OPTIMAL REINSURANCE AND INVESTMENT WITH A COMMON SHOCK AND A RANDOM EXIT TIME

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Abstract. Under the mean-variance framework, we study the continuous-time optimal reinsurance and investment problem with a common shock and a random exit time. To describe the influence of the common shock, we propose a new interdependence mechanism between the insurance market and the financial market. It can reflect both the impact of the occurrence of a common shock and its influence degree on the two markets. Both the termination times of reinsurance and investment are random, and the random exit time is affected simultaneously by exogenous and endogenous random events. The insurer’s objective is to minimize the variance of her terminal wealth under a given level of expected terminal wealth. We derive the explicit optimal reinsurance-investment strategy by employing stochastic optimal control and Lagrange duality techniques. The influences of the market interdependence and the random exit time on the optimal strategy are demonstrated through numerical experiments. The results reveal some meaningful phenomena and provide insightful guidance for reinsurance and investment practice in reality.

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1. Introduction

Usually, insurers simultaneously purchase reinsurance to transfer potential claim risk and invest financial assets to increase wealth. Reinsurance and investment are thus two core topics of paramount importance in insurance and actuarial science. Different decision criteria have been adopted in the study of the optimal reinsurance and investment problem, such as the mean-variance (MV) criterion, the expected utility maximization, and the ruin probability minimization. Since the investment risk significantly affects an insurer’s investment decision, and the MV criterion can directly quantify the investment risk by variance, many studies adopted the MV framework as the decision criterion. See, for example, Yan and Wong [20] and Yang et al. [22, 23]. Although there have been many studies about the optimal reinsurance and investment problem, there are still two important issues worthy of further discussion: one is the general interdependence between the insurance and financial markets, the other is the random exit time of the investment. 

Keywords. Mean-variance criterion, reinsurance, investment, common shock, market interdependence, random exit time.

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market and the financial market, the other is the uncertain exit time. Almost all of the literature assumes that the insurer knows with certainty the final exit time when she makes the reinsurance and investment policy.

In reality, extreme events, such as natural disasters and catastrophic financial events, can cause serious risks in both insurance market and financial market. Huge insurance claims will inevitably have an impact on the financial market. Meanwhile, a financial crisis will also have an impact on the insurance market. Therefore, it is important for us to investigate a common shock affecting both the insurance market and the financial market. The existing results about the interdependence, caused by a common shock, between the insurance market and the financial market can be divided into the following three categories. In the first category, the insurer’s surplus process and the price process of the risky asset are correlated through a correlation coefficient. Specifically, the insurance model follows a Brownian motion with drift, and the risky asset’s price model follows a geometric Brownian motion. Bi and Cai [4] considered this kind of market interdependence. In the second category, the insurer’s surplus process and the risky asset’s price process are modeled by a common counting process. Lindskog and McNeil [13] examined this method for modeling the insurance loss and credit risk. Liang et al. [12] and Yang [21] assumed that the common counting process is a Poisson process. In the third category, the insurer’s surplus process and the price process of the risky asset are affected by a general stochastic process. Bensoussan et al. [3] assumed that general stochastic processes are modeled by a Markov chain and a diffusion process satisfying certain stochastic differential equations (SDEs), and only obtained a sufficient condition for the existence of the equilibrium strategy. When the insurance market and the financial market are affected by a general stochastic process, the resulting reinsurance and investment problem is difficult to cope with.

The above-mentioned studies have the following deficiencies. For the first and third kinds of dependencies, they assume that there is no price jump. In actual financial activities, many unexpected events can cause jumps in the prices of risky assets. For example, on the first day of the Russian–Ukrainian conflict, the sharp rise of risk aversion detonated the global financial market. Except for the sharp drop of Russian stock market, the stock markets of western European countries such as Britain, France and Germany fell deeply. It is more reasonable to consider a price process that experiences jumps. Although studies that consider the second kind of dependency have considered price jumps, they assume that the interdependency does not affect the size of claims and the size of price jumps. In addition, they assume that the number of claims and the number of price jumps are equal, and the sizes of claims and the sizes of price jumps are mutually independent after suffering the common shock. These assumptions are too restrictive and may be inconsistent with reality. To overcome the shortcomings of interdependence settings in current studies, we are devoted to proposing a general interdependence mechanism between the insurance market and the financial market.

Another shortcoming in the aforementioned literature is that the reinsurance and investment horizon is fixed beforehand. Although insurers are sure when they should begin the reinsurance and investment activity, they are usually uncertain about the termination times of reinsurance and investment. There are many exogenous and endogenous random events that can force an insurer to exit the reinsurance and/or investment market. An exit is called an exogenous exit if the insurer exits the market regardless of variations of the insurance and/or financial market environment; otherwise, it is an endogenous exit. The exogenous events could be the insurer’s serious illness, an unexpected huge consumption, etc; the endogenous events could be the sharp increase of the reinsurance premium, the rapid decrease of claims, and a crash in financial assets’ prices.

Quite a few works have considered the optimal investment and/or consumption problem with random exit time. Yaari [19] and Zeng et al. [25] studied the exogenous random exit time; Odean [16] and Kharroubi et al. [11] considered the endogenous random exit time. In practice, the random exit time may simultaneously depend on endogenous and exogenous random events, though this will make the resulting problem more complex. Blanchet-Scalliet et al. [5] first studied a general random exit time caused by both exogenous and endogenous random events. They obtained the optimal investment and consumption strategy by employing dynamic programming and martingale methods. Following this modeling framework, Yu [24] examined the continuous-time optimal investment problem with a random exit time under the MV criterion; Lv et al. [15] studied a similar problem as that in Yu [24], but they considered an incomplete financial market; Bellalah et al. [2] studied the portfolio
selection problem in the presence of information costs and short sales constraints under the expected utility maximization criterion; Huang et al. [10] considered a kind of optimal investment problem under inflation and uncertain time horizon; Wang and Wu [18] studied the MV portfolio selection with discontinuous prices and incomplete market under uncertain time horizon. However, all of these studies assume that the risky assets’ prices experience no jump. As we pointed out before, it is more reasonable to consider price jumps. In addition, the above-named studies only consider one kind of uncertain exit time, i.e., the investment exit time is uncertain. If we consider the reinsurance and investment problem, it is possible that both the reinsurance and investment exit times will be random. As we have analysed above, quite a few exogenous and endogenous random events can result in the reinsurance exit time being random. Of course, considering both types of uncertain exit times would make the problem much more difficult to solve.

Compared with many studies about investment and/or consumption problems with random exit time, there are very few investigations about reinsurance and investment problems with random exit time. To the best of our knowledge, only Gu et al. [7] studied the reinsurance and investment problem with a random exit time, where the random exit time is a stopping time and is completely determined by the risky asset’s price. That is, the random exit time arises fully from endogenous random events caused by the price fluctuation of the risky asset. In other words, it is assumed that exogenous random events and endogenous random events independent of the risky asset’s price would not affect the exit time. This is inconsistent with reality. Meanwhile, Gu et al. [7] considered a single risky asset without price jumps, and did not take into account the interdependence between the insurance market and the financial market.

Considering the state of the art mentioned above, in this article we will study the continuous-time optimal reinsurance and investment problem with a common shock and a random exit time. The insurer’s surplus process is described by an extended compound Poisson model. The insurer purchases reinsurance to reduce her claim risk and invests in the financial market to increase her wealth. The financial market includes one risk-free asset and one risky asset. The risky asset’s price process follows a jump-diffusion process. We use the correlated number of claims and number of price jumps to reflect the interdependence between the insurance market and the financial market caused by a common shock. And the correlation coefficient between the size of claims and the size of price jumps is further adopted to reflect the degree of interdependence. To more comprehensively reflect the effect of the common shock on the insurance market and the financial market, we assume there are additional claims and price jumps in the insurance model and the price process of the risky asset, respectively. To make the problem studied more realistic, we assume that both the termination times of reinsurance and investment are random. The random exit times are simultaneously caused by exogenous and endogenous random events. The dynamic programming principle and the Lagrange duality technique are used to determine the explicit optimal reinsurance-investment strategy and the corresponding efficient frontier. Finally, numerical experiments illustrate the influences of the market interdependence and the random exit time on the optimal reinsurance and investment strategy.

Compared with the existing research results, our main contributions include the following:

- To fully reflect the influence of a common shock, we propose a general interdependence mechanism between the insurance market and the financial market, which significantly extends those in Liang et al. [12] and Yang [21]. In our proposed interdependence mechanism, affected by the common shock, the number of claims and the number of price jumps may not be equal; meanwhile, the size of claims can affect the size of price jumps, and vice versa. Liang et al. [12] and Yang [21] assumed that the number of claims is equal to the number of price jumps, and the size of claims and the size of price jumps are mutually independent.
- The influence degree of a common shock on the insurance market and the financial market is quantified by the additional claims and price jumps. This consideration is more reasonable and can better reflect the effect of the common shock. This also extends the models in Liang et al. Liang et al. [12] and Yang [21], where the size of claims and the size of price jumps, before and after the common shock, are assumed to be independent and identically distributed (i.i.d.) random variables.
- We study the reinsurance and investment problem with a generic random exit time. Besides the surplus process and the price process of the risky asset, the random exit time is also correlated to other endogenous
and exogenous random events. This setting of random exit time is significantly different from that in Gu et al. [7]. Under this setting, the random exit times bring some new uncertainty to the reinsurance and investment problem.

- The explicit optimal reinsurance-investment strategy is obtained. Furthermore, we systematically examine the influences of the common shock and the random exit time on the optimal reinsurance and investment strategy through a series of numerical experiments. The results reveal some interesting phenomena and present useful insights for reinsurance and investment practice in reality.

This article is organized as follows. In Section 2, we introduce the reinsurance and investment model under the influence of a common shock. In Section 3, we formulate the reinsurance-investment problem with random exit time and derive the corresponding optimal reinsurance and investment strategy. In Section 4, numerical experiments are carried out to illustrate the obtained theoretical results. Finally, we draw some conclusions in Section 5.

2. THE REINSURANCE AND INVESTMENT MODEL UNDER A COMMON SHOCK

To describe the influence of a common shock, in this section we propose a new interdependence mechanism between the insurance market and the financial market, and construct the resulting reinsurance and investment model. Throughout this paper, we assume that there are no frictions in the insurance market and the financial market: securities can be traded continuously and there is no transaction cost or tax involved in trading. Let $T > 0$ denote the planned termination time of reinsurance and investment activities.

Due to the presence of random exit times, there are two categories of uncertainties in the economy. The first category is caused by the random fluctuation of the claims and the price of the risky asset (endogenous random events); the second category is caused by the randomness of the exit times of reinsurance and investment activities (endogenous and exogenous random events). In practice, these two kinds of uncertainties may not be independent. To reflect this and to provide a realistic description of the random exit time, we adopt a modeling framework similar to that in Blanchet-Scalliet et al. [5] and following-up researches. We assume that the random exit time simultaneously depends on endogenous and exogenous random events. To this end, let $(\Omega, \mathbb{H}, \mathbb{F}, P)$ describing the uncertainty in the economy be a filtered and complete probability space which includes the above two categories of uncertainties and all the following random variables and stochastic processes. Here $\mathbb{H} := \{\mathcal{H}_t, 0 \leq t \leq T\}$ and $\mathbb{F} := \{\mathcal{F}_t, 0 \leq t \leq T\}$ are right continuous and complete with respect to the real-world probability measure $P$. $\mathcal{F}_t$ is a $\sigma$-algebra generated by poisson processes $N_1(t), N_2(t), N_3(t), N(t)$ and Brownian motion $W(t)$, and $\mathcal{H}_t$ is the corresponding enlargement of $\mathcal{F}_t$ such that for any $t \in [0, T], \mathcal{F}_t \subseteq \mathcal{H}_t$. $\mathbb{F}$ contains all the information about surplus process and price process. $\mathbb{H}$ contains not only all the information about surplus process and price process, but also all the information about random exit times. The detailed explanation about random exit times and the specific definition of $\mathbb{H}$ will be addressed in Section 3.

2.1. Insurance market

Without reinsurance and investment, the insurer’s surplus process $X(t)$ satisfies the following extended compound Poisson model

$$X(t) = \bar{c} + ct - \sum_{i=1}^{N_1(t)} Y_i - \sum_{j=1}^{N(t)} \bar{Y}_j,$$

where the information set of the surplus process is captured by the filtration $\mathbb{F}$. $\bar{c} > 0$ is the initial surplus, $c > 0$ is the rate of insurance premium; $\{Y_i, i = 1, 2, \cdots\}$ and $\{\bar{Y}_j, j = 1, 2, \cdots\}$ respectively are sequences of i.i.d. nonnegative random variables. $Y_i$ is the amount of the $i$th claim independent of the common shock, and $\bar{Y}_j$ is the amount of the $j$th additional claim caused by the common shock. The common random variables of $\{Y_i, i = 1, 2, \cdots\}$ and $\{\bar{Y}_j, j = 1, 2, \cdots\}$ are denoted as $Y$ and $\bar{Y}$, respectively. The distribution function of $Y$ is denoted as $F_Y(\cdot)$ and the density function as $f_Y(\cdot)$, while the distribution function of $\bar{Y}$ is denoted as $F_{\bar{Y}}(\cdot)$ and...
the density function as \( \tilde{f}_Z(\cdot) \). The expectations of \( Y \) and \( \tilde{Y} \) are \( \mu_{11} = E(Y) < \infty \) and \( \bar{\mu}_{11} = E(\tilde{Y}) < \infty \), and their second-order moments are \( \mu_{12} = E(Y^2) < \infty \) and \( \bar{\mu}_{12} = E(\tilde{Y}^2) < \infty \), respectively. \( N_1(t) \) is the number of claims independent of the common shock, and \( N(t) \) is the number of claims caused by the common shock. \( N_1(t) \) and \( N(t) \) are mutually independent \( \mathbb{F} \)-Poisson processes with intensities being \( \lambda_1 \geq 0 \) and \( \lambda \geq 0 \), respectively. Moreover, we assume that \( \{Y_i, i = 1, 2, \cdots\} \) and \( \{\tilde{Y}_j, j = 1, 2, \cdots\} \) and \( N(t) \) are mutually independent.

Remark 2.1. In model (1), the additional cumulative claims \( \sum_{j=1}^{N(t)} \tilde{Y}_j \) are caused by a common shock. If the influence of the common shock is not considered, this part will disappear and model (1) becomes the usual compound Poisson model.

To transfer claim risks, an insurer can consider the reinsurance. We assume that \( a(t) \in [0, 1] \) is the insurer’s retention level. That is, for each claim, the insurer pays \( a(t) \% \) and the reinsurer pays the rest. Let \( c_1 > 0 \) be the rate of reinsurance premium. The claims are then undertaken by the reinsurer with a proportion of \( 1 - a(t) \), and the model (1) becomes

\[
dX^a(t) = [c - (1 - a(t))c_1]dt - a(t)d\left( \sum_{i=1}^{N_1(t)} Y_i \right) - a(t)d\left( \sum_{j=1}^{N(t)} \tilde{Y}_j \right),
\]

where \( X^a(t) \) denotes the surplus process of the insurer after considering reinsurance.

2.2. Financial market

The financial market includes one risk-free asset and one risky asset. The risk-free asset’s price \( B(t) \) satisfies \( dB(t) = rd(t)dt \); here \( r > 0 \) is the interest rate of the risk-free asset. The price process \( S(t) \) of the risky asset is given by

\[
dS(t) = S(t) \left[ \mu dt + \sigma dW(t) + d\left( \sum_{i=1}^{N_2(t)} Z_i \right) + d\left( \sum_{j=1}^{N_g(t)} \tilde{Z}_j \right) \right],
\]

where the information set of the risky asset’s price is captured by the filtration \( \mathbb{F} \). \( \mu \geq r \) represents the appreciation rate of the risky asset, \( \sigma > 0 \) stands for the volatility rate of the risky asset, \( W(t) \) is a standard \( \mathbb{F} \)-Brownian motion. \( \{Z_i, i = 1, 2, \cdots\} \) and \( \{\tilde{Z}_j, j = 1, 2, \cdots\} \) respectively are sequences of i.i.d. random variables. \( Z_i \) is the size of the \( i \)th price jump independent of the common shock, and \( \tilde{Z}_j \) is the size of the \( j \)th additional price jump caused by the common shock. The common random variables of \( \{Z_i, i = 1, 2, \cdots\} \) and \( \{\tilde{Z}_j, j = 1, 2, \cdots\} \) are denoted as \( Z \) and \( \tilde{Z} \), respectively. The distribution function of \( Z \) is denoted as \( \tilde{F}_Z(\cdot) \) and the density function as \( \tilde{f}_Z(\cdot) \), while the distribution function of \( \tilde{Z} \) is denoted as \( \tilde{F}_\tilde{Z}(\cdot) \) and the density function as \( \tilde{f}_\tilde{Z}(\cdot) \). The expectations of \( Z \) and \( \tilde{Z} \) are \( \tilde{\mu}_{21} = E(Z) < \infty \) and \( \bar{\mu}_{21} = E(\tilde{Z}) < \infty \), and their second-order moments are \( \mu_{22} = E(Z^2) < \infty \) and \( \bar{\mu}_{22} = E(\tilde{Z}^2) < \infty \), respectively. \( N_2(t) \) is the number of price jumps independent of the common shock, and \( N_g(t) \) is the number of price jumps caused by the common shock. \( N_2(t) \) and \( N_g(t) \) are mutually independent \( \mathbb{F} \)-Poisson processes with intensities being \( \lambda_2 \geq 0 \) and \( g(\lambda) \geq 0 \), respectively. To ensure that the last term in (3) disappears when the common shock does not exist, we assume that \( g(0) = 0 \). To guarantee the positivity of the risky asset’s price, we assume that \( Z_i > -1 \) and \( \tilde{Z}_i > -1 \). Moreover, we assume that \( \{Z_i, i = 1, 2, \cdots\} \) and \( N_2(t) \), \( \{\tilde{Z}_j, j = 1, 2, \cdots\} \) and \( N_g(t) \), \( \{Y_i, i = 1, 2, \cdots\} \) and \( \{\tilde{Y}_j, j = 1, 2, \cdots\} \), \( N_1(t) \) and \( N_2(t) \), \( N_1(t) \) and \( N_g(t) \) are mutually independent.

To describe the interdependence between the size of claims and the size of price jumps suffering from the common shock, we assume that \( \{\tilde{Y}_j, j = 1, 2, \cdots\} \) and \( \{\tilde{Z}_j, j = 1, 2, \cdots\} \) are related, and their correlation coefficient is \( \rho \), \( \rho \in [0, 1] \). Here, the requirement \( \rho \geq 0 \) reflects the fact that the size of claims and the size of price jumps caused by the common shock are positively correlated.

\footnotetext{This assertion can be proved by solving the stochastic differential equation (3) through using Poisson random measures, compensators of random measures, compensated Poisson measures as well as Itô’s formula. The detailed proof can be provided upon requirement.
Remark 2.2. In model (3), \( \sum_{j=1}^{N_g(t)} Z_j \) represents the additional cumulative price jumps in \([0, t]\) caused by the common shock. If the influence of the common shock is not considered, this term will disappear. When solving the reinsurance and investment problem later on, the correlation between \( Y \) and \( Z \) is mainly reflected in \( \mathbb{E}(YZ) \neq \mathbb{E}(Y)\mathbb{E}(Z) \). With the correlation coefficient \( \rho \) and the distribution information of \( Y \) and \( Z \), we can obtain \( \mathbb{E}(YZ) = \mathbb{E}(Y)\mathbb{E}(Z) + \rho \sqrt{\text{Var}(Y)\text{Var}(Z)} \).

From the insurance market and the financial market introduced above, we can see that the two markets become interdependent when they suffer a common shock; otherwise, they are independent. This reflects the reality that, these two markets usually operate independently; however, some big natural disasters or financial events, such as the September 11 terrorist attacks in the United States in 2001, the global financial crisis in 2008 and the oil spill in the gulf of Mexico in 2010, will inevitably have a common influence on the two markets. The above models and discussion demonstrate that the common shock brings the following characteristics in the insurance market and the financial market:

- The common shock simultaneously affects the insurance market and the financial market, which is reflected by the additional cumulative claims \( \sum_{j=1}^{N(t)} Y_j \) and the additional cumulative price jumps \( \sum_{j=1}^{N_g(t)} Z_j \), respectively.
- The proposed new interdependence mechanism quantifies the mutual influences, caused by the common shock, between the insurance market and the financial market from the following two aspects.
  
  - The number of claims and the number of price jumps are related in the sense that the mean of the number of claims is \( \mathbb{E}[N(t)] = \lambda t \), the mean of the number of price jumps is \( \mathbb{E}[N_g(t)] = g(\lambda) t \), and \( \lambda \) reflects the correlation. When \( \lambda = 0 \), the insurance market and the financial market become mutually independent.
  
  - The size of claims and the jump size of the stock price are related. Concretely, the correlation coefficient \( \rho \) describes the degree of interdependence between the insurance market and the financial market. The larger \( \rho \) is, the stronger the interdependence will be.

Remark 2.3. The proposed interdependence mechanism extends that of Liang et al. [12] and Yang [21]. They studied the following interdependence mechanism. The cumulative claims and price jumps up to time \( t \) are modeled by

\[
\sum_{i=1}^{N_1(t)+N(t)} Y_i \quad \text{and} \quad \sum_{i=1}^{N_2(t)+N(t)} Z_i,
\]

respectively. Here the Poisson process \( N(t) \) represents the influence of the common shock and reflects the interdependence between the insurance market and the financial market. This interdependence setting has the following three shortcomings. Firstly, the number of additional claims and the number of additional price jumps caused by the common shock are set to the same \( N(t) \). Secondly, it is assumed that the common shock does not affect the sizes of claims and the sizes of price jumps, as \( \{Y_i, i = 1, 2, \cdots\} \) and \( \{Z_i, i = 1, 2, \cdots\} \) respectively are i.i.d. random variables. Thirdly, with the presence of the common shock, the sizes of claims still does not affect the sizes of price jumps, and \textit{vice versa}. Our interdependence framework overcomes the above three shortcomings.

2.3. Wealth process

To increase her wealth, an insurer usually invests in the financial market. The dollar amounts invested in the risky asset and the risk-free asset at time \( t \) are denoted as \( \pi(t) \) and \( X^u(t) - \pi(t) \), respectively. Here the insurer can choose the reinsurance-investment strategy \( u := u(t) = (a(t), \pi(t)) \), and \( X^u(t) \) is the resulting wealth process after considering the reinsurance and investment. More specifically, \( X^u(t) \) satisfies the following SDE:

\[
dX^u(t) = [rX^u(t) + c - (1 - a(t))c_1 + (\mu - r)\pi(t)]dt - a(t)d\left( \sum_{i=1}^{N_1(t)} Y_i \right)
\]
\[ -a(t)d\left( \sum_{j=1}^{N(t)} \tilde{Y}_j \right) + \pi(t)dW(t) + \pi(t)d\left( \sum_{i=1}^{N_2(t)} Z_i \right) + \pi(t)d\left( \sum_{j=1}^{N_3(t)} \tilde{Z}_j \right), \tag{4} \]

with \( X^u(0) = x_0 \).

Now, we present the definition of an admissible strategy.

**Definition 2.1.** For \( t \in [0, T] \), a strategy \( u(t) \) is called admissible if it satisfies: (i) \( u(t) \) is \( \mathcal{H}_t \)-predictable; (ii) \( E\left( \int_0^T \pi^1(t) \, dt \right) < +\infty \); (iii) \( u(t) \) is \([0, 1], \forall t \in [0, T] \); (iv) The SDE (4) with respect to \( u(t) \) has a unique strong solution \( X^u(t) \) with \( E[^{\sup_{s \in [t, T]}} X^u(s)^4] < +\infty \).

We denote the set of all admissible reinsurance and investment strategies on the time interval \([0, T] \) by \( U \).

### 3. Problem formulation and optimal reinsurance-investment strategy

In this section, we formulate a MV-type optimization model for the reinsurance-investment problem with a common shock and a random exit time, and derive the optimal reinsurance and investment strategy.

The insurer starts her reinsurance and investment activity at the initial time 0, and determines her optimal reinsurance and investment strategy within the time horizon \( T \). As we illustrated in the introduction, she may be forced to exit the insurance and/or financial market at uncertain times \( \tau_1 \) and \( \tau_2 \), respectively, due to some uncontrollable exogenous and endogenous random events. Here \( \tau_1 \) and \( \tau_2 \) are positive and continuous random variables. Since we simultaneously consider reinsurance and investment, the final random exit time should be \( \tau = \tau_1 \wedge \tau_2 \). Similar to that in Blanchet-Scalliet et al. [5], we do not assume that \( \tau \) is a stopping time of the natural filtration \( \mathbb{F} \) generated by \( N_1(t), N_2(t), N_3(t), N(t) \) and \( W(t) \). That is to say, observing surplus process and price process up to time \( t \) does not include the full information about whether \( \tau \) has occurred or not by time \( t \). In other words, there exist some times \( t \geq 0 \) such that the event \( \{ \tau \leq t \} \notin \mathcal{F}_t \). As what have explained in Blanchet-Scalliet et al. [5], \( \tau \), in general, depends upon both the endogenous and exogenous random events.

Due to this, to obtain the filtration containing all the information about the uncertain exit time \( \tau \), we adopt an approach of filtration enlargement similar to that in Kharroubi et al. [11] and Aksamit et al. [1]. Concretely, \( \mathbb{H} = \{\mathcal{H}_t\}_{t \in [0, T]} \) is defined by

\[ \mathcal{H}_t := \bigcap_{\epsilon > 0} \tilde{\mathcal{H}}_{t+\epsilon}, \]

for all \( 0 \leq t \leq T \), where \( \tilde{\mathcal{H}}_s := \mathcal{F}_s \vee \mathcal{G}_s \) for all \( 0 \leq s \leq T \). \( \mathcal{G}_s = \sigma(\mathbb{1}_{\{\tau \leq s\}}, v \in [0, s]) \) is a \( \sigma \)-algebra generated by the random exit time \( \tau \) up to time \( s \). Here \( \mathbb{1}_{\{\tau \leq s\}} \) is an indicator function, i.e., \( \mathbb{1}_{\{\tau \leq v\}} = 1 \) if \( \tau \leq v \); otherwise, \( \mathbb{1}_{\{\tau \leq v\}} = 0 \). It is clear that \( \mathcal{H}_t \) is the smallest right continuous enlargement of \( \mathcal{F}_t \), and \( \mathcal{H}_t \) contains all the information about surplus process, price process and random exit time up to time \( t \).

The insurer’s objective is to find an optimal reinsurance and investment strategy such that the risk measured by the variance of the terminal wealth, \( \text{Var}[X^u(T \wedge \tau)] \), is minimized under the given expected terminal wealth \( E[X^u(T \wedge \tau)] = l \); here \( l \geq x_0 \) is the prespecified wealth target. Concretely, we consider the following MV-type optimization problem

\[
\min_{u(t) \in U} \text{Var}[X^u(T \wedge \tau)] = E\left\{ (X^u(T \wedge \tau) - l)^2 \right\}
\]

s.t. \( E[X^u(T \wedge \tau)] = l \).

(5)

Note that, if \( \tau < T \) in problem (5), we would have \( u(t) = (a(t), \pi(t)) \equiv (1, 0) \) for \( \tau < t \leq T \), i.e., the insurer does not take any reinsurance and does not invest in the risky asset either.

Problem (5) is a convex programming problem. We can deal with the equality constraint by introducing a Lagrange multiplier \( \xi \in R \). Thus, problem (5) can be solved via the following problem (for every fixed \( \xi \))

\[
\min_{u(t) \in U} \left\{ E\left[ (X^u(T \wedge \tau) - l)^2 \right] + 2\xi E[X^u(T \wedge \tau) - l] \right\}.
\]

(6)
The optimal value and optimal strategy of problem (5) can then be obtained by maximizing the optimal value function of problem (6) over \( \xi \in \mathbb{R} \), according to the Lagrange duality theory. For fixed \( \xi \), problem (6) has exactly the same optimal control as the following problem

\[
\min_{u(t) \in \mathcal{U}} \mathbb{E} \left\{ |X^u(T \wedge \tau) - (l - \xi)|^2 \right\}.
\]  

(7)

Problem (7) contains the random exit time \( \tau \) and is difficult to cope with. This is due to the fact that there exist two kinds of uncertainty related to problem (5), thus problem (7). The first one is caused by the random fluctuation of claims and the price of the risky asset, i.e., markets risk, the second one is caused by the randomness of exit times of reinsurance and investment, i.e., timing risk. And these two kinds of uncertainty are usually interdependent. As we have explained before, the random exit time \( \tau \) is partially affected by \( X(t) \) and \( S(t) \). Like in Blanchet-Scalliet et al. [5], we will apply the method of separation to cope with the random exit time \( \tau \) as follows. Conditioning upon \( \mathcal{F}_t \) allows one to isolate a pure timing uncertainty component. Since \( \mathcal{F}_t \) contains all the information about \( S(t) \) and \( X(t) \) up to time \( t \), \( P(\tau > t | \mathcal{F}_t) \) is the probability that the insurer has not reached his time-horizon at date \( t \), given all possible information about \( S(t) \) and \( X(t) \). We denote by \( F(t) = P(\tau \leq t | \mathcal{F}_t) \) the conditional distribution function of timing uncertainty.

Based on the above discussion, similar to that in Blanchet-Scalliet et al. [5] and Bellalah et al. [2], we make the following assumption on \( F(t) \).

**Assumption 3.1.** \( F(t) \) is an increasing and absolutely continuous process with respect to the Lebesgue measure, with a density process denoted by \( f(t) \), e.g.,

\[
F(t) = \int_0^t f(s) \, ds, \quad 0 \leq t \leq T.
\]

Moreover, we assume that \( F(T) < 1 - \zeta \), where \( \zeta > 0 \) is a constant.

**Remark 3.1.** In Assumption 3.1, the requirement \( F(T) < 1 - \zeta \) ensures that we would not rule out the possible situation \( \tau > T \) and the term \( \mathbb{1}_{\{\tau > T\}} \) in (8) does not disappear. Otherwise, it might be inconsistent with some actual situations.

Under Assumption 3.1, by applying the Fubini theorem we can deduce that problem (7) is equivalent to the following problem

\[
\min_{u(t) \in \mathcal{U}} \mathbb{E} \left\{ |X^u(T \wedge \tau) - (l - \xi)|^2 \right\} = \min_{u(t) \in \mathcal{U}} \mathbb{E} \left\{ \left[ \mathbb{1}_{\{\tau \leq T\}} X^u(\tau) - (l - \xi) \right]^2 + \left[ \mathbb{1}_{\{\tau > T\}} X^u(T) - (l - \xi) \right]^2 \right\} = \min_{u(t) \in \mathcal{U}} \mathbb{E} \left[ \int_0^T (X^u(t) - (l - \xi))^2 \, df(t) + \int_T^{+\infty} (X^u(t) - (l - \xi))^2 \, df(t) \right] = \min_{u(t) \in \mathcal{U}} \mathbb{E} \left[ \int_0^T f(t)(X^u(t) - (l - \xi))^2 \, dt + (1 - F(T))(X^u(T) - (l - \xi))^2 \right].
\]  

(8)

To solve the unconstrained optimization problem in the last line of (8), we firstly solve an auxiliary optimization problem. To this end, we set \( X^u(t) = X^u(t) - (l - \xi) \) and \( m = c + r(l - \xi) \). Then, the SDE (4) becomes

\[
dX^u(t) = \left[ rX^u(t) + m - (1 - a(t))c_1 + (\mu - r)\pi(t) \right] \, dt - a(t) \, \left( \sum_{i=1}^{N_i(t)} Y_i \right)
\]
And we can derive the solution to problem (8) by solving the following auxiliary optimization problem

\[ \min_{u(t) \in \mathcal{U}} E \left[ \int_0^T f(t)(\bar{X}^u(t))^2 dt + (1 - F(T))(\bar{X}^u(T))^2 \right]. \]  

Before solving problem (10), we first define the value function \( V(t, \bar{x}) \) at time \( t \) as

\[ V(t, \bar{x}) = \min_{u(t) \in \mathcal{U}} E_{t, \bar{x}} \left[ \int_t^T f(s)(\bar{X}^u(s))^2 ds + (1 - F(T))(\bar{X}^u(T))^2 \right]. \]  

Here \( E_{t, \bar{x}}[\cdot] = E[\cdot | \bar{X}^u(t) = \bar{x}] \). Obviously, if \( t = 0 \), \( V(0, \bar{x}_0) \) is the optimum value of problem (10) with \( \bar{x}_0 = \bar{X}^u(0) \).

Let \( C^{1,2}([0, T] \times R) \) denote the space of \( \phi(t, \bar{x}) \), such that \( \phi(t, \bar{x}) \) and its derivatives \( \phi_u(t, \bar{x}) \), \( \phi_x(t, \bar{x}) \) and \( \phi_{xx}(t, \bar{x}) \) are continuous on \([0, T] \times R \). \( D^{1,2}([0, T] \times R) \) denotes the space of \( \phi(t, \bar{x}) \), such that \( \phi(t, \bar{x}) \in C^{1,2}([0, T] \times R) \) and \( \phi_x(t, \bar{x}) \) satisfies the linear growth condition: that is, there exists a constant \( \kappa > 0 \), such that \( |\phi_x(t, \bar{x})| \leq \kappa(1 + |\bar{x}|) \).

To more concretely discuss the quantitative effect of a common shock on the insurance market and the financial market, we examine problem (11) in two cases: \( g(\lambda) \leq \lambda \) and \( g(\lambda) > \lambda \). The case \( g(\lambda) \leq \lambda \) means that after suffering a common shock, the number of claims would be greater than the number of price jumps; the case \( g(\lambda) > \lambda \) means that after suffering a common shock, the number of price jumps would be greater than the number of claims. The solution of problem (11) for the case \( g(\lambda) > \lambda \) is similar to that of \( g(\lambda) \leq \lambda \). Therefore, to be consistent with reality, we will mainly focus on the case \( g(\lambda) \leq \lambda \) in the following. And we also provide the final results for the case \( g(\lambda) > \lambda \), for the sake of its completeness.

For presentation convenience, we first introduce the infinitesimal generator for SDE (9). If \( g(\lambda) \leq \lambda \), then for any \( \phi(t, \bar{x}) \in C^{1,2}([0, T] \times R) \) and \( u(t) \in \mathcal{U} \), the usual infinitesimal generator is defined as

\[ \mathcal{A}^u \phi(t, \bar{x}) = \phi_t(t, \bar{x}) + [r \bar{x} + m - (1 - a(t))c_1 + (\mu - r)\pi(t)]\phi_x(t, \bar{x}) + \frac{1}{2} \pi^2(t)\sigma^2 \phi_{xx}(t, \bar{x}) + \lambda_1 E[\phi(t, \bar{x} - a(t)\bar{Y}) - \phi(t, \bar{x})] + \lambda_2 E[\phi(t, \bar{x} + \pi(t)\bar{Z}) - \phi(t, \bar{x})] + (\lambda - g(\lambda))E[\phi(t, \bar{x} - a(t)\bar{Y})] + g(\lambda)E[\phi(t, \bar{x} - a(t)\bar{Y} + \pi(t)\bar{Z}) - \phi(t, \bar{x})]. \]  

Analogously, the usual infinitesimal generator \( \mathcal{B}^u \) for the case \( g(\lambda) > \lambda \) is defined as

\[ \mathcal{B}^u \phi(t, \bar{x}) = \phi_t(t, \bar{x}) + [r \bar{x} + m - (1 - a(t))c_1 + (\mu - r)\pi(t)]\phi_x(t, \bar{x}) + \frac{1}{2} \pi^2(t)\sigma^2 \phi_{xx}(t, \bar{x}) + \lambda_1 E[\phi(t, \bar{x} - a(t)\bar{Y}) - \phi(t, \bar{x})] + \lambda_2 E[\phi(t, \bar{x} + \pi(t)\bar{Z}) - \phi(t, \bar{x})] + g(\lambda - \lambda)E[\phi(t, \bar{x} + \pi(t)\bar{Z}) - \phi(t, \bar{x})] + \lambda E[\phi(t, \bar{x} - a(t)\bar{Y} + \pi(t)\bar{Z}) - \phi(t, \bar{x})]. \]  

Remark 3.2. If \( g(\lambda) < \lambda \), according to the additivity of Poisson process, the Poisson process \( N(t) \) can be written as \( N(t) = N_3(t) + N_g(t) \). Here \( N_3(t) = N(t) - N_g(t) \) is a Poisson process with intensity being \( \lambda - g(\lambda) > 0 \), and \( N_3(t) \) and \( N_g(t) \) are mutually independent Poisson processes. Therefore, if the number of price jumps of the risky asset \( N_g(t) \) increases, the number of claims \( N(t) = N_3(t) + N_g(t) \) would increase together. In (12), the term \( E[\phi(t, \bar{x} - a(t)\bar{Y} + \pi(t)\bar{Z}) - \phi(t, \bar{x})] \) is caused by the claim process and the price jump process related to the Poisson process.
process $N_g(t)$. Therefore, in (12), the term $E[\phi(t, \bar{x} - a(t)\bar{Y} + \pi(t)\bar{Z}) - \phi(t, \bar{x})]$ means that, whenever the number of price jumps of the risky asset increases after a common shock (i.e. $N_g(t)$ increases), the number of claims $N(t)$ would increase together. Similar to the case $g(\lambda) < \lambda$, in (13), the term $E[\phi(t, \bar{x} - a(t)\bar{Y} + \pi(t)\bar{Z}) - \phi(t, \bar{x})]$ means that, whenever the number of claims increases after a common shock (i.e. $N(t)$ increases), the number of price jumps of the risky asset $N_g(t)$ will increase too. In other words, $E[\phi(t, \bar{x} - a(t)\bar{Y} + \pi(t)\bar{Z}) - \phi(t, \bar{x})]$ reflects the interdependence between the insurance market and the financial market after suffering a common shock.

The following verification theorem is essential for ensuring that the reinsurance and investment strategy obtained from problem (11) is optimal.

**Theorem 3.1** (Verification theorem). Suppose that $g(\lambda) \leq \lambda$, $\widehat{W}(t, \bar{x}) \in D^{1,2}([0,T] \times R)$ satisfies the following equation

$$\inf_{u(\cdot) \in \mathcal{U}} \left\{ f(t)\bar{x}^2 + \mathcal{A}^u\widehat{W}(t, \bar{x}) \right\} = 0, \quad (14)$$

and $\widehat{W}(t, \bar{x})$ satisfies the boundary condition $\widehat{W}(T, \bar{x}) = (1 - F(T))\bar{x}^2$; define

$$u^*(t) = \arg\inf_{u(t) \in \mathcal{U}} \left\{ f(t)\bar{x}^2 + \mathcal{A}^u\widehat{W}(t, \bar{x}) \right\}, \quad \inf_{u(t) \in \mathcal{U}} \left\{ f(t)\bar{x}^2 + \mathcal{A}^u\widehat{W}(t, \bar{x}) \right\} = 0, \quad (15)$$

and thus

$$f(t)\bar{x}^2 + \mathcal{A}^{u^*}\widehat{W}(t, \bar{x}) = 0. \quad (16)$$

Then, $V(t, \bar{x}) \geq \widehat{W}(t, \bar{x})$ for any $u(t) \in \mathcal{U}$. Especially, when $u(t) = u^*(t)$, we have $V(t, \bar{x}) = \widehat{W}(t, \bar{x})$, that is, the value function $V(t, \bar{x})$ given by (11) coincides with $\widehat{W}(t, \bar{x})$, and $u^*(t)$ is actually the optimal strategy for problem (11).

**Proof.** Similar to Theorem 4.1 in Guan and Wang [8], we have directly from Dynkin’s formula that

$$E_{t, \bar{x}}\left[ \widehat{W}(T, \bar{X}^u(T)) \right] = \widehat{W}(t, \bar{x}) + E_{t, \bar{x}}\left[ \int_t^T \mathcal{A}^u\widehat{W}(s, \bar{X}^u(s)) \, ds \right]. \quad (17)$$

We first prove $V(t, \bar{x}) \geq \widehat{W}(t, \bar{x})$ for any $u(t) \in \mathcal{U}$.

For any $u(t) \in \mathcal{U}$, we have from (14) that

$$f(t)\bar{x}^2 + \mathcal{A}^u\widehat{W}(t, \bar{x}) \geq 0,$$

so we obtain from (17) that

$$E_{t, \bar{x}}\left[ \widehat{W}(T, \bar{X}^u(T)) + \int_t^T f(s)(\bar{X}^u(s))^2 \, ds \right] \geq \widehat{W}(t, \bar{x}). \quad (18)$$

According to the boundary condition $\widehat{W}(T, \bar{x}) = (1 - F(T))\bar{x}^2$, equation (18) can be rewritten as

$$E_{t, \bar{x}}\left[ (1 - F(T))(\bar{X}^u(T))^2 + \int_t^T f(s)(\bar{X}^u(s))^2 \, ds \right] \geq \widehat{W}(t, \bar{x}), \quad \text{for all } u(t) \in \mathcal{U}.$$

Due to the arbitrariness of $u(t) \in \mathcal{U}$, we can deduce that

$$V(t, \bar{x}) \geq \widehat{W}(t, \bar{x}). \quad (19)$$

Now, we prove that $V(t, \bar{x}) = \widehat{W}(t, \bar{x})$ if $u(t) = u^*(t)$. 
Combining the boundary condition \( \tilde{W}(T, \bar{x}) = (1 - F(T)) \bar{x}^2 \) and (16), we obtain from (17) that

\[
\tilde{W}(t, \bar{x}) = E_t \left[ \tilde{W} \left( T, \tilde{X}^{u*}(T) \right) + \int_t^T f(s) \left( \tilde{X}^{u*}(s) \right)^2 ds \right],
\]

This implies that

\[
\tilde{W}(t, \bar{x}) \geq V(t, \bar{x}). \tag{20}
\]

Equations (19) and (20) ensure that, when \( u(t) = u^*(t) \), we can obtain \( \tilde{W}(t, \bar{x}) = V(t, \bar{x}) \). Therefore, \( u^*(t) \) is actually the optimal strategy, and \( \tilde{W}(t, \bar{x}) \) is the corresponding optimal value function. \( \square \)

Based on Theorem 3.1, we can obtain the optimal solution to problem (11). To this end, we first introduce the following notation:

\[
\begin{aligned}
&l_1 = c_1 - \lambda_1 \mu_{11} - \lambda \mu_{11},
&l_2 = \mu - r + \lambda_2 \mu_{21} + g(\lambda)\mu_{21},
&l_3 = \bar{\sigma}^2 + \lambda_2 \mu_{22} + g(\lambda)\mu_{22},
&l_4 = -\frac{l_1 l_3 + l_2 g(\lambda)E(\bar{Y}\bar{Z})}{l_3 (\lambda \mu_{12} + \lambda_1 \mu_{21}) - \left[g(\lambda)E(\bar{Y}\bar{Z})\right]^2},
&l_5 = -\frac{l_2 (\lambda \mu_{12} + \lambda_1 \mu_{12}) + l_1 g(\lambda)E(\bar{Y}\bar{Z})}{l_3 (\lambda \mu_{12} + \lambda_1 \mu_{12}) - \left[g(\lambda)E(\bar{Y}\bar{Z})\right]^2},
&l_6 = l_1 l_4 + \frac{l_3 l_2 (\lambda \mu_{12} + \lambda_1 \mu_{12}) + l_2 l_5 + \frac{1}{2} l_3 l_5^2 - l_4 l_5 g(\lambda)E(\bar{Y}\bar{Z})}{l_3 (\lambda \mu_{12} + \lambda_1 \mu_{12}) - \left[g(\lambda)E(\bar{Y}\bar{Z})\right]^2},
&l_7 = -\frac{l_1 l_3 + l_2 \Lambda E(\bar{Y}\bar{Z})}{l_3 (\lambda \mu_{12} + \lambda_1 \mu_{12}) - \left[\Lambda E(\bar{Y}\bar{Z})\right]^2},
&l_8 = -\frac{l_2 (\lambda \mu_{12} + \lambda_1 \mu_{12}) + l_1 \Lambda E(\bar{Y}\bar{Z})}{l_3 (\lambda \mu_{12} + \lambda_1 \mu_{12}) - \left[\Lambda E(\bar{Y}\bar{Z})\right]^2},
&l_9 = l_1 l_7 + \frac{l_3 l_5 (\lambda \mu_{12} + \lambda_1 \mu_{12}) + l_2 l_8 + \frac{1}{2} l_3 l_6^2 - l_7 l_8 \Lambda E(\bar{Y}\bar{Z})}{l_3 (\lambda \mu_{12} + \lambda_1 \mu_{12}) - \left[\Lambda E(\bar{Y}\bar{Z})\right]^2}.
\end{aligned}
\tag{21}
\]

The case \( a(t) = 0 \) means that the insurer would transfer all the claims to the reinsurer. Once this happens, the insurer would rather accept the policyholder’s insurance business than take reinsurance, because the reinsurance premium is usually higher than the insurance premium. This case may occur theoretically, but it hardly occurs in practice. The case \( a(t) = 1 \) means that the insurer would bear all the claims by herself and does not carry out reinsurance. This situation is not the focus of this research. Therefore, we mainly consider the case \( 0 < a(t) < 1 \) in what follows. Concretely, we have:

**Theorem 3.2.** If \( g(\lambda) * \lambda \) and \( 0 < l_4 [\bar{x} + \frac{m-c_1}{r} h(t)] < 1 \), the optimal solution to problem (11) is given as follows.

The optimal reinsurance strategy is

\[
a^*(t) = l_4 \left[ \bar{x} + \frac{m-c_1}{r} h(t) \right], \tag{22}
\]

the optimal investment strategy is

\[
\pi^*(t) = l_5 \left[ \bar{x} + \frac{m-c_1}{r} h(t) \right], \tag{23}
\]

and the optimal value function is given by

\[
V(t, \bar{x}) = \tilde{W}(t, \bar{x}) = \frac{1}{2} P(t) \bar{x}^2 + Q(t) \bar{x} + M(t). \tag{24}
\]
Here

\[
\begin{align*}
\tilde{f}(t) &= \int_t^T f(s)e^{2(r+\lambda_0)s} \, ds, \\
\tilde{h}(t) &= \frac{(1 - F(T))\left(e^{2(r+\lambda_0)T} - e^{(r+2\lambda_0)T+rT}\right) + r \int_t^T \tilde{f}(s)e^{r(t-s)} \, ds}{(1 - F(T))e^{2(r+\lambda_0)T} + \tilde{f}(t)},
\end{align*}
\]

and

\[
\begin{align*}
P(t) &= 2(1 - F(T))e^{2(r+\lambda_0)(T-t)} + 2 \int_t^T f(s)e^{2(r+\lambda_0)(s-t)} \, ds, \\
Q(t) &= (m - c_1)e^{-(r+2\lambda_0)t} \int_t^T P(s)e^{(r+2\lambda_0)s} \, ds, \\
M(t) &= \int_t^T \left[(m - c_1)Q(s) + \frac{c_0}{P(s)} Q^2(s)\right] \, ds.
\end{align*}
\]

**Proof.** According to Theorem 3.1, we can assume that a solution to problem (11) has the following parametric form:

\[
\tilde{W}(t, \tilde{x}) = \frac{1}{2}P(t)\tilde{x}^2 + Q(t)\tilde{x} + M(t),
\]

here \(P(t), Q(t)\) and \(M(t)\), to be determined later on, respectively satisfy the boundary conditions \(P(T) = 2(1 - F(T)), Q(T) = 0, M(T) = 0\).

By (26), we have

\[
\begin{align*}
\tilde{W}_t(t, \tilde{x}) &= \frac{1}{2}P'(t)\tilde{x}^2 + Q'(t)\tilde{x} + M'(t), \\
\tilde{W}_x(t, \tilde{x}) &= P(t)\tilde{x} + Q(t), \\
\tilde{W}_{\tilde{x}}(t, \tilde{x}) &= P(t), \\
E\left[\tilde{W}(t, \tilde{x} - a(t)Y) - \tilde{W}(t, \tilde{x})\right] &= \frac{1}{2}\mu_{12}P(t)a^2(t) - [P(t)\tilde{x} + Q(t)]\mu_{11}a(t), \\
E\left[\tilde{W}(t, \tilde{x} - a(t)Y) - \tilde{W}(t, \tilde{x})\right] &= \frac{1}{2}\mu_{12}P(t)a^2(t) - [P(t)\tilde{x} + Q(t)]\mu_{11}a(t), \\
E\left[\tilde{W}(t, \tilde{x} + \pi(t)Z) - \tilde{W}(t, \tilde{x})\right] &= \frac{1}{2}\mu_{22}P(t)\pi^2(t) + [P(t)\tilde{x} + Q(t)]\mu_{21}\pi(t), \\
E\left[\tilde{W}(t, \tilde{x} + \pi(t)Z) - \tilde{W}(t, \tilde{x})\right] &= \frac{1}{2}\mu_{22}P(t)\pi^2(t) + [P(t)\tilde{x} + Q(t)]\mu_{21}\pi(t), \\
E\left[\tilde{W}(t, \tilde{x} + \pi(t)Z - a(t)Y) - \tilde{W}(t, \tilde{x})\right] &= \frac{1}{2}P(t)\left[\mu_{22}\pi^2(t) + \mu_{12}a^2(t)
\right. \\
- 2a(t)\pi(t)E(YZ)\left] + [P(t)\tilde{x} + Q(t)]\left[\mu_{21}\pi(t) - \mu_{11}a(t)\right].
\end{align*}
\]

Substituting (26) and (27) into (14), after some simplifications we have

\[
\begin{align*}
\left[\frac{1}{2}P'(t) + rP(t) + f(t)\right]\tilde{x}^2 + [Q'(t) + rQ(t) + (m - c_1)P(t)]\tilde{x}
+ M'(t) + (m - c_1)Q(t) + \inf_{a \in A} H(a(t), \pi(t)) &= 0,
\end{align*}
\]

where \(H(a(t), \pi(t))\) is given by

\[
H(a(t), \pi(t)) = \frac{1}{2}(\lambda_1\mu_{12} + \lambda\mu_{12})P(t)a^2(t) + l_1[P(t)\tilde{x} + Q(t)]a(t)
+ \frac{1}{2}l_2P(t)\pi^2(t) + l_2[P(t)\tilde{x} + Q(t)]\pi(t) - g(\lambda)P(t)E(YZ)a(t)\pi(t).
\]
Differentiating $H(a(t), \pi(t))$ with respect to $a(t)$ and $\pi(t)$, we have
\[
\frac{\partial H(a(t), \pi(t))}{\partial a(t)} = (\lambda_1 \mu_{12} + \lambda \mu_{12})P(t)a(t) + l_1[P(t)\bar{x} + Q(t)] - g(\lambda)P(t)E(\bar{Y} \bar{Z})\pi(t),
\]
(30)
and
\[
\frac{\partial H(a(t), \pi(t))}{\partial \pi(t)} = l_3P(t)\pi(t) + l_2[P(t)\bar{x} + Q(t)] - g(\lambda)P(t)E(\bar{Y} \bar{Z})a(t).
\]
(31)
Setting $\frac{\partial H(a(t), \pi(t))}{\partial a(t)} = 0$ and $\frac{\partial H(a(t), \pi(t))}{\partial \pi(t)} = 0$, we can derive the following system of equations with respect to $a(t)$ and $\pi(t)$:
\[
\begin{cases}
(\lambda_1 \mu_{12} + \lambda \mu_{12})P(t)a(t) - g(\lambda)P(t)E(\bar{Y} \bar{Z})\pi(t) = -l_1[P(t)\bar{x} + Q(t)], \\
-g(\lambda)P(t)E(\bar{Y} \bar{Z})a(t) + l_3P(t)\pi(t) = -l_2[P(t)\bar{x} + Q(t)].
\end{cases}
\]
(32)
To solve equations (32), we consider the determinant of the coefficient matrix in (32) and have
\[
\begin{vmatrix}
(\lambda_1 \mu_{12} + \lambda \mu_{12})P(t) & -g(\lambda)P(t)E(\bar{Y} \bar{Z}) \\
-g(\lambda)P(t)E(\bar{Y} \bar{Z}) & l_3P(t)
\end{vmatrix} = P^2(t) \begin{vmatrix} l_3(\lambda_1 \mu_{12} + \lambda \mu_{12}) - (g(\lambda)E(\bar{Y} \bar{Z}))^2 \\
l_3P(t) & l_3P(t)
\end{vmatrix}
\]
\[
= P^2(t) \begin{vmatrix} (\sigma^2 + 2\mu_{22} + g(\lambda)\mu_{22})(\lambda_1 \mu_{12} + \lambda \mu_{12}) - (g(\lambda)E(\bar{Y} \bar{Z}))^2 \\
l_3(\lambda_1 \mu_{12} + \lambda \mu_{12}) - (g(\lambda)E(\bar{Y} \bar{Z}))^2 & l_3P(t)
\end{vmatrix}
\]
\[
> P^2(t)(g(\lambda))^2 \left[ E(\bar{Y}^2)E(\bar{Z}^2) - (E(\bar{Y} \cdot \bar{Z}))^2 \right] \geq 0.
\]
Hence, the unique solution to the equation (32) is
\[
\bar{a}(t) = l_4 \left[ \bar{x} + \frac{Q(t)}{P(t)} \right], \quad \bar{\pi}(t) = l_5 \left[ \bar{x} + \frac{Q(t)}{P(t)} \right].
\]
(33)
Now, we prove that $(\bar{a}(t), \bar{\pi}(t))$ is a minimum point of $H(a(t), \pi(t))$.
From (30) and (31), we derive that
\[
\begin{vmatrix}
\frac{\partial^2 H(a(t), \pi(t))}{\partial (a(t))^2} & \frac{\partial H(a(t), \pi(t))}{\partial a(t)} & \frac{\partial H(a(t), \pi(t))}{\partial \pi(t)} \\
\frac{\partial H(a(t), \pi(t))}{\partial a(t)} & \frac{\partial^2 H(a(t), \pi(t))}{\partial (a(t))^2} & \frac{\partial^2 H(a(t), \pi(t))}{\partial (a(t))\partial \pi(t)} \\
\frac{\partial H(a(t), \pi(t))}{\partial \pi(t)} & \frac{\partial^2 H(a(t), \pi(t))}{\partial a(t)\partial \pi(t)} & \frac{\partial^2 H(a(t), \pi(t))}{\partial (\pi(t))^2}
\end{vmatrix}(\bar{a}(t), \bar{\pi}(t)) = l_3P(t).
\]
(34)
Later, we will prove $P(t) > 0$. Hence, the following Hessian matrix of $H(a(t), \pi(t))$ is a positive definite matrix
\[
\begin{pmatrix}
(\lambda_1 \mu_{12} + \lambda \mu_{12})P(t) & -g(\lambda)P(t)E(\bar{Y} \bar{Z}) \\
-g(\lambda)P(t)E(\bar{Y} \bar{Z}) & l_3P(t)
\end{pmatrix}.
\]
And $(\bar{a}(t), \bar{\pi}(t))$ given by (33) must be the global minimum point of $H(a(t), \pi(t))$. 
According to the definition of an admissible reinsurance strategy, if 0 < \( a(t) < 1 \), the optimal reinsurance strategy is \( a^*(t) = \bar{a}(t) \). Inserting (33) into (28), we have

\[
\begin{align*}
\left[ \frac{1}{2} P'(t) + (r + \ell_0)P(t) + f(t) \right] \bar{x}^2 + \left[ Q'(t) + (r + 2\ell_0)Q(t) \right] \bar{x} + (m - c_1)P(t) + (m - c_1)Q(t) + l_0 \frac{Q^2(t)}{P(t)} &= 0.
\end{align*}
\] (35)

To ensure that (35) always holds, the coefficients of \( \bar{x}^2 \), \( \bar{x} \) and the term without \( \bar{x} \) should equal to 0. This and the boundary conditions imply the following ordinary differential equations (ODEs) with associated terminal conditions:

\[
\begin{align*}
\frac{1}{2} P'(t) + (r + \ell_0)P(t) + f(t) &= 0, \quad P(T) = 2(1 - F(T)), \\
Q'(t) + (r + 2\ell_0)Q(t) + (m - c_1)P(t) &= 0, \quad Q(T) = 0, \\
M'(t) + (m - c_1)Q(t) + l_0 \frac{Q^2(t)}{P(t)} &= 0, \quad M(T) = 0.
\end{align*}
\]

Solving these ODEs, we obtain that \( P(t) \), \( Q(t) \) and \( M(t) \) are given by (25). From (25), it is clear that \( P(t) > 0 \). Finally, substituting (25) into (33), we can obtain the optimal reinsurance and investment strategies to problem (11), which are given by (22) and (23), respectively. Following a similar proof as that in Theorem 4.2 in Chen and Yang [6], we can prove that \( a^*(t) \) and \( \pi^*(t) \) given in (22) and (23) satisfy the Definition 2.1.

For any given \( \pi(t) \), we obtain from (29) that \( H(a(t), \pi(t)) \) is a convex function with respect to \( a(t) \). If \( \bar{a}(t) \leq 0 \), we deduce from (29) that for any given \( \pi(t) \), \( H(a(t), \pi(t)) \) is increasing with respect to \( a(t) \) in \( [\bar{a}(t), +\infty) \). These facts and the definition of an admissible reinsurance strategy mean that the optimal reinsurance strategy must be \( a^*(t) = 0 \). On the other hand, if \( \bar{a}(t) \geq 1 \), we can deduce from (29) that for any given \( \pi(t) \), \( H(a(t), \pi(t)) \) is decreasing with respect to \( a(t) \) in \( (-\infty, \bar{a}(t)] \). According to the range of an admissible reinsurance strategy and the convexity of \( H(a(t), \pi(t)) \) with respect to \( a(t) \), we conclude that the optimal reinsurance strategy must be \( a^*(t) = 1 \). As we have explained above, it is not meaningful and necessary for us to decide the optimal investment strategy and optimal value function in these two cases.

Now, we illustrate the meaning of of the optimal reinsurance and investment strategies in Theorem 3.2. From (22), we can see that the term \( g(\lambda) \) in \( l_4 \) and \( h(t) \) represents the influence of the risky asset price fluctuation on the reinsurance strategy after suffering a common shock. In (22), the \( \lambda \) in \( l_4 \) and \( h(t) \) represents the change of the reinsurance strategy after considering the effect of a common shock on the insurance market. Analogously, from (23), we can see that the term \( g(\lambda) \) in \( l_5 \) and \( h(t) \) represents the change of the investment strategy after considering the effect of a common shock on the financial market, and the \( \lambda \) in \( l_5 \) and \( h(t) \) represents the influence of the claim fluctuation on the investment strategy after suffering a common shock. In (22) and (23), the term \( f(t) \) in \( h(t) \) represents the influence of random exit time on the optimal reinsurance strategy and investment strategies.

Similar to the proof of Theorem 3.2, we can obtain the following results for the case \( g(\lambda) > \lambda \). The detailed proof is omitted.

**Theorem 3.3.** If \( g(\lambda) \geq \lambda \) and \( 0 < l_7[\bar{x} + \frac{m - c_1}{r} \bar{h}(t)] < 1 \), the optimal solution to problem (11) is given as follows.

The optimal reinsurance strategy is

\[
a^*(t) = l_7 \left[ \bar{x} + \frac{m - c_1}{r} \bar{h}(t) \right],
\] (36)

the optimal investment strategy is

\[
\pi^*(t) = l_8 \left[ \bar{x} + \frac{m - c_1}{r} \bar{h}(t) \right],
\] (37)
and the optimal value function is given by

\[ V(t, \bar{x}) = \frac{1}{2} \bar{P}(t)\bar{x}^2 + \bar{Q}(t)\bar{x} + \bar{M}(t). \]  

(38)

Here

\[ \bar{f}(t) = \int_t^T f(s)e^{2(r+l_b)s} \, ds, \]

\[ \bar{h}(t) = \frac{(1 - F(T))(e^{2(r+l_b)T} - e^{(r+2l_b)T+rt}) + r\int_t^T \bar{f}(s)e^{r(t-s)} \, ds}{(1 - F(T))e^{2(r+l_b)T} + \bar{f}(t)} \]

and

\[
\begin{align*}
\bar{P}(t) &= 2(1 - F(T))e^{2(r+l_b)(T-t)} + 2\int_t^T f(s)e^{2(r+l_b)(s-t)} \, ds, \\
\bar{Q}(t) &= (m - c_1)e^{-(r+2l_b)t} \int_t^T \bar{P}(s)e^{(r+2l_b)s} \, ds, \\
\bar{M}(t) &= \int_t^T (m - c_1)\bar{Q}(s) + \lambda \bar{Q}^2(s) \, ds.
\end{align*}
\]

(39)

Theorems 3.2 and 3.3 show similar results, but the coefficients of \( E(\bar{Y}Z) \) in \( l_4, l_5, l_6 \) and \( l_7, l_8, l_9 \), respectively, are different. More specifically, for the case \( g(\lambda) \leq \lambda \), the coefficient of \( E(\bar{Y}Z) \) contains \( g(\lambda) \); while for the case \( g(\lambda) > \lambda \), \( g(\lambda) \) becomes \( \lambda \). The reasons are as follows. For the case \( g(\lambda) \leq \lambda \), according to the additivity of Poisson process, we can decompose \( N(t) \) into \( N_3(t) + N_9(t) \). Here \( N_3(t) \) is a Poisson process independent of \( N_9(t) \), with intensity being \( \lambda - g(\lambda) \). That is to say, after suffering a common shock, the price jumps of the risky asset will definitely cause the claims to happen. Therefore, the mutual influence part of the optimal reinsurance strategy and the optimal investment strategy contains a common \( g(\lambda) \), i.e., the coefficient of \( E(\bar{Y}Z) \) contains \( g(\lambda) \). However, for the case \( g(\lambda) > \lambda \), we can decompose \( N_9(t) \) into \( N_4(t) + N(t) \). Here \( N_4(t) \) is a Poisson process independent of \( N(t) \), with intensity being \( g(\lambda) - \lambda \). Similar to the above analysis, we can deduce that after suffering a common shock, the claim will definitely cause the price jumps of the risky asset. Therefore, the coefficient of \( E(\bar{Y}Z) \) becomes \( \lambda \).

Finally, we consider the solution of problem (5). Note that

\[
\begin{align*}
\text{E}\left[ (X^u(T \wedge \tau))^2 \right] &= \text{E}\left[ (X^u(T \wedge \tau) - (l - \xi))^2 \right] \\
&= \text{E}\left[ (X^u(T \wedge \tau) - l)^2 \right] + 2\xi\text{E}[X^u(T \wedge \tau) - l] + \xi^2.
\end{align*}
\]

(40)

Hence, for fixed \( \xi \), we have

\[
\min_{u(t) \in U} \left\{ \text{E}\left[ (X^u(T \wedge \tau) - l)^2 \right] + 2\xi\text{E}[X^u(T \wedge \tau) - l] \right\} = \min_{u(t) \in U} \text{E}\left[ (X^u(T \wedge \tau))^2 \right] - \xi^2 = V(0, \bar{x}_0) - \xi^2,
\]

(41)

where

\[
V(0, \bar{x}_0) - \xi^2 = \xi^2 \left[ 1 - \frac{1}{2}P(0) \right] + \xi[P(0)(x_0 - l) + Q(0)] + \left[ \frac{1}{2}P(0)(x_0 - l) + Q(0) \right](x_0 - l) + M(0).
\]

(42)

Note that \( P(0) > 0 \) and (42) depends on \( \xi \). If \( 0 < P(0) < 2 \), the right-hand side of (42) attains its maximum value at:

\[
\xi^* = \frac{P(0)(x_0 - l) + Q(0)}{2 - P(0)}.
\]

(43)
Otherwise, there does not exist a finite maximum. For the former case, with (43) and the Lagrange duality theorem (see Luenberger [14], page 224, Thm. 1), we have

\[
\max_{\xi \in R} \left\{ \min_{u(t) \in U} \left\{ E \left[ (X^u(T \wedge \tau) - l)^2 \right] + 2\xi E[X^u(T \wedge \tau) - l] \right\} \right\} \\
= V(0, \bar{x}_0) - (\xi)^2 \\
= \left[ \frac{P(0)(x_0 - l) + Q(0)}{2[2 - P(0)]} \right]^2 + \left[ \frac{1}{2} P(0)(x_0 - l) + Q(0) \right](x_0 - l) + M(0).
\]

Based on the above discussion, the optimal strategy of problem (5) can be given as follows. Again, we divide the results into two groups with \(g(\lambda) \leq \lambda\) and \(g(\lambda) > \lambda\), respectively.

**Theorem 3.4.** If \(g(\lambda) \leq \lambda\), \(0 < l_4[x - l + \xi^* + \frac{c + r(l - \xi^*) - c_1}{r} h(t)] < 1\) and \(0 < P(0) < 2\), the optimal solution to problem (5) is given as follows.

The optimal reinsurance strategy is

\[
a^*(t) = l_4 \left[ x - l + \xi^* + \frac{c + r(l - \xi^*) - c_1}{r} h(t) \right], \tag{44}
\]

the optimal investment strategy is

\[
\pi^*(t) = l_5 \left[ x - l + \xi^* + \frac{c + r(l - \xi^*) - c_1}{r} h(t) \right]. \tag{45}
\]

Moreover, the efficient frontier under \(E[X^u(T \wedge \tau)] = l\) is given by

\[
\text{Var}[X^u(T \wedge \tau)] = \left[ \frac{P(0)(x_0 - l) + Q(0)}{2[2 - P(0)]} \right]^2 + \left[ \frac{1}{2} P(0)(x_0 - l) + Q(0) \right](x_0 - l) + M(0). \tag{46}
\]

**Theorem 3.5.** If \(g(\lambda) > \lambda\) and \(0 < P(0) < 2\), we define \(\bar{\xi}^* = \frac{P(0)(x_0 - l) + Q(0)}{2 - P(0)}\). Then when \(0 < l_7[x - l + \bar{\xi}^* + \frac{c + r(l - \bar{\xi}^*) - c_1}{r} h(t)] < 1\), the optimal solution to problem (5) is given as follows.

The optimal reinsurance strategy is

\[
a^*(t) = l_7 \left[ x - l + \bar{\xi}^* + \frac{c + r(l - \bar{\xi}^*) - c_1}{r} h(t) \right]. \tag{47}
\]

the optimal investment strategy is

\[
\pi^*(t) = l_8 \left[ x - l + \bar{\xi}^* + \frac{c + r(l - \bar{\xi}^*) - c_1}{r} h(t) \right]. \tag{48}
\]

Moreover, the efficient frontier under \(E[X^u(T \wedge \tau)] = l\) is given by

\[
\text{Var}[X^u(T \wedge \tau)] = \left[ \frac{P(0)(x_0 - l) + Q(0)}{2[2 - P(0)]} \right]^2 + \left[ \frac{1}{2} P(0)(x_0 - l) + Q(0) \right](x_0 - l) + M(0). \tag{49}
\]

The implications of the optimal strategies in Theorems 3.4 and 3.5 are similar to those in Theorems 3.2 and 3.3, respectively. The only difference lies in the values of \(\bar{x}\) and \(\bar{m}\). In Theorems 3.2 and 3.3, \(\bar{x} = x - (l - \xi)\) and \(\bar{m} = c + r(l - \xi)\); while in Theorem 3.4, \(\bar{x} = x - (l - \xi^*)\) and \(\bar{m} = c + r(l - \xi^*)\); in Theorem 3.5, \(\bar{x} = x - (l - \bar{\xi}^*)\) and \(\bar{m} = c + r(l - \bar{\xi}^*)\).
4. Numerical experiments

In this section, we mainly illustrate the influence of the claim size, the price jump size, and the market interdependence on the optimal reinsurance and investment strategies. We also analyze the similarities and differences between optimal reinsurance and investment strategies with and without a random exit time. To avoid repetition, we will only report the results for the case \( g(\lambda) \leq \lambda \).

With respect to the \( \sigma \)-algebra \( \mathcal{F}_t \), we assume that the distribution functions of \( \tau_1 \) and \( \tau_2 \) are \( F_1(t) \) and \( F_2(t) \), respectively, i.e., \( F_1(t) = \mathbb{P}(\tau_1 \leq t|\mathcal{F}_t) \) and \( F_2(t) = \mathbb{P}(\tau_2 \leq t|\mathcal{F}_t) \). To reflect the interdependence between the random exit of reinsurance and the random exit of investment, similar to many papers, such as Sun et al. [17], we assume that the joint distribution function of \( \tau_1 \) and \( \tau_2 \) satisfies the following Farlie–Gumbel–Morgenstern (FGM) copula function

\[
F(t_1, t_2) = F_1(t_1)F_2(t_2) + \theta F_1(t_1)F_2(t_2)[1 - F_1(t_1)][1 - F_2(t_2)], 0 \leq \theta \leq 1.
\]

Here \( \theta \) reflects the interdependence between \( \tau_1 \) and \( \tau_2 \). Specifically, when \( \theta = 0 \), \( \tau_1 \) and \( \tau_2 \) are mutually independent. According to the definition of \( F(t_1, t_2) \), we can obtain that the distribution function of \( \tau = \tau_1 \land \tau_2 \) with respect to given \( \mathcal{F}_t \) is

\[
F(t) = \mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\tau_1 \land \tau_2 \leq t|\mathcal{F}_t)
= 1 - \mathbb{P}(\tau_1 > t, \tau_2 > t|\mathcal{F}_t)
= 1 - [1 - F_1(t) - F_2(t) + F(t, t)]
= F_1(t) + F_2(t) - F_1(t)F_2(t) - \theta F_1(t)F_2(t)[1 - F_1(t)][1 - F_2(t)].
\]

To carry out the numerical experiment, we further make some technical assumptions. Similar to that in Huang et al. [9] and Bellalah et al. [2], we assume that \( F_1(t) \) and \( F_2(t) \) are given by

\[
F_1(t) = \begin{cases} 1 - e^{-\lambda_1 t}, & \text{for } t > 0, \\ 0, & \text{for } t \leq 0, \end{cases}
F_2(t) = \begin{cases} 1 - e^{-\lambda_2 t}, & \text{for } t > 0, \\ 0, & \text{for } t \leq 0, \end{cases}
\]

respectively. Here \( \lambda_i = \frac{1}{a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3}, 0 \leq a_{i1}, a_{i2}, a_{i3} \leq 1, a_{i1} + a_{i2} + a_{i3} = 1, i = 1, 2 \). \( b_1 = \frac{1}{7}E[N_1(t)Y_i + \sum_{j=1}^{N(t)}Y_j] \) represents the random exit caused by the fluctuation of claims, \( b_2 = \bar{\sigma} + \frac{1}{7}E[N_2(t)Z_i + \sum_{j=1}^{N(t)}Z_j] \) represents the random exit caused by the price fluctuation of the risky asset, and \( b_3 \) is a constant and represents the random exit caused by exogenous random events. This setting of \( \lambda_i \) shows that the average exit time \( E(\tau_i) = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3, i = 1, 2 \), is proportional to the sizes of claims and price jumps and other exogenous random events. However, considering that the random exit time \( \tau_1 \) of reinsurance should be mainly caused by the fluctuation of claims, and the random exit time \( \tau_2 \) of investment should be mainly caused by the price fluctuation of the risky asset, the weight coefficients \( a_{11} \) and \( a_{22} \) are set to relatively large values in the following numerical experiments.

Suppose that \( g(\lambda) = \lambda q, q \in [0, 1] \), and the distribution functions of the claim sizes \( Y \) and \( \bar{Y} \) and the price jump sizes \( Z \) and \( \bar{Z} \) are given by

\[
\tilde{f}_Y(y) = \begin{cases} \eta_1 e^{-\eta_1 y}, & \text{for } y > 0, \\ 0, & \text{for } y \leq 0, \end{cases}
\tilde{f}_\bar{Y}(\bar{y}) = \begin{cases} \eta_1 e^{-\eta_1 \bar{y}}, & \text{for } \bar{y} > 0, \\ 0, & \text{for } \bar{y} \leq 0, \end{cases}
\]

and

\[
\tilde{f}_Z(z) = \begin{cases} \eta_2 e^{-\eta_2 (z+1)}, & \text{for } z > -1, \\ 0, & \text{for } z \leq -1, \end{cases}
\tilde{f}_\bar{Z}(\bar{z}) = \begin{cases} \eta_2 e^{-\eta_2 (\bar{z}+1)}, & \text{for } \bar{z} > -1, \\ 0, & \text{for } \bar{z} \leq -1, \end{cases}
\]
Table 1. Values of basic model parameters.

<table>
<thead>
<tr>
<th></th>
<th>(a_{11})</th>
<th>(a_{12})</th>
<th>(a_{13})</th>
<th>(a_{21})</th>
<th>(a_{22})</th>
<th>(a_{23})</th>
<th>(b_3)</th>
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<th>(\bar{\eta}_1)</th>
<th>(\bar{\eta}_2)</th>
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<td>0.7</td>
<td>0.1</td>
<td>1</td>
<td>0.5</td>
<td>0.9</td>
<td>0.6</td>
<td>0.4</td>
</tr>
</tbody>
</table>

\(T\) | \(\theta\) | \(\rho\) | \(q\) | \(r\) | \(\mu\) | \(\bar{\sigma}\) | \(c\) | \(c_1\) | \(x_0\) |
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</tr>
</thead>
<tbody>
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<td>0.08</td>
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<td>1.1</td>
<td>2</td>
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</table>

Figure 1. Effect of \(\bar{\eta}_1\) on \(a^*(t)\).

respectively. Here \(\eta_1\) and \(\bar{\eta}_1\) are the parameters of the claim independent of the common shock and the claim caused by the common shock, respectively. They are inversely proportional to the mean of the claims. \(\eta_2\) and \(\bar{\eta}_2\) are the parameters of the price jump independent of the common shock and the price jump caused by the common shock, respectively. They are inversely proportional to the mean of the sizes of price jumps.

In the following numerical experiments, unless otherwise stated, we choose the basic parameter setting as out in Table 1.

With the explicit expression of \(P(t)\) in (25), it is easy for us to calculate \(P(0)\) and thus to check whether \(0 < P(0) < 2\) as satisfied. Fortunately, we found that it is easy to ensure \(0 < P(0) < 2\) as long as the values of all the relevant parameters are suitably selected. Especially, for the values of basic model parameters chosen in Table 1, as well as the values of other parameters in all the following numerical experiments, we can always make \(0 < P(0) < 2\) hold! This lays a solid foundation for us hereinafter to apply Theorem 3.4 to determine the optimal reinsurance strategy \(a^*(t)\) and the optimal investment strategy \(\pi^*(t)\).

4.1. Effect of model parameters on the optimal reinsurance strategy

In this subsection, we present the numerical results for the optimal reinsurance strategy \(a^*(t)\), which is determined by (44). Here we set \(t = 2\), \(l = 9\), \(\lambda = 0.8\), \(\lambda_1 = 1\), \(\lambda_2 = 1.2\). The value of wealth process \(X^u(t)\) at time \(t = 2\) is 8, i.e., \(X^u(2) = x = 8\).

Figure 1 shows the influence of \(\bar{\eta}_1\) on \(a^*(t)\) when considering the random exit time (the curves not marked) and not considering the random exit time (the curves marked with NRET), respectively. We can observe from Figure 1 that, whether the random exit time is considered or not, \(a^*(t)\) increases monotonically with respect to \(\bar{\eta}_1\). A larger \(\bar{\eta}_1\) implies that the mean of the claims becomes smaller. As a result, the insurer would maintain a larger retention level. From Figure 1, we also find that \(\bar{\eta}_1\) has a significant influence on \(a^*(t)\). This shows that the insurer is sensitive to the changes in the size of claims.
Figure 2 displays the influence of $\bar{\eta}_2$ on $a^*(t)$ with and without considering the random exit time, respectively. From Figure 2, we can see that, whether the random exit time is considered or not, $a^*(t)$ increases monotonically with respect to $\bar{\eta}_2$. A larger $\bar{\eta}_2$ means that the mean of the sizes of price jumps becomes smaller, i.e., the price risk of the risky asset becomes smaller. Since the investment risk and the claim risk are positively correlated, the insurer would maintain a larger retention level.

Figure 3 illustrates the influence of $\lambda$ on $a^*(t)$ with and without considering the random exit time, respectively. $\lambda$ reflects the interdependence between the number of claims and the number of price jumps. To show the effect of the interdependence, the expected number of claims and that of stock price jumps should be kept the same, i.e., $\lambda + \lambda_1 = \lambda q + \lambda_2$. Therefore, we set $\lambda_1 = \lambda q + \lambda_2 - \lambda$. Similar setting will be also adopted in Figure 7. The larger $\lambda$ is, the more claims and price jumps will be. This will enhance the insurer’s reinsurance willingness. Therefore, from Figure 3, we can see whether the random exit time is considered or not, $a^*(t)$ decreases monotonically with respect to $\lambda$.

Figure 4 illustrates the influence of $\rho$ on $a^*(t)$ with and without considering the random exit time, respectively. From Figure 4, we can see whether the random exit time is considered or not, $a^*(t)$ decreases with respect to $\rho$. Remember that $\rho$ is the correlation coefficient between the size of claims and the size of price jumps after suffering a common shock. A larger $\rho$ implies that the mean of the product of the sizes of claims and the sizes
of price jumps would become greater. As the claims risk and investment risk are positively correlated, the claim risk will increase. Thus, the insurer would decrease her retention level.

4.2. Effect of model parameters on the optimal investment strategy

In this part, we show the numerical results for the optimal investment strategy $\pi^*(t)$, which is determined by (45). Here we set $t = 2, x = 3, l = 6, \lambda = 0.6, \lambda_1 = 0.3, \lambda_2 = 0.1$.

Figure 5 illustrates the influence of $\bar{\eta}_1$ on $\pi^*(t)$ with and without considering the random exit time, respectively. As what we have explained for the results in Figure 1, with the decrease of the mean of claims, the investment risk will also decrease. Hence, from Figure 5, we can see whether the random exit time is considered or not, $\pi^*(t)$ increases with respect to $\bar{\eta}_1$.

Figure 6 presents the effect of $\bar{\eta}_2$ on $\pi^*(t)$ with and without considering the random exit time, respectively. The larger $\bar{\eta}_2$ is, the smaller the mean of the sizes of price jumps will be. Hence, from Figure 6, we can see whether the random exit time is considered or not, $\pi^*(t)$ increases monotonically with respect to $\bar{\eta}_2$.

Figure 7 demonstrates the influence of $\lambda$ on $\pi^*(t)$ with and without considering the random exit time, respectively. From Figure 7, it is clear whether the random exit time is considered or not, $\pi^*(t)$ is decreasing.
with respect to $\lambda$. As what we have explained for the results in Figure 3, a larger $\lambda$ implies that the investment risk will become larger. Therefore, the insurer reduces her investment in the risky asset when $\lambda$ increases.

Figure 8 shows the influence of $\rho$ on $\pi^*(t)$ with and without considering the random exit time, respectively. Similar to that for Figure 4, when $\rho$ increases, the investment risk increases indirectly. Therefore, from Figure 8, it is clear that, whether the random exit time is considered or not, $\pi^*(t)$ decreases monotonically with respect to $\rho$.

Finally, we have the following observations after systematically analyzing the above numerical results:

- The stronger the interdependence between the number of claims and the number of price jumps is, and the stronger the interdependence between the size of claims and the size of price jumps is, the smaller the retention level the insurer will keep, and the less the insurer will invest in the risky asset. These findings can effectively guide insurers’ reinsurance and investment in practice.
- Whether the random exit time is considered or not, the trends of the optimal reinsurance and investment strategies with respect to relevant model parameters are consistent, but the investment amount and the retention level decrease or increase significantly.
- What is more important, compared with the case considering the random exit time, the insurer keeps a smaller retention level and invests more in the risky asset when the random exit time is not considered.
The reasons are as follows: if the random exit time is not considered, the insurer must wait until time $T$ to terminate her reinsurance and investment. In other words, the insurer may encounter more claim risks. Therefore, when the random exit time is not considered, the insurer is more willing to seek the reinsurance protection. Meanwhile, an important way to increase wealth is to invest in risky assets, hence the insurer would invest more in the risky asset when the random exit time is not considered.

5. Conclusions

We have investigated the reinsurance and investment problem with a common shock and a random exit time under the MV criterion in this article. The insurer’s surplus process follows an extended compound Poisson process, and the price process of the risky asset is modeled by a jump-diffusion process. We simultaneously consider the influences of the common shock and the random exit time on the insurance market and the financial market. To reflect the effect of a common shock, we propose a new interdependence mechanism between the insurance market and the financial market. And we adopt a general framework where the random exit time is affected by both exogenous and endogenous random events. Explicit solutions for the optimal reinsurance-investment strategy and the corresponding efficient frontier are obtained by employing stochastic dynamic programming and Lagrange duality techniques. We systematically examine the influences of the market interdependence and the random exit time on reinsurance and investment strategies through numerical experiments and derive some interesting observations.

There remain some issues that are worthy of further investigation in the future: firstly, we can consider the truly endogeneity of the exit time, such as time of ruin; secondly, we can consider other forms of optimization criteria, such as the ruin probability minimization; thirdly, the case with a no-bankruptcy constraint could be considered; finally, non-proportional reinsurance can also be considered in the proposed model.

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References


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