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TOTAL DOMINATOR TOTAL CHROMATIC NUMBERS OF CYCLES AND PATHS

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Abstract. The total dominator total coloring of a graph is a total coloring of the graph such that each object (vertex or edge) of the graph is adjacent or incident to every object of some color class. The minimum number of the color classes of a total dominator total coloring of a graph is called the total dominator total chromatic number of the graph. In (A.P. Kazemi, F. Kazemnejad and S. Moradi, *Contrib. Discrete Math.* (2022).), the authors initiated to study the total dominator total coloring of a graph and found some useful results, and presented some problems. Finding the total dominator total chromatic numbers of cycles and paths were two of them which we consider them here.

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1. Introduction

All graphs considered here are non-empty, finite, undirected and simple. For standard graph theory terminology not given here we refer to [14]. Let $G = (V, E)$ be a graph with the *vertex set* V of *order* $n(G)$ and the *edge set* E of *size* $m(G)$. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V$ are $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The *degree* of a vertex v is also $deg_G(v) = |N_G(v)|$. The *minimum* and *maximum degree* of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If $\delta(G) = \Delta(G) = k$, then G is called *k-regular*. An *independent set* of G is a subset of vertices of G, no two of which are adjacent. And a *maximum independent set* is an independent set of the largest cardinality in G. This cardinality is called the *independence number* of G, and is denoted by $\alpha(G)$. Also a *mixed independent set* of G is a subset of $V(G) \cup E(G)$, no two objects of which are adjacent or incident, and a *maximum mixed independent set* is a mixed independent set of the largest cardinality in G. This cardinality is called the *mixed independence number* of G, and is denoted by $\alpha_{\rm mix}(G)$. Two isomorphic graphs G and H are shown by $G \cong H$.

We write C_n and P_n for a *cycle* and a *path* of order n, respectively, while $G[S]$ is *induced subgraph* of G by a vertex set S. The *line graph* $L(G)$ of G is a graph with the vertex set $E(G)$ and two vertices of $L(G)$ are adjacent when they are incident in G. The *total graph* $T(G)$ of a graph $G = (V, E)$ is the graph whose vertex set is $V \cup E$ and two vertices are adjacent whenever they are either adjacent or incident in G [1]. It is obvious that if G has

Keywords. Total dominator total coloring, total dominator total chromatic number, total domination number, total mixed domination number, total graph.

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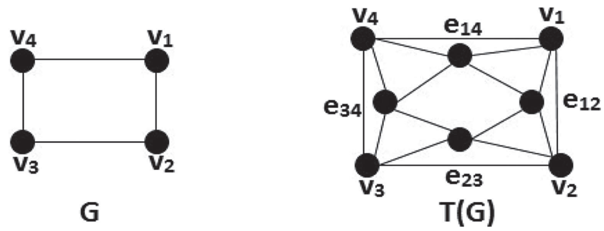

FIGURE 1. The illustration of G (left) and $T(G)$ (right).

order n and size m, then $T(G)$ has order $n+m$ and size $3m+|E(L(G))|$, and also $T(G)$ contains both G and $L(G)$ as two induced subgraphs and it is the largest graph formed by adjacent and incidence relation between graph elements. Since $deg_{T(G)}(v_i) = 2deg_G(v_i)$ and $deg_{T(G)}(e_{ij}) = deg_G(v_i) + deg_G(v_j)$, if G is k-regular, then $T(G)$ is $2k$-regular. Also we have $\alpha_{\rm mix}(G) = \alpha(T(G))$.

Here, we fix a notation for the vertex set and the edge set of line and total of a graph which we use thorough this paper. For a graph $G=(V,E)$ with the vertex set $V=\{v_i|\ 1\leq i\leq n\}$, we have $V(L(G))=\mathcal{E}$ and $E(L(G))=\{e_{ij}e_{ik}\mid e_{ij},e_{ik}\in\mathcal{E}$ and $j\neq k\}$, $V(T(G))=V\cup\mathcal{E}$ and $E(T(G))=E\cup E(L(G))\cup\{e_{ij}v_i, e_{ij}v_j\mid e_{ij}\in\mathcal{E}\}$, where $\mathcal{E}=\{e_{ij}\mid v_iv_j\in E\}$. In Figure 1 a graph G and its total graph are shown for an example.

DOMINATION. Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes *et al.* [3, 4]. A famous type of domination is total domination, and the literature on this subject has been surveyed and detailed in the recent book [6]. A *total dominating set*, briefly TDS, S of a graph $G=(V,E)$ is a subset of the vertex set of G such that for each vertex v, $N_G(v)\cap S\neq\emptyset$. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a TDS of G. Similarly, a subset $S\subseteq V\cup E$ of a graph G is called a *total mixed dominating set*, briefly TMDS, if each object of $(V\cup E)$ is either adjacent or incident to an object of S, and the *total mixed domination number* $\gamma_{\rm tm}(G)$ of G is the minimum cardinality of a TMDS of G [12]. A min-TDS/ min-TMDS of G denotes a TDS/ TMDS of G with minimum cardinality. Also we agree that *a vertex v dominates an edge e* or *an edge e dominates a vertex v* mean $v\in e$. Similarly, we agree that *an edge dominates another edge* means they have a common vertex. The next theorem can be easily obtained.

Theorem 1.1. [12] *For any graph G without isolated vertex, $\gamma_{\rm tm}(G)=\gamma_t(T(G))$.*

GRAPH COLORING. Graph coloring is used as a model for a vast number of practical problems involving allocation of scarce resources (*e.g.*, scheduling problems), and has played a key role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization. Graph colorability is NP-complete in the general case, although the problem is solvable in polynomial time for many classes [2]. A *proper coloring* of a graph G is a function from the vertices of the graph to a set of colors such that any two adjacent vertices have different colors, and the minimum number of colors needed in a proper coloring of a graph is called the *chromatic number* $\chi(G)$ of G. In a similar way, a *total coloring* of G assigns a color to each vertex and to each edge so that colored objects have different colors when they are adjacent or incident, and the minimum number of colors needed in a total coloring of a graph is called the *total chromatic number* $\chi_T(G)$ of G [14].

A *color class* in a coloring of a graph is a set consisting of all those objects assigned the same color. For simply, if f is a coloring of G with the coloring classes V_1, V_2, $\cdots$, V_ℓ, we write $f=(V_1,V_2,\cdots,V_\ell)$. Motivated by the relation between coloring and total dominating, the concept of total dominator coloring in graphs introduced in [9] by Kazemi, and extended in [5, 7, 8, 10, 13].

Definition 1.2. [9] A *total dominator coloring*, briefly TDC, of a graph G with a possitive minimum degree is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some color class. The *total dominator chromatic number* $\chi_d^t(G)$ of G is the minimum number of color classes in a TDC of G.

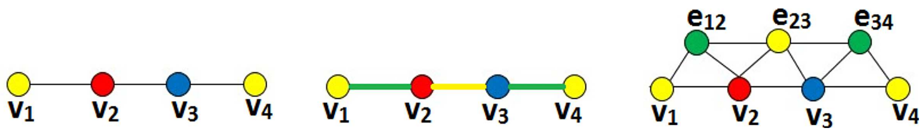

FIGURE 2. A min-TDC of P_4 (left) and a min-TDTC of P_4 (Middle) with its corresponding min-TDC of $T(P_4)$ (right).

In [11], the authors initiated studying of a new concept called total dominator total coloring in graphs which is obtained from the concept of total dominator coloring of a graph by replacing total coloring of a graph instead of (vertex) coloring of it.

Definition 1.3. [11] A *total dominator total coloring*, briefly TDTC, of a graph G with a positive minimum degree is a total coloring of G in which each object of the graph is adjacent or incident to every object of some color class. The *total dominator total chromatic number* $\chi_d^{tt}(G)$ of G is the minimum number of color classes in a TDTC of G.

It can be easily obtained the next theorem.

Theorem 1.4. [11] *For any graph G without isolated vertex, $\chi_d^{tt}(G) = \chi_d^t(T(G))$.*

For any TDC (TDTC) $f = (V_1, V_2, \cdots, V_\ell)$ of a graph G, a vertex (an object) v is called a *common neighbor* of V_i or we say V_i *totally dominates* v, and we write $v \succ_t V_i$, if vertex (object) v is adjacent (adjacent or incident) to every vertex (object) in V_i. Otherwise we write $v \not\succ_t V_i$. The set of all common neighbors of V_i with respect to f is called the *common neighborhood* of V_i in G and denoted by $CN_{G,f}(V_i)$ or simply by $CN(V_i)$. Also every TDC or TDTC of G with $\chi_d^t(G)$ or $\chi_d^{tt}(G)$ color classes is called a *min*-TDC or a *min*-TDTC, respectively. For an example see Figure 2. Also for any TDC $(V_1, V_2, \cdots, V_\ell)$ and any TDTC $(W_1, W_2, \cdots, W_\ell)$ of a graph G, we have

$$\bigcup_{i=1}^{\ell} CN(V_i) = V(G) \text{ and } \bigcup_{i=1}^{\ell} CN(W_i) = V(G) \cup E(G). \tag{1.1}$$

GOAL. In [11], the authors initiated to study the total dominator total coloring of a graph and found some useful results, and presented some problems. Finding the total dominator total chromatic numbers of cycles and paths were two of them which we consider them here.

We recall the following proposition from [9] which is useful for our investigation. Propositions 2.3 and 3.3 show that the upper bound given in Proposition 1.5 is tight.

Proposition 1.5. [9] *For any connected graph G with $\delta(G) \geq 1$,*

$$\chi_d^t(G) \leq \gamma_t(G) + \min_S \chi(G[V(G) - S]), \tag{1.2}$$

where $S \subseteq V(G)$ is a min-TDS of G. And so $\chi_d^t(G) \leq \gamma_t(G) + \chi(G)$.

2. CYCLES

Here, we calculate the total dominator total chromatic number of cycles. First we recall a proposition from [12] and calculate the mixed independence number of a cycle.

Proposition 2.1. [12] *For any cycle C_n of order $n \geq 3$,*

$$\gamma_{tm}(C_n) = \begin{cases} \lceil \frac{4n}{7} \rceil + 1 & \text{if } n \equiv 5 \pmod 7, \\ \lceil \frac{4n}{7} \rceil & \text{if } n \not\equiv 5 \pmod 7. \end{cases}$$

Lemma 2.2. *For any cycle C_n of order $n \geq 3$, $\alpha_{\rm mix}(C_n) = \lfloor \frac{2n}{3} \rfloor$.*

Proof. Let $C_n : v_1 v_2 \cdots v_n$ be a cycle of order $n \geq 3$. Then $V(T(C_n)) = V \cup \mathcal{E}$ where $\mathcal{E} = \{e_{i(i+1)} \mid 1 \leq i \leq n\}$. Since every vertex in an independent set belongs to exactly three triangles and also $T(C_n)$ has $2n$ distinct triangles, $\alpha(T(C_n)) > \lfloor \frac{2n}{3} \rfloor$ implies that two vertices of a triangle belong to the independent set, which is not possible. Hence $\alpha(T(C_n)) \leq \lfloor \frac{2n}{3} \rfloor$. On the other hand, since

$$\begin{array}{ll} \{v_{3i-2}, e_{(3i-1)(3i)} \mid 1 \leq i \leq \lfloor \frac{n}{3} \rfloor\} & \text{when } n \equiv 0, 1 \pmod 3, \\ \{v_{3i-2} \mid 1 \leq i \leq \lceil \frac{n}{3} \rceil\} \cup \{e_{(3i-1)(3i)} \mid 1 \leq i \leq \lfloor \frac{n}{3} \rfloor\} & \text{when } n \equiv 2 \pmod 3, \end{array}$$

are independent sets of cardinality $\lfloor \frac{2n}{3} \rfloor$, we have $\alpha_{\rm mix}(C_n) = \alpha(T(C_n)) = \lfloor \frac{2n}{3} \rfloor$. □

Proposition 2.3. *For any cycle C_n of order $n \geq 3$,*

$$\chi_d^{tt}(C_n) = \begin{cases} \gamma_{tm}(C_n) + 1 & \text{if } n = 3, 4, 5, \\ \gamma_{tm}(C_n) + 2 & \text{if } n = 6, 9, 12, \\ \gamma_{tm}(C_n) + 3 & \text{if } n \geq 7 \text{ and } n \neq 9, 12. \end{cases}$$

which by considering Proposition 2.1 implies

$$\chi_d^{tt}(C_n) = \begin{cases} n & \text{if } 3 \leq n \leq 8, \\ n-1 & \text{if } n = 9, \end{cases}$$

and for $n \geq 10$,

$$\chi_d^{tt}(C_n) = \begin{cases} \lceil \frac{4n}{7} \rceil + 4 & \text{if } n \equiv 5 \pmod 7 \text{ and } n \neq 12, \\ \lceil \frac{4n}{7} \rceil + 3 & \text{if } n \not\equiv 5 \pmod 7 \text{ or } n = 12. \end{cases}$$

Proof. Let $C_n : v_1 v_2 \cdots v_n v_1$ be a cycle of order $n \geq 3$. Then $V(T(C_n)) = V \cup \mathcal{E}$ where $\mathcal{E} = \{e_{i(i+1)} \mid 1 \leq i \leq n\}$. We know from [12] that for $n \geq 3$, the sets

$$\begin{array}{ll} S_0 = \{v_{7i+2}, v_{7i+3}, e_{(7i+5)(7i+6)}, e_{(7i+6)(7i+7)} \mid 0 \leq i \leq \lfloor \frac{n}{7} \rfloor - 1\} & \text{if } n \equiv 0 \pmod 7, \\ S_1 = S_0 \cup \{e_{(n-1)n}\} & \text{if } n \equiv 1 \pmod 7, \\ S_2 = S_3 = S_0 \cup \{v_{n-1}, v_n\} & \text{if } n \equiv 2, 3 \pmod 7, \\ S_4 = S_0 \cup \{v_{n-2}, v_{n-1}, v_n\} & \text{if } n \equiv 4 \pmod 7, \\ S_5 = S_0 \cup \{v_{n-3}, v_{n-2}, v_{n-1}, v_n\} & \text{if } n \equiv 5 \pmod 7, \\ S_6 = S_0 \cup \{v_{n-4}, v_{n-3}, e_{(n-2)(n-1)}, e_{(n-1)n}\} & \text{if } n \equiv 6 \pmod 7. \end{array}$$

are min-TDSs of $T(C_n)$, and since $\chi(T(C_n) - S_r) \leq 3$ for $0 \leq r \leq 6$, we have

$$\chi_d^{tt}(C_n) \leq \gamma_{tm}(C_n) + 3, \tag{2.1}$$

by Proposition 1.5. For $n \geq 20$ or $n = 15, 18$, since $T(C_n)$ has $2n$ distinct triangles with the vertex sets $\{v_i, v_{i+1}, e_{i(i+1)}\}$ or $\{e_{i(i+1)}, v_{i+1}, e_{(i+1)(i+2)}\}$ for $1 \leq i \leq n$, and at least two color classes are needed for totally dominating the vertices of the consecutive triangles with the vertex sets $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, e_{i(i+1)}, e_{(i+1)(i+2)}, e_{(i+2)(i+3)}\}$ or $\{v_{i+1}, v_{i+2}, v_{i+3}, e_{i(i+1)}, e_{(i+1)(i+2)}, e_{(i+2)(i+3)}, e_{(i+3)(i+4)}\}$, we conclude that the number of used color classes in $T(C_n)$ is at least $2\lfloor \frac{2n}{5} \rfloor \geq \gamma_t(T(C_n)) + 3$, and so $\chi_d^{tt}(C_n) = \chi_d^t(T(C_n)) = \gamma_{tm}(C_n) + 3$ by (2.1). Therefore, by considering the following facts in which $f = (V_1, V_2, \cdots, V_\ell)$ is an arbitrary min-TDC of $T(C_n)$, $|V_1| \geq |V_2| \geq \cdots \geq |V_\ell|$, $A_i = \{V_k \mid |V_k| = i\}$ and $|A_i| = a_i$ for $1 \leq i \leq \alpha$ where $\alpha := \alpha_{\rm mix}(C_n) = \lfloor \frac{2n}{3} \rfloor$, we continue our proof when $3 \leq n \leq 19$ except $n = 15, 18$.

- ⋆ `Fact 1.` $|V_k| \leq \lfloor \frac{2n}{3} \rfloor$ for $1 \leq k \leq \ell$, and $\sum_{i=1}^{\ell} |V_i| = 2n$.
- ⋆ `Fact 2.` For any $v \in V \cup \mathcal{E}$, if $v \succ_t V_k$ for some $1 \leq k \leq \ell$, then $|V_k| \leq 2$.

- ⋆ `Fact 3.` For any vertex $v \in M$, if $v \succ_t V_k$ for some $1 \le k \le \ell$ and $|V_k| = 2$, then $CN(V_k) \cap M = \{v\}$, where $M \in \{\mathcal{E}, V\}$, and since $CN(V_k) \cap V \neq \emptyset$ if and only if $CN(V_k) \cap \mathcal{E} \neq \emptyset$, we have $|CN(V_k)| = 2$.
- ⋆ `Fact 4.` For any color class V_k of cardinality one, $|CN(V_k) \cap V| = |CN(V_k) \cap \mathcal{E}| = 2$.
- ⋆ `Fact 5.` $2a_1 + a_2 \ge n$ (by `Facts 3, 4`).
- ⋆ `Fact 6.` $\gamma_{tm}(C_n) \le a_1 + a_2 \le \ell$. Because the set S is a TDS of $T(C_n)$ where $|S \cap V_i| = 1$ for each $V_i \in A_1 \cup A_2$ by `Fact 2` (for left), and $a_1 + \cdots + a_\alpha = \ell$ (for right).
- ⋆ `Fact 7.` $n - \ell \le a_1 \le \lfloor \frac{\alpha\ell - 2n}{\alpha - 1} \rfloor$. Because

$$\begin{aligned} 2n - a_1 &= |V(T(C_n))| - |\{V_i \mid |V_i| = 1 \text{ for } 1 \le i \le \ell\}| \\ &= \Sigma_{|V_i| \ge 2} |V_i| \\ &\le (\ell - a_1)\alpha \end{aligned}$$

implies the upper bound, and for the lower bound

$$\begin{aligned} 2\ell - n &\ge 2(a_1 + a_2) - n \quad (\text{by } \texttt{Fact 6}) \\ &\ge a_2 \qquad\qquad\qquad (\text{by } \texttt{Fact 5}), \end{aligned} \tag{2.2}$$

implies

$$\begin{aligned} a_1 &\ge \tfrac{n - a_2}{2} \ (\text{by } \texttt{Fact 5}) \\ &\ge n - \ell \text{ by (2.2)}. \end{aligned}$$

- ⋆ `Fact 8.` $\max\{0, n - 2a_1, \gamma_{tm}(C_n) - a_1\} \le a_2 \le \min\{2\ell - n, \ell - a_1\}$ (by `Facts 5-7`).
- ⋆ `Fact 9.` If $J = \{k \mid |V_k| = 2 \text{ and } |CN(V_k)| \neq 0\}$, then the number of isolated vertices of $T(C_n)[\bigcup_{|V_k|=1} V_k]$ is at most $|J|$ (because $(|V_k|, |CN(V_k)|) = (2, 2)$ implies $T(C_n)[CN(V_k)] \cong K_2$, by `Fact 3`).
- ⋆ `Fact 10.` If $|\bigcup_{|V_i|=1} CN(V_i)| = 4a_1$, then $a_1 \le |J| \le a_2$.

`Fact 1` implies $\ell \ge n$ for $3 \le n \le 4$, and since the coloring functions

$$(\{v_1, e_{23}\}, \{v_3, e_{12}\}, \{v_2, e_{13}\}) \text{ and } (\{e_{12}, e_{34}\}, \{e_{23}, e_{14}\}, \{v_1, v_3\}, \{v_2, v_4\})$$

are respectively TDCs of $T(C_3)$ and $T(C_4)$, we have $\chi_d^{tt}(C_n) = \chi_d^t(T(C_n)) = n$ for $3 \le n \le 4$. So we assume $5 \le n \le 19$ except $n = 15, 18$, and continue our proof in the following two cases by this assumption that $\mathcal{H}_k$ denotes a graph of order k with positive minimum degree.

Case 1. $5 \le n \le 8$. Since for $5 \le n \le 8$ the coloring function g with the criterion $g(e_{(i+1)(i+2)}) = g(v_i) = i$ when $1 \le i \le n$ is a TDC of $T(C_n)$, we have

$$\chi_d^{tt}(C_n) \le n \quad \text{for } 5 \le n \le 8. \tag{2.3}$$

Since also $\ell \ge n$ for $5 \le n \le 8$, by the following reasons, (2.3) implies $\chi_d^{tt}(C_n) = n$.

- $n = 5$. Let $\ell = 4$. Then $(a_1, a_2) = (1, 3)$. Because $2a_1 + a_2 \ge 5$, $a_1 + a_2 = 4$, $a_1 = 1$ and $\max\{0, 5 - 2a_1, 4 - a_1\} \le a_2 \le \min\{4, 5 - a_1\}$ by `Facts 5-8`. But $(a_1, a_2) = (1, 3)$ implies $\sum_{i=1}^{4} |V_i| = 7 \neq 2n$, which contradicts `Fact 1`. Thus $\ell \ge 5$.
- $n = 6$. Let $\ell = 5$. Then $(a_1, a_2) = (1, 4)$, $(2, 2)$, $(2, 3)$. Because $2a_1 + a_2 \ge 6$, $4 \le a_1 + a_2 \le 5$, $1 \le a_1 \le 2$ and $\max\{0, 6 - 2a_1, 4 - a_1\} \le a_2 \le \min\{4, 5 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (1, 4)$, $(2, 3)$ imply $\sum_{i=1}^{5} |V_i| \neq 2n$ and $(a_1, a_2) = (2, 2)$ implies $|V_1| > \alpha = 4$, which contradict `Fact 1`, we have $\ell \ge 6$.
- $n = 7$. Let $\ell = 6$. Then $(a_1, a_2) = (1, 5)$, $(2, 3)$, $(2, 4)$, $(3, 1)$, $(3, 2)$, $(3, 3)$. Because $2a_1 + a_2 \ge 7$, $4 \le a_1 + a_2 \le 6$, $1 \le a_1 \le 3$ and $\max\{0, 7 - 2a_1, 4 - a_1\} \le a_2 \le \min\{5, 6 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (1, 5)$, $(2, 4)$, $(3, 3)$ imply $\sum_{i=1}^{6} |V_i| \neq 2n$ and $(a_1, a_2) = (2, 3)$, $(3, 1)$, $(3, 2)$ imply $|V_1| > \alpha = 4$, which contradict `Fact 1`, we have $\ell \ge 7$.

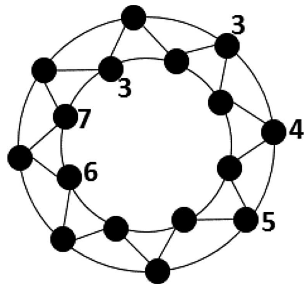

FIGURE 3. The illustration of $T(C_8)$ when $(a_1, a_2) = (4, 1)$ and $T(C_8)[V_4 \cup \cdots \cup V_7] \cong 2K_2$.

- $n = 8$. Let $\ell = 7$. Then $(a_1, a_2) = (1, 6)$, $(2, 4)$, $(2, 5)$, $(3, 2)$, $(3, 3)$, $(3, 4)$, $(4, 1)$, $(4, 2)$, $(4, 3)$. Because $2a_1 + a_2 \geq 8$, $5 \leq a_1 + a_2 \leq 7$, $1 \leq a_1 \leq 4$ and $\max\{0, 8 - 2a_1, 5 - a_1\} \leq a_2 \leq \min\{6, 7 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$ imply $\sum_{i=1}^{7} |V_i| \neq 2n$ and $(a_1, a_2) = (2, 4)$, $(3, 3)$, $(4, 2)$ imply $|V_1| > \alpha = 5$, which contradict `Fact 1`, and $(a_1, a_2) \neq (3, 2)$ by `Fact 10`, we have $(a_1, a_2) = (4, 1)$, that is, $(|V_1|, \cdots, |V_7|) = (5, 5, 2, 1, 1, 1, 1)$. Since the number of isolated vertices of $T(C_8)[V_4 \cup \cdots \cup V_7]$ is at most 1 (by `Fact 9`) and so $T(C_8)[V_4 \cup \cdots \cup V_7] \cong K_1 \cup \mathcal{H}_3$, or $\mathcal{H}_4$, we have

$$\begin{aligned} |\bigcup_{i=3}^{7} CN(V_i)| &\leq |CN(V_3)| + |\bigcup_{i=4}^{7} CN(V_i)| \\ &\leq 2 + \max\{1 \times 3 + 9, 2 \times 7\} \\ &= |V(T(C_8))|. \end{aligned}$$

Since $|\bigcup_{i=3}^{7} CN(V_i)| = |V(T(C_8)|$ if and only if $T(C_8)[V_4 \cup \cdots \cup V_7] \cong 2K_2$, by assumptions $T(C_8)[V_4 \cup V_5] \cong K_2$ and $T(C_8)[V_6 \cup V_7] \cong K_2$, as you can see in Figure 3, we have
  - $|CN(V_3)| = 2$ and $|CN(V_4) \cup CN(V_5))| = |(CN(V_6) \cup CN(V_7)| = 7$,
  - $CN(V_3) \cap (\bigcup_{i=4}^{7} CN(V_i)) = \emptyset$ and $(CN(V_4) \cup CN(V_5)) \cap (CN(V_6) \cup CN(V_7)) = \emptyset$,
  - $|V_3 \cap (\bigcup_{i=4}^{5} CN(V_i))| = 1$ and $|V_3 \cap (\bigcup_{i=6}^{7} CN(V_i))| = 1$.

  But then the induced subgraph $T(C_8)[V_1 \cup V_2]$ with chromatic number 2 contains a clique K_3 as a subgraph, which is not possible. Thus $\ell \geq 8$, as desired.

Case 2. $9 \leq n \leq 19$ except $n = 15, 18$.

- $n = 9$. Let $\ell = 7$. Then $(a_1, a_2) = (2, 5)$, $(3, 3)$, $(3, 4)$, $(4, 2)$, $(4, 3)$. Because $2a_1 + a_2 \geq 9$, $6 \leq a_1 + a_2 \leq 7$, $2 \leq a_1 \leq 4$ and $\max\{0, 9 - 2a_1, 6 - a_1\} \leq a_2 \leq \min\{5, 7 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (2, 5)$, $(3, 4)$, $(4, 3)$ imply $\sum_{i=1}^{7} |V_i| \neq 2n$ and $(a_1, a_2) = (4, 2)$, $(3, 3)$ imply $|V_1| > \alpha = 6$, which contradict `Fact 1`, we have $\ell \geq 8$. Now since the coloring function, shown in Figure 4,

$$(\{v_1, v_6, v_8, e_{23}, e_{45}\}, \{v_7, v_9, e_{12}, e_{34}, e_{56}\}, \{v_2, e_{19}\}, \{v_3\}, \{v_4\}, \{v_5, e_{67}\}, \{e_{78}\}, \{e_{89}\}),$$

  is a TDC of $T(C_9)$, we have $\chi_d^{tt}(C_9) = 8$.
- $n = 10$. Let $\ell = 8$. Then $(a_1, a_2) = (2, 6)$, $(3, 4)$, $(3, 5)$, $(4, 2)$, $(4, 3)$, $(4, 4)$, $(5, 1)$, $(5, 2)$, $(5, 3)$. Because $2a_1 + a_2 \geq 10$, $6 \leq a_1 + a_2 \leq 8$,$2 \leq a_1 \leq 5$ and $\max\{0, 10 - 2a_1, 6 - a_1\} \leq a_2 \leq \min\{6, 8 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (2, 6)$, $(3, 5)$, $(4, 4)$, $(5, 3)$ imply $\sum_{i=1}^{8} |V_i| \neq 2n$ and $(a_1, a_2) = (3, 4)$, $(4, 3)$, $(5, 1)$, $(5, 2)$ imply $|V_1| > \alpha = 6$, which contradict `Fact 1`, and $(a_1, a_2) \neq (4, 2)$ by `Fact 10`. Thus $\ell \geq 9$, and in fact $\chi_d^{tt}(C_{10}) = 9$ by (2.1).
- $n = 11$. Let $\ell = 9$. Then $(a_1, a_2) = (2, 7)$, $(3, 5)$, $(3, 6)$, $(4, 3)$, $(4, 4)$, $(4, 5)$, $(5, 2)$, $(5, 3)$, $(5, 4)$, $(6, 1)$, $(6, 2)$, $(6, 3)$. Because $2a_1 + a_2 \geq 11$, $7 \leq a_1 + a_2 \leq 9$,$2 \leq a_1 \leq 6$ and $\max\{0, 11 - 2a_1, 7 - a_1\} \leq a_2 \leq \min\{7, 9 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (2, 7)$, $(3, 6)$, $(4, 5)$, $(5, 4)$, $(6, 3)$ imply $\sum_{i=1}^{9} |V_i| \neq 2n$ and $(a_1, a_2) = (3, 5)$, $(4, 4)$, $(5, 3)$, $(6, 2)$ imply $|V_1| > \alpha = 7$, which contradict `Fact 1`, and $(a_1, a_2) \neq (4, 3)$ by `Fact 10`, we assume $(a_1, a_2) = (5, 2)$, $(6, 1)$.

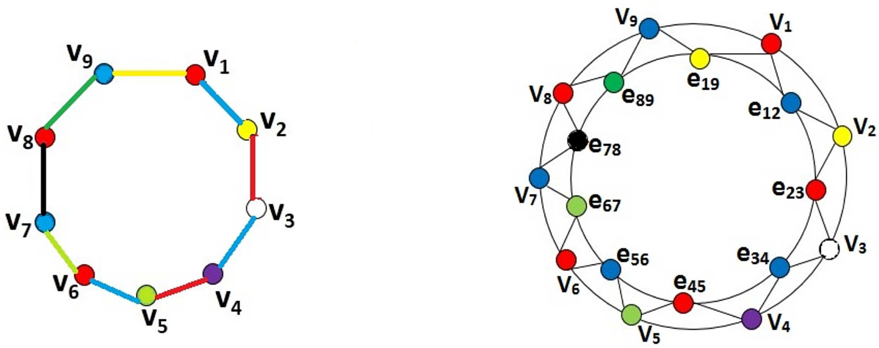

FIGURE 4. A min-TDTC of C_9 (left) and the corresponding min-TDC of $T(C_9)$ (right).

- ∘ $(a_1, a_2) = (5, 2)$. Then, since the number of isolated vertices of $T(C_{11})[V_5 \cup \cdots \cup V_9]$ is at most 2, (by `Fact 9`) and so $T(C_{11})[V_5 \cup \cdots \cup V_9] \cong \overline{K_2} \cup H_3$, $K_1 \cup H_4$ or H_5, we have

$$\begin{aligned} |\bigcup_{i=3}^{9} CN(V_i)| &\leq |\bigcup_{i=3}^{4} CN(V_i)| + |\bigcup_{i=5}^{9} CN(V_i)| \\ &\leq 4 + \max\{2 \times 3 + 9, 1 \times 3 + 2 \times 7, 7 + 9\} \\ &< |V(T(C_{11}))|, \end{aligned}$$

a contradiction with (1.1).

- ∘ $(a_1, a_2) = (6, 1)$. Then the number of isolated vertices of $T(C_{11})[V_4 \cup \cdots \cup V_9]$ is at most 1 (by `Fact 9`). If the subgraph has one isolated vertex, then $T(C_{11})[V_4 \cup \cdots \cup V_9] \cong K_1 \cup \mathcal{H}_5$ which implies

$$\begin{aligned} |\bigcup_{i=3}^{9} CN(V_i)| &\leq |CN(V_3)| + |\bigcup_{i=4}^{9} CN(V_i)| \\ &\leq 2 + (3 + 7 + 9) \\ &< |V(T(C_{11}))|, \end{aligned}$$

a contradiction with (1.1). So we assume $T(C_{11})[V_4 \cup \cdots \cup V_9]$ has no isolated vertex. Then $CN(V_i) \cap CN(V_j) \neq \emptyset$ for some $3 \leq i < j \leq 9$. Since obviously $|CN(V_i) \cap CN(V_j)| \geq 2$ implies $|\bigcup_{i=3}^{9} CN(V_i)| < |V(T(C_{11}))|$, we assume $|CN(V_i) \cap CN(V_j)| = 1$ for some $3 \leq i < j \leq 9$, and so $|\bigcup_{i=3}^{9} CN(V_i)| = |V(T(C_{11}))|$. But then the subgraph $T(C_{11})[V_1 \cup V_2]$ with chromatic number 2 contains K_3 as a subgraph, which is not possible.

Thus $\ell \geq 10$, and in fact $\chi_d^{tt}(C_{11}) = 10$ by (2.1).

- $n = 12$. Let $\ell = 9$. Then $(a_1, a_2) = (3, 6)$, $(4, 4)$, $(4, 5)$, $(5, 3)$, $(5, 4)$, $(6, 2)$, $(6, 3)$. Because $2a_1 + a_2 \geq 12$, $8 \leq a_1 + a_2 \leq 9$, $3 \leq a_1 \leq 6$ and $\max\{0, 12 - 2a_1, 8 - a_1\} \leq a_2 \leq \min\{6, 9 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (3, 6)$, $(4, 5)$, $(5, 4)$, $(6, 3)$ imply $\sum_{i=1}^{9} |V_i| \neq 2n$ and $(a_1, a_2) = (4, 4)$, $(5, 3)$, $(6, 2)$ imply $|V_1| > \alpha = 8$, which contradict `Fact 1`, we have $\ell \geq 10$. Now since $f = (V_1, V_2, \cdots, V_{10})$ is a TDC of $T(C_{12})$ where $V_1 = \{v_4, v_6, v_{11}, e_{1(12)}, e_{23}, e_{78}, e_{9(10)}\}$, $V_2 = \{v_5, v_{10}, v_{12}, e_{12}, e_{34}, e_{67}, e_{89}\}$, $V_3 = \{v_2\}$, $V_4 = \{v_1, v_3\}$, $V_5 = \{e_{45}\}$, $V_6 = \{e_{56}\}$, $V_7 = \{v_7, v_9\}$, $V_8 = \{v_8\}$, $V_9 = \{e_{(10)(11)}\}$, $V_{10} = \{e_{(11)(12)}\}$, we have $\chi_d^{tt}(C_{12}) = 10$.
- $n = 13$. Let $\ell = 10$. Then $(a_1, a_2) = (3, 7)$, $(4, 5)$, $(4, 6)$, $(5, 3)$, $(5, 4)$, $(5, 5)$, $(6, 2)$, $(6, 3)$, $(6, 4)$, $(7, 1)$, $(7, 2)$, $(7, 3)$. Because $2a_1 + a_2 \geq 13$, $8 \leq a_1 + a_2 \leq 10$, $3 \leq a_1 \leq 7$ and $\max\{0, 13 - 2a_1, 8 - a_1\} \leq a_2 \leq \min\{7, 10 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4)$, $(7, 3)$ imply $\sum_{i=1}^{10} |V_i| \neq 2n$, and $(a_1, a_2) = (4, 5)$, $(5, 4)$, $(6, 3)$, $(7, 1)$, $(7, 2)$ imply $|V_1| > \alpha = 8$, which contradict `Fact 1`, and $(a_1, a_2) \neq (5, 3)$ by `Fact 10`, we assume $(a_1, a_2) = (6, 2)$. Then, since the number of isolated vertices of $T(C_{13})[V_5 \cup \cdots \cup V_{10}]$ is at most 2 (by `Fact 9`) and so $T(C_{13})[V_5 \cup \cdots \cup V_{10}] \cong \overline{K_2} \cup \mathcal{H}_4$, $\overline{K_1} \cup \mathcal{H}_5$ or $\mathcal{H}_6$, we have

$$\begin{aligned} |\bigcup_{i=3}^{10} CN(V_i)| &\leq |\bigcup_{i=3}^{4} CN(V_i)| + |\bigcup_{i=5}^{10} CN(V_i)| \\ &\leq 4 + \max\{2 \times 3 + 2 \times 7, 3 + 7 + 9, 3 \times 7\} \\ &< |V(T(C_{13}))|, \end{aligned}$$

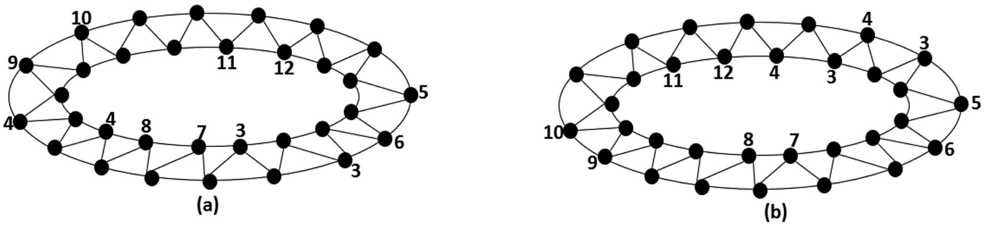

FIGURE 5. The illustration of $T(C_{16})$ when $(a_1, a_2) = (8, 2)$ and $T(C_{16})[V_5 \cup \cdots \cup V_{12}] \cong 4K_2$.

a contradiction with (1.1). Thus $\ell \geq 11$, and in fact $\chi_d^{tt}(C_{13}) = 11$ by (2.1).

- $n = 14$. Let $\ell = 10$. Then $(a_1, a_2) = (4, 6)$, $(5, 4)$, $(5, 5)$, $(6, 2)$, $(6, 3)$, $(6, 4)$, $(7, 1)$, $(7, 2)$, $(7, 3)$. Because $2a_1 + a_2 \geq 14$, $8 \leq a_1 + a_2 \leq 10$, $4 \leq a_1 \leq 7$ and $\max\{0, 14 - 2a_1, 8 - a_1\} \leq a_2 \leq \min\{6, 10 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (4, 6)$, $(5, 5)$, $(6, 4)$, $(7, 3)$ imply $\sum_{i=1}^{10} |V_i| \neq 2n$ and $(a_1, a_2) = (5, 4)$, $(6, 3)$, $(7, 1)$, $(7, 2)$ imply $|V_1| > \alpha = 9$, which contradict `Fact 1`, and $(a_1, a_2) \neq (6, 2)$ by `Fact 10`, we have $\ell \geq 11$, and in fact $\chi_d^{tt}(C_{14}) = 11$ by (2.1).
- $n = 16$. Let $\ell = 12$. Then $(a_1, a_2) = (4, 8)$, $(5, 6)$, $(5, 7)$, $(6, 4)$, $(6, 5)$, $(6, 6)$, $(7, 3)$, $(7, 4)$, $(7, 5)$, $(8, 2)$, $(8, 3)$, $(8, 4)$, $(9, 1)$, $(9, 2)$, $(9, 3)$. Because $2a_1 + a_2 \geq 16$, $10 \leq a_1 + a_2 \leq 12$, $4 \leq a_1 \leq 9$ and $\max\{0, 16 - 2a_1, 10 - a_1\} \leq a_2 \leq \min\{8, 12 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (4, 8)$, $(5, 7)$, $(6, 6)$, $(7, 5)$, $(8, 4)$, $(9, 3)$ imply $\sum_{i=1}^{12} |V_i| \neq 2n$ and $(a_1, a_2) = (5, 6)$, $(6, 5)$, $(7, 4)$, $(8, 3)$, $(9, 1)$,$(9, 2)$ imply $|V_1| > \alpha = 10$, which contradict `Fact 1`, and $(a_1, a_2) \neq (6, 4)$ by `Fact 10`, we assume $(a_1, a_2) = (7, 3)$, $(8, 2)$.
  - $(a_1, a_2) = (7, 3)$. Then, since the number of isolated vertices of $T(C_{16})[V_6 \cup \cdots \cup V_{12}]$ is at most 3 (by `Fact 9`) and so $T(C_{16})[V_6 \cup \cdots \cup V_{12}] \cong \overline{K_3} \cup \mathcal{H}_4$, $\overline{K_2} \cup \mathcal{H}_5$, $\overline{K_1} \cup \mathcal{H}_6$ or $\mathcal{H}_7$, we have

$$\begin{aligned} |\bigcup_{i=3}^{12} CN(V_i)| &\leq |\bigcup_{i=3}^{6} CN(V_i)| + |\bigcup_{i=7}^{12} CN(V_i)| \\ &\leq 6 + \max\{3 \times 3 + 2 \times 7, 2 \times 3 + 7 + 9, 3 + 3 \times 7, 2 \times 7 + 9\} \\ &< |V(T(C_{16}))|, \end{aligned}$$

    a contradiction with (1.1).
  - $(a_1, a_2) = (8, 2)$. Then, since the number of isolated vertices of $T(C_{16})[V_5 \cup \cdots \cup V_{12}]$ is at most 2 (by `Fact 9`) and so $T(C_{16})[V_5 \cup \cdots \cup V_{12}] \cong \overline{K_2} \cup \mathcal{H}_6$, $\overline{K_1} \cup \mathcal{H}_7$ or $\mathcal{H}_8$, we have

$$\begin{aligned} |\bigcup_{i=3}^{12} CN(V_i)| &\leq |\bigcup_{i=3}^{4} CN(V_i)| + |\bigcup_{i=5}^{12} CN(V_i)| \\ &\leq 4 + \max\{2 \times 3 + 3 \times 7, 3 + 2 \times 7 + 9, 4 \times 7\} \\ &= |V(T(C_{16}))|. \end{aligned}$$

    Since $|\bigcup_{i=3}^{12} CN(V_i)| = |V(T(C_{16}))|$ if and only if $T(C_{16})[V_5 \cup \cdots \cup V_{12}] \cong 4K_2$, we may assume $T(C_{16})[V_5 \cup \cdots \cup V_{12}][V_{2i-1} \cup V_{2i}] \cong K_2$ for $3 \leq i \leq 6$. Then
    - $|CN(V_i)| = 2$ for $3 \leq i \leq 4$,
    - $CN(V_i) \cap (\bigcup_{j=5}^{12} CN(V_j)) = \emptyset$ for $3 \leq i \leq 4$,
    - $|CN(V_{2i-1}) \cup CN(V_{2i})| = 7$ for $3 \leq i \leq 6$,
    - $(CN(V_{2i-1}) \cup CN(V_{2i})) \cap (CN(V_{2j-1}) \cup CN(V_{2j})) = \emptyset$ for $3 \leq i < j \leq 6$,
    - $1 \leq |V_3 \cap (\bigcup_{j=5}^{12} CN(V_j))| \leq 2$ and $1 \leq |V_4 \cap (\bigcup_{j=5}^{12} CN(V_j))| \leq 2$.

    But this implies that V_3, $\cdots$, V_{12} has one of the positions shown in Figure 5. Then, since the induced subgraph $T(C_{16})[V_1 \cup V_2]$ with chromatic number 2 contains a clique K_3 as a subgraph, we reach to contradiction.
- $n = 17$. Let $\ell = 12$. Then $(a_1, a_2) = (5, 7)$, $(6, 5)$, $(6, 6)$, $(7, 3)$, $(7, 4)$, $(7, 5)$, $(8, 2)$, $(8, 3)$, $(8, 4)$, $(9, 1)$, $(9, 2)$, $(9, 3)$. Because $2a_1 + a_2 \geq 17$, $10 \leq a_1 + a_2 \leq 12$ and $5 \leq a_1 \leq 9$ and $\max\{0, 17 - 2a_1, 10 - a_1\} \leq a_2 \leq \min\{7, 12 - a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (5, 7)$, $(6, 6)$, $(7, 5)$, $(8, 4)$, $(9, 3)$ imply $\sum_{i=1}^{12} |V_i| \neq 2n$, and

$(a_1, a_2) = (6,5), (7,4), (8,3), (9,1), (9,2)$ imply $|V_1| > \alpha = 11$, which contradict `Fact 1`, and $(a_1, a_2) \neq (7,3)$ by `Fact 10`, we assume $(a_1, a_2) = (8,2)$. Then, since the number of isolated vertices of $T(C_{17})[V_5 \cup \cdots \cup V_{12}]$ is at most 2 (by `Fact 9`) and so $T(C_{17})[V_5 \cup \cdots \cup V_{12}] \cong \overline{K_2} \cup \mathcal{H}_6$, $\overline{K_1} \cup \mathcal{H}_7$ or $\mathcal{H}_8$, we have

$$\begin{aligned}|\textstyle\bigcup_{i=3}^{12} CN(V_i)| &\leq |\textstyle\bigcup_{i=3}^{4} CN(V_i)| + |\textstyle\bigcup_{i=5}^{12} CN(V_i)| \\ &\leq 4 + \max\{2\times 3 + 3\times 7, 3 + 2\times 7 + 9, 4\times 7\} \\ &< |V(T(C_{17}))|,\end{aligned}$$

a contradiction with (1.1). Thus $\ell \geq 13$, and in fact $\chi_d^{tt}(C_{17}) = 13$ by (2.1).

- $n = 19$. Let $\ell = 14$. Then $(a_1, a_2) = (5,9), (6,7), (6,8), (7,5), (7,6), (7,7), (8,4), (8,5), (8,6), (9,3), (9,4), (9,5), (10,2), (10,3), (10,4), (11,1), (11,2), (11,3)$. Because $2a_1 + a_2 \geq 19$, $12 \leq a_1 + a_2 \leq 14$, $5 \leq a_1 \leq 11$ and $\max\{0, 19-2a_1, 12-a_1\} \leq a_2 \leq \min\{9, 14-a_1\}$ by `Facts 5-8`. Since $(a_1, a_2) = (5,9), (6,8), (7,7), (8,6), (9,5), (10,4), (11,3)$ imply $\sum_{i=1}^{14} |V_i| \neq 2n$ and $(a_1, a_2) = (6,7), (7,6), (8,5), (9,4), (10,3), (11,1), (11,2)$ imply $|V_1| > \alpha = 12$, which contradict `Fact 1`, and $(a_1, a_2) \neq (7,5)$ by `Fact 10`, we assume $(a_1, a_2) = (8,4), (9,3), (10,2)$.
    - $(a_1, a_2) = (8,4)$. Then, since the number of isolated vertices of $T(C_{19})[V_7 \cup \cdots \cup V_{14}]$ is at most 4 (by `Fact 9`) and so $T(C_{19})[V_7 \cup \cdots \cup V_{14}] \cong \overline{K_4} \cup \mathcal{H}_4$, $\overline{K_3} \cup \mathcal{H}_5$, $\overline{K_2} \cup \mathcal{H}_6$, $K_1 \cup \mathcal{H}_7$ or $\mathcal{H}_8$, we have

    $$\begin{aligned}|\textstyle\bigcup_{i=3}^{14} CN(V_i)| &\leq |\textstyle\bigcup_{i=3}^{6} CN(V_i)| + |\textstyle\bigcup_{i=7}^{14} CN(V_i)| \\ &\leq 8 + \max\{4\times 3 + 2\times 7, 3\times 3 + 7 + 9, 2\times 3 + 3\times 7, \\ &\quad 3 + 2\times 7 + 9, 4\times 7\} \\ &< |V(T(C_{19}))|,\end{aligned}$$

    a contradiction with (1.1).
    - $(a_1, a_2) = (9,3)$. Then, since the number of isolated vertices of $T(C_{19})[V_6 \cup \cdots \cup V_{14}]$ is at most 3 (by `Fact 9`) and so $T(C_{19})[V_6 \cup \cdots \cup V_{14}] \cong \overline{K_3} \cup \mathcal{H}_6$, $\overline{K_2} \cup \mathcal{H}_7$, $K_1 \cup \mathcal{H}_8$ or $\mathcal{H}_9$, we have

    $$\begin{aligned}|\textstyle\bigcup_{i=3}^{14} CN(V_i)| &\leq |\textstyle\bigcup_{i=3}^{5} CN(V_i)| + |\textstyle\bigcup_{i=6}^{14} CN(V_i)| \\ &\leq 6 + \max\{3\times 3 + 3\times 7, 2\times 3 + 2\times 7 + 9, 3 + 4\times 7, \\ &\quad 3\times 7 + 9\} \\ &< |V(T(C_{19}))|,\end{aligned}$$

    a contradiction with (1.1).
    - $(a_1, a_2) = (10,2)$. Then, since the number of isolated vertices of $T(C_{19})[V_5 \cup \cdots \cup V_{14}]$ is at most 2 (by `Fact 9`), we have $T(C_{19})[V_5 \cup \cdots \cup V_{14}] \cong \overline{K_2} \cup \mathcal{H}_8$, $K_1 \cup \mathcal{H}_9$ or $\mathcal{H}_{10}$. Let $T(C_{19})[V_5 \cup \cdots \cup V_{14}] \cong \overline{K_2} \cup \mathcal{H}_8$. Then

    $$\begin{aligned}|\textstyle\bigcup_{i=3}^{14} CN(V_i)| &\leq |\textstyle\bigcup_{i=3}^{4} CN(V_i)| + |\textstyle\bigcup_{i=5}^{14} CN(V_i)| \\ &\leq 4 + 2\times 3 + 4\times 7 \\ &= |V(T(C_{19}))|.\end{aligned} \tag{2.4}$$

    Since $|\bigcup_{i=3}^{14} CN(V_i)| = 38$ if and only if equality holds in (2.4), $V_3, \cdots, V_{14}$ are in the position shown in Figure 6. But then we reach to this contradiction that $T(P_{19})[V_1 \cup V_2]$ with chromatic number 2 contains K_3 as a subgraph. Now let $T(C_{19})[V_5 \cup \cdots \cup V_{14}] \cong K_1 \cup \mathcal{H}_9$. Then

    $$\begin{aligned}|\textstyle\bigcup_{i=3}^{14} CN(V_i)| &\leq |\textstyle\bigcup_{i=3}^{4} CN(V_i)| + |\textstyle\bigcup_{i=5}^{14} CN(V_i)| \\ &\leq 4 + 1\times 3 + 3\times 7 + 9 \\ &< |V(T(C_{19}))|,\end{aligned}$$

    a contradiction with (1.1). Finally let $T(C_{19})[V_5 \cup \cdots \cup V_{14}] \cong \mathcal{H}_{10}$, which implies

    $$\begin{aligned}|\textstyle\bigcup_{i=3}^{14} CN(V_i)| &\leq |\textstyle\bigcup_{i=3}^{4} CN(V_i)| + |\textstyle\bigcup_{i=5}^{14} CN(V_i)| \\ &\leq 4 + 5\times 7 \\ &= 39.\end{aligned} \tag{2.5}$$

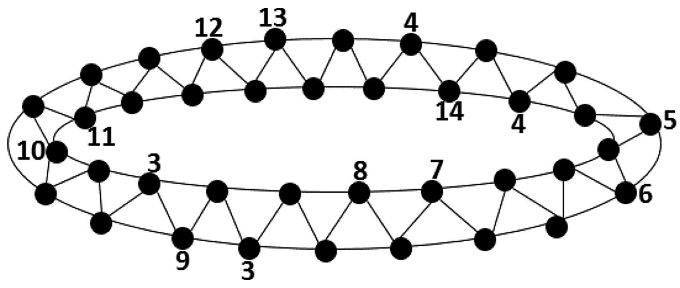

FIGURE 6. The illustration of $T(C_{19})$ when $(a_1, a_2) = (10, 2)$, $T(C_{19})[V_5 \cup \cdots \cup V_{14}] \cong \overline{K_2} \cup \mathcal{H}_8$ and $|\bigcup_{i=3}^{14} CN(V_i)| = 38$.

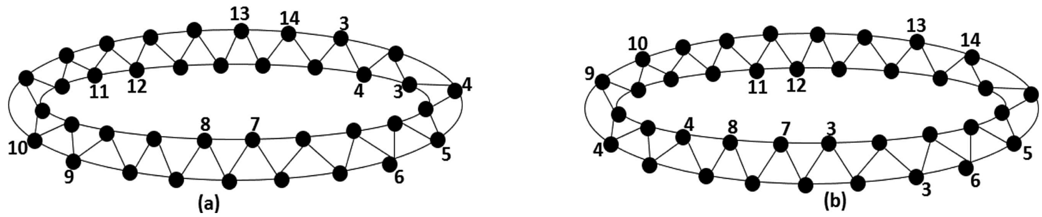

FIGURE 7. The illustration of $T(C_{19})$ when $(a_1, a_2) = (10, 2)$, $T(C_{19})[V_5 \cup \cdots \cup V_{14}] \cong 5K_2$ and $|\bigcup_{i=3}^{14} CN(V_i)| = 38$.

Since $|\bigcup_{i=3}^{14} CN(V_i)| = 38$ if and only if $\mathcal{H}_{10} \cong 5K_2$, by assumptions $\mathcal{H}_{10}[V_{2i-1} \cup V_{2i}] \cong K_2$ for $3 \le i \le 7$, we have

- $|CN(V_i)| = 2$ for $3 \le i \le 4$,
- $CN(V_i) \cap (\bigcup_{j=5}^{14} CN(V_j)) = \emptyset$ for $3 \le i \le 4$,
- $|CN(V_{2i-1}) \cup CN(V_{2i})| = 7$ for $3 \le i \le 7$,
- $|(CN(V_{2i-1}) \cup CN(V_{2i})) \cap (CN(V_{2j-1}) \cup CN(V_{2j}))| \le 1$ for $3 \le i < j \le 7$,
- $1 \le |V_3 \cap (\bigcup_{j=5}^{14} CN(V_j))| \le 2$ and $1 \le |V_4 \cap (\bigcup_{j=5}^{14} CN(V_j))| \le 2$ for some $3 \le j \ne k \le 6$.

That is $V_3, \cdots, V_{14}$ are in the position shown in Figure 7. But then we reach to this contradiction that $T(P_{19})[V_1 \cup V_2]$ with chromatic number 2 contains K_3 as a subgraph.

Thus $\ell \ge 15$, and in fact $\chi_d^{tt}(C_{19}) = 15$ by (2.1).

□

3. Paths

Here, we calculate the total dominator total chromatic number of paths. First we recall a proposition and calculate the mixed indepence number of a path.

Proposition 3.1. [12] *For any path P_n of order $n \ge 2$,*

$$\gamma_{tm}(P_n) = \begin{cases} \lfloor \frac{4n}{7} \rfloor & \text{if } n \equiv 4 \pmod 7, \\ \lceil \frac{4n}{7} \rceil & \text{if } n \not\equiv 4 \pmod 7. \end{cases}$$

Lemma 3.2. *For any path P_n of order $n \ge 3$, $\alpha_{\text{mix}}(P_n) = \lceil \frac{2n-1}{3} \rceil$.*

Proof. Let $P_n : v_1v_2\cdots v_n$ be a path of order $n \geq 3$. Then $V(T(P_n)) = V \cup \mathcal{E}$ where $\mathcal{E} = \{e_{i(i+1)} \mid 1 \leq i \leq n-1\}$. Since the number of triangles that a vertex w from an independent set of $T(P_n)$ belongs to them is

$$\begin{array}{l} 1 \text{ if and only if } w \in \{v_1, v_n\} \text{ or} \\ 2 \text{ if and only if } w \in \{e_{12}, e_{(n-1)n}\} \text{ or} \\ 3 \text{ if and only if } w \notin \{v_1, e_{12}, v_n, e_{(n-1)n}\}, \end{array}$$

we conclude $\{v_{3i+1} \mid 0 \leq i \leq \lceil \frac{n}{3} \rceil - 1\} \cup \{e_{(3i+2)(3i+3)} \mid 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1\}$ is a maximum independent set of cardinality $\lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{3} \rceil = \lceil \frac{2n-1}{3} \rceil$, and so $\alpha_{\text{mix}}(P_n) = \alpha(T(P_n)) = \lceil \frac{2n-1}{3} \rceil$. □

Proposition 3.3. *For any path P_n of order $n \geq 2$,*

$$\chi_d^{tt}(P_n) = \begin{cases} \gamma_{tm}(P_n) + 1 \text{ if } n = 2, 3, \\ \gamma_{tm}(P_n) + 2 \text{ if } n = 4, 5, 6, 8, 9, 10, 13, 16, \\ \gamma_{tm}(P_n) + 3 \text{ if } n = 7, \ \ n \neq 13, 16 \text{ or } n \geq 11 \end{cases}$$

which by considering Proposition 3.1 implies

$$\chi_d^{tt}(P_n) = \begin{cases} n + 1 \text{ if } n = 2, \\ n \qquad \text{if } 3 \leq n \leq 7, \\ n - 1 \text{ if } 8 \leq n \leq 9, \end{cases}$$

and for $n \geq 10$,

$$\chi_d^{tt}(P_n) = \begin{cases} \lfloor \frac{4n}{7} \rfloor + 3 \text{ if } n \equiv 4 \pmod 7 \text{ or } n = 10, 13, 16, \\ \lceil \frac{4n}{7} \rceil + 3 \text{ if } n \not\equiv 4 \pmod 7 \text{ and } n \neq 10, 13, 16. \end{cases}$$

Proof. Let $P_n : v_1v_2\cdots v_n$ be a path of order $n \geq 2$. Then $V(T(P_n)) = V \cup \mathcal{E}$ where $\mathcal{E} = \{e_{i(i+1)} \mid 1 \leq i \leq n-1\}$. Since $T(P_2) \cong K_3$ and $T(P_3)$ contains K_3 as a subgraph and $(\{v_1, e_{23}\}, \{v_3, e_{12}\}, \{v_2\})$ is a TDC of $T(P_3)$, we have $\chi_d^{tt}(P_n) = \chi_d^t(T(P_n)) = 3$ for $n = 2, 3$. We know from [12] that for $n \geq 4$ the sets

$$\begin{array}{ll} S_0 = \{v_{7i+2}, v_{7i+3}, e_{(7i+5)(7i+6)}, e_{(7i+6)(7i+7)} \mid 0 \leq i \leq \lfloor \frac{n}{7} \rfloor - 1\} & \text{if } n \equiv 0 \pmod 7, \\ S_1 = S_0 \cup \{e_{(n-1)n}\} & \text{if } n \equiv 1 \pmod 7, \\ S_2 = S_3 = S_0 \cup \{v_{n-1}, v_n\} & \text{if } n \equiv 2, 3 \pmod 7, \\ S_4 = S_0 \cup \{v_{n-2}, v_{n-1}\} & \text{if } n \equiv 4 \pmod 7, \\ S_5 = S_0 \cup \{v_{n-3}, v_{n-2}, v_{n-1}\} & \text{if } n \equiv 5 \pmod 7, \\ S_6 = S_0 \cup \{v_{n-4}, v_{n-3}, e_{(n-2)(n-1)}, e_{(n-1)n}\} & \text{if } n \equiv 6 \pmod 7 \end{array}$$

are minimum total dominating sets of $T(P_n)$. Since $\chi(T(P_n) - S_r) = 3$ for each $0 \leq r \leq 6$, Proposition 1.5 implies

$$\chi_d^{tt}(P_n) \leq \gamma_{tm}(P_n) + 3. \tag{3.1}$$

For $n = 19, 12$ or $n \geq 24$, since $T(P_n)$ has $2n-3$ triangles with the vertex sets $\{v_i, v_{i+1}, e_{i(i+1)}\}$ when $1 \leq i \leq n-1$ or $\{e_{(i-1)i}, v_i, e_{i(i+1)}\}$ when $2 \leq i \leq n-1$, and since at least two color classes are needed for totally dominating the vertices of every five consecutive triangles with the vertex sets

$$\begin{array}{l} \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, e_{i(i+1)}, e_{(i+1)(i+2)}, e_{(i+2)(i+3)}\} \text{ or} \\ \{v_i, v_{i+1}, v_{i+2}, e_{(i-1)i}, e_{i(i+1)}, e_{(i+1)(i+2)}, e_{(i+2)(i+3)}\}, \end{array}$$

we conclude the number of used color classes in $T(P_n)$ is at least $2\lfloor \frac{2n-3}{5} \rfloor \geq \gamma_t(T(P_n)) + 3 = \gamma_{tm}(P_n) + 3$, and so $\chi_d^{tt}(P_n) = \gamma_{tm}(P_n) + 3$ by (3.1). Therefore we continue our proof in the following two remained cases by this assumption that $\mathcal{H}_k$ denotes a graph of order k with positive minimum degree.

Case 1. $n = 18, 20, 21, 23$. Let $f = (V_1, V_2, \cdots, V_\ell)$ be a min-TDC of $T(P_n)$ and let $A = \{v_1, v_2, v_3\}$, $B = \{v_{n-2}, v_{n-1}, v_n\}$. Then $v_1 \succ_t V_k$ for some k implies $V_k = \{w\}$ where $w \in \{v_2, e_{12}\}$. Let $w \succ_t V_m$ for some m.

Since $V_m \subseteq \{v_1, v_3, e_{12}, e_{23}\}$ when $w = v_2$ and $V_m \subseteq \{v_1, v_2, e_{23}\}$ when $w = e_{12}$, we have $V_m \neq V_k$. Since a similar result holds by considering v_n instead of v_1, we conclude $|\{V_k \mid v_i \succ_t V_k \text{ for some } v_i \in A \cup B\}| \geq 4$. Let $\{V_k \mid v_i \succ_t V_k \text{ for some } v_i \in A\} = \{V_1, V_2\}$ and $\{V_k \mid v_i \succ_t V_k \text{ for some } v_i \in B\} = \{V_3, V_4\}$ in which $|V_1|+|V_2| \leq 3$ and $|V_3|+|V_4| \leq 3$. Then $\bigcup_{i=1}^{4} V_i \subseteq A \cup B \cup \{e_{12}, e_{23}, e_{(n-2)(n-1)}, e_{(n-1)n}\}$. Now since the subgraph of $T(P_n) - (A \cup B)$ induced by $\{v_4, e_{34}, e_{45}\}$ is a complete graph and $\{v_4, e_{34}, e_{45}\} \cap (\bigcup_{i=1}^{4} V_i) = \emptyset$, we have $\ell \geq 7$. Without loss of generality, we may assume $e_{34} \in V_5$, $e_{45} \in V_6$ and $v_4 \in V_7$. Let $C = \{v_i, e_{i(i+1)} \mid 6 \leq i \leq n-4\}$ and $C_V = \{V_k \mid v \succ_t V_k \text{ for some } v \in C\}$. Then for every $v \in C$, $v \not\succ_t V_k$ when $1 \leq k \leq 7$. We know the subgraph of $T(P_n)$ induced by C has $2(n-10)$ distinct triangles with the vertex sets $\{v_i, v_{i+1}, e_{i(i+1)}\}$ or $\{e_{(i-1)i}, v_i, e_{i(i+1)}\}$ for $6 \leq i \leq n-4$. Since at least two color classes are needed for totally dominating the vertices of every five consecutive triangles with the vertex sets $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, e_{i(i+1)}, e_{(i+1)(i+2)}, e_{(i+2)(i+3)}\}$ or $\{v_i, v_{i+1}, v_{i+2}, e_{(i-1)i}, e_{i(i+1)}, e_{(i+1)(i+2)}, e_{(i+2)(i+3)}\}$, we conclude that the number of used color classes in $T(P_n)[C]$ is at least $2\lfloor \frac{2(n-10)}{5} \rfloor$. So $\chi_d^t(T(P_n)) \geq 7 + 2\lfloor \frac{2(n-10)}{5} \rfloor \geq \gamma_t(T(P_n)) + 3$, which implies $\chi_d^{tt}(P_n) = \gamma_{tm}(P_n) + 3$ by (3.1).

Case 2. $4 \leq n \leq 17$. Let $f = (V_1, V_2, \cdots, V_\ell)$ be a min-TDC of $T(P_n)$ such that $|V_1| \geq |V_2| \geq \cdots \geq |V_\ell|$ and for $1 \leq i \leq \lceil \frac{2n-1}{3} \rceil$, let $A_i = \{V_k \mid |V_k| = i\}$ be a set of cardinality a_i (we recall $\lceil \frac{2n-1}{3} \rceil = \alpha_{\text{mix}}(P_n) = \alpha(T(P_n))$ which is denoted simply by α). By considering the following facts we continue our proof in the following subcases.

- ⋆ `Fact 1.` $|V_k| \leq \lceil \frac{2n-1}{3} \rceil$ for $1 \leq k \leq \ell$ and $\sum_{i=1}^{\ell} |V_i| = 2n-1$.
- ⋆ `Fact 2.` For any $v \in V \cup \mathcal{E}$, if $v \succ_t V_k$ for some $1 \leq k \leq \ell$, then $|V_k| \leq 2$.
- ⋆ `Fact 3.` For any vertex $v \in M$, if $v \succ_t V_k$ for some $1 \leq k \leq \ell$ and $|V_k| = 2$, then $CN(V_k) \cap M = \{v\}$, where $M \in \{\mathcal{E}, V\}$, and since $CN(V_k) \cap V \neq \emptyset$ if and only if $CN(V_k) \cap \mathcal{E} \neq \emptyset$, we have $|CN(V_k)| = 2$.
- ⋆ `Fact 4.` For any color class V_k of cardinality one, $1 \leq |CN(V_k) \cap V| \leq 2$ and $1 \leq |CN(V_k) \cap \mathcal{E}| \leq 2$.
- ⋆ `Fact 5.` $a_1 \geq 2$. Because for $i = 1, n$, $v_i \succ_t V_{k_i}$ implies $|V_{k_i}| = 1$ and $V_{k_1} \neq V_{k_n}$.
- ⋆ `Fact 6.` $\gamma_{tm}(P_n) \leq a_1 + a_2 \leq \ell$. Because the set S is a TDS of $T(P_n)$ where $|S \cap V_i| = 1$ for each $V_i \in A_1 \cup A_2$ by `Fact 2` (for left), and $a_1 + \cdots + a_\alpha = \ell$ (for right).
- ⋆ `Fact 7.` $2a_1 + a_2 \geq n$ (by `Facts 3,4`).
- ⋆ `Fact 8.` $\max\{n - \ell, 2\} \leq a_1 \leq \lfloor \frac{\alpha\ell - 2n + 1}{\alpha - 1} \rfloor$. Because

$$\begin{aligned} 2n - 1 - a_1 &= |V(T(P_n))| - |\{V_i \mid |V_i| = 1 \text{ for } 1 \leq i \leq \ell\}| \\ &= \Sigma_{|V_i| \geq 2} |V_i| \\ &\leq (\ell - a_1)\alpha \end{aligned}$$

implies the upper bound, and for the lower bound

$$\begin{aligned} 2\ell - n &\geq 2(a_1 + a_2) - n && (\text{by } \texttt{Fact 6}) \\ &\geq a_2 && (\text{by } \texttt{Fact 7}), \end{aligned} \tag{3.2}$$

implies

$$\begin{aligned} a_1 &\geq \tfrac{n - a_2}{2} \ (\text{by } \texttt{Fact 7}) \\ &\geq n - \ell \text{ by } (3.2). \end{aligned}$$

- ⋆ `Fact 9.` $\max\{0, n - 2a_1, \gamma_{tm}(P_n) - a_1\} \leq a_2 \leq \min\{2\ell - n, \ell - a_1\}$ (by `Facts 5-8`).
- ⋆ `Fact 10.` If $J = \{k \mid |V_k| = 2 \text{ and } |CN(V_k)| \neq 0\}$, then the number of isolated vertices of $T(P_n)[\bigcup_{|V_k|=1} V_k]$ is at most $|J|$ (because $(|V_k|, |CN(V_k)|) = (2, 2)$ implies $T(P_n)[CN(V_k)] \cong K_2$, by `Fact 3`).
- ⋆ `Fact 11.` $V(T(P_n))$ has an unique partition to three maximal independent sets $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$ such that

$$\begin{aligned} &|\mathcal{W}_1| = |\mathcal{W}_2| = |\mathcal{W}_3| = \lceil \tfrac{2n-1}{3} \rceil && \text{if } n \equiv 2 \pmod 3, \\ &|\mathcal{W}_1| = |\mathcal{W}_2| + 1 = |\mathcal{W}_3| + 1 = \lceil \tfrac{2n-1}{3} \rceil && \text{if } n \equiv 1 \pmod 3, \\ &|\mathcal{W}_1| = |\mathcal{W}_2| = |\mathcal{W}_3| + 1 = \lceil \tfrac{2n-1}{3} \rceil && \text{if } n \equiv 0 \pmod 3. \end{aligned}$$

- $n = 4, 5$. Since $\ell = n-1$ implies $a_1 = 1$, which contradicts `Fact 5`, we have $\ell \geq n$. Now since $(\{v_2\}, \{v_3\}, \{e_{12}, e_{34}\}, \{v_1, e_{23}, v_4\})$ is a TDC of $T(P_4)$ and also $(\{v_2\}, \{v_3\}, \{v_4\}, \{v_5, e_{12}, e_{34}\}, \{v_1, e_{23}, e_{45}\})$ is a TDC of $T(P_5)$, we obtain $\chi_d^{tt}(P_n) = n$ when $n = 4, 5$.
- $n = 6$. Let $\ell = 5$. Then $(a_1, a_2) = (2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 2)$. Because `Facts 5-9` imply $2a_1 + a_2 \geq 6$, $4 \leq a_1 + a_2 \leq 5$, $2 \leq a_1 \leq 3$ and $\max\{0, 6-2a_1, 4-a_1\} \leq a_2 \leq \min\{4, 5-a_1\}$. Since $(a_1, a_2) = (2, 3)$, $(3, 2)$ imply $\sum_{i=1}^{5} |V_i| \neq 2n-1$ and $(a_1, a_2) = (2, 2)$, $(3, 1)$ imply $|V_1| > \alpha = 4$, which contradict `Fact 1`, we have $\ell \geq 6$. Now since the coloring function $(\{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{e_{12}, e_{34}, e_{56}\}, \{v_1, e_{23}, e_{45}, v_6\})$ is a TDC of $T(P_6)$, we obtain $\chi_d^{tt}(P_6) = 6 = n$.
- $n = 7$. Let $\ell = 6$. Then $(a_1, a_2) = (2, 3)$, $(2, 4)$, $(3, 1)$, $(3, 2)$, $(3, 3)$, $(4, 0)$, $(4, 1)$, $(4, 2)$. Because `Facts 5-9` imply $2a_1 + a_2 \geq 7$, $4 \leq a_1 + a_2 \leq 6$, $2 \leq a_1 \leq 4$ and $\max\{0, 7-2a_1, 4-a_1\} \leq a_2 \leq \min\{5, 6-a_1\}$. Since $(a_1, a_2) = (2, 4)$, $(3, 3)$, $(4, 2)$ imply $\sum_{i=1}^{6} |V_i| \neq 2n-1$ and $(a_1, a_2) = (3, 2)$, $(4, 1)$ imply $|V_1| > \alpha = 5$, which contradict `Fact 1`, we have $(a_1, a_2) = (2, 3)$, $(3, 1)$, $(4, 0)$.
  - $(a_1, a_2) = (4, 0)$. Then, since the number of isolated vertices of $T(P_7)[V_3 \cup \cdots \cup V_6]$ is zero, we have $T(P_7)[V_3 \cup \cdots \cup V_6] \cong \mathcal{H}_4$. Since $|\bigcup_{i=3}^{6} CN(V_i)| < |V(T(P_7))| = 13$ when $\mathcal{H}_4 \ncong 2K_2$, we assume $\mathcal{H}_4 \cong 2K_2$. By assumptions $T(P_7)[V_3 \cup V_4] \cong T(P_7)[V_5 \cup V_6] \cong K_2$, we have $|(\bigcup_{i=3}^{4} CN(V_i)) \cap (\bigcup_{i=5}^{6} CN(V_i))| = 1$. This guarantees that
  $$\begin{aligned}(V_3, V_4, V_5, V_6) = {} & (\{v_2\}, \{v_3\}, \{v_5\}, \{v_6\}), \\ & \text{or } (\{v_2\}, \{v_3\}, \{e_{56}\}, \{e_{67}\}).\end{aligned}$$
  But then, since $T(P_7)[V_1 \cup V_2]$ with chromatic number two contains a complete subgraph K_3 with the vertex set $\{v_4, e_{34}, e_{45}\}$, we reach to the contradiction.
  - $(a_1, a_2) = (3, 1)$. Then, since the number of isolated vertices of $T(P_7)[V_4 \cup \cdots \cup V_6]$ is one and so $T(P_7)[V_3 \cup \cdots \cup V_6] \cong K_1 \cup K_2$ or $\mathcal{H}_3$, we have
  $$\begin{aligned}|\textstyle\bigcup_{i=3}^{6} CN(V_i)| &\leq |CN(V_3)| + |\textstyle\bigcup_{i=4}^{6} CN(V_i)| \\ &\leq 2 + \max\{3+7, 9\} \\ &< |V(T(P_7))|,\end{aligned}$$
  a contradiction with (1.1).
  - $(a_1, a_2) = (2, 3)$. Then, since the number of isolated vertices of $T(P_7)[V_5 \cup V_6]$ is two or zero and so $T(P_7)[V_3 \cup \cdots \cup V_6] \cong \overline{K_2}$ or K_2, we have
  $$\begin{aligned}|\textstyle\bigcup_{i=3}^{6} CN(V_i)| &\leq |\textstyle\bigcup_{i=3}^{4} CN(V_i)| + |\textstyle\bigcup_{i=5}^{6} CN(V_i)| \\ &\leq 4 + \max\{2 \times 3, 7\} \\ &< |V(T(P_7))|,\end{aligned}$$
  a contradiction with (1.1).

  Thus $\ell \geq 7$, which implies $\chi_d^{tt}(P_7) = 7 = \gamma_{tm}(P_7) + 3$ by (3.1).
- $n = 8$. Let $\ell = 6$. Then $(a_1, a_2) = (2, 4)$, $(3, 2)$, $(3, 3)$. Because `Facts 5-9` imply $2a_1 + a_2 \geq 8$, $5 \leq a_1 + a_2 \leq 6$, $3 \leq a_1 \leq 5$ and $\max\{0, 8-2a_1, 5-a_1\} \leq a_2 \leq \min\{4, 6-a_1\}$. Since $(a_1, a_2) = (2, 4)$, $(3, 3)$ imply $\sum_{i=1}^{6} |V_i| \neq 2n-1$ and $(a_1, a_2) = (3, 2)$ imply $|V_1| > \alpha = 5$, which contradict `Fact 1`, we have $\ell \geq 7$. Since now $(\{v_1, e_{23}, e_{45}, e_{67}, v_8\}, \{e_{12}, e_{34}, v_5, e_{78}\}, \{v_4, e_{56}\}, \{v_3\}, \{v_2\}, \{v_6\}, \{v_7\})$ is a TDC of $T(P_8)$, we obtain $\chi_d^{tt}(P_8) = 7 = n-1$.
- $n = 9$. Let $\ell = 7$. Then $(a_1, a_2) = (2, 5)$, $(3, 3)$, $(3, 4)$, $(4, 2)$, $(4, 3)$, $(5, 1)$, $(5, 2)$. Because `Facts 5-9` imply $2a_1 + a_2 \geq 9$, $6 \leq a_1 + a_2 \leq 7$, $2 \leq a_1 \leq 5$ and $\max\{0, 9-2a_1, 6-a_1\} \leq a_2 \leq \min\{5, 7-a_1\}$. Since $(a_1, a_2) = (2, 5)$, $(3, 4)$, $(4, 3)$,$(5, 2)$ imply $\sum_{i=1}^{7} |V_i| \neq 2n-1$ and $(a_1, a_2) = (3, 3)$, $(4, 2)$, $(5, 1)$ imply $|V_1| > \alpha = 6$, which contradict `Fact 1`, we have $\ell \geq 8$. Now since
$$(\{v_2\}, \{v_3\}, \{e_{45}\}, \{e_{56}\}, \{v_7\}, \{v_8\}, \{v_1, v_4, v_6, v_9, e_{23}, e_{78}\}, \{e_{12}, e_{34}, e_{67}, e_{89}, v_5\})$$
is a TDC of $T(P_9)$, we have $\chi_d^{tt}(P_9) = 8 = n-1$.

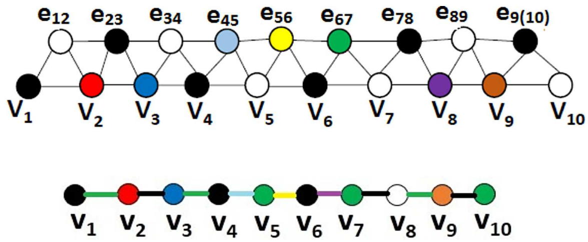

FIGURE 8. A min-TDC of $T(P_{10})$ (up) and the corresponding min-TDTC of P_{10} (down).

- $n = 10$. Let $\ell = 7$. Then $(a_1, a_2) = (3, 4), (4, 2), (4, 3), (5, 1), (5, 2)$. Because `Facts 5-9` imply $2a_1 + a_2 \geq 10$, $6 \leq a_1 + a_2 \leq 7$, $3 \leq a_1 \leq 5$ and $\max\{0, 10 - 2a_1, 7 - a_1\} \leq a_2 \leq \min\{4, 7 - a_1\}$. Since $(a_1, a_2) = (3, 4)$, $(4, 3)$, $(5, 2)$ imply $\sum_{i=1}^{7} |V_i| \neq 2n - 1$ and $(a_1, a_2) = (4, 2)$, $(5, 1)$ imply $|V_1| > \alpha = 7$, which contradict `Fact 1`, we have $\ell \geq 8$. Now since $(V_1, V_2, \{e_{45}, e_{67}\}, \{e_{56}\}, \{v_2\}, \{v_3\}, \{v_8\}, \{v_9\})$ is a TDC of $T(P_{10})$ where $V_1 = \{v_1, v_4, v_6, e_{23}, e_{78}, e_{9(10)}\}$, $V_2 = \{v_5, v_7, v_{10}, e_{12}, e_{34}, e_{89}\}$ and it is shown in Figure 8, we have $\chi_d^{tt}(P_{10}) = 8 = \lfloor \frac{4n}{7} \rfloor + 3$.
- $n = 11$. Let $\ell = 8$. Then $(a_1, a_2) = (3, 5), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3)$. Because `Facts 5-9` imply $2a_1 + a_2 \geq 11$, $6 \leq a_1 + a_2 \leq 8$, $3 \leq a_1 \leq 5$ and $\max\{0, 11 - 2a_1, 6 - a_1\} \leq a_2 \leq \min\{5, 8 - a_1\}$. Since $(a_1, a_2) = (3, 5), (4, 4), (5, 3)$ imply $\sum_{i=1}^{8} |V_i| \neq 2n - 1$ and $(a_1, a_2) = (4, 3), (5, 2)$ imply $|V_1| > \alpha = 7$, which contradict `Fact 1`, we have $(a_1, a_2) = (5, 1)$, that is, $(|V_1|, \cdots, |V_8|) = (7, 7, 2, 1, 1, 1, 1, 1)$. Then $V_3 \cup \cdots \cup V_8$ is a maximal independent set by `Fact 11`, and all vertices of the set are totally dominated by no color class. Thus $\ell \geq 9$, which implies $\chi_d^{tt}(P_{11}) = 9 = \gamma_{tm}(P_{11}) + 3$ by (3.1).
- $n = 12$. Let $\ell = 9$. Then $(a_1, a_2) = (3, 6), (4, 4), (4, 5), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (7, 0), (7, 1), (7, 2)$. Because `Facts 5-9` imply $2a_1 + a_2 \geq 12$, $7 \leq a_1 + a_2 \leq 9$, $3 \leq a_1 \leq 7$ and $\max\{0, 12 - 2a_1, 7 - a_1\} \leq a_2 \leq \min\{6, 9 - a_1\}$. Since $(a_1, a_2) = (3, 6), (4, 5), (5, 4), (6, 3), (7, 2)$ imply $\sum_{i=1}^{9} |V_i| \neq 2n - 1$ and $(a_1, a_2) = (4, 4)$, $(5, 3)$, $(6, 2)$, $(7, 1)$ imply $|V_1| > \alpha = 8$, which contradict `Fact 1`, we have $(a_1, a_2) = (5, 2), (6, 1), (7, 0)$. If $(a_1, a_2) = (7, 0)$, then $V_3 \cup \cdots \cup V_9 = \mathcal{W}_3$ which is a maximal independent set by `Fact 11`, and so no vertex of the set is totally dominated by a color class.
  - $(a_1, a_2) = (5, 2)$. Then, since the number of isolated vertices of $T(P_{12})[V_5 \cup \cdots \cup V_9]$ is at most 2 (by `Fact 10`) and so $T(P_{12})[V_5 \cup \cdots \cup V_9] \cong \overline{K_2} \cup H_3$, $K_1 \cup \mathcal{H}_4$ or $\mathcal{H}_5$,

$$\begin{aligned} |\bigcup_{i=3}^{9} CN(V_i)| &\leq |\bigcup_{i=3}^{4} CN(V_i)| + |\bigcup_{i=5}^{9} CN(V_i)| \\ &\leq 4 + \max\{2 \times 3 + 9, 1 \times 3 + 2 \times 7, 1 \times 7 + 1 \times 9\} \\ &< |V(T(P_{12}))|, \end{aligned}$$

    a contradiction with (1.1).
  - $(a_1, a_2) = (6, 1)$. Then, since the number of isolated vertices of $T(P_{12})[V_4 \cup \cdots \cup V_9]$ is at most 1 (by `Fact 10`) and so $T(P_{12})[V_4 \cup \cdots \cup V_9] \cong K_1 \cup \mathcal{H}_5$ or $\mathcal{H}_6$,

$$\begin{aligned} |\bigcup_{i=3}^{9} CN(V_i)| &\leq |\bigcup_{i=3}^{4} CN(V_i)| + |\bigcup_{i=5}^{9} CN(V_i)| \\ &\leq 2 + \max\{1 \times 3 + 1 \times 7 + 1 \times 9, 3 \times 7\} \\ &= |V(T(P_{12}))|. \end{aligned} \tag{3.3}$$

    We see that equality holds in (3.3) if and only if $T(P_{12})[V_4 \cup \cdots \cup V_9] \cong 3K_2 \cong \bigcup_{i=1}^{3} T(P_{12})[V_{2i-1} \cup V_{2i}]$ such that $(CN(V_{2i-1}) \cup CN(V_{2i})) \cap (CN(V_{2j-1}) \cup CN(V_{2j}))$ for each $1 \leq i < j \leq 3$. This implies $V_4 = \{v_2\}$, $V_5 = \{v_3\}$, $V_6 = \{e_{56}\}$, $V_7 = \{e_{67}\}$, $V_8 = \{v_9\}$, $V_9 = \{v_{10}\}$. Since the vertices v_{12} and $e_{(11)(12)}$ did not totally dominated by a color class of cardinality one, we must have $CN(V_3) = \{v_{12}, e_{(11)(12)}\}$ which is not possible.

  Thus $\ell \geq 10$, which implies $\chi_d^{tt}(P_{12}) = 10 = \gamma_{tm}(P_{12}) + 3$ by (3.1).

- $n = 13$. Let $\ell = 9$. Then $(a_1, a_2) = (4, 5)$, $(5, 3)$, $(5, 4)$, $(6, 2)$,$(6, 3)$, $(7, 1)$, $(7, 2)$. Because `Facts 5-9` imply $2a_1 + a_2 \geq 13$, $8 \leq a_1 + a_2 \leq 9$, $4 \leq a_1 \leq 7$ and $\max\{0, 13 - 2a_1, 8 - a_1\} \leq a_2 \leq \min\{7, 10 - a_1\}$. Since $(a_1, a_2) = (4, 5)$, $(5, 4)$, $(6, 3)$, $(7, 2)$ imply $\sum_{i=1}^{9} |V_i| \neq 2n - 1$, and $(a_1, a_2) = (5, 3)$, $(6, 2)$, $(7, 1)$ imply $|V_1| > \alpha = 9$, which contradict `Fact 1`. Thus $\ell \geq 10$. Now since $(V_1, V_2, \cdots, V_{10})$ is a TDC of $T(P_{13})$ where

$$\begin{array}{l} V_1 = \{v_1, v_6, v_8, v_{13}, e_{23}, e_{45}, e_{9(10)}, e_{(11)(12)}\}, \\ V_2 = \{v_5, v_7, v_9, e_{12}, e_{34}, e_{(10)(11)}, e_{(12)(13)}\}, \\ V_3 = \{v_4, e_{56}\},\ V_4 = \{v_{10}, e_{89}\},\ V_5 = \{v_2\},\ V_6 = \{v_3\}, \\ V_7 = \{e_{67}\},\ V_8 = \{e_{78}\},\ V_9 = \{v_{11}\},\ V_{10} = \{v_{12}\}, \end{array}$$

  we have $\chi_d^{tt}(P_{13}) = 10$.
- $n = 14$. Let $\ell = 10$. Then $(a_1, a_2) = (4, 6)$, $(5, 4)$, $(5, 5)$, $(6, 2)$, $(6, 3)$, $(6, 4)$, $(7, 1)$, $(7, 2)$, $(7, 3)$. Because `Facts 5-9` imply $2a_1 + a_2 \geq 14$, $8 \leq a_1 + a_2 \leq 10$, $4 \leq a_1 \leq 7$ and $\max\{0, 14 - 2a_1, 8 - a_1\} \leq a_2 \leq \min\{6, 10 - a_1\}$. Since $(a_1, a_2) = (4, 6)$, $(5, 5)$, $(6, 4)$, $(7, 3)$ imply $\sum_{i=1}^{10} |V_i| \neq 2n - 1$, and $(a_1, a_2) = (5, 4)$, $(6, 3)$, $(7, 2)$ imply $|V_1| > \alpha = 9$, which contradict `Fact 1`, we have $(a_1, a_2) = (6, 2)$, $(7, 1)$, that is,

$$\begin{aligned} (|V_1|, \cdots, |V_{10}|) = &\ (9, 8, 2, 2, 1, 1, 1, 1, 1, 1), \\ &\text{or } (9, 9, 2, 1, 1, 1, 1, 1, 1, 1). \end{aligned}$$

  If $(|V_1|, \cdots, |V_{10}|) = (9, 9, 2, 1, 1, 1, 1, 1, 1, 1)$, then $V_3 \cup \cdots \cup V_{10}$ is a maximal independent set by `Fact 11`, and so no vertex of the set is totally dominated by a color class by `Fact 2`. If $(|V_1|, \cdots, |V_{10}|) = (9, 8, 2, 2, 1, 1, 1, 1, 1, 1)$, then $V_3 \cup \cdots \cup V_{10} = \mathcal{W}_j \cup \{w\}$ for some $1 \leq j \leq 3$ and some $w \in \bigcup_{j \neq i=1}^{3} \mathcal{W}_i$ by `Fact 11`. Since $deg(w) = 4$, there exist at least five vertices in $\mathcal{W}_j$ which are not totally dominated by a color class, a contradiction.
- $n = 15$. Let $\ell = 11$. Then $(a_1, a_2) = (4, 7)$, $(5, 5)$, $(5, 6)$, $(6, 3)$, $(6, 4)$, $(6, 5)$, $(7, 2)$, $(7, 3)$, $(7, 4)$, $(8, 1)$, $(8, 2)$, $(8, 3)$, $(9, 0)$, $(9, 1)$, $(9, 2)$. Because `Facts 5-9` imply $2a_1 + a_2 \geq 15$, $9 \leq a_1 + a_2 \leq 11$, $4 \leq a_1 \leq 9$ and $\max\{0, 15 - 2a_1, 9 - a_1\} \leq a_2 \leq \min\{7, 11 - a_1\}$. Since $(a_1, a_2) = (4, 7)$, $(5, 6)$, $(6, 5)$, $(7, 4)$, $(8, 3)$, $(9, 2)$ imply $\sum_{i=1}^{11} |V_i| \neq 2n - 1$ and $(a_1, a_2) = (5, 5)$, $(6, 4)$, $(7, 3)$, $(8, 2)$, $(9, 1)$ imply $|V_1| > \alpha = 10$, which contradict `Fact 1`, we have $(a_1, a_2) = (6, 3)$, $(7, 2)$, $(8, 1)$, $(9, 0)$. If $(a_1, a_2) = (9, 0)$. Then $V_3 \cup \cdots \cup V_{11} = \mathcal{W}_3$ is a maximal independent set (by `Fact 11`), and so no vertex of the set is totally dominated by a color class.
  - $(a_1, a_2) = (8, 1)$, that is, $(|V_1|, \cdots, |V_{11}|) = (10, 9, 2, 1, 1, 1, 1, 1, 1, 1, 1)$. Then either $V_3 \cup \cdots \cup V_{11} = \mathcal{W}_2$ which is a maximal independent set (by `Fact 11`) and so no vertex of the set is totally dominated by a color class, or $V_3 \cup \cdots \cup V_{11} = \mathcal{W}_3 \cup \{w\}$ for some vertex $w \in \mathcal{W}_1 \cup \mathcal{W}_2$, and since $|N(w) \cap \mathcal{W}_3| \leq 2$, there exist at least seven vertices in $V_3 \cup \cdots \cup V_{11}$ which are totally dominated by no color class.
  - $(a_1, a_2) = (7, 2)$. Then, since the number of isolated vertices of $T(P_{15})[\bigcup_{i=5}^{11} V_i]$ is at most 2 (by `Fact 10`) and so $T(P_{15})[\bigcup_{i=5}^{11} V_i] \cong \overline{K_2} \cup H_5$, $K_1 \cup \mathcal{H}_6$ or $\mathcal{H}_7$, we have

$$\begin{aligned} |\bigcup_{i=3}^{11} CN(V_i)| &\leq |\bigcup_{i=3}^{4} CN(V_i)| + |\bigcup_{i=5}^{11} CN(V_i)| \\ &\leq 4 + \max\{2 \times 3 + 7 + 9, 1 \times 3 + 3 \times 7, 2 \times 7 + 1 \times 9\} \\ &< |V(T(P_{15}))|, \end{aligned}$$

    a contradiction with (1.1).
  - $(a_1, a_2) = (6, 3)$. Then, since the number of isolated vertices of $T(P_{15})[\bigcup_{i=6}^{11} V_i]$ is at most 3 (by `Fact 10`) and so $T(P_{15})[\bigcup_{i=6}^{11} V_i] \cong \overline{K_3} \cup H_3$, $\overline{K_2} \cup H_4$, $K_1 \cup \mathcal{H}_5$ or $\mathcal{H}_6$, we have

$$\begin{aligned} |\bigcup_{i=3}^{11} CN(V_i)| &\leq |\bigcup_{i=3}^{5} CN(V_i)| + |\bigcup_{i=6}^{11} CN(V_i)| \\ &\leq 6 + \max\{3 \times 3 + 9, 2 \times 3 + 2 \times 7, 3 + 7 + 9, 3 \times 7\} \\ &< |V(T(P_{15}))|, \end{aligned}$$

    a contradiction with (1.1).

  Thus $\ell \geq 12$, which implies $\chi_d^{tt}(P_{15}) = 12 = \gamma_{tm}(P_{15}) + 3$ by (3.1).

- $n = 16$. Let $\ell = 11$. Then $(a_1, a_2) = (5,6)$, $(6,4)$, $(6,5)$, $(7,3)$, $(7,4)$, $(8,2)$, $(8,3)$, $(9,1)$, $(9,2)$. Because `Facts 5-9` imply $2a_1+a_2 \geq 16$, $10 \leq a_1+a_2 \leq 11$, $5 \leq a_1 \leq 9$ and $\max\{0, 16-2a_1, 10-a_1\} \leq a_2 \leq \min\{8, 11-a_1\}$. Since $(a_1, a_2) = (5,6)$, $(6,5)$, $(7,4)$, $(8,3)$, $(9,2)$ imply $\sum_{i=1}^{11} |V_i| \neq 2n-1$ and $(a_1, a_2) = (6,4)$, $(7,3)$, $(8,2)$, $(9,1)$ imply $|V_1| > \alpha = 11$, which contradict `Fact 1`, we have $\ell \geq 12$. Now since $(V_1, V_2, \cdots, V_{12})$ is a TDC of $T(P_{16})$ where
$$\begin{array}{l} V_1 = \{v_1, v_4, v_6, v_{11}, v_{13}, v_{16}, e_{23}, e_{78}, e_{9(10)}, e_{(14)(15)}\}, \\ V_2 = \{v_5, v_7, v_{10}, v_{12}, e_{12}, e_{34}, e_{89}, e_{(13)(14)}, e_{(15)(16)}\}, \\ V_3 = \{v_2\},\ V_4 = \{v_3\},\ V_5 = \{e_{45}, e_{67}\},\ V_6 = \{e_{56}\},\ V_7 = \{v_8\}, \\ V_8 = \{v_9\},\ V_9 = \{e_{(10)(11)}, e_{(12)(13)}\},\ V_{10} = \{e_{(11)(12)}\},\ V_{11} = \{v_{14}\},\ V_{12} = \{v_{15}\}, \end{array}$$
we have $\chi_d^{tt}(P_{16}) = 12$.
- $n = 17$. Let $\ell = 12$. Then $(a_1, a_2) = (5,7)$, $(6,5)$, $(6,6)$, $(7,3)$, $(7,4)$, $(7,5)$, $(8,2)$, $(8,3)$, $(8,4)$, $(9,1)$, $(9,2)$, $(9,3)$. Because `Facts 5-9` imply $2a_1+a_2 \geq 17$, $10 \leq a_1+a_2 \leq 12$, $5 \leq a_1 \leq 9$ and $\max\{0, 17-2a_1, 10-a_1\} \leq a_2 \leq \min\{7, 12-a_1\}$. Since $(a_1, a_2) = (5,7)$, $(6,6)$, $(7,5)$, $(8,4)$, $(9,3)$ imply $\sum_{i=1}^{12} |V_i| \neq 2n-1$ and $(a_1, a_2) = (6,5)$, $(7,4)$, $(8,3)$, $(9,2)$ imply $|V_1| > \alpha = 11$, which contradict `Fact 1`, we have $(a_1, a_2) = (7,3)$, $(8,2)$, $(9,1)$. If $(a_1, a_2) = (9,1)$, then $V_3 \cup \cdots \cup V_{12}$ is a maximal independent set (by `Fact 11`) and so no vertex of the set is totally dominated by a color class. Now let $(a_1, a_2) = (8,2)$, that is, $(|V_1|, \cdots, |V_{12}|) = (11, 10, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1)$. Then $V_3 \cup \cdots \cup V_{12} = \mathcal{W}_j \cup \{w\}$ for some $1 \leq j \leq 3$ and some $w \in \bigcup_{j \neq i=1}^{3} \mathcal{W}_i$, and since $|N(w) \cap \mathcal{W}_j| \leq 2$, there exist at least nine vertices in $\mathcal{W}_j$ which are totally dominated by no color class. Finally let $(a_1, a_2) = (7,3)$. Then, since the number of isolated vertices of $T(P_{17})[\bigcup_{i=6}^{12} V_i]$ is at most 3 (by `Fact 10`) and so $T(P_{17})[\bigcup_{i=6}^{12} V_i] \cong \overline{K_3} \cup H_4$, $\overline{K_2} \cup H_5$, $K_1 \cup \mathcal{H}_6$ or $\mathcal{H}_7$, we have
$$\begin{array}{rl} |\bigcup_{i=3}^{12} CN(V_i)| & \leq |\bigcup_{i=3}^{5} CN(V_i)| + |\bigcup_{i=6}^{12} CN(V_i)| \\ & \leq 6 + \max\{3 \times 3 + 2 \times 7, 2 \times 3 + 7 + 9, 3 + 3 \times 7, 2 \times 7 + 9\} \\ & < |V(T(P_{17}))|, \end{array}$$
a contradiction with (1.1). Thus $\ell \geq 13$, which implies $\chi_d^{tt}(P_{17}) = 13 = \gamma_{tm}(P_{17}) + 3$ by (3.1).

□

4. A Problem

By comparing the total dominator chromatic numbers of some graphs G such as paths, cycles with their total dominator total chromatic numbers, we see that $\chi_d^{tt}(G) - \chi_d^t(G) \to \infty$ when $n \to \infty$ for them. So, we end our paper with the following important problem.

Problem 4.1. *Find some real number $r > 1$ such that for any graph G, $\chi_d^{tt}(G) \geq r\chi_d^t(G)$.*

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