TOUGHNESS AND ISOLATED TOUGHNESS CONDITIONS FOR PATH-FACTOR CRITICAL COVERED GRAPHS

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Abstract. Given a graph $G$ and an integer $k \geq 2$. A spanning subgraph $H$ of $G$ is called a $P_{2k}$-factor of $G$ if every component of $H$ is a path with at least $k$ vertices. A graph $G$ is said to be $P_{2k}$-factor covered if for any $e \in E(G)$, $G$ admits a $P_{2k}$-factor including $e$. A graph $G$ is called a ($P_{2k}, n$)-factor critical graph if $G - V'$ is $P_{2k}$-factor covered for any $V' \subseteq V(G)$ with $|V'| = n$. In this paper, we study the toughness and isolated toughness conditions for ($P_{2k}, n$)-factor critical covered graphs, where $k = 2, 3$. Let $G$ be a $(n + 1)$-connected graph. It is shown that (i) $G$ is a ($P_{22}, n$)-factor critical graph if its toughness $\tau(G) > \frac{n+2}{2}$; (ii) $G$ is a ($P_{22}, n$)-factor critical graph if its isolated toughness $I(G) > \frac{n+2}{2}$; (iii) $G$ is a ($P_{23}, n$)-factor critical graph if $\tau(G) > \frac{n+2}{2}$ and $|V(G)| \geq n + 3$; (iv) $G$ is a ($P_{23}, n$)-factor critical graph if $I(G) > \frac{n+2}{2}$ and $|V(G)| \geq n + 3$. Furthermore, we claim that these conditions are best possible in some sense.

Keywords. Graph, Path-factor, Toughness, Isolated toughness, ($P_{22}, n$)-factor critical covered graph, ($P_{23}, n$)-factor critical covered graph.

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1. Introduction

All graphs considered here are finite and simple. We refer to [5] for the notation and terminologies not defined here. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Given a vertex $v \in V(G)$, let $d_G(v)$ denote the degree of $v$, that is, the number of edges incident to $v$ in $G$. If $d_G(v) = 0$ for some vertex $v$ in $G$, then $v$ is said to be an isolated vertex in $G$. Let $i(G)$ be the number of isolated vertices of $G$. For any subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ induced by $S$, and $G - S := G[V(G) \setminus S]$ is the resulting graph after deleting the vertices of $S$ from $G$. The number of connected components of a graph $G$ is denoted by $\omega(G)$. We write $\kappa(G)$ for the vertex connectivity of $G$.

Next, we introduce two parameters for a graph, namely, the toughness and the isolated toughness. The toughness of $G$ was first introduced by Chvátal [6] as

$$
\tau(G) = \min \left\{ \frac{|X|}{\omega(G-X)} : X \subseteq V(G), \omega(G-X) \geq 2 \right\},
$$

if $G$ is not complete; otherwise, $\tau(G) = +\infty$.

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The isolated toughness of $G$ was defined by Yang et al. [20] as

$$I(G) = \min \left\{ \frac{|X|}{i(G-X)} : X \subseteq V(G), i(G-X) \geq 2 \right\},$$

if $G$ is not complete; otherwise, $I(G) = +\infty$.

A subgraph $H$ of $G$ is called a spanning subgraph of $G$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. For a family of connected graphs $\mathcal{F}$, a spanning subgraph $H$ of a graph $G$ is called an $\mathcal{F}$-factor of $G$ if each component of $H$ is isomorphic to some graph in $\mathcal{F}$. A spanning subgraph $H$ of a graph $G$ is called a $P_{\geq k}$-factor of $G$ if every component of $H$ is isomorphic to a path of order at least $k$, where $k \geq 2$ is an integer. For example, a $P_{\geq 3}$-factor means a graph factor in which every component is a path of order at least three.

Since Tutte proposed the well-known Tutte 1-factor theorem [17], there are many results on path-factor [3, 8, 10, 12, 14, 15, 18, 27] and path-factor covered graphs [7, 9, 19, 22–26, 28]. For more other results on graph factors, we refer the reader to the survey papers and books [1, 16, 21].

Akiyama et al. [2] provided a criterion for a graph having a $P_{\geq 2}$-factor as follows.

**Theorem 1.1** (Akiyama et al. [2]). A graph $G$ has a $P_{\geq 2}$-factor if and only if $i(G-X) \leq 2|X|$ for all $X \subseteq V(G)$.

In order to characterize a graph possessing a $P_{\geq 3}$-factor, Kaneko [11] put forward the concept of a sun as follows. A graph $R$ is called factor-critical if $R - v$ has a 1-factor for each $v \in V(R)$. Let $R$ be a factor-critical graph and $V(R) = \{v_1, v_2, \ldots, v_n\}$. By adding new vertices $\{u_1, u_2, \ldots, u_n\}$ together with new edges $\{v_iu_i : 1 \leq i \leq n\}$ to $R$, the resulting graph is called a sun. Note that, according to Kaneko [11], we regard $K_1$ and $K_2$ also as a sun, respectively. Usually, the suns other than $K_1$ and $K_2$ are also called big suns. It is called a sun component of $G$ if the component of $G$ is isomorphic to a sun. We denote by $\text{sun}(G)$ the number of sun components in $G$.


**Theorem 1.2** (Kaneko [11]). A graph $G$ has a $P_{\geq 3}$-factor if and only if $\text{sun}(G-X) \leq 2|X|$ for all $X \subseteq V(G)$.

Zhang and Zhou [22] first defined a graph $G$ to be $P_{\geq k}$-factor covered if $G$ admits a $P_{\geq k}$-factor containing $e$ for any $e \in E(G)$. In the same paper, they obtained a characterization for $P_{\geq 2}$-factor covered graphs and $P_{\geq 3}$-factor covered graphs, respectively.

**Theorem 1.3** (Zhang and Zhou [22]). Let $G$ be a connected graph. Then $G$ is a $P_{\geq 2}$-factor covered graph if and only if $i(G-X) \leq 2|X| - \varepsilon_1(X)$ for all $X \subseteq V(G)$, where $\varepsilon_1(X)$ is defined by

$$\varepsilon_1(X) = \begin{cases} 2 & \text{if } X \neq \emptyset \text{ and } X \text{ is not an independent set;} \\ 1 & \text{if } X \text{ is a nonempty independent set and there exists a nontrivial component of } G - X; \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 1.4** (Zhang and Zhou [22]). Let $G$ be a connected graph. Then $G$ is a $P_{\geq 3}$-factor covered graph if and only if $\text{sun}(G-X) \leq 2|X| - \varepsilon_2(X)$ for all $X \subseteq V(G)$, where $\varepsilon_2(X)$ is defined by

$$\varepsilon_2(X) = \begin{cases} 2 & \text{if } X \neq \emptyset \text{ and } X \text{ is not an independent set;} \\ 1 & \text{if } X \text{ is a nonempty independent set and there exists a non \text{--} sun component of } G - X; \\ 0 & \text{otherwise.} \end{cases}$$

A graph $G$ is a $(P_{\geq k}, n)$-factor critical covered graph if for any $V' \subseteq V(G)$ with $|V'| = n$, $G - V'$ is $P_{\geq k}$-factor covered. In this paper, we study $(P_{\geq k}, n)$-factor critical covered graphs and get some sufficient conditions for graphs to be $(P_{\geq k}, n)$-factor critical covered graphs depending on toughness and isolated toughness, which are given in Sections 2 and 3.
2. \((P_{\geq 2}, n)\)-FACTOR CRITICAL COVERED GRAPHS

In this section, using toughness and isolated toughness, we obtain two sufficient conditions for the existence of \((P_{\geq 2}, n)\)-factor critical covered graphs.

**Theorem 2.1.** Let \(G\) be a \((n + 1)\)-connected graph, where \(n \geq 0\) is an integer. If its toughness \(\tau(G) > \frac{n+2}{3}\), then \(G\) is a \((P_{\geq 2}, n)\)-factor critical covered graph.

**Proof.** If \(G\) is a complete graph, then it is easily seen that \(G\) is a \((P_{\geq 2}, n)\)-factor critical covered graph by \(\kappa(G) \geq n + 1\). Next, we consider that \(G\) is a non-complete graph.

For any \(V' \subseteq V(G)\) with \(|V'| = n\), we write \(H = G - V'\). Clearly, \(H\) is connected and \(|V(H)| \geq 2\). To justify Theorem 2.1, it suffices to verify that \(H\) is \(P_{\geq 2}\)-factor covered. Next, we assume that \(H\) is not \(P_{\geq 2}\)-factor covered. Then by Theorem 1.3, there exists a subset \(X \subseteq V(H)\) such that

\[
i(H - X) \geq 2|X| - \varepsilon_1(X) + 1. \tag{1}
\]

**Claim 2.1.** \(X \neq \emptyset\).

**Proof.** On the contrary, we assume that \(X = \emptyset\). Then \(\varepsilon_1(X) = 0\). It follows from (1) that

\[
i(H) = i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 = 1. \tag{2}
\]

Note that \(H\) is connected since \(\kappa(H) \geq \kappa(G) - |V'| \geq 1\). Combining this with (2), \(1 \leq i(H) \leq \omega(H) = 1\), that is,

\[
i(H) = \omega(H) = 1. \tag{3}
\]

It follows from (3) that \(H\) is an isolated vertex, which contradicts that \(|V(H)| \geq 2\). Hence, Claim 2.1 is verified.

Next, we will distinguish two cases below to complete the proof of Theorem 2.1.

**Case 1.** \(|X| = 1\).

In this case, we obtain \(\varepsilon_1(X) \leq 1\). By the definition of \(\varepsilon_1(X)\), we have \(\varepsilon_1(X) = 1\) if there is a nontrivial component of \(H - X\). Otherwise, \(\varepsilon_1(X) = 0\). If \(\varepsilon_1(X) = 1\), then it follows from (1) that \(i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 = 2\), and thus

\[
\omega(H - X) \geq i(H - X) + 1 \geq 3. \tag{4}
\]

If \(\varepsilon_1(X) = 0\), then by (1), we have

\[
\omega(H - X) \geq i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 = 3. \tag{5}
\]

Combining (4), (5) with the definition of \(\tau(G)\), we obtain

\[
\frac{n + 2}{3} < \tau(G) \leq \frac{|X \cup V'|}{\omega(G - X \cup V')} = \frac{|V'| + |X|}{\omega(H - X)} \leq \frac{n + 1}{3},
\]

which is a contradiction.

**Case 2.** \(|X| \geq 2\).

By the definition of \(\varepsilon_1(X)\), we obtain \(\varepsilon_1(X) \leq 2\). It follows from (1) and \(H = G - V'\) that

\[
\omega(H - X) \geq i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq 2|X| - 1 \geq 3. \tag{6}
\]

According to (6), we have

\[
|X| \leq \frac{\omega(H - X) + 1}{2}. \tag{7}
\]
In terms of (6), (7) and the definition of \( \tau(G) \), we obtain

\[
\tau(G) \leq \frac{|X \cup V'|}{\omega(G - X \cup V')}
= \frac{|V'| + |X|}{\omega(H - X)}
\leq \frac{n + \omega(H - X) + 1}{2 \omega(H - X)}
= \frac{1}{2} + \frac{n}{2 \omega(H - X)} + \frac{1}{2 \omega(H - X)}
\leq \frac{n + 2}{3},
\]

which contradicts that \( \tau(G) > \frac{n+2}{3} \). This completes the proof of Theorem 2.1.

\[\square\]

**Remark 2.2.** The condition \( \tau(G) > \frac{n+2}{3} \) in Theorem 2.1 is sharp. Set \( G = K_{n+2} \lor 3K_1 \), where \( n \) is a nonnegative integer, and \( \lor \) denotes “join”. It is easily seen that \( \tau(G) = \frac{n+2}{3} \) and \( \kappa(G) = n+2 > n+1 \). For any \( V' \subseteq V(K_{n+2}) \) with \( |V'| = n \), let \( G' = G - V' \). We select \( X = V(K_{n+2}) \setminus V' \subseteq V(G) \), and so \( |X| = 2 \). Note that \( X \) is not an independent set. Then we admit \( \varepsilon_1(X) = 2 \). Thus, we acquire \( i(G' - X) = 3 > 2 = 2|X| - \varepsilon_1(X) \). Using Theorem 1.3, \( G' \) is not \( P_{\geq 2}\)-factor critical covered, that is, \( G \) is not \((P_{\geq 2}, n)\)-factor critical covered.

Note that a \((P_{\geq k}, 0)\)-factor critical covered graph is simply called a \( P_{\geq k}\)-factor covered graph, where \( k = 2, 3, \ldots \). Dai [7] verified that a connected graph \( G \) is a \( P_{\geq 2}\)-factor covered graph if \( I(G) > \frac{2}{3} \). We generalize the above result and give an isolated toughness condition for a graph being a \((P_{\geq 2}, n)\)-factor critical covered graph.

**Theorem 2.3.** Let \( G \) be a \((n + 1)\)-connected graph, where \( n \geq 1 \) is an integer. If its isolated toughness \( I(G) > \frac{n+1}{2} \), then \( G \) is a \((P_{\geq 2}, n)\)-factor critical covered graph.

**Proof.** If \( G \) is a complete graph, then it is easily seen that \( G \) is a \((P_{\geq 2}, n)\)-factor critical covered graph by \( \kappa(G) \geq n + 1 \). Next, we consider that \( G \) is a non-complete graph.

For any \( V' \subseteq V(G) \) with \( |V'| = n \), we write \( H = G - V' \). Clearly, \( H \) is connected and \( |V(H)| \geq 2 \). To justify Theorem 2.3, it suffices to verify that \( H \) is \( P_{\geq 2}\)-factor covered. Next, we assume that \( H \) is not \( P_{\geq 2}\)-factor covered. Then by Theorem 1.3, there exists a subset \( X \subseteq V(H) \) such that

\[
i(H - X) \geq 2|X| - \varepsilon_1(X) + 1. \tag{8}
\]

**Claim 2.2.** \( X \neq \emptyset \).

**Proof.** On the contrary, we assume that \( X = \emptyset \). Then \( \varepsilon_1(X) = 0 \). It follows from (8) that

\[
i(H) = i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 = 1. \tag{9}
\]

Note that \( H \) is connected since \( \kappa(H) \geq \kappa(G) - |V'| \geq 1 \). Combining this with (9), \( 1 \leq i(H) \leq \omega(H) = 1 \), that is,

\[
i(H) = \omega(H) = 1. \tag{10}
\]

It follows from (10) that \( H \) is an isolated vertex, which contradicts that \( |V(H)| \geq 2 \). Hence, Claim 2.2 is verified. \[\square\]
Case 1. $|X| = 1$.
In this case, we obtain $\varepsilon_1(X) \leq 1$. Then by (8), we get
$$i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq 2.$$ (11)
Combining (11) with the fact that $I(G) > \frac{n+1}{2}$, we obtain
$$\frac{n + 1}{2} < I(G) \leq \frac{|X \cup V'|}{i(G - X \cup V')} = \frac{|V'| + |X|}{i(H - X)} \leq \frac{n + 1}{2},$$
a contradiction.

Case 2. $|X| \geq 2$.
By the definition of $\varepsilon_1(X)$, we obtain $\varepsilon_1(X) \leq 2$. It follows from (8) and $H = G - V'$ that
$$i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq 2|X| - 1 \geq 3.$$ (12)
According to (12), we have
$$|X| \leq \frac{i(H - X) + 1}{2}.$$ (13)
In terms of (12), (13) and the definition of $I(G)$, we obtain
$$I(G) \leq \frac{|X \cup V'|}{i(G - X \cup V')} = \frac{|V'| + |X|}{i(H - X)} \leq \frac{n + \frac{i(H - X) + 1}{2}}{i(H - X)}$$
$$= \frac{1}{2} + \frac{n}{i(H - X)} + \frac{1}{2i(H - X)}$$
$$\leq \frac{1}{2} + \frac{n}{3} + \frac{1}{6}$$
$$= \frac{n + 2}{3},$$
which contradicts that $I(G) > \frac{n+2}{3}$. This completes the proof of Theorem 2.3. □

Remark 2.4. The condition $I(G) > \frac{n+1}{2}$ in Theorem 2.3 is sharp. Set $G = K_{n+1} \lor (2K_1 \lor K_2)$, where $n$ is a positive integer. It is easily seen that $I(G) = \frac{n+1}{2}$ and $\kappa(G) = n + 1$. For any $V' \subseteq V(K_{n+1})$ with $|V'| = n$, let $G' = G - V'$.

We select $X = V(K_{n+1}) \setminus V' \subseteq V(G)$, and so $|X| = 1$. Note that $X$ is not an independent set, and $G' - X$ has a nontrivial component. Then we admit $\varepsilon_1(X) = 1$. Thus, we acquire $i(G' - X) = 2 > 1 = 2|X| - \varepsilon_1(X)$. Using Theorem 1.3, $G'$ is not $P_{\geq 2}$-factor covered, that is, $G$ is not $(P_{\geq 2}, n)$-factor critical covered.

3. $(P_{\geq 3}, n)$-FACTOR CRITICAL COVERED GRAPHS

In this section, we obtain two sufficient conditions for $(P_{\geq 3}, n)$-factor critical covered graphs by toughness and isolated toughness. Bazgan et al. [4] verified that a connected graph $G$ with $|V(G)| \geq 3$ has a $P_{\geq 3}$-factor if $\tau(G) \geq 1$. We generalize the above result and give a toughness condition for a graph being a $(P_{\geq 3}, n)$-factor critical covered graph.
Theorem 3.1. Let $G$ be a $(n + 1)$-connected graph with $|V(G)| \geq n + 3$, where $n \geq 0$ is an integer. If its toughness $\tau(G) > \frac{n + 2}{3}$, then $G$ is a $(P_{\geq 3}, n)$-factor critical covered graph.

Proof. If $G$ is a complete graph, then it is easily seen that $G$ is a $(P_{\geq 3}, n)$-factor critical covered graph by $\kappa(G) \geq n + 1$. Next, we consider that $G$ is a non-complete graph.

For any $V' \subseteq V(G)$ with $|V'| = n$, we write $H = G - V'$. Clearly, $H$ is connected. To justify Theorem 3.1, it suffices to verify that $H$ is $P_{\geq 3}$-factor covered. Next, we assume that $H$ is not $P_{\geq 3}$-factor covered. Then by Theorem 1.4, there exists a subset $X \subseteq V(H)$ such that

$$\text{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1.$$  \hspace{1cm} (14)

Claim 3.1. $X \neq \emptyset$.

Proof. On the contrary, we assume that $X = \emptyset$. Then $\varepsilon_2(X) = 0$. It follows from (14) that

$$\text{sun}(H) = \text{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 = 1.$$  \hspace{1cm} (15)

Note that $H$ is connected since $\kappa(H) \geq \kappa(G) - |V'| \geq 1$. Combining this with (15), $1 \leq \text{sun}(H) \leq \omega(H) = 1$, that is,

$$\text{sun}(H) = \omega(H) = 1.$$  \hspace{1cm} (16)

It follows from (16) and $|V(G)| \geq n + 3$ that $H$ is a sun with $|V(H)| \geq 3$. By the definition of big sun, we denote $R$ the factor-critical subgraph of $H$. Note that $|V(H)| = 2|V(R)| \geq 6$. Let $r \in V(R)$ and $R' := R \setminus \{r\}$. Choose $X' = V' \cup V(R')$, then $\omega(G - X') = |V(R')| + 1 = |V(R)| \geq 3$. Using the definition of $\tau(G)$, we obtain

$$\tau(G) \leq \frac{|X'|}{\omega(G - X')} = \frac{|V'| + |V(R')|}{|V(R)|} = 1 + \frac{n - 1}{|V(R)|} \leq \frac{n + 2}{3},$$

which contradicts that $\tau(G) > \frac{n + 2}{3}$ in Theorem 3.1. Hence, Claim 3.1 is verified. \hfill \square

Next, we will distinguish two cases below to completes the proof of Theorem 3.1.

Case 1. $|X| = 1$.

In this case, we obtain $\varepsilon_2(X) \leq 1$. By the definition of $\varepsilon_2(X)$, we have $\varepsilon_2(X) = 1$ if there is a non-sun component of $H - X$. Otherwise, $\varepsilon_2(X) = 0$. If $\varepsilon_2(X) = 1$, then it follows from (14) that $\text{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 = 2$, and thus

$$\omega(H - X) \geq \text{sun}(H - X) + 1 \geq 3.$$  \hspace{1cm} (17)

If $\varepsilon_2(X) = 0$, then by (14), we have

$$\omega(H - X) \geq \text{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 = 3.$$  \hspace{1cm} (18)

Combining (17), (18) with the the definition of $\tau(G)$, we obtain

$$\frac{n + 2}{3} < \tau(G) \leq \frac{|X \cup V'|}{\omega(G - X \cup V')} = \frac{|V'| + |X|}{\omega(H - X)} \leq \frac{n + 1}{3},$$

which is a contradiction.

Case 2. $|X| \geq 2$.

By the definition of $\varepsilon_2(X)$, we obtain $\varepsilon_2(X) \leq 2$. It follows from (14) and $H = G - V'$ that

$$\omega(H - X) \geq \text{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| - 1 \geq 3.$$  \hspace{1cm} (19)
According to (19), we have
\[ |X| \leq \frac{\omega(H - X) + 1}{2}. \]  
(20)

In terms of (19), (20) and the definition of \( \tau(G) \), we obtain
\[ \tau(G) \leq \frac{|X \cup V'|}{\omega(G - X \cup V')} = \frac{|V'| + |X|}{\omega(H - X)} = \frac{n + \omega(H - X) + 1}{2 \omega(H - X)} = \frac{1}{2} + \frac{n}{3} + \frac{1}{6} = \frac{n + 2}{3}, \]
which contradicts that \( \tau(G) > \frac{n+2}{3} \). This completes the proof of Theorem 3.1. \( \square \)

**Remark 3.2.** The condition \( \tau(G) > \frac{n+2}{3} \) in Theorem 3.1 is sharp. Set \( G = K_{n+2} \lor 3K_1 \), where \( n \) is a nonnegative integer. It is easily seen that \( \tau(G) = \frac{n+2}{3} \) and \( \kappa(G) = n + 2 > n + 1 \). For any \( V' \subseteq V(K_{n+2}) \) with \( |V'| = n \), let \( G' = G - V' \). We select \( X = V(K_{n+2}) \setminus V' \subseteq V(G) \), and so \( |X| = 2 \). Note that \( X \) is not an independent set. Then we admit \( \varepsilon_2(X) = 2 \). Thus, we acquire \( \text{sun}(G' - X) = 3 > 2 = 2|X| - \varepsilon_2(X) \). Using Theorem 1.4, \( G' \) is not \( P_{\geq 3} \)-factor covered, that is, \( G \) is not \( (P_{\geq 3}, n) \)-factor critical covered.

Zhou [28] verified that a connected graph \( G \) with \( |V(G)| \geq 3 \) is a \( P_{\geq 3} \)-factor covered graph if \( I(G) > \frac{5}{3} \). We generalize the above result and give an isolated toughness condition for \( (P_{\geq 3}, n) \)-factor critical covered graphs.

**Theorem 3.3.** Let \( G \) be a \( (n+1) \)-connected graph with \( |V(G)| \geq n+3 \), where \( n \geq 1 \) is an integer. If its isolated toughness \( I(G) > \frac{n+3}{2} \), then \( G \) is a \( (P_{\geq 3}, n) \)-factor critical covered graph.

**Proof.** If \( G \) is a complete graph, then it is easily seen that \( G \) is a \( (P_{\geq 3}, n) \)-factor critical covered graph by \( \kappa(G) \geq n+1 \). Next, we consider that \( G \) is a non-complete graph.

For any \( V' \subseteq V(G) \) with \( |V'| = n \), we write \( H = G - V' \). Clearly, \( H \) is connected. To justify Theorem 3.3, it suffices to verify that \( H \) is \( P_{\geq 3} \)-factor covered. Next, we assume that \( H \) is not \( P_{\geq 3} \)-factor covered. Then by Theorem 1.4, there exists a subset \( X \subseteq V(H) \) such that
\[ \text{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1. \]  
(21)

**Claim 3.2.** \( X \neq \emptyset \).

**Proof.** On the contrary, we assume that \( X = \emptyset \). Then \( \varepsilon_2(X) = 0 \). It follows from (21) that
\[ \text{sun}(H) = \text{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 = 1. \]  
(22)

Note that \( H \) is connected since \( \kappa(H) \geq \kappa(G) - |V'| \geq 1 \). Combining this with (22), \( 1 \leq \text{sun}(H) \leq \omega(H) = 1 \), that is,
\[ \text{sun}(H) = \omega(H) = 1. \]  
(23)
It follows from (23) and $|V(G)| \geq n + 3$ that $H$ is a sun with $|V(H)| \geq 3$. By the definition of big sun, we denote by $R$ the factor-critical subgraph of $H$. Note that $|V(H)| = 2|V(R)| \geq 6$. Choose $X' = V' \cup V(R)$, then $i(G - X') = |V(R)| \geq 3$. Using the definition of $I(G)$, we obtain

$$I(G) \leq \frac{|X'|}{i(G - X')} = \frac{|V'| + |V(R)|}{|V(R)|} = 1 + \frac{n}{|V(R)|} \leq \frac{n + 3}{3},$$

which contradicts that $I(G) > \frac{n + 3}{2}$ in Theorem 3.3. Hence, Claim 3.2 is verified.

Assume that there exist $a$ isolated vertices, $b$ isolated edges and $c$ big sun components $Q_1, Q_2, \ldots, Q_c$, where $|V(Q_i)| \geq 6$ for $1 \leq i \leq c$, in $H - X$. By (21), we get

$$\text{sun}(H - X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1. \tag{24}$$

**Case 1.** $|X| = 1$.

Clearly, $\varepsilon_2(X) \leq 1$ by the definition of $\varepsilon_2(X)$. In terms of (24), we obtain

$$\text{sun}(H - X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| = 2. \tag{25}$$

We choose one vertex from every $K_2$ component of $H - X$, and use $Y_1$ to denote the set of such vertices. For every $Q_i$, we denote the factor-critical subgraph of $Q_i$ by $R_i$. We choose one vertex $y_i \in V(R_i)$ for $1 \leq i \leq c$, and write $Y_2 = \{y_1, y_2, \ldots, y_c\}$. Apparently, we obtain

$$i(G - (V' \cup X \cup Y_1 \cup Y_2)) = a + b + c \geq 2. \tag{26}$$

In terms of (25), (26) and the definition of $I(G)$, we obtain

$$I(G) \leq \frac{|V' \cup X \cup Y_1 \cup Y_2|}{i(G - V' \cup X \cup Y_1 \cup Y_2)} = \frac{n + 1 + b + c}{a + b + c} = \frac{n + 1 + \text{sun}(H - X) - a}{\text{sun}(H - X)} \leq 1 + \frac{n + 1}{\text{sun}(H - X)} \leq 1 + \frac{n + 1}{2} = \frac{n + 3}{2},$$

which contradicts that $I(G) > \frac{n + 3}{2}$.

**Case 2.** $|X| \geq 2$.

Clearly, $\varepsilon_2(X) \leq 2$ by the definition of $\varepsilon_2(X)$. In terms of (24), we obtain

$$\text{sun}(H - X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| - 1 \geq 3. \tag{27}$$

It follows immediately that

$$|X| \leq \frac{a + b + c + 1}{2}. \tag{28}$$
We choose one vertex from every $K_2$ component of $H - X$, and use $Y_0$ to denote the set of such vertices. We use $R_i$ to denote the factor-critical subgraph of $Q_i$ for each $Q_i$, and set $Y_i = V(R_i)$, where $i = 1, 2, \ldots, c$. Obviously, $|Y_0| = b$ and $i(H_i - Y_i) = |Y_i| = |V(H_i)| / 2$. Let $Y := \bigcup_{i=0}^{c} Y_i$. Then by (27), we obtain
\[ i(G - (V' \cup X \cup Y)) = a + b + \sum_{i=1}^{c} |Y_i| = a + b + \sum_{i=1}^{c} |V(H_i)| / 2 \geq a + b + c \geq 3. \] (29)

Combining (28), (29) with the definition of $I(G)$, we have
\[ \frac{n + 3}{2} < I(G) \]
\[ \leq \frac{|V' \cup X \cup Y|}{i(G - V' \cup X \cup Y)} \]
\[ = \frac{|V'| + |X| + b + \sum_{i=1}^{c} |V(H_i)|}{a + b + \sum_{i=1}^{c} |V(H_i)| / 2} \]
\[ \leq 1 + \frac{|V'| + |X|}{a + b + \sum_{i=1}^{c} |V(H_i)| / 2} \]
\[ \leq 1 + \frac{n + |X|}{a + b + c} \]
\[ \leq 1 + \frac{n}{a + b + c} + \frac{a + b + c + 1}{2(a + b + c)} \]
\[ \leq 1 + \frac{n}{3} + \frac{2}{3} \]
\[ = \frac{n + 5}{3}, \]

a contradiction. This completes the proof of Theorem 3.3. \qed

**Remark 3.4.** The condition $I(G) > \frac{n + 3}{2}$ in Theorem 3.3 cannot be replaced by $I(G) \geq \frac{n + 3}{2}$. Set $G = K_{n+1} \cup (2K_2 \cup K_1)$, where $n$ is a positive integer. It is easily seen that $I(G) = \frac{n + 3}{2}$ and $\kappa(G) = n + 1$. For any $V' \subseteq V(K_{n+1})$ with $|V'| = n$, let $G' = G - V'$. We select $X = V(K_{n+1}) \setminus V' \subseteq V(G)$, and so $|X| = 1$. Then we admit $0 \leq \varepsilon_2(X) \leq 1$. Thus, we acquire $\text{sun}(G' - X) = 3 > 2 \geq 2|X| - \varepsilon_2(X)$. Using Theorem 1.4, $G'$ is not $P_{2,3}$-factor covered, that is, $G$ is not $(P_{2,3}, n)$-factor critical covered.

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**References**


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