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INDEPENDENT ROMAN BONDAGE OF GRAPHS

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Abstract. An independent Roman dominating function (IRD-function) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the conditions that (i) every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$, and (ii) the set of all vertices assigned non-zero values under f is independent. The weight of an IRD-function is the sum of its function values over all vertices, and the independent Roman domination number $i_R(G)$ of G is the minimum weight of an IRD-function on G. In this paper, we initiate the study of the independent Roman bondage number $b_{iR}(G)$ of a graph G having at least one component of order at least three, defined as the smallest size of set of edges $F \subseteq E(G)$ for which $i_R(G - F) > i_R(G)$. We begin by showing that the decision problem associated with the independent Roman bondage problem is NP-hard for bipartite graphs. Then various upper bounds on $b_{iR}(G)$ are established as well as exact values on it for some special graphs. In particular, for trees T of order at least three, it is shown that $b_{iR}(T) \le 3$, while for connected planar graphs the upper bounds are in terms of the maximum degree with refinements depending on the girth of the graph.

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1. Introduction

We consider simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. For a vertex v of V, let $N_G(v)$ denote the set of neighbors of v and let $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = |N_G(v)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. When no confusion arises, we simply write N, d, δ and Δ instead of N_G, d_G, $\delta(G)$ and $\Delta(G)$, respectively. A *universal vertex* in a graph G is a vertex adjacent to the remaining vertices of G. A *leaf* is a vertex of degree one while its neighbor is called a *support vertex*. If v is a support vertex, then we denote by $L(v)$ the set of leaves adjacent to v. For definitions and notations not given here we refer to [13].

As usual, the *path* (*cycle*, *complete graph*, *complete bipartite graph*, respectively) of order n is denoted by P_n (C_n, K_n, $K_{p,q}$, respectively). A *tree* is a connected acyclic graph. A *star* of order n is the graph $K_{1,n-1}$. A tree

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T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $DS_{p,q}$.

A set $S \subseteq V(G)$ is a *dominating set* if every vertex not in S has at least one neighbor in S. The *domination number* of a graph G is the minimum cardinality of a dominating set of G. In 1990, Fink *et al.* [10] introduced the *bondage number* $b(G)$ to measure the vulnerability or the stability of the domination in an interconnection network G under edge failure. They defined the bondage number of a graph G as the minimum number of edges whose removal from G increases the domination number. Since then the concept of bondage has been studied for several graph parameters, for instance see [2,14,15,17,23].

A *Roman dominating function* (RD-function) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex $u \in V(G)$ with $f(u) = 0$ has a neighbor v with $f(v) = 2$. The *weight* of an RD-function f is the value $f(V(G)) = \omega(f) = \sum_{u \in V(G)} f(u)$, and the *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RD-function on G. Roman domination was introduced in 2004 by Cockayne *et al.* [9] and is now well-studied in graph theory. The literature on Roman domination and its variations has been surveyed and detailed in two book chapters and three surveys [4–8]. For an RD-function f on a graph G, let $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$. Since there is a 1–1 correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V, we will simply write $f = (V_0, V_1, V_2)$.

An RD-function $f = (V_0, V_1, V_2)$ on a graph G is an *independent Roman dominating function* (IRD-function) if the set $V_1 \cup V_2$ is independent, that is no two vertices in $V_1 \cup V_2$ are adjacent. The *independent Roman domination number* $i_R(G)$ is the minimum weight of an IRD-function on G. Moreover, an IRD-function of a graph G with minimum weight is called an i_R-function of G. Independent Roman domination was defined in [9] but Abadi *et al.* were the first to study this topic. Independent (Roman) domination has been studied by several authors [3,19–22].

In this paper, we initiate the study of the independent Roman bondage number $b_{iR}(G)$ of a graph G defined as the smallest set of edges $F \subseteq E(G)$ for which $i_R(G - F) > i_R(G)$. Since the independent Roman domination number of a connected graph of order two does not increase after deleting the unique edge, we will assume that $\Delta(G) \geq 2$. It is worth noting that the concept of Roman bondage was first studied by Jafari Rad and Volkmann in 2011 for the Roman domination number of a graph G (see [15,16]), where the corresponding parameter has been denoted by $b_R(G)$.

Among the results established in this paper, we start by showing that the decision problem associated with the independent Roman bondage number is NP-hard for bipartite graphs. Then several upper bounds are given for $b_{iR}(G)$. In particular for trees of order at least three it is shown that this parameter is at most three, while for connected planar graphs the upper bounds are in terms of the maximum degree with improvements depending on the girth of the graph. Furthermore, exact values on the independent Roman bondage number are also given for some special graphs including paths, cycles and complete bipartite graphs.

Clearly, for every connected graph G of order at least two, $i_R(G) \geq 2$. The characterization of connected graphs G with $i_R(G) \in \{2, 3\}$, which will be useful in the sequel, is given by the following result whose proof is omitted because of its easiness.

Proposition 1. *Let G be a connected graph of order $n \geq 2$. Then*

(1) $i_R(G) = 2$ *if and only if* $\Delta(G) = n - 1$.
(2) $i_R(G) = 3$ *if and only if* $\Delta(G) = n - 2$.

2. NP-hardness result

Our goal in this section is to show that the decision problem associated with the independent Roman bondage is NP-hard in bipartite graphs.

Independent Roman bondage problem (IR-bondage)

Instance: *A nonempty graph G and a positive integer k.*

Question: *Is* $b_{iR}(G) \le k$?

We will show that the NP-hardness of the IR-bondage problem by transforming the 3-SAT problem to it in polynomial time. We recall that the 3-SAT problem specified below was proven to be NP-complete in [12].

3-SAT problem

Instance: A collection $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, \dots, m$.

Question: Is there a truth assignment for U that satisfies all the clauses in $\mathcal{C}$?

Now, we are ready to show that the IR-bondage problem is NP-hard for bipartite graphs.

Theorem 2. *The IR-bondage problem is NP-hard for bipartite graphs.*

Proof. Let $U = \{u_1, u_2, \dots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$. We shall build a graph G and a positive integer k such that $\mathcal{C}$ is satisfiable if and only if $b_{iR}(G) \le k$.

For each variable $u_i \in U$, we associate a bipartite graph $H_i = K_{3,5}$ with bipartite sets $X = \{x_i, y_i, z_i\}$ and $Y = \{p_i, u_i, w_i, \bar{u}_i, q_i\}$, where all edges between the vertices of X and those of Y exist except three edges, namely, $x_i\bar{u}_i, y_iu_i$ and z_iq_i. Also for each clause $C_j = \{p_i, q_i, r_i\} \in \mathcal{C}$, we associate a single vertex c_j by adding the edge-set $E_j = \{c_jp_i, c_jq_i, c_jr_i\}$. Finally, we add a path $P = s_1s_2s_3$, by joining s_1 and s_3 to every vertex c_j. Let G be the resulting graph and set $k = 1$. Clearly, G is a bipartite graph of order $8n + m + 3$, and hence the construction of such a graph G can be accomplished in polynomial time. All that remain to be shown is that $\mathcal{C}$ is satisfiable if and only if $b_{iR}(G) = 1$. This can be done by proving the following four claims.

Claim 1. $i_R(G) \ge 4n + 2$. For any $\gamma_{iR}(G)$-function $f = (V_0, V_1, V_2)$, we have $f(V(H_i)) \ge 4$. Moreover, if $i_R(G) = 4n + 2$, then $f(V(H_i)) = 4, f(s_2) = 2, f(s_1) = f(s_3) = 0, |\{u_i, \overline{u_i}\} \cap V_2| \le 1$ for each $i \in \{1, 2, \dots, n\}$, while $\sum_{j=1}^{m} f(c_j) = 0$.

Proof of Claim 1. Let $f = (V_0, V_1, V_2)$ be an i_R-function of G, and let H_i' be the subgraph of H_i induced by the vertices $V(H_i) - \{u_i, \overline{u_i}\}$. If $u_i, \overline{u_i} \in V_2$, then $f(V(H_i)) \ge 4$. Hence assume that $|V_2 \cap \{u_i, \overline{u_i}\}| = 1$. If $q_i \in V_0$ or $w_i \in V_0$, then there is at least one vertex $t \in \{x_i, y_i, z_i\}$ such that $t \in V_2$; otherwise $q_i, w_i \in V_1$. In either case, $f(V(H_i')) \ge 2$ and thus $f(V(H_i)) \ge 4$. Finally, assume that $\{u_i, \overline{u_i}\} \cap V_2 = \emptyset$, and let f' be a restriction of f on H_i'. Then f' is an IRD-function of H_i' and $f'(V(H_i')) \ge i_R(H_i')$. Since the maximum degree of H_i' is at most $|V(H_i')| - 3$, by Proposition 1, $i_R(H_i') > 3$. Hence $f'(V(H_i')) \ge 4$ and thus $f(V(H_i)) \ge 4$. On the other hand, if $s_1 \in V_0$ or $s_3 \in V_0$, then there is at least one vertex t in $\{c_1, c_2, \dots, c_m, s_2\}$ such that $t \in V_2$; otherwise $\{s_1, s_3\} \subseteq V_1$. In either case, $f(N_G[V(P)]) \ge 2$, and hence $i_R(G) \ge 4n + 2$.

Suppose now that $i_R(G) = 4n + 2$. Then $f(V(H_i)) = 4$ and since $f(N_G[q_i]) \ge 1$, at most one of u_i and $\overline{u_i}$ is in V_2 for each $i \in \{1, 2, \dots, n\}$, while $f(N_G[V(P)]) = 2$. It follows that $s_2 \in V_2$, since $f(N_G[s_2]) \ge 1$. Consequently, $c_j \in V_0$ for each $j \in \{1, 2, \dots, m\}$. □

Claim 2. $i_R(G) = 4n + 2$ if and only if $\mathcal{C}$ is satisfiable.

Proof of Claim 2. Suppose that $i_R(G) = 4n + 2$ and let $f = (V_0, V_1, V_2)$ be an i_R-function of G. By Claim 1, $f(V(H_i)) = 4, \{u_i, \overline{u_i}\} \cap V_1 = \emptyset$ and $|\{u_i, \overline{u_i}\} \cap V_2| \le 1$ for each $i \in \{1, 2, \dots, n\}$ and $\sum_{j=1}^{m} f(c_j) = f(s_1) = f(s_3) = 0$. Define a mapping $t : U \to \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if } \; u_i \in V_2 \text{ or } \{u_i, \overline{u_i}\} \cap V_2 = \emptyset, \\ F & \text{if } \; \overline{u_i} \in V_2, \end{cases} \tag{1}$$

for $i \in \{1, 2, \dots, n\}$. We shall show that t is a satisfying truth assignment for $\mathcal{C}$. It is sufficient to show that every clause in $\mathcal{C}$ is satisfied by t. To this end, we arbitrary choose a clause $C_j \in \mathcal{C}$ with $j \in \{1, 2, \dots, m\}$.

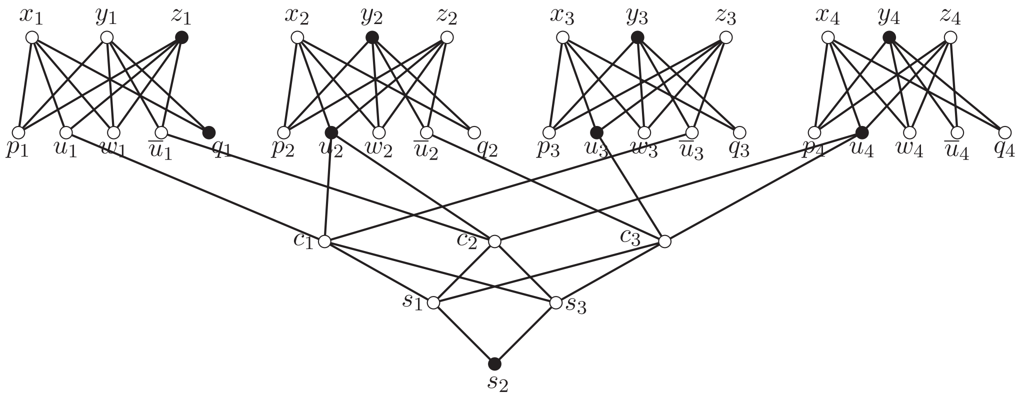

FIGURE 1. An instance of the independent Roman bondage problem resulting from an instance of 3SAT. Here $k = 1$ and $\gamma_{iR}(G) = 18$, where the bold vertex p means there is an IRD-function f with $f(p) = 2$.

By Claim 1, $f(c_j) = f(s_1) = f(s_3) = 0$. There exists some i with $i \in \{1, 2, \dots, n\}$ such that $f(u_i) = 2$ or $f(\overline{u_i}) = 2$, where c_j is adjacent to u_i or $\overline{u_i}$. Suppose that c_j is adjacent to u_i, where $f(u_i) = 2$ or $\{u_i, \overline{u_i}\} \cap V_2 = \emptyset$. Since u_i is adjacent to c_j in G, the literal u_i is in the clause C_j by the construction of G, it follows that $t(u_i) = T$ by (1), which implies that the clause C_j is satisfied by t. Suppose that c_j is adjacent to $\overline{u_i}$ where $f(\overline{u_i}) = 2$. Since $\overline{u_i}$ is adjacent to c_j in G, the literal $\overline{u_i}$ is in the clause C_j. Since $f(\overline{u_i}) = 2$, it follows that $t(u_i) = F$ by (1). Thus, t assigns $\overline{u_i}$ the truth value T, that is, t satisfies the clause C_j. By the arbitrariness of j with $j \in \{1, 2, \dots, m\}$, t satisfies all the clauses in $\mathcal{C}$, that is, $\mathcal{C}$ is satisfiable.

Conversely, suppose that $\mathcal{C}$ is satisfiable, and let $t : U \to \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. Define a function f on $V(G)$ as follows: if $t(u_i) = T$, then let $f(u_i) = f(y_i) = 2$, and if $t(u_i) = F$, then let $f(\overline{u_i}) = f(x_i) = 2$. Let $f(s_2) = 2$, and any other vertex of G is assigned a 0 under f. Clearly, $f(G) = 4n + 2$. Since t is a satisfying truth assignment for $\mathcal{C}$, for each $j \in \{1, 2, \dots, m\}$, at least one of literals in C_j is truth under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex r with $f(r) = 2$ since c_j is adjacent to each literal in C_j by the construction of G. Moreover, one can easily check that f is an IRD-function of G, and thus $i_R(G) \le f(G) = 4n + 2$. By Claim 1, $i_R(G) \ge 4n + 2$, and so $i_R(G) = 4n + 2$. □

Claim 3. For any $e \in E(G)$, $i_R(G - e) \le 4n + 3$.

Proof of Claim 3. For any edge $e \in E(G)$, it is sufficient to construct an IRD-function f on $G - e$ with weight $4n+3$. We first assume that e is incident with either s_1, s_3 or some c_j. Without loss of generality, let e be incident with s_1 or $e \in \{c_j u_i, c_j\overline{u_i}\}$. Let $f(s_3) = 2, f(s_1) = 1$ and $f(u_i) = f(y_i) = 2$ for each $i \in \{1, 2, \dots, n\}$ and $f(x) = 0$ otherwise. If e is not incident neither with u_i nor y_i, then let $f(s_1) = 1$, $f(s_3) = 2$, $f(u_i) = f(y_i) = 2$ and $f(x) = 0$ otherwise. If e is not incident neither with $\overline{u_i}$ nor x_i, then let $f(s_1) = 1$, $f(s_3) = 2$, $f(\overline{u_i}) = f(x_i) = 2$ and $f(x) = 0$ otherwise. If $e \in \{u_i x_i, \overline{u_i} y_i\}$, then let $f(s_1) = 1, f(s_3) = 2$, $f(q_i) = f(z_i) = 2$ and $f(x) = 0$ otherwise. In any case, f is an IRD-function of $G - e$ with weight $4n + 3$ and hence $i_R(G - e) \le 4n + 3$. □

Claim 4. $i_R(G) = 4n + 2$ if and only if $b_{iR}(G) = 1$.

Proof of Claim 4. Assume $i_R(G) = 4n + 2$ and take $e = s_1 s_2$. Suppose that $i_R(G - e) = i_R(G)$. Let f' be an i_R-function of $G - e$. It is clear that f' is also an i_R-function on G, unless $f'(s_1) > 0$ and $f'(s_2) > 0$, which is impossible because of $i_R(G-e) = i_R(G) = 4n+2$. By Claim 1, we have $f'(c_j) = 0$ for each $j \in \{1, 2, \dots, m\}$ and

$f'(s_2) = 2$. But then s_1 and all its neighbors in $G - e$ are assigned a 0 under f', which leads to a contradiction. Hence, $i_R(G) < i_R(G - e)$ and so $b_{iR}(G) = 1$.

Now assume that $b_{iR}(G) = 1$. By Claim 1, we have that $i_R(G) \geq 4n + 2$. Let e' be an edge such that $i_R(G-e') > i_R(G)$. By Claim 3, $i_R(G-e') \leq 4n+3$. Therefore, $4n+2 \leq i_R(G) < i_R(G-e') \leq 4n+3$, yielding $4n + 2 = i_R(G)$. □

By Claims 2 and 4, we proved that $b_{iR}(G) = 1$ if and only if there is a truth assignment for U that satisfies all clauses in $\mathcal{C}$ and the proof is complete. □

3. Exact values of $b_{iR}(G)$

In this section we determine the independent Roman bondage number for some special graphs. We begin by recalling some useful results given in [1, 15].

Proposition 3 ([1])**.**

(1) $i_R(K_n) = \gamma_R(K_n) = 2$.
(2) $i_R(P_n) = i_R(C_n) = \gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$.
(3) $i_R(K_{m,n}) = \min\{m, n\} + 1$.

Proposition 4 ([15])**.** *Let G be a graph of order at least 3 with exactly $t \geq 1$ universal vertices. Then $b_R(G) = \lceil \frac{t}{2} \rceil$.*

Proposition 5 ([15])**.** *For $n \geq 3$,* $b_R(P_n) = \begin{cases} 2 & \text{if} \quad n \equiv 2 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$

Proposition 6 ([15])**.** *For $n \geq 3$,* $b_R(C_n) = \begin{cases} 3 & \text{if} \quad n \equiv 2 \pmod 3, \\ 2 & \text{otherwise.} \end{cases}$

The following result is straightforward and its proof is omitted.

Observation 7. *Let G be a connected graph of order at least three. If $i_R(G) = \gamma_R(G)$, then $b_{iR}(G) \leq b_R(G)$.*

Proposition 8. *If G is a graph of order at least 3 with exactly $t \geq 1$ universal vertices, then $b_{iR}(G) = \lceil \frac{t}{2} \rceil$. In particular, $b_{iR}(K_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 3$.*

Proof. Since $t \geq 1$, we observe that $i_R(G) = \gamma_R(G) = 2$. Moreover, Observation 7, Propositions 3-(1) and 4 imply that $b_{iR}(G) \leq \lceil \frac{t}{2} \rceil$. Now, to prove the inverse inequality, let $F \subseteq E(G)$ be an arbitrary subset of edges with $|F| < \lceil \frac{t}{2} \rceil$, and let $G' = G - F$. Since G' still has a universal vertex, we have $i_R(G') = 2$. Hence $b_{iR}(G) \geq \lceil \frac{t}{2} \rceil$, and the desired equality follows. □

Proposition 9. *For $n \geq 3$,* $b_{iR}(P_n) = \begin{cases} 2 & \text{if} \quad n \equiv 2 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$

Proof. By Observation 7 and Propositions 3-(2) and 5, $b_{iR}(P_n) \leq 2$ if $n \equiv 2 \pmod 3$ and $b_{iR}(P_n) = 1$ if $n \not\equiv 2 \pmod 3$. Hence to get the desired result, it is enough to show that $b_{iR}(P_n) \geq 2$ if $n \equiv 2 \pmod 3$. Assume that $n \equiv 2 \pmod 3$, and let $P_n = v_1 v_2 \dots v_n$. For an arbitrary edge e of P_n, $P_n - e$ is the union of two disjoint paths P and Q such that $n(P) + n(Q) = n$. Clearly, $i_R(P_n - e) = i_R(P) + i_R(Q)$. Moreover, there are non-negative integers m_1, m_2 such that $m_1 + m_2 = n$ and $n(P) = 3m_1$, $n(Q) = 3m_2 + 2$, or $n(P) = 3m_1 + 1$, $n(Q) = 3m_2 + 1$ or $n(P) = 3m_1 + 2$, $n(Q) = 3m_2$. If $n(P) = 3m_1$ and $n(Q) = 3m_2 + 2$, then Proposition 3-(2) leads to

$$i_R(P_n - e) = i_R(P) + i_R(Q) = 2m_1 + 2m_2 + 2 = 2n + 2 = i_R(P_n)$$

and so $b_{iR}(P_n) \geq 2$ in this case. Thus $b_{iR}(P_n) = 2$ in the first case. Using a similar argument for the remaining cases also shows that $b_{iR}(P_n) = 2$. □

Proposition 10. *For $n \geq 3$, $b_{iR}(C_n) = \begin{cases} 3 & \text{if} \quad n \equiv 2 \pmod 3, \\ 2 & \text{otherwise.} \end{cases}$*

Proof. It follows from $i_R(P_n) = i_R(C_n)$ that $b_{iR}(C_n) \geq 2$. Also, by Observation 7 and Propositions 3-(2) and 6 we have $b_{iR}(C_n) \leq 3$ if $n \equiv 2 \pmod 3$ and $b_{iR}(C_n) = 2$ if $n \not\equiv 2 \pmod 3$. Thus to achieve the proof it is enough to show that $b_{iR}(C_n) \geq 3$ if $n \equiv 2 \pmod 3$. Assume that $n \equiv 2 \pmod 3$ and let $C_n = v_1 v_2 \dots v_n v_1$. Let e and e' be two arbitrary edges of C_n. Clearly, $C_n - \{e, e'\}$ is the union of two disjoint paths P and Q such that $n(P) + n(Q) = n$. Therefore $i_R(C_n - \{e, e'\}) = i_R(P) + i_R(Q)$. As in the proof of Proposition 9, we can see that $b_{iR}(C_n) \geq 3$, and hence $b_{iR}(C_n) = 3$ when $n \equiv 2 \pmod 3$. □

Proposition 11. *For $n > m \geq 1$, $b_{iR}(K_{m,n}) = m$.*

Proof. Let $X = \{u_1, \dots, u_m\}$ and $Y = \{y_1, \dots, y_n\}$ be the bipartite sets of $K_{m,n}$. Let $F = \{u_i y_1 \mid 1 \leq i \leq m\}$ and let G' be the spanning graph of $K_{m,n}$ obtained from $K_{m,n}$ by removing all edges of F. Since G' is the union of two disjoint graphs, namely K_1 and $K_{m,n-1}$, we deduce from Proposition 3-(3) that $i_R(G') = m+2 > i_R(K_{m,n})$. Hence $b_{iR}(K_{m,n}) \leq m$. To prove the inverse inequality, let $F \subseteq E(K_{m,n})$ be an arbitrary subset of edges with $|F| < m$, and let G' be obtained from $K_{m,n}$ by removing all edges of F. Then clearly $d_{G'}(u_i) = n$ for some i, say $i = 1$ and the function g defined on $V(G')$ by $g(u_1) = 2$, $g(u_i) = 1$ for $i \geq 2$ and $g(x) = 0$ otherwise, is an IRD-function of G' of weight $m + 1$, implying that $b_{iR}(K_{m,n}) \geq m$. Therefore $b_{iR}(K_{m,n}) = m$, and the proof is complete. □

4. Bounds on $b_{iR}(G)$

In this section, we first present an upper for the independent Roman bondage number of general graphs and then we show that the independent Roman bondage number of any tree with at least three vertices is at most three.

Theorem 12. *Let G be a connected graph. If xyz is a path of length 2 in G, then*

$$b_{iR}(G) \leq \mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) - 3 - \ell(x, z),$$

where $\ell(x, z) = 1$ if $xz \in E(G)$ and $\ell(x, z) = 0$ otherwise.

Proof. Let $F \subseteq E(G)$ be the set of all edges incident with either x, y or z except the edge yz. Obviously, $|F| = \mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) - 3$ when $xz \notin E(G)$ and $|F| = \mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) - 4$ when $xz \in E(G)$. Let G' be the graph obtained from G by removing all edges of F. We claim that $i_R(G') > i_R(G)$, leading to $b_{iR}(G) \leq |F|$. Let $g = (V_0, V_1, V_2)$ be an i_R-function of G'. Since x is isolated in G' and y, z induce a path on two vertices, we have $g(x) = 1$ and $g(y) + g(z) = 2$. Without loss of generality, assume that $g(y) = 2$ and $g(z) = 0$. If $d_G(y) = 2$, then the function f defined on $V(G)$ by $f(x) = f(z) = 0$ and $f(w) = g(w)$ for any other vertex w, is an IRD-function of G of weight less than $i_R(G')$. Hence we can assume $d_G(y) \geq 3$.

If $|V_2 \cap N_G(y)| = 0$, then the function f defined by $f(x) = f(z) = 0$, $f(u) = 0$ for $u \in V_1 \cap N_G(y)$ and $f(w) = g(w)$ for any other vertex w, is again an IRD-function of G of weight less than $\gamma_{iR}(G')$. Hence, assume that $|V_2 \cap N_G(y)| \geq 1$.

If $|(V_2 - \{y\}) \cap N_G(x)| \geq 1$ and $|(V_2 - \{y\}) \cap N_G(z)| \geq 1$, then the function f defined by $f(x) = f(z) = f(y) = 0$ and $f(w) = g(w)$ for any other vertex w, is an IRD-function of G of weight less than $\gamma_{iR}(G')$. If $|(V_2 - \{y\}) \cap N_G(x)| \geq 1$, $|(V_2 - \{y\}) \cap N_G(z)| = 0$ and $|V_1 \cap N_G(z)| \geq 1$, then the function f defined by $f(x) = f(z) = f(y) = 0$, $f(u) = 2$ for some vertex $u \in V_1 \cap N_G(z)$ and $f(w) = g(w)$ for any other vertex w, is an IRD-function of G of weight less than $\gamma_{iR}(G')$. If $|(V_2 - \{y\}) \cap N_G(x)| \geq 1$ and $|(V_1 \cup (V_2 - \{y\})) \cap N_G(z)| = 0$, then the function f defined by $f(x) = f(y) = 0$, $f(z) = 1$ and $f(w) = g(w)$ for any other vertex w, is an IRD-function of G of weight less than $\gamma_{iR}(G')$. From the above, we can assume now that $|(V_2 - \{y\}) \cap N_G(x)| = 0$.

If $|V_1 \cap N_G(x)| \geq 1$ and $|V_2 \cap (N_G(z) - \{y\})| \geq 1$, then the function f defined by $f(x) = f(z) = f(y) = 0$, $f(u) = 2$ for some vertex $u \in V_1 \cap N_G(x)$ and $f(w) = g(w)$ otherwise, is an IRD-function of G of weight less than $\gamma_{iR}(G')$. If $|(V_1 \cup (V_2 - \{y\})) \cap N_G(x)| = 0$ and $|(V_2 - \{y\}) \cap N_G(z)| \geq 1$, then the function f defined by $f(y) = f(z) = 0$, $f(x) = 1$ and $f(u) = g(u)$ otherwise, is an IRD-function of G of weight less than $\gamma_{iR}(G')$. Hence we assume that $|V_2 \cap (N_G(z) - \{y\})| = 0$.

If $|V_1 \cap N_G(x)| \geq 1$ and $|V_1 \cap N_G(z)| \geq 1$, then the function f defined by $f(x) = f(y) = f(z) = 0$, $g(w) = 2$ for some vertex $w \in V_1 \cap N_G(x)$, $g(w') = 2$ for some vertex $w' \in V_1 \cap N_G(z)$ and $f(u) = g(u)$ otherwise, is an IRD-function of G of weight less than $\gamma_{iR}(G')$.

If $|V_1 \cap N_G(x)| = 0$ and $|V_1 \cap N_G(z)| \geq 1$, then the function f defined by $f(x) = 1$, $f(y) = f(z) = 0$, $g(w) = 2$ for some vertex $w \in V_1 \cap N_G(z)$ and $f(u) = g(u)$ otherwise, is an IRD-function of G of weight less than $\gamma_{iR}(G')$. Finally, if $|V_1 \cap N_G(x)| = 0$ and $|V_1 \cap N_G(z)| = 0$, then certainly $xz \notin E(G)$ and thus the function f defined by $f(x) = f(z) = 1$, $f(y) = 0$ and $f(u) = g(u)$ for the remaining vertices, is an IRD-function of G of weight less than $\gamma_{iR}(G')$. In any case considered above we have shown that $i_R(G') > i_R(G)$, which completes the proof. □

Theorem 12 leads to the following corollary.

Corollary 13. *Let G be a connected graph of order $n \geq 3$ with $\Delta(G) \geq 2$ and let xyz be a path in G where x is a vertex with minimum degree $\delta(G)$. Then*

$$b_{iR}(G) \leq \delta(G) + 2\Delta(G) - 3 - \ell(x, z),$$

where $\ell(x, z) = 1$ if $xz \in E(G)$ and $\ell(x, z) = 0$ otherwise. Moreover, this bound is sharp for cycles C_n with $n \equiv 2 \pmod 3$.

Jafari Rad and Volkmann [15] proved that for any tree T of order at least 3, $b_R(T) \leq 3$. Although the independent bondage number of a tree T with at least three vertices is, as we will see, also bounded above by 3, the proof of this result is rather difficult than the proof given for the Roman bondage number.

Theorem 14. *If T is a tree of order $n \geq 3$, then*

$$b_{iR}(T) \leq 3.$$

Furthermore, this bound is sharp.

Proof. Clearly $\mathrm{diam}(T) \geq 2$, since $n \geq 3$. If $\mathrm{diam}(T) = 2$, then T is a star, where for any edge e of T, $i_R(T - e) = 3 > i_R(T) = 2$ and so $b_{iR}(T) = 1$. Hence assume that $\mathrm{diam}(T) \geq 3$, and let $v_1v_2 \ldots v_k$ ($k \geq 4$) be a diametral path in T such that $d_T(v_2)$ is as large as possible. If $\mathrm{diam}(T) = 3$, then T is a double star $\mathrm{DS}_{p,q}$ for some integers $q \geq p \geq 1$. If $p = 1$, then $i_R(T - v_2v_3) = 4 > i_R(T) = 3$ and thus $b_{iR}(T) = 1$. Now if $p \geq 2$, then removing edges v_1v_2 and v_3v_4 provides a forest H with three components, one of which is a double star $\mathrm{DS}_{p-1,q-1}$ and the two others are two single vertices. In this case, $i_R(H) = 2 + (2 + (p - 1)) = 3 + p > i_R(T) = 2 + p$ yielding $b_{iR}(T) \leq 2$. In the following we can assume that $\mathrm{diam}(T) \geq 4$, and thus $k \geq 5$. Root T at v_k, and let T_x denote the subtree induced by a vertex x and its descendants in the rooted tree T.

If $d_T(v_3) = 2$, then let H be the forest obtained from T by removing edges v_3v_4 and v_3v_2. Clearly any $i_R(H)$-function f such that $f(v_2)$ is as large as possible, assigns 1 to v_3 and 2 to v_2, and the function g defined on $V(T)$ by $g(v_3) = 0$ and $g(x) = f(x)$ for the remaining vertices, is an IRD-function of T of weight less than $\omega(f)$ and thus $b_{iR}(T) \leq 2$. Hence we may assume that $d_T(v_3) \geq 3$. Let $N_T(v_3) - \{v_2, v_4\} = \{u_1, \ldots, u_k\}$. We proceed with the following cases.

Case 1. $d_T(v_2) \geq 3$ and v_3 is a support vertex.

Without loss of generality, let u_1 be a leaf neighbor of v_3. Let H be the forest obtained from T by removing edges v_1v_2 and v_3u_1, and let f be a $i_R(H)$-function. Since v_1 and u are isolated in H, we have $f(v_1) = f(u_1) = 1$. Now, if $f(v_2) = 2$, then $f(v_3) = 0$ and the function g defined on $V(T)$ by $g(v_1) = 0$ and $g(x) = f(x)$ for the remaining vertices, is an IRD-function of T of weight less than $\omega(f)$ and thus $b_{iR}(T) \leq 2$. If $f(v_3) = 2$, then $f(v_2) = 0$ and the function g defined on $V(T)$ by $g(u_1) = 0$ and $g(x) = f(x)$ otherwise, is an IRD-function of T of weight less than $\omega(f)$ yielding $b_{iR}(T) \leq 2$. Hence we can assume that $f(v_2) \leq 1$ and $f(v_3) \leq 1$. Then all leaves adjacent to v_2 in H should be assigned a 1 under f except one leaf that should be assigned a 2 (because of v_2). In this case, the function g defined on $V(T)$ by $g(x) = 0$ for $x \in N_T(v_2)$, $g(v_2) = 2$ and $g(x) = f(x)$ for the remaining vertices, is an IRD-function of T of weight less than $\omega(f)$ and thus $b_{iR}(T) \leq 2$.

Case 2. $d_T(v_2) \geq 3$ and v_3 has a child of degree 2.

Without loss of generality, let u_1 be a child of v_3 with degree two and let u_1' be the leaf neighbor of u_1. Let H be the forest obtained from T by removing edges v_2v_1 and v_3u_1. Suppose f is a $i_R(H)$-function such that $f(v_2)$ is maximized. Since v_1 is isolated in H and vertices u_1, u_1' induce a K_2 component of H, we have $f(v_1) = 1$ and either $f(u_1) = 2$ or $f(u_1') = 2$, say $f(u_1') = 2$ and thus $f(u_1) = 0$. Now, if $f(v_3) = 2$, then the function g defined on $V(T)$ by $g(u_1') = 1$ and $g(x) = f(x)$ otherwise, is an IRD-function of T of weight less than $\omega(f)$. Hence let $f(v_3) \leq 1$. Since v_2 has a leaf neighbor in H, by the choice of f we have $f(v_2) = 2$ and thus the function g defined on $V(T)$ by $g(v_1) = g(v_3) = 0$ and $g(x) = f(x)$ otherwise, is an IRD-function of T of weight less than $\omega(f)$. In either case, $b_{iR}(T) \leq 2$.

Case 3. $d_T(v_2) \geq 3$ and any child of v_3 has degree at least 3.

By the assumption, each u_i has at least two leaf neighbors say u_i', u_i''. Let H be the forest obtained from T by removing edges v_2v_1 and u_1u_1' and let $f = (V_0, V_1, V_2)$ be an i_R-function of H such that $f(v_2) + f(u_1)$ is maximized. Clearly, $f(v_1) = f(u_1') = 1$. If $f(v_2) = f(u_1) = 2$, then the function g defined on $V(T)$ by $g(v_1) = g(u_1') = 0$ and $g(x) = f(x)$ otherwise, is an IRD-function of T of $\omega(f) - 2$. Also, if $f(v_2) = 2$ and $f(u_1) \leq 1$ (the case $f(v_2) \leq 1$ and $f(u_1) = 2$ is similar), then the function g defined on $V(T)$ by $g(v_1) = 0$ and $g(x) = f(x)$ otherwise, is an IRD-function of T of weight less than $\omega(f) - 1$ (note that since u_1 has a leaf neighbor in H, we must have $f(u_1) = 0$). Hence we can assume that $f(v_2) \leq 1$ and $f(u_1) \leq 1$. Since each of v_2 and u_1 has a leaf neighbor in H, by our choice of f, we must have $f(v_2) = f(u_1) = 0$. Again the choice of f implies that $f(v_3) = 2$. It follows that $f(v_2) = f(u_i) = 0$ for each i, $N_T(v_2) - \{v_3\} \subseteq V_1$ and $N_T(u_i) - \{v_3\} \subseteq V_1$ for each $i \in \{1, \ldots, k\}$. If $(N_T(v_4) - \{v_3\}) \cap V_2 \neq \emptyset$, then the function g defined on $V(T)$ by $g(v_3) = 0$, $g(v_2) = g(u_i) = 2$ for all $i \in \{1, \ldots, k\}$, $g(x) = 0$ for $x \in L(v_2) \cup \big(\cup_{i=1}^{k} L(u_i)\big)$ and $g(x) = f(x)$ otherwise, is an IRD-function of T of weight less than $\omega(f)$. Therefore we may assume that $(N_T(v_4) - \{v_3\}) \cap V_2 = \emptyset$. If $(N_T(v_4) - \{v_3\}) \cap V_1 \neq \emptyset$, then the function g defined on $V(T)$ by $g(v_4) = 2$, $g(v_3) = 0$, $g(x) = 0$ for $x \in (N_T(v_4) - \{v_3\}) \cap V_1$, $g(v_2) = g(u_i) = 2$ for all $i \in \{1, \ldots, k\}$, $g(x) = 0$ for $x \in L(v_2) \cup (\cup_{i=1}^{k} L(u_i))$ and $g(x) = f(x)$ otherwise, is an IRD-function of T of weight less than $\omega(f)$. Hence let $(N(v_4) - \{v_3\}) \cap V_1 = \emptyset$. In this case, define an IRD-function g on $V(T)$ by $g(v_4) = 1$, $g(v_3) = 0$, $g(v_2) = g(u_i) = 2$ for all $i \in \{1, \ldots, k\}$, $g(x) = 0$ for $x \in L(v_2) \cup (\cup_{i=1}^{k} L(u_i))$ and $g(x) = f(x)$ otherwise. In either case, we have shown that $i_R(T) < i_R(H)$ and thus $b_{iR}(T) \leq 2$.

Case 4. $d_T(v_2) = 2$.

We conclude from the choice of diametral path that each child of v_3 with depth one has degree 2. Assume first that $d_T(v_3) = 3$, and let H be the forest obtained from T by removing edges v_3v_2, v_3v_4, v_3u_1. Let f be an i_R-function of H such that $f(v_2)$ is maximized. Since v_3 is isolated in H, we have $f(v_3) = 1$ and by the choice of f we have $f(v_2) = 2$. In this case, the function g defined on $V(T)$ by $g(v_3) = 0$ and $g(x) = f(x)$ otherwise, is an IRD-function of T of weight $i_R(H) - 1$, implying that $b_{iR}(T) \leq 3$. Hence we can assume in the following that $d_T(v_3) \geq 4$. Let $z_1, \ldots, z_t$ be the leaves adjacent to v_3, if any, and $a_1, \ldots, a_s$ be the children of v_3 besides v_2 that are support vertices (all of degree two) and let a_i' be the leaf neighbor of a_i. Let us examine the following situations.

Subcase 4.1. v_4 is a support vertex.

Let w be a leaf neighbor of v_4 and consider the forest H obtained from T by removing edges wv_4, v_3v_2. Let f be an i_R-function of H such that $f(v_1)$ is maximized. Obviously, $f(w) = 1$ and $f(v_1) = 2$. Now if $f(v_4) = 2$, then reassigning a 0 to w we get an IRD-function T of weight $i_R(H) - 1$. Hence let $f(v_4) \leq 1$. If $f(v_3) = 2$, then reassigning a 1 to v_1 provides again an IRD-function of T of weight $i_R(H) - 1$. Hence we assume that $f(v_4) \leq 1$ and $f(v_3) \leq 1$. In this case, we must have $f(z_i) = 1$ for each $i \in \{1, \ldots, t\}$ and $f(a_j) + f(a_j') = 2$ for every j. Define the function g on $V(T)$ by $g(v_3) = 2$, $g(x) = 0$ for $x \in N(v_3)$, $g(a_j') = 1$ for $j \in \{1, \ldots, s\}$ and $g(x) = f(x)$ otherwise. Clearly since $d_T(v_3) \geq 4$, the function g is an IRD-function of T of weight less than $i_R(H)$. In either case, $b_{iR}(T) \leq 2$.

Subcase 4.2. v_4 has a child w which is a support vertex of degree two.

Let w' be a leaf neighbor of w and consider the forest H obtained from T by removing edges wv_4, v_3v_2. Let f be an i_R-function of H such that $f(v_1) + f(w')$ is maximized. Then we must have $f(v_1) = f(w') = 2$.

Now if $f(v_4) = 2$, then reassigning a 1 to w' provides an IRD-function T of weight smaller than $i_R(H)$. Likewise, if $f(v_3) = 2$, then reassigning a 1 to v_1 provides an IRD-function of T of weight smaller than $i_R(H)$. Hence we can assume that $f(v_4) \le 1$ and $f(v_3) \le 1$. As seen in Subcase 4.1, we can define an IRD-function of T of weight less than $i_R(H)$, and therefore $b_{iR}(T) \le 2$.

Subcase 4.3. $d_T(v_4) = 2$.

Let H be the forest obtained from T by removing edges v_3v_4 and v_4v_5, and let f be an i_R-function of H such that $f(v_3)$ is maximized. Then we must have $f(v_4) = 1$ and $f(v_3) = 2$. In this case, reassigning a 0 to v_4 provides an IRD-function T of weight less than $i_R(H)$. Therefore $b_{iR}(T) \le 2$ and the study of this subcase is closed.

Taking in account the situations previously discussed, we may assume $d_T(v_4) \ge 3$, where each child of v_4 has either depth 1 and degree at least three or depth 3. We will also assume that v_5 is not a leaf. Moreover, according to Cases 1–3, we can assume that each child w of v_4 with depth 3 has degree least four, where each child of w is either a leaf or a support vertex of degree two. Let z be a child of v_4 different from v_3. Remove the edges zv_4, v_3v_2 and let H be the resulting graph. Let f be an i_R-function of H. If $f(v_3) = 2$ or $f(v_4) \le 1$ and $f(v_3) \le 1$, then using a similar argument to that applied in Subcase 4.1 we can define an IRD-function of T of weight less than $i_R(H)$. Hence let $f(v_4) = 2$. Then f must assign 0 to every neighbor of v_4 in H and 1 to each leaf descendant of v_4 which is at distance two from v_4 in H. Also we may assume that f assign 2 to each leaf descendant of v_4 which is at distance 3 from v_4 in H. If v_5 has a neighbor u other than v_4 with $f(u) \ge 1$, then consider the function g defined on $V(T)$ by $g(u) = \min\{f(u)+1, 2\}$, $f(v_4) = 0$, $f(x) = 2$ for $x \in N_T(v_4) - \{v_5\}$, $f(x) = 0$ for every vertex x at distance two from v_4 in T_{v_4} and $f(x) = 1$ for every leaf x at distance three from v_4 in T_{v_4}, and $g(x) = f(x)$ otherwise. One can see that g is an IRD-function of T of weight less than $i_R(H)$. In the following we assume that all neighbors of v_5 but v_4 are assigned a 0 under f. Consider the function g defined on $V(T)$ by $g(v_5) = 1$, $f(v_4) = 0$, $f(x) = 2$ for $x \in N_T(v_4) - \{v_5\}$, $f(x) = 0$ for every vertex x at distance two from v_4 in T_{v_4}, $f(x) = 1$ for x for every leaf x at distance three from v_4 in T_{v_4}, and $g(x) = f(x)$ otherwise. Then g is an IRD-function of T of weight less than $i_R(H)$, implying that $b_{iR}(T) \le 2$.

To see the sharpness of the bound above, let T be a tree obtained from $k \ge 2$ disjoint paths $P_i = v_i^1v_i^2v_i^3v_i^4v_i^5$ be adding $k-1$ edges between the vertices $v_1^3, v_2^3, \ldots, v_k^3$ so that the resulting graphs is connected. It is easy to see that $i_R(T) = \gamma_R(T) = 4k$ and $b_{iR}(T) = 3$. □

5. Planar graphs

In this section we establish some bounds for the independent Roman bondage number of planar graphs. We first recall some definitions and then we recall some well-known results. A *cut-edge* of a graph is an edge whose deletion increases the number of connected components of the graph. A connected graph G is *2-connected* if G cannot be disconnected by the removal of any vertex. The *girth* of a graph G is the length of a shortest cycle contained in G.

Proposition 15 ([11])**.** *Let G be a planar graph of order n with girth $g(G) < \infty$. If c is the number of cut-edges in G, then*

$$|E(G)| \le \frac{g(G)(n-2) - c}{g(G) - 2}.$$

Proposition 16. *Let G be a planar graph. Then $\delta(G) \le 5$. Moreover, if G has girth $g \ge 4$, then $\delta(G) \le 3$, while if $g \ge 6$, then $\delta(G) \le 2$.*

Proposition 17 ([18])**.** *Let v be a vertex of a planar graph G with $d(v) \ge 3$, and let $E_v = \{xy \mid x, y \in N(v), xy \notin E(G)\}$. Then there exists a subset $S \subseteq E_v$ such that $H = G + S$ is still a planar graph and $H[N(v)]$ is 2-connected.*

Now we are ready to state our first upper bound on $b_{iR}(G)$ in terms of the maximum degree.

Theorem 18. *Let G be a connected planar graph of order at least three without vertices of degree 5. Then*

$$b_{iR}(G) \le \Delta(G) + 7.$$

Proof. Let $V_{\leq 4} = \{u_1, \dots, u_m\}$ be the set of vertices of G with degree at most four. It follows from Proposition 16 and our assumption that $V_{\leq 4} \neq \emptyset$. Suppose for the sake of contradiction that $b_{iR}(G) \geq \Delta(G) + 8$. First let $u_i u_j \in E(G)$ for some two different indices $i, j \in \{1, \dots, m\}$. In addition, we can assume that there is a path $u_i u_j z$ of length 2 in G, since G is a connected graph of order at least three. If $d_G(u_j) = 2$, then we deduce from Theorem 12 that $b_{iR}(G) \leq \Delta(G) + 3$, while if $d_G(u_j) \geq 3$, then Theorem 12 leads to $b_{iR}(G) \leq \Delta(G) + 5$. In either case, we get a contradiction. Hence we assume that $u_i u_j \notin E(G)$. If u_i and u_j have a common neighbor z, then by considering the path of length two $u_i z u_j$, Theorem 12 leads to $b_{iR}(G) \leq d_G(z) + 5 \leq \Delta(G) + 5$, a contradiction again. Hence $V_{\leq 4}$ is independent and no two vertices of $V_{\leq 4}$ have a common neighbor.

Assume, without loss of generality, that the k first vertices (possibly none) of $V_{\leq 4}$ are of degree at least 3. Applying Proposition 17 on such k vertices, let $H_0 = G$ and $H_i = H_{i-1} + T_i$ for $i \in \{1, \dots, k\}$, where T_i is a subset of $E_{u_i} = \{xy \mid x, y \in N(u_i), xy \notin E(H_{i-1})\}$ such that $H_{i-1} + T_i$ is still a planar graph and $H_i[N(u_i)]$ is 2-connected. Let u be any vertex of $V_{\leq 4}$ and let $v \in N_G(u)$. Observe that since $V_{\leq 4}$ is independent, $v \notin V_{\leq 4}$. If $d_G(u) \leq 2$, then by considering a path of length two including u, v and some neighbor z of v different from u, we deduce from Theorem 12 and our earlier assumption that

$$\Delta(G) + 8 \leq b_{iR}(G) \leq d_G(u) + d_G(v) + d_G(z) - 3 \leq d_G(v) + \Delta(G) - 1,$$

and thus $d_G(v) \geq 9$. Therefore $d_{H_k}(v) \geq 9$. Moreover, if $d_G(u) \in \{3, 4\}$, then a similar argument shows that $d_G(v) \geq 7$ and thus $d_{H_k}(v) \geq 7$. Now since H_k remains planar (by Prop. 17), $V_{\leq 4}$ is independent and no two vertices of $V_{\leq 4}$ have a common neighbor, it follows that the subgraph induced by $V(H_k) - V_{\leq 4}$ is a planar graph with minimum degree at least 6, contradicting Proposition 16. Therefore $b_{iR}(G) \leq \Delta(G) + 7$, and the proof is complete. □

Theorem 19. *Let G be a connected planar graph of order at least three. Then*

$$b_{iR}(G) \leq \Delta(G) + 8.$$

Proof. Let $V_{\leq 3} = \{v \in V(G) \mid d_G(v) \leq 3\}$, $V_4 = \{v \in V(G) \mid d_G(v) = 4\}$, $V_5 = \{v \in V(G) \mid d_G(v) = 5\}$ and $V_{\leq 5} = V_{\leq 3} \cup V_4 \cup V_5$. If $V_5 = \emptyset$, then the result is immediate from Theorem 18. Hence we assume that $V_5 \neq \emptyset$. Suppose for the sake of contradiction that $b_{iR}(G) \geq \Delta(G) + 9$. As in the proof of Theorem 18 we can see that $d_G(u, v) \geq 3$ for any distinct vertices $u, v \in V_{\leq 5}$ and that $d_G(v) \geq 7$ for $v \in N_G(u)$ and $u \in V_{\leq 5}$. Moreover, by considering the graph H_k as in the proof of Theorem 18, the graph H_k remains planar (by Prop. 17), and since $V_{\leq 5}$ is an independent set in which no two vertices have a common neighbor, it follows that the subgraph induced by $V(H_k) - V_{\leq 5}$ is a planar graph with minimum degree at least 6 which definitely contradicts Proposition 16. This completes the proof. □

Let G be a nontrivial connected graph with maximum degree Δ. For every $i \in \{1, \dots, \Delta\}$, let $n_i(G) = n_i$ be the number of vertices of degree i and $n_{\geq i}(G) = n_{\geq i}$ the number of vertices of degree at least i. We observe that $n = \sum_{k=1}^{\Delta} n_k$ and $2|E(G)| = \sum_{k=1}^{\Delta} k n_k$. Restricted to planar graphs with girth at least four, our next result is an improvement of the bounds of Theorems 18 and 19.

Theorem 20. *Let G be a connected planar graph of order $n \geq 3$ with maximum degree Δ. If $g(G) \geq 4$, then $b_{iR}(G) \leq \Delta(G) + 4$.*

Proof. If G is a tree, then the result follows from Theorem 14. Hence we assume that G contains a cycle of length at least four. Proposition 16 and the assumption $g(G) \geq 4$ yield $\delta(G) \leq 3$. If $\Delta \leq 4$, then by Corollary 13 we get $b_{iR}(G) \leq 2\Delta \leq \Delta + 4$. Hence let $\Delta \geq 5$. It follows from Proposition 15 that $2|E(G)| \leq 4n - 8 = 4n_1 + 4(\sum_{k=2}^{\Delta} n_k) - 8$ and thus

$$3n_1 + 2n_2 + n_3 \geq n_5 + 2n_6 + \cdots + (\Delta - 4)n_\Delta + 8. \tag{2}$$

We consider the following cases.

Case 1. $\Delta = 5$.

If $\delta \leq 2$, then the results comes from Corollary 13. Hence we assume that $\delta = 3$. If there are two vertices x, y with $d_G(x) = 3$, $d_G(y) \leq 4$ and $d(x, y) \leq 2$, then Theorem 12 leads to $b_{iR}(G) \leq \Delta + 4$. So we may assume that $d(x, y) \geq 3$ when $d_G(x) = 3$ and $d_G(y) \leq 4$, and this implies that $n_{\geq 5} \geq 3n_3$ which contradicts (2).

Case 2. $\Delta = 6$.

If $\delta = 1$, then the results comes from Corollary 13. Hence we assume that $\delta \in \{2, 3\}$. If there are two vertices x, y with $d_G(x) = 3$, $d_G(y) \leq 4$ and $d(x, y) \leq 2$ or $d_G(x) = 2$, $d_G(y) \leq 5$ and $d(x, y) \leq 2$, then Theorem 12 leads to $b_{iR}(G) \leq \Delta + 4$. So we may assume that $d(x, y) \geq 3$ when $d_G(x) = 3$ and $d_G(y) \leq 4$ or $d_G(x) = 2$ and $d_G(y) \leq 5$, and this implies that $n_{\geq 5} \geq 2n_2 + 3n_3$. By (2) we obtain

$$2n_2 + 3n_3 \geq n_5 + 2n_6 + \cdots + (\Delta - 4)n_\Delta + 8 \geq 2n_2 + 3n_3 + 8,$$

a contradiction.

Case 3. $\Delta \geq 7$.

If there are two vertices x, y with $d_G(x) = i$, $d_G(y) \leq 7 - i$ and $d(x, y) \leq 2$ for some $i \in \{1, 2, 3\}$, then Theorem 12 leads to $b_{iR}(G) \leq \Delta + 4$. So we may assume that $d(x, y) \geq 3$ when $d_G(x) = i$ and $d_G(y) \leq 7 - i$ for each $i \in \{1, 2, 3\}$. In this case, $n_{\geq 5} \geq n_1 + 2n_2 + 3n_3$ and $n_{\geq 7} \geq n_1$. By (2) we obtain

$$n_1 + 2n_2 + 3n_3 \geq n_{\geq 5} + 2n_{\geq 7} + 8 \geq 3n_1 + 2n_2 + 3n_3 + 8,$$

a contradiction.

□

The proofs of the following results are similar to the proof of Theorem 20 and therefore we omit them.

Theorem 21. *Let G be a connected planar graph of order $n \geq 3$ with maximum degree Δ. If $g(G) \geq k$ for $k \in \{5, 6\}$, then $b_{iR}(G) \leq \Delta(G) + 8 - k$.*

Theorem 22. *Let G be a connected planar graph of order $n \geq 3$ with maximum degree Δ. If $g(G) \geq 8$, then $b_{iR}(G) \leq \Delta(G) + 1$.*

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