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OPTIMALITY CONDITIONS AND DUALITY RESULTS FOR A ROBUST BI-LEVEL PROGRAMMING PROBLEM

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Abstract. Robust bi-level programming problems are a newborn branch of optimization theory. In this study, we have considered a bi-level model with constraint-wise uncertainty at the upper-level, and the lower-level problem is fully convex. We use the optimal value reformulation to transform the given bi-level problem into a single-level mathematical problem and the concept of robust counterpart optimization to deal with uncertainty in the upper-level problem. Necessary optimality conditions are beneficial because any local minimum must satisfy these conditions. As a result, one can only look for local (or global) minima among points that hold the necessary optimality conditions. Here we have introduced an extended non-smooth robust constraint qualification (RCQ) and developed the KKT type necessary optimality conditions in terms of convexifactors and subdifferentials for the considered uncertain two-level problem. Further, we establish as an application the robust bi-level Mond-Weir dual (MWD) for the considered problem and produce the duality results. Moreover, an example is proposed to show the applicability of necessary optimality conditions.

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Introduction

Bi-level programming problems (BLPPs) are sequence-based problems, also known as leader-follower problems, where the leader endeavors to optimize his/her decisions on the basis of the follower's reaction. The upper-level represents the objective and constraints of the leader, while lower-level represents the objective and constraints of the follower. The concept of BLPPs can be traced back to seminal publications of von Stackelberg [36, 37]. In 1973, Bracken and McGill [5] developed the first mathematical bi-level model, which led many researchers to develop exciting theories and applications such as optimistic and pessimistic approaches, single-level reformulation, optimality conditions, duality results, algorithms, etc. For more literature, readers may refer to the book [13] and the references therein.

The famous way to develop the optimality conditions and duality theorems of a BLPP is to transform it to an equivalent single-level mathematical programming problem so that we can use the existing literature of mathematical programming. There are many ways to do so, such as primal KKT reformulation, classical KKT

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reformulation, optimal value reformulation, Ψ function reformulation, etc. [11, 12] developed optimality conditions by using KKT reformulation. Optimality conditions with the help of optimal value reformulation can be found in [7, 14, 15]. For comparison in KKT reformulation and optimal value reformulation, readers may refer to [40]. Optimality conditions by using Ψ reformulation are studied in [18, 27]. By utilising an approximation of KKT type conditions, [25] have lately worked on the Pareto and weak Pareto solutions of the multiobjective optimization. In general, for a mathematical programming problem, if the functions involved are convex and differentiable, the optimality conditions are developed in terms of gradients. Now, if the functions lose smoothness, the optimality conditions are established in terms of convex subgradients. Further, for the case where functions are non-convex and non-smooth, the generalized subdifferentials are used. Convexifactors are generalized version of subdifferentials. Usually, they are subsets of many eminent subdifferentials such as Clarke subdifferentials, Mordukhovich subdifferentials and Michel Penot subdifferentials. Thus, the results obtained in terms of convexifactors are sharp. Jennane *et al.* [22] obtained necessary optimality conditions of a non-smooth multiobjective BLPP in terms of tangential subdifferentials. Optimality conditions for BLPPs in terms of convexifactors are studied in [16, 18, 26].

In mathematical optimization theory, the principle of duality posits that optimization problems can be seen from one of two perspectives: primal or dual. With the help of the bifunction, image space analysis, and polynomial ring methods, [24] investigated the strong duality of a standard convex optimization problem without relying on constraint qualifications. Despite the vast development of theoretical aspects of BLPPs over the last 48 years, the work on duality theory is scarce. Aboussoror and Adly [2] formulated a Fenchel–Lagrange dual and established the strong duality result, and developed the necessary and sufficient optimality conditions for a BLPP. In 2011, Suneja and Kohli [33] formulated a Wolfe dual (WD) and a MWD corresponding to the BLPP and established the relationship between duals and the bi-level problem via duality results, respectively. Gadhi *et al.* [18] developed a MWD and corresponding duality theorems for a multiobjective BLPP. Recently, Van Su *et al.* [35] established strong and weak duality theorems corresponding to the WD and MWD problem for the non-smooth multiobjective BLPP with equilibrium constraints.

It has been seen in real-world problems that even relatively minor fluctuations of uncertain data can severely impair the feasibility and, hence, the significance of the nominal optimal solution (*i.e.*, the optimal solution corresponding to the nominal data) as discussed in [4]. Therefore, an approach that generates "immunized against uncertainty" solutions is needed in applications. The only conventional approach of this kind is provided by stochastic programming, which substitutes the original constraints with their "chance versions" and assigns data fluctuation a probability distribution. This places a requirement on a candidate solution to satisfy the constraints with probability $\geq 1-\epsilon,\ \epsilon \ll 1$ being a predetermined tolerance. However, there is no straightforward method to associate the data fluctuations with a probability distribution. To handle optimization problems with uncertain data, robust optimization can be seen as a supplement to stochastic programming. Here, the "uncertain-but-bounded" model of data fluctuation allows the uncertain data to flow through a specified uncertainty set. It requires that a candidate solution be robustly feasible – to satisfy the constraints regardless of how the data from this set are realized. One associates the original uncertain problem with its robust counterpart to obtain the robust optimal solution [8].

In bi-level problems, uncertainty could occur in various ways, *e.g.*, decision uncertainty (uncertainty in the decision of leader or uncertainty in the reaction of the follower) and data uncertainty (uncertainty in the lower-level problem, or uncertainty in the upper-level problem). Many real-world applications modeling robust bi-level problems have recently been studied, such as renewable energy location [29], supply distribution [32], resource recovery planning [38], electric vehicle charging stations [41], etc. Recently, Goerigk *et al.* [20] examined the bi-level combinatorial problems under convex uncertainty. Chuong and Jeyakumar [9] derived a strong duality between affinely adjustable bi-level robust linear program and its dual with the help of generalized Farkas Lemma. Buchheim *et al.* [6] studied the complexity of robust bi-level problems with uncertainty in the lower-level's objective. Beck and Schmidt [3] have investigated the effect of uncertainty of the upper-level decision on the lower-level. Swain and Ojha [34] studied the robust counterparts of the uncertain mean-variance problems under box and ellipsoidal uncertainties by converting the problem into BLPP.

In [26], Kohli has considered the following bi-level model and developed the necessary optimality conditions using an upper estimate of the Clarke subdifferential and the idea of a convexifactor.

$$
\begin{aligned}
\text{(BLPP)} \qquad & \min_{x,y} \; F(x,\ y) \\
& \text{s.t.} \quad G_j(x,\ y) \le 0, \quad j \in J, \\
& \quad y \in \Psi(x),
\end{aligned}
$$

where for each $x \in \mathbb{R}^{n_1}$, $\Psi(x)$ is the set of optimal solutions to the below stated parametric convex optimization problem

$$
\begin{aligned}
& \min_{y} \quad f(x,\ y) \\
& \text{s.t.} \quad g_i(x,\ y) \le 0, \quad i \in I.
\end{aligned}
$$

Later, Gadhi [19] corrected the flaws in [26] and presented an alternative proof for the main result. Chen *et al.* [8] studied an uncertain constrained multiobjective optimization problem modeled as:

$$
\begin{aligned}
\text{(UCMOP)} \qquad & \min \;\; f(x) = \{f_1(x), f_2(x), \ldots, f_k(x)\} \\
& \text{s.t.} \;\; -g_i(x,\ \nu) \in S, \quad i = \{1, 2, \ldots, m\}.
\end{aligned}
$$

Chen *et al.* [8] used the robust approach to develop the optimality conditions for this model. Motivated by the work of [8,19,26] in this paper, we have considered a bi-level programming problem with uncertainty at the upper-level constraint. We consider here the case that the probability distribution which this data uncertainty follows is not known, but it is known that it belongs to an uncertainty set. Thus we have adopted the robust counterpart approach to deal with uncertainty. First, we transform the robust counterpart bi-level problem into a single-level problem with the help of optimal value reformulation. We have extended the Abadie constraint qualification to an extended non-smooth robust constraint qualification. We have developed the optimality conditions in terms of subdifferentials and convexifactors by using the concepts used by Kohli [26] and Gadhi [19] to deal with two levels and Chen *et al.* [8] to deal with the data uncertainty. Furthermore, Mond-Weir type dual is introduced, and the relation between the two problems is established via weak and strong duality theorems. To the best of the author's knowledge, the optimality conditions for robust bi-level problems still need to be developed. Hence the results obtained here are new and will help researchers develop new theories in this exciting new field of robust BLPP.

The remainder of the article as pursues: Section 1 provides the basic concepts and used definitions. Section 2 presents the robust BLPP and its reformulation to the single-level robust counterpart problem. In Section 3, we have developed the necessary optimality conditions for the considered problem under the appropriate assumptions. Section 4 is devoted to constructing MWD and establishing the weak and strong duality results. Finally, conclusion is given in Sections 5 and 6 some future directions are discussed.

1. Preliminaries

For $\{\emptyset\} \neq P \subseteq \mathbb{R}^n$, the closure, convex hull, convex cone (containing $(0, \ldots, 0)$) and negative polar cone formed by P are defined, respectively, by $cl\ P$, conv P, cone P and P^-.

A function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called locally Lipschitz around $\hat{x}_1 \in \text{dom } h$ if there exist an open ball N of $\hat{x}_1$ and $\kappa \ge 0$ such that

$$|h(x_1) - h(x_2)| \le \kappa ||x_1 - x_2||, \qquad \forall\ x_1,\ x_2 \in N.$$

Definition 1.1. [10] Let $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be a locally Lipschitz function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then Clarke subdifferential of F at $(x_1,\ x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is given as

$$\partial^c F(x_1,\ x_2) = \{(\xi_1,\ \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \text{ such that}$$

$$F^0((x_1,\ x_2),\ (l_1,\ l_2)) \geq \langle (\xi_1,\ \xi_2),\ (l_1,\ l_2)\rangle,$$
$$\forall\ (l_1,\ l_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\},$$

where $F^0((x_1,\ x_2),\ (l_1,\ l_2))$ is Clarke generalized directional derivative of F at $(x_1,\ x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ in direction $(l_1,\ l_2)$ and is given by

$$F^0((x_1,\ x_2),\ (l_1,\ l_2)) = \lim_{(u,\ v)\to(x_1,\ x_2)} \sup_{t\to 0^+} \frac{F((u,\ v)+t(l_1,\ l_2)) - F(u,\ v)}{t},$$
$$\text{where } (u,\ v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \text{ and } t > 0.$$

"

Definition 1.2. "[15] Let $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be Lipschitz continuous around $(\hat{x}_1,\ \hat{x}_2)$, its Fréchet subdifferential at $(x_1,\ x_2)$ is defined as

$$\hat{\partial} F(x_1, x_2) = \Bigg\{ (l_1,\ l_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \text{ such that}$$
$$\liminf_{(u,\ w)\to(x_1,\ x_2)} \frac{F(u,\ w) - F(x_1,\ x_2) - \langle (l_1,\ l_2),\ (u,\ w)-(x_1,\ x_2)\rangle}{\|\ (u,\ w)-(x_1,\ x_2)\ \|} \geq 0 \Bigg\}.$$

If function F is convex then its Fréchet subdifferential reduces to the subdifferential of convex analysis.

Let $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R} \bigcup \{\pm\infty\}$ be an extended real-valued function and let $(x_1,\ x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ where $F(x_1,\ x_2)$ is finite. The lower and upper Dini directional derivatives of F at $(x_1,\ x_2)$ in the direction $(l_1,\ l_2)$ are defined, respectively, by"

$$F_d^-((x_1,\ x_2),\ (l_1,\ l_2)) = \liminf_{t\to 0^+} \frac{F((x_1,\ x_2)+t(l_1,\ l_2)) - F(x_1,\ x_2)}{t}$$

and

$$F_d^+((x_1,\ x_2),\ (l_1,\ l_2)) = \limsup_{t\to 0^+} \frac{F((x_1,\ x_2)+t(l_1,\ l_2)) - F(x_1,\ x_2)}{t}\ .\text{"}$$

Definition 1.3. "[17,23] Let $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R} \bigcup \{\pm\infty\}$ be an extended real-valued function

– The function F is said to admit an upper convexifactor (UCF) $\partial^* F(x_1,\ x_2)$ at $(x_1,\ x_2)$ if and only if $\partial^* F(x_1,\ x_2) \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is closed and for each $(l_1,\ l_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$F_d^-((x_1,\ x_2),\ (l_1,\ l_2)) \leq \sup_{(x_1^*,\ x_2^*)\in\partial^* F(x_1,\ x_2)} \langle (x_1^*,\ x_2^*),\ (l_1,\ l_2)\rangle.$$

– The function F is said to admit a lower convexifactor (LCF) $\partial_* F(x_1,\ x_2)$ at $(x_1,\ x_2)$ if and only if $\partial_* F(x_1,\ x_2) \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is closed and for each $(l_1,\ l_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$F_d^+((x_1,\ x_2),\ (l_1,\ l_2)) \geq \inf_{(x_1^*,\ x_2^*)\in\partial_* F(x_1,\ x_2)} \langle (x_1^*,\ x_2^*),\ (l_1,\ l_2)\rangle.$$

– F is said to admit a convexifactor $\partial^* F(x_1,\ x_2)$ at $(x_1,\ x_2)$ if and only if $\partial^* F(x_1,\ x_2)$ is both an upper and lower convexifactor of F at $(x_1,\ x_2)$.
– F is said to have an upper semi-regular convexifactor (USRCF) $\partial^* F(x_1,\ x_2)$ at $(x_1,\ x_2)$ if $\partial^* F(x_1,\ x_2)$ is an upper convexifactor at $(x_1,\ x_2)$ and for each $(l_1,\ l_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$F_d^+((x_1,\ x_2),\ (l_1,\ l_2)) \leq \sup_{(x_1^*,\ x_2^*)\in\partial^* F(x_1,\ x_2)} \langle (x_1^*,\ x_2^*),\ (l_1,\ l_2)\rangle.\text{"}$$

For a locally Lipschitz function F at $(x_1,\ x_2)$ Clarke subdifferential is a convexifactor.

Definition 1.4. "[33] Let $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be a real-valued function and $(u,\ v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Assume that F admits convexifactor $\partial^* F(u,\ v)$, F is said to be

– ∂^*-convex at $(u,\ v)$ iff $\forall\ (x_1,\ x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$:

$$\langle \rho, (x_1,\ x_2) - (u,\ v) \rangle \leq F(x_1,\ x_2) - F(u,\ v), \quad \forall \rho \in \partial^* F(u,\ v).$$

– ∂^*-quasiconvex at $(u,\ v)$ iff $\forall\ (x_1,\ x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$:

$$F(x_1,\ x_2) - F(u,\ v) \leq 0 \ \Rightarrow\ \langle \rho, (x_1,\ x_2) - (u,\ v) \rangle \leq 0, \quad \forall \rho \in \partial^* F(u,\ v).$$

– ∂^*-pseudoconvex at $(u,\ v)$ iff $\forall\ (x_1,\ x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$:

$$F(x_1,\ x_2) - F(u,\ v) < 0 \ \Rightarrow\ \langle \rho, (x_1,\ x_2) - (u,\ v) \rangle < 0, \quad \forall \rho \in \partial^* F(u,\ v)."$$

Definition 1.5. "[30] Let $P : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set valued mapping and $\hat{x}_1 \in \text{dom } P$. P is inner semicompact at $\hat{x}_1$ with $P(\hat{x}_1) \neq \emptyset$ iff for every sequence $x_{1_k} \to \hat{x}_1$ with $P(x_{1_k}) \neq \emptyset$ there is a sequence $y_k \in P(x_{1_k})$ that contains a convergent subsequence as $k \to \infty$."

2. Non-smooth Robust Bi-level Model

In this section, we have considered an uncertain BLPP (Q). With the help of optimal value reformulation and the concept of robust counterpart, we transform (Q) to a single-level robust counterpart problem (RBPP). Later in this section, we have introduced the extended non-smooth RCQ. Let the uncertain BLPP (Q) be defined as:

$$(Q) \qquad \begin{aligned} &\min_{x_1,\ x_2} F_U(x_1,\ x_2) \\ &\text{s.t.}\ \ G_i(x_1,\ \nu_i) \leq 0, \quad \forall i \in I, \\ &\qquad x_2 \in \Upsilon(x_1), \end{aligned}$$

where for some sequentially compact topological space Ω_i, $\nu_i \in \Omega_i$ is an uncertain parameter. For each $x_1 \in \mathbb{R}^{n_1}$ the parametric optimization problem (Q_{x_1}) has the set of optimal solutions $\Upsilon(x_1)$

$$(Q_{x_1}) \qquad \begin{aligned} &\min_{x_2} F_L(x_1,\ x_2) \\ &\text{s.t.}\ g_j(x_1,\ x_2) \leq 0, \quad \forall j \in J, \end{aligned}$$

where, $F_U : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, is locally Lipschitz function, $G_i : \mathbb{R}^{n_1} \times \Omega_i \to \mathbb{R},\ i \in I = \{1, 2, \ldots, n\}$ is a function which is convex in x_1, continuous in ν and is upper semicontinuous in (x_1, ν), $F_L : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and $g_j : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R},\ j \in J = \{1, 2, \ldots, m\}$ are continuous convex functions and $\Upsilon(x_1) = \arg\min_{x_2}\{F_L(x_1, x_2) :\ g(x_1, x_2) \leq 0\}$; n, n_1, n_2 and m are whole numbers.

For examining uncertainty problems where the decision-maker does not know about the uncertain parameters probability distribution, a well-known robust approach called as a robust counterpart is used.

The robust counterpart of (Q) is:

$$(\text{RP}') \qquad \begin{aligned} &\min_{x_1, x_2} F_U(x_1,\ x_2) \\ &\text{s.t.}\ \ G_i(x_1,\ \nu_i) \leq 0, \quad \forall \nu_i \in \Omega_i,\ \forall i \in I, \\ &\qquad x_2 \in \Upsilon(x_1), \end{aligned}$$

where the uncertain constraints are applied to all alternative values of the parameters within their specified uncertainty sets $\Omega_i,\ i=1,2,\ldots,n$.

The problem (RP$'$) can be viewed as the severest possible scenario of (Q). The robust counterpart is a model that resolves the uncertain worst-case scenario without using uncertain variables. Therefore, optimizing (Q) with (RP$'$) is the robust technique (worst technique) for (Q). A feasible solution to the robust counterpart problem is the robust feasible solution to the uncertain problem (Q) which should, by definition, fulfill all realizations of the constraints from $\Omega_i,\ i=1,2,\ldots,n$ (uncertainty sets). An optimal solution of (RP$'$) is a robust feasible solution with the best possible value of the objective. Using optimal value function reformulation, we get

$$\text{(RBPP)}\qquad \begin{aligned} &\min_{x_1,x_2}\ F_U(x_1,\ x_2)\\ &\text{s.t. }\ G_i(x_1,\ \nu_i)\le 0,\quad \forall \nu_i\in\Omega_i,\ \forall\ i\in I,\\ &\qquad g_j(x_1,\ x_2)\le 0,\quad \forall j\in J,\\ &\qquad F_L(x_1,\ x_2)\le \varphi(x_1),\end{aligned}$$

where,

$$\varphi(x_1)=\inf_{x_2}\{F_L(x_1,\ x_2): g_j(x_1,\ x_2)\le 0,\quad \forall j\in J\}$$

is the optimal value function. Since $F_L(.,.)$ and $g_j(.,.),\ j\in J$ are convex functions, therefore, $\varphi(.)$ is also a convex function.

Let

$$X=\{x_1\in\mathbb{R}^{n_1}: G_i(x_1,\ \nu_i)\le 0,\quad \forall\nu_i\in\Omega_i,\ \forall i\in I\},$$

represents the upper-level constraint set and

$$S(x_1)=\{x_2\in\mathbb{R}^{n_2}:\ g_j(x_1,\ x_2)\le 0,\quad \forall j\in J\},$$

be the feasible set of lower-level problem for a fixed x_1 and

$$\begin{aligned} H=\{(x_1,\ x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}:&F_L(x_1,x_2)-\varphi(x_1)\le 0,\ g_j(x_1,x_2)\le 0,\ j\in J,\\ &G_i(x_1,\nu_i)\le 0,\ \forall\nu_i\in\Omega_i,\ i\in I\},\end{aligned}$$

be the feasible set of (RBPP).

Remark 2.1. If the second variable of the upper-level constraints become certain $\forall i\in I$ then the given problem (Q) reduces to the one studied by [26] and [19].

Definition 2.2. A robust feasible point $(\hat{x}_1,\ \hat{x}_2)\in H$ is called a robust local minimizer of (RBPP) if there exist open balls N_{x_1} of $\hat{x}_1$ and N_{x_2} of $\hat{x}_2$ such that $\forall\ (x_1,x_2)\in H\cap(N_{x_1}\times N_{x_2})$ one has $F_U(x_1,x_2)\ge F_U(\hat{x}_1,\hat{x}_2)$.

On the lines of [39, Lemma 1.1], we have the following Lemma:

Lemma 2.3. *As long as (Q) admits a solution, $(\hat{x}_1,\ \hat{x}_2)$ is a solution to (Q) iff it is a robust local minimizer of (*RBPP).

Lemma 2.4. "[19] *Let S be a nonempty, convex and compact set and K be a convex cone. If*

$$\sup_{z\in S}\langle z,\ d\rangle\ge 0,\quad \forall d\in K^-$$

then

$$0\in S+cl\ K."$$

Let Ψ, $\Psi_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be real-valued functions defined as follows:

$$\Psi_i(x_1) = \max_{\nu_i \in \Omega_i} G_i(x_1,\ \nu_i) \tag{1}$$

and

$$\Psi(x_1) = \max_{i \in \{1,\ldots,n\}} \Psi_i(x_1). \tag{2}$$

Ψ_i is well defined, since G_i is upper semicontinuous and $\Omega_i \neq \emptyset$, convex and compact $\forall j \in J$.

Lemma 2.5. [8] *"Let I represents a finite index set and $h_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in I)$ are locally Lipschitz functions. Set $h(x_1) = \max_{i \in I} h_i(x_1)$. Then $\partial h(x_1) \subseteq \operatorname{conv}\{\partial h_i(x_1) :\ i \in I(x_1)\}\ \forall\ x_1 \in \mathbb{R}^n$, where $I(x_1) = \{i \in I :\ h_i(x_1) = h(x_1)\}$."*

Extended Non-smooth RCQ

For a nonlinear programming problem dealing with the smooth functions, a well-known constraint qualification (CQ) known as Abadie CQ is used, which is weaker than most CQs and was introduced by [1]. However, since the single-level problem in our case is non-smooth and has an uncertain parameter, we should indeed extend the Abadie CQ. So, the extended non-smooth RCQ for all $\nu_i \in \Omega_i(\hat{x}_1)$ is defined as

$$\begin{aligned}\Bigg(\bigcup_{j\in J_m} \operatorname{conv}\, \partial^* g_j(\hat{x}_1,\ \hat{x}_2) \bigcup \bigcup_{i \in I_n} \operatorname{conv}\, \partial\mathit{\Psi}(\hat{x}_1) \times \{0\} \\ \bigcup \partial F_L(\hat{x}_1,\ \hat{x}_2) - \partial\varphi(\hat{x}_1) \times \{0\}\Bigg)^- \subseteq T(H,\ (\hat{x}_1,\ \hat{x}_2)),\end{aligned}$$

where $I_n = \{i \in I : \Psi_i(\hat{x}_1) = 0\}$, $J_m = \{j \in J :\ g_j(\hat{x}_1, \hat{x}_2) = 0\}$, $\Omega_i(\hat{x}_1) = \{\nu_i \in \Omega_i : G_i(\hat{x}_1,\ \nu_i) = \Psi_i(\hat{x}_1)\}$ and $T(H, (\hat{x}_1, \hat{x}_2))$ represents the tangent cone to H at $(\hat{x}_1,\ \hat{x}_2)$ defined as

$$\begin{aligned}T(H, (\hat{x}_1, \hat{x}_2)) = \{(l_1,\ l_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} :\ \exists t_n \downarrow 0 \text{ and } (d_{n_1},\ d_{n_2}) \to (l_1,\ l_2) \\ \text{such that } (\hat{x}_1,\ \hat{x}_2) + t_n(d_{n_1},\ d_{n_2}) \in H\}.\end{aligned}$$

3. Necessary condition

The core of optimization theory is the concept of optimality conditions. The existence of necessary and sufficient optimality conditions enables the generation of efficient numerical approaches for the practical solution of a given optimization problem. Necessary optimality conditions are beneficial because any local minimum must satisfy these conditions. As a result, one can only look for local (or global) minima among points that hold the necessary optimality conditions.

In this section, we present the necessary optimality condition for (Q) by using the optimistic approach. Before proceeding with the main results of this paper we need to define the lower-level regularity condition. The lower-level regularity of (Q_{x_1}) at $(\hat{x}_1,\ \hat{x}_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, is given as:

$$\left.\begin{array}{l} 0 \in \sum\limits_{j\in J_m} \eta_j \partial_{x_2} g_j(\hat{x}_1,\ \hat{x}_2), \\ \eta_j \geq 0,\ \eta_j g_j(\hat{x}_1,\ \hat{x}_2) = 0,\ j \in J_m \end{array}\right\} \Longrightarrow [\eta_j = 0,\ j \in J_m].$$

Theorem 3.1 (Necessary Condition)**.** *Let $(\hat{x}_1, \hat{x}_2) \in H$ be a robust local minimizer of (Q). Assume that F_U is locally Lipschitz and admit bounded (USRCF) $\partial^* F_U(\hat{x}_1,\ \hat{x}_2)$ at $(\hat{x}_1, \hat{x}_2)$, Ω_i is convex and that the function $G_i(x_1, .)$ is concave on Ω_i, for each $x_1 \in X$ and for each $i \in I$. Furthermore, we suppose that the extended non-smooth robust constraint qualification holds at $(\hat{x}_1, \hat{x}_2)$.*

Suppose that the argminimum map Υ is inner semicompact at $\hat{x}_1$, that for each vector $x_2 \in \Upsilon(\hat{x}_1)$, $(\hat{x}_1,\ x_2)$ is lower-level regular. Then, there exist $\hat{\nu}_i \in \Omega_i(\hat{x}_1)$, $\lambda_0 \geq 0$, $\mu_j \geq 0$, $j \in J_m$, $\tau_i \geq 0$, $i \in I_n$, $\lambda_j \geq 0$, $j \in J$ and also $x_2^* \in \Upsilon(\hat{x}_1)$ along with the closedness of cone A, *where*

$$A := \left(\bigcup_{i\in I_n} \text{conv } \partial\Psi(\hat{x}_1) \times \{0\}\right) \cup \left(\bigcup_{j\in J_m} \text{conv } \partial^* g_j(\hat{x}_1,\ \hat{x}_2)\right) \cup (\partial F_L(\hat{x}_1,\ \hat{x}_2) - \partial\varphi(\hat{x}_1) \times \{0\}),$$

such that the following conditions satisfy:

(1) $0 \in \text{conv } \partial^* F_U(\hat{x}_1,\ \hat{x}_2) + \sum_{i\in I_n} \tau_i \text{conv } \partial_{x_1} G_i(\hat{x}_1,\ \hat{\nu}_i) + \sum_{j\in J_m} \mu_j \text{conv } \partial^* g_j(\hat{x}_1,\ \hat{x}_2) + \lambda_0(\partial F_L(\hat{x}_1, \hat{x}_2) - \partial_{x_1} F_L(\hat{x}_1,\ x_2^*) \times \{0\}) - \lambda_0\left(\sum_{j\in J} \lambda_j \partial_{x_1} g_j(\hat{x}_1,\ x_2^*) \times \{0\}\right).$
(2) $0 \in \partial_{x_2} F_L(\hat{x}_1,\ x_2^*) + \sum_{j\in J} \lambda_j \partial_{x_2} g_j(\hat{x}_1,\ x_2^*)$.
(3) $\lambda_j \geq 0,\ 0 = \lambda_j g_j(\hat{x}_1,\ x_2^*), \quad \forall\ j = 1, 2, \ldots, m.$
(4) $\mu_j \geq 0,\ 0 = \mu_j g_j(\hat{x}_1,\ \hat{x}_2), \quad \forall\ j = 1, 2, \ldots, m.$
(5) $\tau_i \geq 0,\ 0 = \tau_i G_i(\hat{x}_1,\ \hat{\nu}_i), \quad \forall\ i = 1, 2, \ldots, n.$

Here, $\partial, \partial_{x_1}, \partial_{x_2}$ symbolizes, respectively, the full and partial subdifferentials of convex analysis.

Proof. Let $(l_1,\ l_2) \in T(H, (\hat{x}_1, \hat{x}_2))$, from the definition of tangent cone, $\exists t_n \downarrow 0$ and $(l_{n_1},\ l_{n_2}) \to (l_1,\ l_2)$ such that $(\hat{x}_1,\ \hat{x}_2) + t_n(l_{n_1},\ l_{n_2}) \in H\ \forall n$.

Since $(\hat{x}_1,\ \hat{x}_2)$ is robust local minimum of F_U over H, therefore, we have

$$\frac{F_U((\hat{x}_1, \hat{x}_2) + t_n(l_{n_1},\ l_{n_2})) - F_U(\hat{x}_1, \hat{x}_2)}{t_n} \geq 0, \quad \text{for sufficiently large } n. \tag{3}$$

Now,

$$\begin{aligned} &\frac{F_U((\hat{x}_1, \hat{x}_2) + t_n(l_{n_1},\ l_{n_2})) - F_U(\hat{x}_1, \hat{x}_2)}{t_n} \\ &\quad = \frac{F_U((\hat{x}_1, \hat{x}_2) + t_n(l_{n_1},\ l_{n_2})) - F_U((\hat{x}_1, \hat{x}_2) + t_n(l_1,\ l_2))}{t_n} \\ &\qquad + \frac{F_U((\hat{x}_1, \hat{x}_2) + t_n(l_1,\ l_2)) - F_U(\hat{x}_1, \hat{x}_2)}{t_n}. \end{aligned} \tag{4}$$

Since F_U is locally Lipschitz, thus,

$$\frac{F_U((\hat{x}_1, \hat{x}_2) + t_n(l_{n_1},\ l_{n_2})) - F_U((\hat{x}_1, \hat{x}_2) + t_n(l_1,\ l_2))}{t_n} \to 0 \text{ as } n \to \infty.$$

Take on both sides of the equation (4) Limit supremum and from above and (3), we get

$$\limsup_{t_n\to 0^+} \frac{F_U((\hat{x}_1, \hat{x}_2) + t_n(l_{n_1},\ l_{n_2})) - F_U(\hat{x}_1, \hat{x}_2)}{t_n} = (F_U)_d^+((\hat{x}_1, \hat{x}_2), (l_1,\ l_2)) \geq 0.$$

That is, we have

$$(F_U)_d^+((\hat{x}_1, \hat{x}_2), (l_1,\ l_2)) \geq 0, \quad \forall (l_1,\ l_2) \in T(H,\ (\hat{x}_1,\ \hat{x}_2)).$$

Since $\partial^* F_U(\hat{x}_1,\ \hat{x}_2)$ is upper semiregular at $(\hat{x}_1,\ \hat{x}_2)$, we get

$$\sup_{\eta\in\partial^* F_U(\hat{x}_1, \hat{x}_2)} \langle \eta, (l_1,\ l_2)\rangle \geq 0, \quad \forall (l_1,\ l_2) \in T(H, (\hat{x}_1,\ \hat{x}_2)). \tag{5}$$

Since extended non-smooth RCQ holds at $(\hat{x}_1,\ \hat{x}_2)$, we have

$$\sup_{\eta\in \text{conv }\ \partial^* F_U(\hat{x}_1,\hat{x}_2)} \langle \eta,\ (l_1,\ l_2)\rangle \geq 0, \quad \forall (l_1,\ l_2)\in (\text{cone } A)^- \tag{6}$$

where $(\text{cone } A)^-$ is the negative polar cone of A and A is defined as

$$A=\left(\bigcup_{i\in I_n}\text{conv } \partial\Psi(\hat{x}_1)\times\{0\}\right)\cup\left(\bigcup_{j\in J_m}\text{conv } \partial^* g_j(\hat{x}_1,\ \hat{x}_2)\right) \cup(\partial F_L(\hat{x}_1,\hat{x}_2)-\partial\varphi(\hat{x}_1)\times\{0\}).$$

Since $\partial^* F_U(\hat{x}_1,\hat{x}_2)$ is a closed set, conv $\partial^* F_U(\hat{x}_1,\hat{x}_2)$ is a compact set [21, Theorem 1.4.3]. By Lemma 2.4, we get

$$(0,\ 0)\in \text{conv } \partial^* F_U(\hat{x}_1,\hat{x}_2)+cl \text{ cone } A.$$

Since cone A is closed, we have

$$(0,\ 0)\in \text{conv } \partial^* F_U(\hat{x}_1,\ \hat{x}_2)+\text{cone } A.$$

Therefore,

$$\begin{aligned}0\in \text{conv } \partial^* F_U(\hat{x}_1,\ \hat{x}_2)+\sum_{i\in I_n}\text{cone } \partial\Psi(\hat{x}_1)\times\{0\}+\sum_{j\in J_m}\text{cone } \partial^* g_j(\hat{x}_1,\ \hat{x}_2)\\ +\text{cone } (\partial F_L(\hat{x}_1,\ \hat{x}_2)-\partial\varphi(\hat{x}_1)\times\{0\}).\end{aligned} \tag{7}$$

From Lemma 2.5, we obtain

$$\partial\Psi(\hat{x}_1)\subseteq \text{conv}\{\partial\Psi_i(\hat{x}_1):\ i\in I_n\}, \tag{8}$$

by [28, Theorem 2.4, p. 290], we get

$$\partial\Psi_i(\hat{x}_1)=\{\xi_i:\exists \nu_i\in\Omega_i(\hat{x}_1) \text{ such that } \xi_i\in\partial_{x_1}G_i(\hat{x}_1,\nu_i)\}, \tag{9}$$

for all $i\in I$, where $\Omega_i(\hat{x}_1)=\{\nu_i\in\Omega_i : G_i(\hat{x}_1,\nu_i)=\Psi_i(\hat{x}_1)\}\neq\emptyset$ is convex and compact set. Thus, (8) and (9) imply that $\exists\ \hat{\nu}=(\hat{\nu}_1,\hat{\nu}_2,\ldots,\hat{\nu}_n)\in\Omega$ such that

$$G_i(\hat{x}_1,\ \hat{\nu}_i)=\Psi_i(\hat{x}_1),\ i\in I$$

and

$$\partial\Psi(\hat{x}_1)\subseteq \text{conv }\{\partial_{x_1}G_i(\hat{x}_1,\hat{\nu}_i):\ i\in I_n\}.$$

From above and (7), we obtain

$$\begin{aligned}0\in \text{conv } \partial^* F_U(\hat{x}_1,\ \hat{x}_2)+\sum_{i\in I_n}\text{cone conv } \partial_{x_1}G_i(\hat{x}_1,\ \hat{\nu}_i)\times\{0\}\\ +\sum_{j\in J_m}\text{cone } \partial^* g_j(\hat{x}_1,\hat{x}_2)+\text{cone } (\partial F_L(\hat{x}_1,\hat{x}_2)-\partial\varphi(\hat{x}_1)\times\{0\}).\end{aligned}$$

Consequently, we can find scalars $\lambda_0\geq 0, \mu_j\geq 0, j\in J_m$ and $\tau_i\geq 0,\ i\in I_n$ such that

$$\begin{aligned}0\in \text{conv } \partial^* F_U(\hat{x}_1,\ \hat{x}_2)+\sum_{i\in I_n}\tau_i \text{ conv } \partial_{x_1}G_i(\hat{x}_1,\ \hat{\nu}_i)\times\{0\}\\ +\sum_{j\in J_m}\mu_j \text{ conv } \partial^* g_j(\hat{x}_1,\ \hat{x}_2)+\lambda_0 \text{ conv } (\partial F_L(\hat{x}_1,\ \hat{x}_2)-\partial\varphi(\hat{x}_1)\times\{0\})\end{aligned}$$

and the complementary slackness conditions (4) and (5) holds. Since $\partial F_L(\hat{x}_1,\ \hat{x}_2)$ and $\partial\varphi(\hat{x}_1)$ are convex, we obtain

$$\begin{aligned}0 \in \text{conv } \partial^* F_U(\hat{x}_1, \hat{x}_2) + \sum_{i\in I_n} \tau_i \text{ conv}\partial_{x_1} G_i(\hat{x}_1,\ \nu_i) \times \{0\} \\ + \sum_{j\in J_m} \mu_j \text{ conv } \partial^* g_j(\hat{x}_1,\ \hat{x}_2) + \lambda_0\ (\partial F_L(\hat{x}_1,\ \hat{x}_2) - \partial\varphi(\hat{x}_1) \times \{0\}).\end{aligned} \tag{10}$$

Moreover, with the help of the following relationship for convex, continuous function $F_L(x_1,\ x_2)$ that holds, [30, Corollary 3.44]

$$\partial F_L(x_1, x_2) \subset \partial_{x_1} F_L(x_1, x_2) \times \partial_{x_2} F_L(x_1, x_2). \tag{11}$$

By using (11) and [31, Theorem 8], the upper estimate of Clarke subdifferential of the value function at $\hat{x}_1$ is as follows:

$$\partial\varphi(\hat{x}_1) := \left[\bigcup_{x_2\in\Upsilon(\hat{x}_1)} \left\{ \bigcup_{(\lambda_1,\ldots,\lambda_m)\in\Lambda(\hat{x}_1,\ x_2)} \left(\partial_{x_1} F_L(\hat{x}_1,\ x_2) + \sum_{j\in J} \lambda_j \partial_{x_1} g_j(\hat{x}_1,\ x_2)\right)\right\}\right], \tag{12}$$

where $\Lambda(\hat{x}_1,\ x_2)$ is defined by

$$\begin{aligned}\Lambda(\hat{x}_1,\ x_2) := \Big\{(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m : 0 \in \partial_{x_2} F_L(\hat{x}_1,\ x_2) + \sum_{j\in J} \lambda_j \partial_{x_2} g_j(\hat{x}_1,\ x_2), \\ \lambda_j \geq 0,\ \lambda_j g_j(\hat{x}_1,\ x_2) = 0, \quad j \in J\Big\}.\end{aligned} \tag{13}$$

Combining (10), (12) and (13), we have the necessary conditions (1), (2) and (3). □

3.1. Application of necessary condition

Example 3.2. Let $(x_1,\ x_2) \in \mathbb{R}^2$ and $\nu \in \Omega = [0,\ 1]$. Clearly Ω is convex and compact. Consider the following robust bi-level optimization problem

$$\begin{aligned}(Q) \qquad & \min_{x_1,\ x_2} F_U(x_1,\ x_2) = |x_1| + |x_2| + 2x_1 + x_2 \\ & \text{s.t. } G_1(x_1,\ \nu) = x_1 - \nu \leq 0, \\ & \qquad x_2 \in \Upsilon(x_1),\end{aligned}$$

where for each $x_1 \in \mathbb{R}^{n_1}$, the parametric optimization problem (Q_{x_1}) has the set of optimal solutions $\Upsilon(x_1)$

$$\begin{aligned}(Q_{x_1}) \qquad & \min_{x_2}\ F_L(x_1,\ x_2) = |x_1| + x_2 \\ & \text{s.t. } \ g_1(x_1,\ x_2) = -x_2 \leq 0, \\ & \qquad g_2(x_1,\ x_2) = x_2 - 1 \leq 0.\end{aligned}$$

The optimal value function is

$$\varphi = \inf_{x_2}\{|x_1| + x_2 :\ x_2 \in [0, 1]\}$$

and

$$\Psi = \max_{\nu\in[0,1]} \{x_1 - \nu\} = x_1.$$

The feasible set of (RBPP) is

$$H = \{(x_1,\ 0) : x_1 \leq 0\}.$$

Clearly $\hat{x} = (\hat{x_1},\ \hat{x_2}) = (0,\ 0) \in H$ is the robust local minimizer of (Q).

– The set $\partial^* F_U(\hat{x}) = \{(1,\ 0), (1,\ 2), (3,\ 0), (3,\ 2)\}$ is upper semi-regular convexifactor of F_U.
– The function $x_1 - \nu$ is concave on Ω for x_1=0.
– The non-smooth robust constraint qualification holds at $\hat{x}$. Indeed, since

$$\begin{aligned} C(\hat{x}) = \Big(& \underset{j\in J_m}{\cup} \text{conv } \partial^* g_j(\hat{x}_1,\ \hat{x}_2) \bigcup \underset{i\in I_n}{\cup} \text{conv } \partial\Psi(\hat{x}_1) \times \{0\} \\ & \bigcup \partial F_L(\hat{x}_1,\ \hat{x}_2) - \partial\varphi(\hat{x}_1) \times \{0\}\Big) = \{(0,\ -1), (0,\ 1), (1,\ 0), (0,\ 1)\} \\ & (C(\hat{x}))^- \subseteq T(H,\ (0,\ 0)) = \{(x_1,\ 0) : x_1 \le 0\} \end{aligned}$$

– The argminimum map Υ is inner semicompact at $\hat{x_1} = 0$.
– (Q_{x_1}) is lower-level regular at (0, 0) since

$$\left.\begin{array}{l} \eta_1(-1) + \eta_2(1) = 0 \\ \eta_1, \eta_2 \ge 0,\ \eta_1(-x_2) + \eta_2(x_2 - 1) = 0, \end{array}\right\} \Longrightarrow [\eta_1,\ \eta_2 = 0].$$

– For $(\lambda_0, \mu_1, \mu_2, \tau_1, \lambda_1, \lambda_2)$= (1, 1, 0, 0, 1, 0), the necessary conditions (1)–(5) holds.

4. Robust bi-level Mond-Weir dual

MWD and WD are the two widely used dual in literature. Due to the weaker assumptions used, MWD has an advantage over WD. Here we have formulated a robust bi-level Mond-Weir dual (RBMWD) corresponding to the considered robust BLPP. Moreover, we have developed the relationship between the solutions of these two problems in terms of weak and strong duality theorems.

$$\begin{aligned} \text{(RBMWD)} \qquad & \max \quad F_U(u, v) \\ & \text{s.t.} \quad 0 \in \text{ conv } \partial^* F_U(u,\ v) + \sum_{i\in I_n} \alpha_i \text{ conv } \partial_{x_1} G_i(u, \nu_i) \times \{0\} \\ & \qquad + \sum_{j\in J_m} \beta_j \text{ conv } \partial^* g_j(u,\ v) + \gamma\Big(\partial F_L(u, v) \\ & \qquad - \partial\varphi(u) \times \{0\}\Big), \\ & \qquad F_L(u, v) - \varphi(u) \ge 0, \\ & \qquad \alpha_i G_i(u, \nu_i) \ge 0, \quad \nu_i \in \Omega_i, \\ & \qquad \beta_j g_j(u, v) \ge 0, \\ & \qquad \gamma \ge 0, \quad \alpha_i \ge 0, \quad i \in I_n, \quad \beta_j \ge 0, \quad j \in J_m. \end{aligned}$$

Theorem 4.1 (Weak Duality)**.** *Let $(x_1,\ x_2)$ be a feasible solution of (Q) and $(u,\ v,\ \nu,\ \alpha,\ \beta,\ \gamma)$ be a feasible solution of (RBMWD). Let $F_U(.,\ .)$ be ∂^*-pseudoconvex at $(u,\ v)$ and $F_L(.,\ .) - \varphi(.)$, $G_i(.,\ \nu_i)$, $i \in I_n$ and $g_j(.,\ .)$, $j \in J_m$ are ∂^*-quasiconvex at $(u,\ v)$, then*

$$F_U(x_1,\ x_2) \ge F_U(u,\ v).$$

Proof. By contrary, suppose that $F_U(x_1,\ x_2) < F_U(u,\ v)$. Since F_U is ∂^*-pseudoconvex at $(u,\ v)$, we have

$$\langle \xi_1, (x_1 - u, x_2 - v)\rangle < 0, \qquad \forall\ \xi_1 \in \partial^* F_U(u,\ v). \tag{14}$$

Since $(u,v,\nu,\alpha,\beta,\gamma)$ is feasible for (RBMWD), we obtain $\xi_1 \in \text{conv}\partial^* F_U(u,\ v)$, $\xi_2 \in \text{conv}\ (\partial^* F_L(u,\ v) - \partial^*\varphi(u) \times \{0\})$, $\eta_i \in \text{conv}\ \partial_{x_1} G_i(u,\ \nu_i)$ and $\mu_j \in \text{conv}\ \partial^* g_j(u,\ v)$ such that

$$0 = \xi_1 + \gamma\xi_2 + \sum_{i\in I_n} \alpha_i\eta_i + \sum_{j\in J_m} \beta_j\mu_j. \tag{15}$$

Since (x_1,x_2) is feasible for (Q) and by Lemma 2.3 (x_1,x_2) is feasible for (RBPP). $(u,v,\nu,\alpha,\beta,\gamma)$ is feasible for (RBMWD) thus we have $F_L(x_1,x_2) - \varphi(x_1) \le F_L(u,v) - \varphi(u)$. From ∂^*-quasiconvexity of $F_L(.,.) - \varphi(.)$, we get

$$\langle \gamma\xi_2,\ (x_1 - u,\ x_2 - v)\rangle \le 0, \quad \forall\ \xi_2 \in \partial^*(F_L(u,\ v) - \varphi(u)) \text{ and } \gamma \ge 0. \tag{16}$$

Similarly for $\alpha_i \ge 0$ and $\beta_j \ge 0$, $\alpha_i G_i(x_1,\nu_i) \le \alpha_i G_i(u,\nu_i),\ i \in I_n$ and $\beta_j g_j(x_1,\ x_2) \le \beta_j g_j(u,v),\ j \in J_m$. By ∂^*-quasiconvexity of $G_i(.,\nu_i)$ and $g_j(.,.)$, we get

$$\left\langle \sum_{i\in I_n} \alpha_i\eta_i,\ (x_1 - u, 0)\right\rangle \le 0, \tag{17}$$

$$\left\langle \sum_{j\in J_m} \beta_j\mu_j,\ (x_1 - u,\ x_2 - v)\right\rangle \le 0. \tag{18}$$

From (14), (16) and (18), we have

$$\left\langle \xi_1 + \gamma\xi_2 + \sum_{j\in J_m} \beta_j\mu_j,\ (x_1 - u, x_2 - v)\right\rangle < 0, \tag{19}$$

from (17) and (19), we get the contradiction to (15). □

Theorem 4.2 (Strong Duality). *Let $(\hat{x}_1,\ \hat{x}_2)$ be a robust minimizer of (Q) where the extended robust non-smooth constraint qualification holds. Then, there exists $\hat{\nu} \in \Omega$, $(\hat{\alpha},\hat{\beta},\hat{\gamma}) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}_+$ such that $(\hat{x}_1,\hat{x}_2,\hat{\nu},\hat{\alpha},\hat{\beta},\hat{\gamma})$ is feasible for (RBMWD) and values of two objective functions are equal. Moreover, if the hypotheses of Theorem 4.1 holds, then $\left(\hat{x}_1,\hat{x}_2,\hat{\nu},\hat{\alpha},\hat{\beta},\hat{\gamma}\right)$ is a robust maximizer of (RBMWD).*

Proof. Let $(\hat{x}_1,\hat{x}_2)$ be a robust minimizer of (Q) where the extended non-smooth robust constraint qualification holds. Using Theorem 3.1 $\exists\ \hat{\nu} \in \Omega$ and $(\hat{\alpha},\hat{\beta},\hat{\gamma}) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}_+$ such that

$$\begin{aligned} 0 \in \text{conv}\ \partial^* F_U(\hat{x}_1,\hat{x}_2) &+ \sum_{i\in I_n} \hat{\alpha}_i\ \text{conv}\ \partial_{x_1} G_i(\hat{x}_1,\ \hat{\nu}) \times \{0\} \\ &+ \sum_{j\in J_m} \hat{\beta}_i\ \text{conv}\ \partial^* g_j(\hat{x}_1,\ \hat{x}_2) + \hat{\gamma}\ (\partial F_L(\hat{x}_1,\ \hat{x}_2) - \partial\varphi(\hat{x}_1) \times \{0\}). \end{aligned}$$

By Lemma 2.3 $(\hat{x}_1,\hat{x}_2)$ is feasible for (RBPP) and from

$$\varphi(x_1) = \inf_{x_2}\{F_L(x_1,x_2) : g_j(x_1,x_2) \le 0, \quad \forall j \in J\}$$

we always have $F_L(x_1,\ x_2) - \varphi(x_1) \ge 0,\ \forall\ (x_1,\ x_2)$, where $x_2 \in S(x_1)$; thus the (RBPP) inequality constraint $F_L(x_1,\ x_2) - \varphi(x_1) \le 0,\ x_2 \in S(x_1)$ and $x_1 \in \mathbb{R}^{n_1}$ is equivalent to $F_L(x_1,\ x_2) - \varphi(x_1) = 0$. Therefore, we have

$$\sum_{j\in J_m} \hat{\beta}_i g_j(\hat{x}_1,\hat{x}_2) = 0, \sum_{i\in I_n} \hat{\alpha}_i G_i(\hat{x}_1,\hat{\nu}) = 0 \text{ and } F_L(\hat{x}_1,\hat{x}_2) - \varphi(\hat{x}_1) = 0.$$

Hence $\left(\hat{x}_1,\hat{x}_2,\hat{\nu},\hat{\alpha},\hat{\beta},\hat{\gamma}\right)$ is feasible for (RBMWD) and the values of the two objective functions are equal.

Assume that $\left(\hat{x}_1, \hat{x}_2, \hat{\nu}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}\right)$ is not a robust maximizer of (RBMWD). Therefore there exist $(u, v, \nu, \alpha, \beta, \gamma)$ feasible for dual such that

$$F_U(u,\ v) > F_U(\hat{x}_1, \hat{x}_2).$$

But this contradicts the weak duality Theorem 4.1. Hence $\left(\hat{x}_1, \hat{x}_2, \hat{\nu}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}\right)$ is a robust maximizer of (RBMWD). □

5. Results and conclusion

This paper's contribution is as follows: we have considered a bi-level model whose upper-level constraints include some uncertainty, and the lower-level problem is fully convex. An equivalent single-level mathematical problem is established by using the optimal value reformulation. To deal with uncertainty at the upper-level problem, we follow the concept of robust counterpart optimization. To this end, we develop the KKT type necessary optimality conditions. Constraint qualifications play an essential role in developing the necessary optimality conditions and strong duality results. As Abadie CQ is weaker than most existing CQs, naturally, one tries to use it, but as our reformulated problem is non-smooth and uncertain, we have extended ACQ and introduced a non-smooth RCQ. Moreover, an example is given to validate our necessary condition. Further, we have developed the robust bi-level MWD and established the relationship between the solutions to both the problems with the help of weak and strong duality theorems.

6. Future directions

For future research, developing sufficient conditions for robust BLPPs will be challenging. Formulating sufficient conditions in a non-smooth setting alone is a difficult task, and here along with non-smoothness, uncertainty is also added. So one should be very careful while examining these types of problems. From a computational perspective, one could also develop efficient algorithms based on duality results.

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