covered graph if for any $V'\subseteq V(G)$ with $|V'|=n$, $G-V'$ is $P_{\geq k}$-factor covered. In this paper, we study $(P_{\geq k},n)$-factor critical covered graphs and get some sufficient conditions for graphs to be $(P_{\geq k},n)$-factor critical covered graphs depending on toughness and isolated toughness, which are given in Sections 2 and 3.

2. $(P_{\geq 2}, n)$-FACTOR CRITICAL COVERED GRAPHS

In this section, using toughness and isolated toughness, we obtain two sufficient conditions for the existence of $(P_{\geq 2}, n)$-factor critical covered graphs.

Theorem 2.1. *Let G be a $(n+1)$-connected graph, where $n \geq 0$ is an integer. If its toughness $\tau(G) > \frac{n+2}{3}$, then G is a $(P_{\geq 2}, n)$-factor critical covered graph.*

Proof. If G is a complete graph, then it is easily seen that G is a $(P_{\geq 2}, n)$-factor critical covered graph by $\kappa(G) \geq n+1$. Next, we consider that G is a non-complete graph.

For any $V' \subseteq V(G)$ with $|V'| = n$, we write $H = G - V'$. Clearly, H is connected and $|V(H)| \geq 2$. To justify Theorem 2.1, it suffices to verify that H is $P_{\geq 2}$-factor covered. Next, we assume that H is not $P_{\geq 2}$-factor covered. Then by Theorem 1.3, there exists a subset $X \subseteq V(H)$ such that

$$i(H - X) \geq 2|X| - \varepsilon_1(X) + 1. \tag{1}$$

Claim 2.1. $X \neq \emptyset$.

Proof. On the contrary, we assume that $X = \emptyset$. Then $\varepsilon_1(X) = 0$. It follows from (1) that

$$i(H) = i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 = 1. \tag{2}$$

Note that H is connected since $\kappa(H) \geq \kappa(G) - |V'| \geq 1$. Combining this with (2), $1 \leq i(H) \leq \omega(H) = 1$, that is,

$$i(H) = \omega(H) = 1. \tag{3}$$

It follows from (3) that H is an isolated vertex, which contradicts that $|V(H)| \geq 2$. Hence, Claim 2.1 is verified. □

Next, we will distinguish two cases below to complete the proof of Theorem 2.1.

Case 1. $|X| = 1$.

In this case, we obtain $\varepsilon_1(X) \leq 1$. By the definition of $\varepsilon_1(X)$, we have $\varepsilon_1(X) = 1$ if there is a nontrivial component of $H - X$. Otherwise, $\varepsilon_1(X) = 0$. If $\varepsilon_1(X) = 1$, then it follows from (1) that $i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 = 2$, and thus

$$\omega(H - X) \geq i(H - X) + 1 \geq 3. \tag{4}$$

If $\varepsilon_1(X) = 0$, then by (1), we have

$$\omega(H - X) \geq i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 = 3. \tag{5}$$

Combining (4), (5) with the the definition of $\tau(G)$, we obtain

$$\frac{n+2}{3} < \tau(G) \leq \frac{|X \cup V'|}{\omega(G - X \cup V')} = \frac{|V'| + |X|}{\omega(H - X)} \leq \frac{n+1}{3},$$

which is a contradiction.

Case 2. $|X| \geq 2$.

By the definition of $\varepsilon_1(X)$, we obtain $\varepsilon_1(X) \leq 2$. It follows from (1) and $H = G - V'$ that

$$\omega(H - X) \geq i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq 2|X| - 1 \geq 3. \tag{6}$$

According to (6), we have

$$|X| \leq \frac{\omega(H - X) + 1}{2}. \tag{7}$$

In terms of (6), (7) and the definition of $\tau(G)$, we obtain

$$\begin{aligned}\tau(G) &\le \frac{|X\cup V'|}{\omega(G-X\cup V')}\\ &= \frac{|V'|+|X|}{\omega(H-X)}\\ &\le \frac{n+\frac{\omega(H-X)+1}{2}}{\omega(H-X)}\\ &= \frac{1}{2}+\frac{n}{\omega(H-X)}+\frac{1}{2\omega(H-X)}\\ &\le \frac{1}{2}+\frac{n}{3}+\frac{1}{6}\\ &= \frac{n+2}{3},\end{aligned}$$

which contradicts that $\tau(G) > \frac{n+2}{3}\cdot$ This completes the proof of Theorem 2.1. □

Remark 2.2. The condition $\tau(G) > \frac{n+2}{3}$ in Theorem 2.1 is sharp. Set $G = K_{n+2}\vee 3K_1$, where n is a nonnegative integer, and $\vee$ denotes "join". It is easily seen that $\tau(G) = \frac{n+2}{3}$ and $\kappa(G) = n+2 > n+1$. For any $V' \subseteq V(K_{n+2})$ with $|V'| = n$, let $G' = G - V'$. We select $X = V(K_{n+2}) \setminus V' \subseteq V(G)$, and so $|X| = 2$. Note that X is not an independent set. Then we admit $\varepsilon_1(X) = 2$. Thus, we acquire $i(G'-X) = 3 > 2 = 2|X| - \varepsilon_1(X)$. Using Theorem 1.3, G' is not $P_{\ge 2}$-factor covered, that is, G is not $(P_{\ge 2}, n)$-factor critical covered.

Note that a $(P_{\ge k}, 0)$-factor critical covered graph is simply called a $P_{\ge k}$-factor covered graph, where $k = 2,\ 3$. Dai [7] verified that a connected graph G is a $P_{\ge 2}$-factor covered graph if $I(G) > \frac{2}{3}\cdot$ We generalize the above result and give an isolated toughness condition for a graph being a $(P_{\ge 2}, n)$-factor critical covered graph.

Theorem 2.3. *Let G be a $(n+1)$-connected graph, where $n \ge 1$ is an integer. If its isolated toughness $I(G) > \frac{n+1}{2}$, then G is a $(P_{\ge 2}, n)$-factor critical covered graph.*

Proof. If G is a complete graph, then it is easily seen that G is a $(P_{\ge 2}, n)$-factor critical covered graph by $\kappa(G) \ge n+1$. Next, we consider that G is a non-complete graph.

For any $V' \subseteq V(G)$ with $|V'| = n$, we write $H = G - V'$. Clearly, H is connected and $|V(H)| \ge 2$. To justify Theorem 2.3, it suffices to verify that H is $P_{\ge 2}$-factor covered. Next, we assume that H is not $P_{\ge 2}$-factor covered. Then by Theorem 1.3, there exists a subset $X \subseteq V(H)$ such that

$$i(H-X) \ge 2|X| - \varepsilon_1(X) + 1. \tag{8}$$

Claim 2.2. $X \ne \emptyset$.

Proof. On the contrary, we assume that $X = \emptyset$. Then $\varepsilon_1(X) = 0$. It follows from (8) that

$$i(H) = i(H-X) \ge 2|X| - \varepsilon_1(X) + 1 = 1. \tag{9}$$

Note that H is connected since $\kappa(H) \ge \kappa(G) - |V'| \ge 1$. Combining this with (9), $1 \le i(H) \le \omega(H) = 1$, that is,

$$i(H) = \omega(H) = 1. \tag{10}$$

It follows from (10) that H is an isolated vertex, which contradicts that $|V(H)| \ge 2$. Hence, Claim 2.2 is verified. □

Case 1. $|X| = 1$.
In this case, we obtain $\varepsilon_1(X) \le 1$. Then by (8), we get

$$i(H - X) \ge 2|X| - \varepsilon_1(X) + 1 \ge 2. \tag{11}$$

Combining (11) with the fact that $I(G) > \frac{n+1}{2}$, we obtain

$$\frac{n+1}{2} < I(G) \le \frac{|X \cup V'|}{i(G - X \cup V')} = \frac{|V'| + |X|}{i(H - X)} \le \frac{n+1}{2},$$

a contradiction.

Case 2. $|X| \ge 2$.
By the definition of $\varepsilon_1(X)$, we obtain $\varepsilon_1(X) \le 2$. It follows from (8) and $H = G - V'$ that

$$i(H - X) \ge 2|X| - \varepsilon_1(X) + 1 \ge 2|X| - 1 \ge 3. \tag{12}$$

According to (12), we have

$$|X| \le \frac{i(H - X) + 1}{2}\cdot \tag{13}$$

In terms of (12), (13) and the definition of $I(G)$, we obtain

$$\begin{aligned} I(G) &\le \frac{|X \cup V'|}{i(G - X \cup V')} \\ &= \frac{|V'| + |X|}{i(H - X)} \\ &\le \frac{n + \frac{i(H-X)+1}{2}}{i(H - X)} \\ &= \frac{1}{2} + \frac{n}{i(H - X)} + \frac{1}{2i(H - X)} \\ &\le \frac{1}{2} + \frac{n}{3} + \frac{1}{6} \\ &= \frac{n+2}{3}, \end{aligned}$$

which contradicts that $I(G) > \frac{n+2}{3}\cdot$ This completes the proof of Theorem 2.3. □

Remark 2.4. The condition $I(G) > \frac{n+1}{2}$ in Theorem 2.3 is sharp. Set $G = K_{n+1} \vee (2K_1 \cup K_2)$, where n is a positive integer. It is easily seen that $I(G) = \frac{n+1}{2}$ and $\kappa(G) = n + 1$. For any $V' \subseteq V(K_{n+1})$ with $|V'| = n$, let $G' = G - V'$. We select $X = V(K_{n+1}) \setminus V' \subseteq V(G)$, and so $|X| = 1$. Note that X is not an independent set, and $G' - X$ has a nontrivial component. Then we admit $\varepsilon_1(X) = 1$. Thus, we acquire $i(G' - X) = 2 > 1 = 2|X| - \varepsilon_1(X)$. Using Theorem 1.3, G' is not $P_{\ge 2}$-factor covered, that is, G is not $(P_{\ge 2}, n)$-factor critical covered.

3. $(P_{\ge 3}, n)$-FACTOR CRITICAL COVERED GRAPHS

In this section, we obtain two sufficient conditions for $(P_{\ge 3}, n)$-factor critical covered graphs by toughness and isolated toughness. Bazgan *et al.* [4] verified that a connected graph G with $|V(G)| \ge 3$ has a $P_{\ge 3}$-factor if $\tau(G) \ge 1$. We generalize the above result and give a toughness condition for a graph being a $(P_{\ge 3}, n)$-factor critical covered graph.

Theorem 3.1. *Let G be a $(n+1)$-connected graph with $|V(G)| \geq n+3$, where $n \geq 0$ is an integer. If its toughness $\tau(G) > \frac{n+2}{3}$, then G is a $(P_{\geq 3}, n)$-factor critical covered graph.*

Proof. If G is a complete graph, then it is easily seen that G is a $(P_{\geq 3}, n)$-factor critical covered graph by $\kappa(G) \geq n+1$. Next, we consider that G is a non-complete graph.

For any $V' \subseteq V(G)$ with $|V'| = n$, we write $H = G - V'$. Clearly, H is connected. To justify Theorem 3.1, it suffices to verify that H is $P_{\geq 3}$-factor covered. Next, we assume that H is not $P_{\geq 3}$-factor covered. Then by Theorem 1.4, there exists a subset $X \subseteq V(H)$ such that

$$\mathrm{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1. \tag{14}$$

Claim 3.1. $X \neq \emptyset$.

Proof. On the contrary, we assume that $X = \emptyset$. Then $\varepsilon_2(X) = 0$. It follows from (14) that

$$\mathrm{sun}(H) = \mathrm{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 = 1. \tag{15}$$

Note that H is connected since $\kappa(H) \geq \kappa(G) - |V'| \geq 1$. Combining this with (15), $1 \leq \mathrm{sun}(H) \leq \omega(H) = 1$, that is,

$$\mathrm{sun}(H) = \omega(H) = 1. \tag{16}$$

It follows from (16) and $|V(G)| \geq n+3$ that H is a sun with $|V(H)| \geq 3$. By the definition of big sun, we denote R the factor-critical subgraph of H. Note that $|V(H)| = 2|V(R)| \geq 6$. Let $r \in V(R)$ and $R' := R \setminus \{r\}$. Choose $X' = V' \cup V(R')$, then $\omega(G - X') = |V(R')| + 1 = |V(R)| \geq 3$. Using the definition of $\tau(G)$, we obtain

$$\tau(G) \leq \frac{|X'|}{\omega(G - X')} = \frac{|V'| + |V(R')|}{|V(R)|} = 1 + \frac{n-1}{|V(R)|} \leq \frac{n+2}{3},$$

which contradicts that $\tau(G) > \frac{n+2}{3}$ in Theorem 3.1. Hence, Claim 3.1 is verified. □

Next, we will distinguish two cases below to completes the proof of Theorem 3.1.

Case 1. $|X| = 1$.

In this case, we obtain $\varepsilon_2(X) \leq 1$. By the definition of $\varepsilon_2(X)$, we have $\varepsilon_2(X) = 1$ if there is a non-sun component of $H - X$. Otherwise, $\varepsilon_2(X) = 0$. If $\varepsilon_2(X) = 1$, then it follows from (14) that $\mathrm{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 = 2$, and thus

$$\omega(H - X) \geq \mathrm{sun}(H - X) + 1 \geq 3. \tag{17}$$

If $\varepsilon_2(X) = 0$, then by (14), we have

$$\omega(H - X) \geq \mathrm{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 = 3. \tag{18}$$

Combining (17), (18) with the the definition of $\tau(G)$, we obtain

$$\frac{n+2}{3} < \tau(G) \leq \frac{|X \cup V'|}{\omega(G - X \cup V')} = \frac{|V'| + |X|}{\omega(H - X)} \leq \frac{n+1}{3},$$

which is a contradiction.

Case 2. $|X| \geq 2$.

By the definition of $\varepsilon_2(X)$, we obtain $\varepsilon_2(X) \leq 2$. It follows from (14) and $H = G - V'$ that

$$\omega(H - X) \geq \mathrm{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| - 1 \geq 3. \tag{19}$$

According to (19), we have

$$|X| \le \frac{\omega(H-X)+1}{2}. \tag{20}$$

In terms of (19), (20) and the definition of $\tau(G)$, we obtain

$$\begin{aligned}
\tau(G) &\le \frac{|X \cup V'|}{\omega(G - X \cup V')} \\
&= \frac{|V'| + |X|}{\omega(H-X)} \\
&= \frac{n + \frac{\omega(H-X)+1}{2}}{\omega(H-X)} \\
&= \frac{1}{2} + \frac{n}{\omega(H-X)} + \frac{1}{2\omega(H-X)} \\
&\le \frac{1}{2} + \frac{n}{3} + \frac{1}{6} \\
&= \frac{n+2}{3},
\end{aligned}$$

which contradicts that $\tau(G) > \frac{n+2}{3}\cdot$ This completes the proof of Theorem 3.1. □

Remark 3.2. The condition $\tau(G) > \frac{n+2}{3}$ in Theorem 3.1 is sharp. Set $G = K_{n+2} \vee 3K_1$, where n is a nonnegative integer. It is easily seen that $\tau(G) = \frac{n+2}{3}$ and $\kappa(G) = n+2 > n+1$. For any $V' \subseteq V(K_{n+2})$ with $|V'| = n$, let $G' = G - V'$. We select $X = V(K_{n+2}) \setminus V' \subseteq V(G)$, and so $|X| = 2$. Note that X is not an independent set. Then we admit $\varepsilon_2(X) = 2$. Thus, we acquire $\text{sun}(G' - X) = 3 > 2 = 2|X| - \varepsilon_2(X)$. Using Theorem 1.4, G' is not $P_{\ge 3}$-factor covered, that is, G is not $(P_{\ge 3}, n)$-factor critical covered.

Zhou [28] verified that a connected graph G with $|V(G)| \ge 3$ is a $P_{\ge 3}$-factor covered graph if $I(G) > \frac{5}{3}\cdot$ We generalize the above result and give an isolated toughness condition for $(P_{\ge 3}, n)$-factor critical covered graphs.

Theorem 3.3. *Let G be a $(n+1)$-connected graph with $|V(G)| \ge n+3$, where $n \ge 1$ is an integer. If its isolated toughness $I(G) > \frac{n+3}{2}$, then G is a $(P_{\ge 3}, n)$-factor critical covered graph.*

Proof. If G is a complete graph, then it is easily seen that G is a $(P_{\ge 3}, n)$-factor critical covered graph by $\kappa(G) \ge n+1$. Next, we consider that G is a non-complete graph.

For any $V' \subseteq V(G)$ with $|V'| = n$, we write $H = G - V'$. Clearly, H is connected. To justify Theorem 3.3, it suffices to verify that H is $P_{\ge 3}$-factor covered. Next, we assume that H is not $P_{\ge 3}$-factor covered. Then by Theorem 1.4, there exists a subset $X \subseteq V(H)$ such that

$$\text{sun}(H - X) \ge 2|X| - \varepsilon_2(X) + 1. \tag{21}$$

Claim 3.2. $X \ne \emptyset$.

Proof. On the contrary, we assume that $X = \emptyset$. Then $\varepsilon_2(X) = 0$. It follows from (21) that

$$\text{sun}(H) = \text{sun}(H - X) \ge 2|X| - \varepsilon_2(X) + 1 = 1. \tag{22}$$

Note that H is connected since $\kappa(H) \ge \kappa(G) - |V'| \ge 1$. Combining this with (22), $1 \le \text{sun}(H) \le \omega(H) = 1$, that is,

$$\text{sun}(H) = \omega(H) = 1. \tag{23}$$

It follows from (23) and $|V(G)| \geq n+3$ that H is a sun with $|V(H)| \geq 3$. By the definition of big sun, we denote by R the factor-critical subgraph of H. Note that $|V(H)| = 2|V(R)| \geq 6$. Choose $X' = V' \cup V(R)$, then $i(G - X') = |V(R)| \geq 3$. Using the definition of $I(G)$, we obtain

$$I(G) \leq \frac{|X'|}{i(G-X')} = \frac{|V'| + |V(R)|}{|V(R)|} = 1 + \frac{n}{|V(R)|} \leq \frac{n+3}{3},$$

which contradicts that $I(G) > \frac{n+3}{2}$ in Theorem 3.3. Hence, Claim 3.2 is verified. □

Assume that there exist a isolated vertices, b isolated edges and c big sun components $Q_1, Q_2, \ldots, Q_c$, where $|V(Q_i)| \geq 6$ for $1 \leq i \leq c$, in $H - X$. By (21), we get

$$\mathrm{sun}(H - X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1. \tag{24}$$

Case 1. $|X| = 1$.
Clearly, $\varepsilon_2(X) \leq 1$ by the definition of $\varepsilon_2(X)$. In terms of (24), we obtain

$$\mathrm{sun}(H - X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| = 2. \tag{25}$$

We choose one vertex from every K_2 component of $H - X$, and use Y_1 to denote the set of such vertices. For every Q_i, we denote the factor-critical subgraph of Q_i by R_i. We choose one vertex $y_i \in V(R_i)$ for $1 \leq i \leq c$, and write $Y_2 = \{y_1, y_2, \ldots, y_c\}$. Apparently, we obtain

$$i(G - (V' \cup X \cup Y_1 \cup Y_2)) = a + b + c \geq 2. \tag{26}$$

In terms of (25), (26) and the definition of $I(G)$, we obtain

$$\begin{aligned} I(G) &\leq \frac{|V' \cup X \cup Y_1 \cup Y_2|}{i(G - V' \cup X \cup Y_1 \cup Y_2)} \\ &= \frac{n+1+b+c}{a+b+c} \\ &= \frac{n+1+\mathrm{sun}(H-X) - a}{\mathrm{sun}(H-X)} \\ &\leq 1 + \frac{n+1}{\mathrm{sun}(H-X)} \\ &\leq 1 + \frac{n+1}{2} \\ &= \frac{n+3}{2}, \end{aligned}$$

which contradicts that $I(G) > \frac{n+3}{2}$.

Case 2. $|X| \geq 2$.
Clearly, $\varepsilon_2(X) \leq 2$ by the definition of $\varepsilon_2(X)$. In terms of (24), we obtain

$$\mathrm{sun}(H - X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| - 1 \geq 3. \tag{27}$$

It follows immediately that

$$|X| \leq \frac{a+b+c+1}{2}. \tag{28}$$

We choose one vertex from every K_2 component of $H - X$, and use Y_0 to denote the set of such vertices. We use R_i to denote the factor-critical subgraph of Q_i for each Q_i, and set $Y_i = V(R_i)$, where $i = 1, 2, \ldots, c$. Obviously, $|Y_0| = b$ and $i(H_i - Y_i) = |Y_i| = \frac{|V(H_i)|}{2}$. Let $Y := \bigcup_{i=0}^{c} Y_i$. Then by (27), we obtain

$$i(G - (V' \cup X \cup Y)) = a + b + \sum_{i=1}^{c} |Y_i| = a + b + \sum_{i=1}^{c} \frac{|V(H_i)|}{2} \geq a + b + c \geq 3. \tag{29}$$

Combining (28), (29) with the definition of $I(G)$, we have

$$\begin{aligned}
\frac{n+3}{2} &< I(G) \\
&\leq \frac{|V' \cup X \cup Y|}{i(G - V' \cup X \cup Y)} \\
&= \frac{|V'| + |X| + b + \sum_{i=1}^{c} \frac{|V(H_i)|}{2}}{a + b + \sum_{i=1}^{c} \frac{|V(H_i)|}{2}} \\
&\leq 1 + \frac{|V'| + |X|}{a + b + \sum_{i=1}^{c} \frac{|V(H_i)|}{2}} \\
&\leq 1 + \frac{n + |X|}{a + b + c} \\
&\leq 1 + \frac{n}{a + b + c} + \frac{a + b + c + 1}{2(a + b + c)} \\
&\leq 1 + \frac{n}{3} + \frac{2}{3} \\
&= \frac{n+5}{3},
\end{aligned}$$

a contradiction. This completes the proof of Theorem 3.3. □

Remark 3.4. The condition $I(G) > \frac{n+3}{2}$ in Theorem 3.3 cannot be replaced by $I(G) \geq \frac{n+3}{3}$. Set $G = K_{n+1} \vee (2K_2 \cup K_1)$, where n is a positive integer. It is easily seen that $I(G) = \frac{n+3}{3}$ and $\kappa(G) = n + 1$. For any $V' \subseteq V(K_{n+1})$ with $|V'| = n$, let $G' = G - V'$. We select $X = V(K_{n+1}) \setminus V' \subseteq V(G)$, and so $|X| = 1$. Then we admit $0 \leq \varepsilon_2(X) \leq 1$. Thus, we acquire $\mathrm{sun}(G' - X) = 3 > 2 \geq 2|X| - \varepsilon_2(X)$. Using Theorem 1.4, G' is not $P_{\geq 3}$-factor covered, that is, G is not $(P_{\geq 3}, n)$-factor critical covered.

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