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MINIMIZING THE NUMBER OF EDGES IN ($P_k \cup K_3$)-SATURATED CONNECTED GRAPHS

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Abstract. For a graph H, a graph G is H*-saturated* if it contains no copy of H as a (not necessarily induced) subgraph, but the addition of any edge missing from G creates a copy of H in the resultant graph. The connected saturation number $\mathrm{sat}'(n, H)$ is defined as the minimum number of edges in H-saturated connected graphs on n vertices. In this paper we consider the $(P_k \cup K_3)$-saturated connected graphs on n vertices and focus on the determination of $\mathrm{sat}'(n, P_k \cup K_3)$. We prove that $n + 2 \leq \mathrm{sat}'(n, P_k \cup K_3) \leq n + \frac{3k-6}{2}$ for $n > \frac{3k+4}{2}$ with $k \geq 4$ and characterize the extremal graphs at which the upper bounds are attained. Moreover, the exact values of $\mathrm{sat}'(n, P_k \cup K_3)$ are determined with $k \in \{3, 4\}$ and we get $\mathrm{sat}(n, P_2 \cup K_3) = \mathrm{sat}'(n, P_2 \cup K_3) = n$.

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1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let $G = (V, E)$ be a graph, as usual, denote by $V(G)$, $E(G)$, $n(G)$, $m(G)$ and $\overline{G}$ the vertex set, edge set, the number of vertices, the number of edges and the complement of G, respectively. For any $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the degree and the set of neighbors of v in G, respectively. As usual, let $\Delta(G)$ and $\delta(G)$ be the maximum degree and the minimum degree, respectively, of graph G. Denote by K_n, P_n, C_n and S_n the complete graph, path, cycle and the star graph on n vertices, respectively. The *distance* $d_G(x, y)$ of two vertices x, y is the length of a shortest $x - y$ path in G. The *eccentricity*, denoted by $\mathrm{ecc}_G(v)$, of a vertex v in a graph G is $\max\{d(u, v) : u \in V(G)\}$. For $A \subseteq V(G)$, let $G[A]$ be the subgraph of G induced by A. Given vertex sets $A, B \subseteq V(G)$, we denote by $E_G(A, B)$, or $E(A, B)$ for simplicity, the edge subset of G formed by edges with one endvertex in A and the other in B, and its cardinality will be written as $e_G(A, B)$ or $e(A, B)$ for short, and let $A \setminus B$ denote $A - B$. For any edge $e \in E(\overline{G})$, we write by $G + e$ the graph obtained from G by adding the new edge e. Given any two vertex-disjoint graphs G and H, their *union* $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and their *join* $G \vee H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$.

Keywords. Graph saturation, Saturation number, Cyclomatic number.

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For two graphs G and H, G is *H-saturated* if it contains no copy of H as a (not necessarily induced) subgraph, but the addition of any edge missing from G creates a copy of H in the resultant graph. Note that G mentioned here is not necessarily connected. The *saturation number* $\mathrm{sat}(n,H)$ is defined as the minimum number of edges in H-saturated graphs on n vertices. This can be viewed as the dual of the celebrated *Turán number* $ex(n,H)$, the maximum number of edges in H-saturated graphs on n vertices. For some recent results on the Turán number, please see [24]. Moreover, for a (not necessarily connected) graph H, we denote by $\mathrm{sat}'(n,H)$ and $\mathrm{sat}''(n,H)$ the minimum number of edges in H-saturated connected graphs and H-saturated disconnected graphs, respectively, on n vertices. Clearly, $\mathrm{sat}(n,H)=\min\{\mathrm{sat}'(n,H),\mathrm{sat}''(n,H)\}$ where $\mathrm{sat}'(n,H)$ and $\mathrm{sat}''(n,H)$ are called *connected saturation number* and *disconnected saturation number*, respectively.

Saturation number was first studied by Erdös *et al.* [10], who proved that $\mathrm{sat}(n,K_p)=(p-2)(n-p-2)+\binom{p-2}{2}$ with the extremal graphs $K_{p-2}\vee\overline{K_{n-p+2}}$. Kászonyi and Tuza [20] considered $\mathrm{sat}(n,H)$ for $H\in\{S_k,mK_2,P_m\}$ and determined the extremal graphs, respectively. As a generalization, Faudree *et al.* [15] proved $\mathrm{sat}(n,tK_p)=(t-1)\binom{p+1}{2}+\binom{p-2}{2}+(p-2)(n-p+2)$ and constructed the extremal graphs $K_{p-2}\vee\big((t-1)K_{p+1}\cup\overline{K_{n-pt-t+3}}\big)$. Moreover, they also determined $\mathrm{sat}(n,K_p\cup K_q)$ and $\mathrm{sat}(n,F_{t,p,\ell})$ with the extremal graphs where $F_{t,p,\ell}$ is the generalized friendship graph composed of t copies of K_p intersecting in a common K_ℓ for positive integers t, p and ℓ.

For the path, Kászonyi and Tuza [20] found $\mathrm{sat}(n,P_m)=n-\lfloor\frac{n}{a_m}\rfloor$ where $a_m=3\cdot2^{k-1}-2$ if $m=2k$ and $a_m=2^{k+1}-2$ if $m=2k+1$, and they also characterized the family of extremal graphs. Frick and Singleton [16] proved $\mathrm{sat}(n,P_n)=\lceil\frac{3n-2}{2}\rceil$ for $n\geq54$ and several small order cases. For $t\geq2$, let $F=P_{k_1}\cup P_{k_2}\cup\cdots\cup P_{k_t}$ be a linear forest where P_k denotes a path on k vertices and $k_1\geq k_2\geq\cdots\geq k_t$. Chen *et al.* [8] investigated the saturation numbers for forests and provided the upper and lower bounds on $\mathrm{sat}(n,H)$ with $H\in\{F,tP_k,P_k\cup P_\ell\}$. Furthermore, they obtained the exact values of $\mathrm{sat}(n,P_m\cup tP_2)$ with $m\in\{3,4,5\}$. Moreover, some enumerative properties of star-free graphs are reported in [19]. Fan and Wang [12] proved that $\mathrm{sat}(n,P_5\cup tP_2)=\min\big\{\lceil\frac{5n-4}{6}\rceil,3t+12\big\}$ for $n\geq3t+8$ with the extremal graphs $K_6\cup(t-1)K_3\cup\overline{K_{n-3t-3}}$ for $n>\frac{18t+76}{5}$. Recently Yan [25] showed that $\mathrm{sat}(n,P_6\cup tP_2)=\min\big\{n-\lfloor\frac{n}{10}\rfloor,3t+18\big\}$ with the extremal graphs $K_7\cup(t-1)K_3\cup\overline{K_{n-3t-4}}$ for $n>\frac{10}{3}t+20$. The known results about C_k-saturated graphs are mainly for small values of k. Please refer to [5,10,22,23] for the exact values of $\mathrm{sat}(n,C_k)$ with $k\leq5$ and $\mathrm{sat}(n,C_n)$. For $k\geq6$, some lower bounds and upper bounds on $\mathrm{sat}(n,C_k)$ are established in [1,17,18,21,26]. Please see an informative survey [9] for some detailed results in graph saturation.

By now there are some results on $\mathrm{sat}(n,H_1\cup H_2)$ if H_1 and H_2 have a same type such as the above cases when they are both paths or complete graphs. But for all we know, there are no results on $\mathrm{sat}(n,H_1\cup H_2)$ when H_1 and H_2 are of distinct types. In this paper we consider the $(P_k\cup K_3)$-saturated connected graphs on n vertices and determine the bounds on $\mathrm{sat}'(n,P_k\cup K_3)$. For convenience, a $(P_k\cup K_3)$-saturated (connected) graph with minimum number of edges is called a *minimum $(P_k\cup K_3)$-saturated (connected) graph*. The paper is organized as follows. In Section 2 we give the upper bounds on the saturation number and characterize the extremal graphs at which the upper bounds are attained. In Section 3 we give a lower bound on the saturation number $\mathrm{sat}'(n,P_k\cup K_3)\geq n+2$ for $n>\frac{3k+4}{2}$ with $k\geq4$. In Section 4 we determine the values of $\mathrm{sat}'(P_k\cup K_3)$ with $k\in\{2,3,4\}$. We end with a few remarks in Section 5.

2. Upper bound on $\mathrm{sat}'(n,P_k\cup K_3)$

Let $C_{k+2}=v_1v_2\ldots v_{k+2}v_1$ be a cycle. Let G^* be the graph obtained by attaching $n-k-2$ pendant vertices to v_{k+2} of C_{k+2} where X is the set of these newly-attached pendant vertices. Next we introduce two kinds of graphs based on the parity of k. If k is even, let G^k_{even} be the graph obtained from G^* by adding all edges v_iv_{k+2-i}, v_iv_{k+1-i} for $i=1,2,\ldots,\frac{k}{2}-1$ and v_iv_{k+3-i} for $i=2,\ldots,\frac{k}{2}$ (see Fig. 1). If k is odd, let G^k_{odd} be the graph obtained from G^* by adding all edges v_iv_{k+2-i} for $i=1,2,\ldots,\lfloor\frac{k}{2}\rfloor$, v_iv_{k+1-i} for $i=1,2,\ldots,\lfloor\frac{k}{2}\rfloor-1$

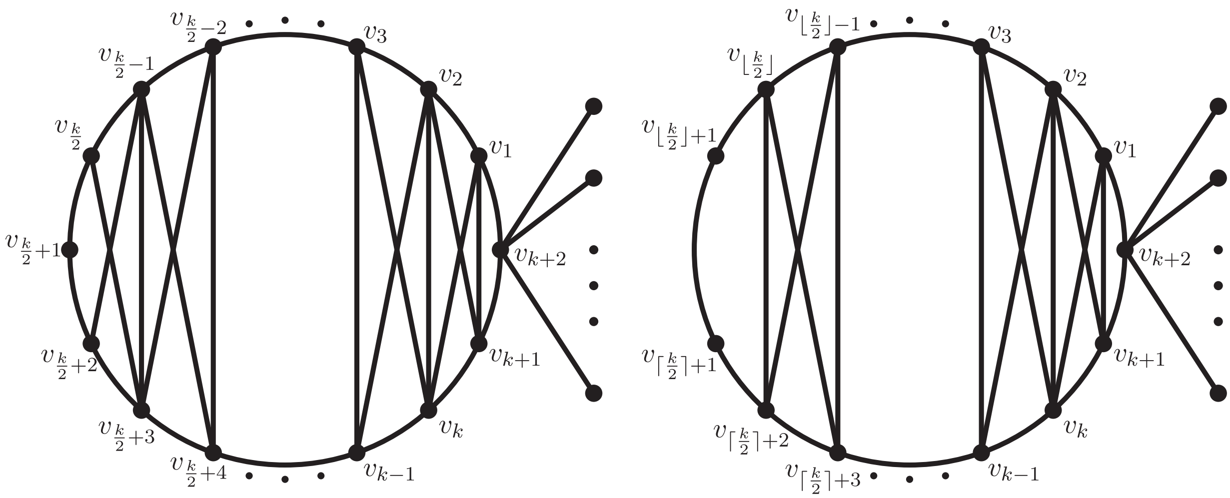

FIGURE 1. Graph G^k_{even} (*left*) and graph G^k_{odd} (*right*).

and $v_i v_{k+3-i}$ for $i = 2, \ldots, \lfloor\frac{k}{2}\rfloor$ (see Fig. 1). It can be easily checked that $m(G^k_{\text{even}}) = n + 3\left(\frac{k}{2} - 1\right)$ and $m(G^k_{\text{odd}}) = n + 3\left(\lfloor\frac{k}{2}\rfloor - 1\right) + 1$.

In the following result we provide an upper bound on $\text{sat}'(n, P_k \cup K_3)$.

Theorem 2.1. *Let $k \geq 4$ be an integer. Then*

$$\text{sat}'(n, P_k \cup K_3) \leq \begin{cases} n + 3\left(\frac{k}{2} - 1\right), & \text{if } k \text{ is even,} \\ n + 3\left(\lfloor\frac{k}{2}\rfloor - 1\right) + 1, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. For proving the result, it suffices to show that G^k_{even} and G^k_{odd} defined as above are $(P_k \cup K_3)$-saturated. We can easily check that G^k_{even} and G^k_{odd} are $(P_k \cup K_3)$-free. Next we show that there exists a copy of $P_k \cup K_3$ as a subgraph of $G^k_{\text{even}} + e$ and $G^k_{\text{odd}} + e$ by adding any edge e missing from G^k_{even} and G^k_{odd}, respectively, depending on the parity of k. Assuming that k is even, we divide into three cases based on the position of non-edge e.

Case A1. $e = uv$ with $u, v \in X$.

It is clear that $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = uvv_{k+2}u$ and $P_k = v_2 v_3 \ldots v_{k+1}$.

Case A2. $e \in E(X, V(C_{k+2}) \setminus \{v_{k+2}\})$.

Let $Y = \left\{v_1, v_2, \ldots, v_{\frac{k}{2}+1}\right\}$. We choose any vertex $v \in X$. By the symmetry of G^k_{even}, without loss of generality, we can consider any edge $e \in E(v, Y)$.

Subcase A2.1. $e = vv_1$.

It is clear that $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_1 v_{k+2} v v_1$ and $P_k = v_2 v_3 \cdots v_{k+1}$.

Subcase A2.2. $e = vv_j$ with $j \in \left\{2, 3, \ldots, \frac{k}{2} + 1\right\}$.

In this subcase $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_1 v_{k+1} v_k v_1$ and $P_k = v_{k+2} v v_j v_{j-1} \cdots v_2 v_{k-1} v_{k-2} v_{j+1}$.

Case A3. $e = v_i v_j$ is a non-edge with $v_i, v_j \in V(C_{k+2})$.

By the symmetry of G^k_{even}, we only need to consider that one of the endpoint of e is v_i with $i \in \left\{k+2, 1, 2, \ldots, \frac{k}{2}, \frac{k}{2} + 1\right\}$. We choose any vertex $v \in X$.

Subcase A3.1. $e = v_{k+2} v_j$.

We only need to consider that $j \in \{2, 3, \ldots, \frac{k}{2}, \frac{k}{2}+1\}$ by the symmetry of G^k_{even}. In this subcase $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_1v_{k+1}v_kv_1$ and $P_k = vv_{k+2}v_jv_{j-1}\cdots v_2v_{k-1}v_{k-2}\cdots v_{j+1}$.

Subcase A3.2. $e = v_iv_j$ with $i \in \{1, 2, 3, \ldots, \frac{k}{2}-3\}$.

If $j \in \{i+2, i+3, \ldots, k-i-1, k-i\}$, $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{i+1}v_{k+1-i}v_{k+2-i}v_{i+1}$ and $P_k = vv_{k+2}v_1v_{k+1}\cdots v_{i-1}v_{k+3-i}v_iv_jv_{j+1}\cdots v_{k-i}v_{i+2}v_{i+3}\cdots v_{j-1}$ (particularly $P_k = vv_{k+2}v_1v_{k+1}\cdots v_{i-1}v_{k+3-i}v_iv_{i+2}v_{i+3}\cdots v_{k-i}$ if $j = i+2$).

If $j \in \{1, 2, \ldots, i-2\}$, $G^k_{\text{even}}+e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{i-1}v_{k+3-i}v_{k+2-i}v_{i-1}$ and $P_k = v_{k+1-i}v_{k-i}\cdots v_{i-1}v_iv_jv_{j+1}\cdots v_{i-2}v_{k+4-i}v_{k+5-i}\cdots v_{k+2-j}v_{j-1}v_{k+3-j}\cdots v_1v_{k+1}v_{k+2}v$ (particularly $P_k = v_{k+1-i}v_{k-i}\cdots v_{i-1}v_iv_1v_2\cdots v_{i-2}v_{k+4-i}v_{k+5-i}\cdots v_{k+1}v_{k+2}v$ if $j = 1$).

If $j \in \{k+4-i, k+5-i, \ldots, k+1\}$, $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{i-1}v_{k+3-i}v_{k+2-i}v_{i-1}$ and $P_k = v_{k+1-i}v_{k-i}\cdots v_{i-1}v_iv_jv_{j-1}\cdots v_{k+4-i}v_{i-2}v_{i-3}\cdots v_{k+2-j}v_{j+1}$ $v_{k+1-j}\cdots v_{k+1}v_1v_{k+2}v$ (particularly $P_k = v_{k+1-i}v_{k-i}\cdots v_{i-1}v_iv_{k+1}v_k\cdots v_{k+4-i}v_{i-2}v_{i-3}\cdots v_1v_{k+2}v$ if $j = k+1$).

Subcase A3.3. $e = v_{\frac{k}{2}-2}v_j$.

If $j = \frac{k}{2}$, $e = v_{\frac{k}{2}-2}v_{\frac{k}{2}}$. $G^k_{\text{even}}+e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\frac{k}{2}-1}v_{\frac{k}{2}+3}v_{\frac{k}{2}+4}v_{\frac{k}{2}-1}$ and $P_k = vv_{k+2}v_1v_{k+1}\cdots v_{\frac{k}{2}-3}v_{\frac{k}{2}+5}v_{\frac{k}{2}-2}v_{\frac{k}{2}}v_{\frac{k}{2}+1}v_{\frac{k}{2}+2}$.

If $j = \frac{k}{2}+1$, $e = v_{\frac{k}{2}-2}v_{\frac{k}{2}+1}$. $G^k_{\text{even}}+e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\frac{k}{2}-1}v_{\frac{k}{2}}v_{\frac{k}{2}+3}v_{\frac{k}{2}-1}$ and $P_k = vv_{k+2}v_{k+1}v_1\cdots v_{\frac{k}{2}+4}v_{\frac{k}{2}-2}v_{\frac{k}{2}+1}v_{\frac{k}{2}+2}$.

If $j = \frac{k}{2}+2$, $e = v_{\frac{k}{2}-2}v_{\frac{k}{2}+2}$. $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\frac{k}{2}-1}v_{\frac{k}{2}+3}v_{\frac{k}{2}+4}v_{\frac{k}{2}-1}$ and $P_k = vv_{k+2}v_1v_{k+1}\cdots v_{\frac{k}{2}-3}v_{\frac{k}{2}+5}v_{\frac{k}{2}-2}v_{\frac{k}{2}+2}v_{\frac{k}{2}+1}v_{\frac{k}{2}}$.

If $j \in \{\frac{k}{2}+6, \frac{k}{2}+7, \ldots, k+1\}$, $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\frac{k}{2}-3}v_{\frac{k}{2}+5}v_{\frac{k}{2}+4}v_{\frac{k}{2}-3}$ and $P_k = v_{\frac{k}{2}+3}v_{\frac{k}{2}+2}v_{\frac{k}{2}+1}v_{\frac{k}{2}}v_{\frac{k}{2}-1}v_{\frac{k}{2}-2}v_jv_{j-1}\cdots v_{\frac{k}{2}+6}v_{\frac{k}{2}-4}v_{\frac{k}{2}-5}\cdots v_{k+2-j}v_{j+1}$ $v_{k+1-j}\cdots v_{k+1}v_1v_{k+2}v$ (particularly $P_k = v_{\frac{k}{2}+3}v_{\frac{k}{2}+2}v_{\frac{k}{2}+1}v_{\frac{k}{2}}v_{\frac{k}{2}-1}v_{\frac{k}{2}-2}v_{k+1}v_k\cdots v_{\frac{k}{2}+6}v_{\frac{k}{2}-4}v_{\frac{k}{2}-5}\cdots$ $v_1v_{k+2}v$ if $j = k+1$).

Subcase A3.4. $e = v_{\frac{k}{2}-1}v_j$.

If $j = \frac{k}{2}+1$, $e = v_{\frac{k}{2}-1}v_{\frac{k}{2}+1}$. $G^k_{\text{even}}+e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\frac{k}{2}-1}v_{\frac{k}{2}+1}v_{\frac{k}{2}}v_{\frac{k}{2}-1}$ and $P_k = vv_{k+2}v_1v_{k+1}\cdots v_{\frac{k}{2}-2}v_{\frac{k}{2}+4}v_{\frac{k}{2}+3}v_{\frac{k}{2}+2}$.

If $j \in \{\frac{k}{2}+5, \frac{k}{2}+6, \ldots, k+1\}$, $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\frac{k}{2}-2}v_{\frac{k}{2}+4}v_{\frac{k}{2}+3}v_{\frac{k}{2}-2}$ and $P_k = v_{\frac{k}{2}+2}v_{\frac{k}{2}+1}v_{\frac{k}{2}}v_{\frac{k}{2}-1}v_jv_{j-1}\cdots v_{\frac{k}{2}+5}v_{\frac{k}{2}-3}v_{\frac{k}{2}-4}\cdots v_{k+2-j}v_{j+1}v_{k+1-j}$ $\cdots v_{k+1}v_1v_{k+2}v$ (particularly $P_k = v_{\frac{k}{2}+2}v_{\frac{k}{2}+1}v_{\frac{k}{2}}v_{\frac{k}{2}-1}v_{k+1}v_k\cdots v_{\frac{k}{2}+5}v_{\frac{k}{2}-3}v_{\frac{k}{2}-4}\cdots v_1v_{k+2}v$ if $j = k+1$).

Subcase A3.5. $e = v_{\frac{k}{2}}v_j$.

If $j = \frac{k}{2}+2$, $e = v_{\frac{k}{2}}v_{\frac{k}{2}+2}$. $G^k_{\text{even}}+e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\frac{k}{2}}v_{\frac{k}{2}+2}v_{\frac{k}{2}+1}v_{\frac{k}{2}}$ and $P_k = vv_{k+2}v_1v_{k+1}\cdots v_{\frac{k}{2}-1}v_{\frac{k}{2}+3}$.

If $j \in \{\frac{k}{2}+4, \frac{k}{2}+5, \ldots, k+1\}$, $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\frac{k}{2}-1}v_{\frac{k}{2}+3}v_{\frac{k}{2}+2}v_{\frac{k}{2}-1}$ and $P_k = v_{\frac{k}{2}+1}v_{\frac{k}{2}}v_jv_{j-1}\cdots v_{\frac{k}{2}+4}v_{\frac{k}{2}-2}v_{\frac{k}{2}-3}\cdots v_{k+2-j}v_{j+1}v_{k+1-j}\cdots v_{k+1}v_1$ $v_{k+2}v$ (particularly $P_k = v_{\frac{k}{2}+1}v_{\frac{k}{2}}v_{k+1}v_k\cdots v_{\frac{k}{2}+4}v_{\frac{k}{2}-2}v_{\frac{k}{2}-3}\cdots v_1v_{k+2}v$ if $j = k+1$).

Subcase A3.6. $e = v_{\frac{k}{2}+1}v_j$.

If $j = \frac{k}{2}+3$, $e = v_{\frac{k}{2}+1}v_{\frac{k}{2}+3}$. $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\frac{k}{2}+1}v_{\frac{k}{2}+3}v_{\frac{k}{2}+2}v_{\frac{k}{2}+1}$ and $P_k = vv_{k+2}v_1v_{k+1}\cdots v_{\frac{k}{2}-2}v_{\frac{k}{2}+4}v_{\frac{k}{2}-1}v_{\frac{k}{2}}$.

If $j \in \{\frac{k}{2}+4, \frac{k}{2}+5, \ldots, k+1\}$, $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\frac{k}{2}}v_{\frac{k}{2}-1}v_{\frac{k}{2}+3}v_{\frac{k}{2}}$ and $P_k = v_{\frac{k}{2}+2}v_{\frac{k}{2}+1}v_jv_{j-1}\cdots v_{\frac{k}{2}+4}v_{\frac{k}{2}-2}v_{\frac{k}{2}-3}\cdots v_{k+2-j}v_{j+1}v_{k+1-j}\cdots v_{k+1}v_1v_{k+2}v$ (particularly $P_k = v_{\frac{k}{2}+2}v_{\frac{k}{2}+1}v_{k+1}v_k\cdots v_{\frac{k}{2}+4}v_{\frac{k}{2}-2}v_{\frac{k}{2}-3}\cdots v_1v_{k+2}v$ if $j = k+1$).

Therefore, we have shown that $G^k_{\text{even}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph for any non-edge e, which means that G^k_{even} is $(P_k \cup K_3)$-saturated. Similar to the argument for G^k_{even}, we will verify that

G^k_{odd} also is $(P_k \cup K_3)$-saturated. Now, assuming that k is odd, we divide into three cases based on the position of non-edge e.

Case B1. $e = uv$ with $u, v \in X$.

It is clear that $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = uvv_{k+2}u$ and $P_k = v_2v_3 \cdots v_{k+1}$.

Case B2. $e \in E(X, V(C_{k+2}) \backslash \{v_{k+2}\})$.

Let $Y = \left\{v_1, v_2, \ldots, v_{\lfloor \frac{k}{2} \rfloor + 1}\right\}$. We choose any vertex $v \in X$. By the symmetry of G^k_{odd}, without loss of generality, we can consider any edge $e \in E(v, Y)$.

Subcase B2.1. $e = vv_1$.

It is clear that $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_1v_{k+2}vv_1$ and $P_k = v_2v_3 \cdots v_{k+1}$.

Subcase B2.2. $e = vv_j$ with $j \in \left\{2, 3, \ldots, \lfloor \frac{k}{2} \rfloor + 1\right\}$.

In this subcase $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_1v_{k+1}v_kv_1$ and $P_k = v_{k+2}vv_jv_{j-1} \cdots v_2v_{k-1}v_{k-2}v_{j+1}$.

Case B3. $e = v_iv_j$ is a non-edge with $v_i, v_j \in V(C_{k+2})$.

By the symmetry of G^k_{odd}, we only need to consider that one of the endpoint of e is v_i with $i \in \left\{k+2, 1, 2, \ldots, \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1\right\}$. We choose any vertex $v \in X$.

Subcase B3.1. $e = v_{k+2}v_j$.

We only need to consider that $j \in \left\{2, 3, \ldots, \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1\right\}$ by the symmetry of G^k_{odd}. In this subcase $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_1v_{k+1}v_kv_1$ and $P_k = vv_{k+2}v_jv_{j-1} \cdots v_2v_{k-1}v_{k-2} \cdots v_{j+1}$.

Subcase B3.2. $e = v_iv_j$ with $i \in \left\{1, 2, 3, \ldots, \lfloor \frac{k}{2} \rfloor - 2\right\}$.

If $j \in \{i+2, i+3, \ldots, k-i-1, k-i\}$, $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{i+1}v_{k+1-i}v_{k+2-i}v_{i+1}$ and $P_k = vv_{k+2}v_1v_{k+1} \cdots v_{i-1}v_{k+3-i}v_iv_jv_{j+1} \cdots v_{k-i}v_{i+2}v_{i+3} \cdots v_{j-1}$ (particularly $P_k = vv_{k+2}v_1v_{k+1} \cdots v_{i-1}v_{k+3-i}v_iv_{i+2}v_{i+3} \cdots v_{k-i}$ if $j = i+2$).

If $j \in \{1, 2, \ldots, i-2\}$, $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{i-1}v_{k+3-i}v_{k+2-i}v_{i-1}$ and $P_k = v_{k+1-i}v_{k-i} \cdots v_{i+1}v_iv_jv_{j+1} \cdots v_{i-2}v_{k+4-i}v_{k+5-i} \cdots v_{k+2-j}v_{j-1}v_{k+3-j} \cdots v_1v_{k+1}v_{k+2}v$ (particularly $P_k = v_{k+1-i}v_{k-i} \cdots v_{i+1}v_iv_1v_2 \cdots v_{i-2}v_{k+4-i}v_{k+5-i} \cdots v_{k+1}v_{k+2}v$ if $j = 1$).

If $j \in \{k+4-i, k+5-i, \ldots, k+1\}$, $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{i-1}v_{k+3-i}v_{k+2-i}v_{i-1}$ and $P_k = v_{k+1-i}v_{k-i} \cdots v_{i+1}v_iv_jv_{j-1} \cdots v_{k+4-i}v_{i-2}v_{i-3} \cdots v_{k+2-j} v_{j+1}v_{k+1-j} \cdots v_{k+1}v_1v_{k+2}v$ (particularly $P_k = v_{k+1-i}v_{k-i} \cdots v_{i+1}v_iv_{k+1}v_k \cdots v_{k+4-i}v_{i-2}v_{i-3} \cdots v_1v_{k+2}v$ if $j = k+1$).

Subcase B3.3. $e = v_{\lfloor \frac{k}{2} \rfloor - 1}v_j$.

If $j = \lfloor \frac{k}{2} \rfloor + 1$, $e = v_{\lfloor \frac{k}{2} \rfloor - 1}v_{\lfloor \frac{k}{2} \rfloor + 1}$. $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\lfloor \frac{k}{2} \rfloor}v_{\lceil \frac{k}{2} \rceil + 2}v_{\lceil \frac{k}{2} \rceil + 3}v_{\lfloor \frac{k}{2} \rfloor}$ and $P_k = vv_{k+2}v_1v_{k+1} \cdots v_{\lfloor \frac{k}{2} \rfloor - 2}v_{\lceil \frac{k}{2} \rceil + 4}v_{\lfloor \frac{k}{2} \rfloor - 1}v_{\lfloor \frac{k}{2} \rfloor + 1}v_{\lceil \frac{k}{2} \rceil + 1}$.

If $j = \lceil \frac{k}{2} \rceil + 1$, $e = v_{\lfloor \frac{k}{2} \rfloor - 1}v_{\lceil \frac{k}{2} \rceil + 1}$. $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\lfloor \frac{k}{2} \rfloor}v_{\lceil \frac{k}{2} \rceil + 2}v_{\lceil \frac{k}{2} \rceil + 3}v_{\lfloor \frac{k}{2} \rfloor}$ and $P_k = vv_{k+2}v_1v_{k+1} \cdots v_{\lfloor \frac{k}{2} \rfloor - 2}v_{\lceil \frac{k}{2} \rceil + 4}v_{\lfloor \frac{k}{2} \rfloor - 1}v_{\lceil \frac{k}{2} \rceil + 1}v_{\lfloor \frac{k}{2} \rfloor + 1}$.

If $j \in \left\{\lceil \frac{k}{2} \rceil + 5, \lceil \frac{k}{2} \rceil + 6, \ldots, k+1\right\}$, $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\lfloor \frac{k}{2} \rfloor - 2}v_{\lceil \frac{k}{2} \rceil + 4}v_{\lceil \frac{k}{2} \rceil + 3}v_{\lfloor \frac{k}{2} \rfloor - 2}$ and $P_k = v_{\lceil \frac{k}{2} \rceil + 2}v_{\lceil \frac{k}{2} \rceil + 1}v_{\lfloor \frac{k}{2} \rfloor + 1}v_{\lfloor \frac{k}{2} \rfloor}v_{\lfloor \frac{k}{2} \rfloor - 1}v_jv_{j-1} \cdots v_{\lceil \frac{k}{2} \rceil + 5}v_{\lfloor \frac{k}{2} \rfloor - 3} v_{\lfloor \frac{k}{2} \rfloor - 4} \cdots v_{k+2-j}v_{j+1}v_{k+1-j} \cdots v_{k+1}v_1v_{k+2}v$ (particularly $P_k = v_{\lceil \frac{k}{2} \rceil + 2}v_{\lceil \frac{k}{2} \rceil + 1}v_{\lfloor \frac{k}{2} \rfloor + 1}v_{\lfloor \frac{k}{2} \rfloor}v_{\lfloor \frac{k}{2} \rfloor - 1}v_{k+1} v_k \cdots v_{\lceil \frac{k}{2} \rceil + 5}v_{\lfloor \frac{k}{2} \rfloor - 3}v_{\lfloor \frac{k}{2} \rfloor - 4} \cdots v_1v_{k+2}v$ if $j = k+1$).

Subcase B3.4. $e = v_{\lfloor \frac{k}{2} \rfloor}v_j$.

If $j = \lceil \frac{k}{2} \rceil + 1$, $e = v_{\lfloor \frac{k}{2} \rfloor}v_{\lceil \frac{k}{2} \rceil + 1}$. $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\lfloor \frac{k}{2} \rfloor}v_{\lceil \frac{k}{2} \rceil + 1}v_{\lfloor \frac{k}{2} \rfloor + 1}v_{\lfloor \frac{k}{2} \rfloor}$ and $P_k = vv_{k+2}v_1v_{k+1} \cdots v_{\lfloor \frac{k}{2} \rfloor - 1}v_{\lceil \frac{k}{2} \rceil + 3}v_{\lceil \frac{k}{2} \rceil + 2}$.

If $j \in \left\{\lceil \frac{k}{2} \rceil + 4, \lceil \frac{k}{2} \rceil + 5, \ldots, k+1\right\}$, $G^k_{\text{odd}} + e$ contains a copy of $P_k \cup K_3$ as a subgraph with $K_3 = v_{\lfloor \frac{k}{2} \rfloor - 1}v_{\lceil \frac{k}{2} \rceil + 3}v_{\lceil \frac{k}{2} \rceil + 2}v_{\lfloor \frac{k}{2} \rfloor - 1}$ and $P_k = v_{\lceil \frac{k}{2} \rceil + 1}v_{\lfloor \frac{k}{2} \rfloor + 1}v_{\lfloor \frac{k}{2} \rfloor}v_jv_{j-1} \cdots v_{\lceil \frac{k}{2} \rceil + 4}v_{\lfloor \frac{k}{2} \rfloor - 2}v_{\lfloor \frac{k}{2} \rfloor - 3} \cdots$

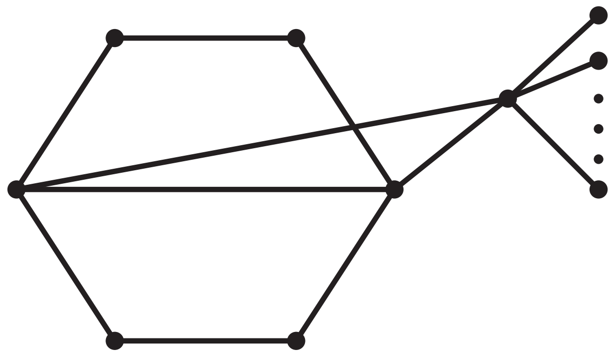

FIGURE 2. Graph $\widetilde{G}$.

$v_{k+2-j}v_{j+1}v_{k+1-j}\cdots v_{k+1}v_1v_{k+2}v$ (particularly $P_k = v_{\lceil\frac{k}{2}\rceil+1}v_{\lfloor\frac{k}{2}\rfloor+1}v_{\lfloor\frac{k}{2}\rfloor}v_{k+1}v_k\cdots v_{\lceil\frac{k}{2}\rceil+4}v_{\lfloor\frac{k}{2}\rfloor-2}$ $v_{\lfloor\frac{k}{2}\rfloor-3}\cdots v_1v_{k+2}v$ if $j=k+1$).

Subcase B3.5. $e = v_{\lfloor\frac{k}{2}\rfloor+1}v_j$.

If $j = \lceil\frac{k}{2}\rceil+2$, $e = v_{\lfloor\frac{k}{2}\rfloor+1}v_{\lceil\frac{k}{2}\rceil+2}$. $G^k_{\text{odd}}+e$ contains a copy of $P_k\cup K_3$ as a subgraph with $K_3 = v_{\lfloor\frac{k}{2}\rfloor+1}v_{\lceil\frac{k}{2}\rceil+2}v_{\lceil\frac{k}{2}\rceil+1}v_{\lfloor\frac{k}{2}\rfloor+1}$ and $P_k = vv_{k+2}v_1v_{k+1}\cdots v_{\lfloor\frac{k}{2}\rfloor-1}v_{\lceil\frac{k}{2}\rceil+3}v_{\lfloor\frac{k}{2}\rfloor}$.

If $j \in \{\lceil\frac{k}{2}\rceil+3, \lceil\frac{k}{2}\rceil+4,\ldots,k+1\}$, $G^k_{\text{odd}}+e$ contains a copy of $P_k\cup K_3$ as a subgraph with $K_3 = v_{\lceil\frac{k}{2}\rceil+2}v_{\lfloor\frac{k}{2}\rfloor}v_{\lfloor\frac{k}{2}\rfloor-1}v_{\lceil\frac{k}{2}\rceil+2}$ and $P_k = v_{\lceil\frac{k}{2}\rceil+1}v_{\lfloor\frac{k}{2}\rfloor+1}v_jv_{j-1}\cdots v_{\lceil\frac{k}{2}\rceil+3}v_{\lfloor\frac{k}{2}\rfloor-2}v_{\lfloor\frac{k}{2}\rfloor-3}\cdots v_{k+2-j}$ $v_{j+1}v_{k+1-j}\cdots v_{k+1}v_1v_{k+2}v$ (particularly $P_k = v_{\lceil\frac{k}{2}\rceil+1}v_{\lfloor\frac{k}{2}\rfloor+1}v_{k+1}v_k\cdots v_{\lceil\frac{k}{2}\rceil+3}v_{\lfloor\frac{k}{2}\rfloor-2}v_{\lfloor\frac{k}{2}\rfloor-3}\cdots v_1$ $v_{k+2}v$ if $j=k+1$).

Note that $m(G^k_{\text{even}}) = n+3\big(\frac{k}{2}-1\big)$ and $m(G^k_{\text{odd}}) = n+3\big(\lfloor\frac{k}{2}\rfloor-1\big)+1$ from the respective structures of G^k_{even} and G^k_{odd}. Then we complete the proof of Theorem 2.1. □

Recalling that $\text{sat}(n,P_k\cup K_3) = \min\{\text{sat}'(n,P_k\cup K_3),\text{sat}''(n,P_k\cup K_3)\}$, we arrive at the following result.

Corollary 2.1. *Let $k\geq 4$ be an integer. Then*

$$\text{sat}(n,P_k\cup K_3) \leq \begin{cases} n+3\big(\frac{k}{2}-1\big), & \text{if } k \text{ is even},\\ n+3\big(\lfloor\frac{k}{2}\rfloor-1\big)+1, & \text{if } k \text{ is odd}.\end{cases}$$

Note that the statement of Theorem 2.1 cannot be extended to the case $k\geq 3$ as explained in the following.

Remark 2.1. $k\geq 4$ is necessary in the statement of Theorem 2.1. It can be routinely checked that G^k_{odd} as constructed in Figure 2 is not a $(P_k\cup K_3)$-saturated connected graph for $k=3$. For $k=3$, a $(P_k\cup K_3)$-saturated connected graph $\widetilde{G}$ of order n is constructed with $n+2$ edges in Figure 2.

3. Lower bound on $\text{sat}'(n,P_k\cup K_3)$

In this section we focus on determining the lower bound on $\text{sat}'(n,P_k\cup K_3)$. Before doing it, we need a preliminary result. A graph G is *k-self-centered* if $\text{ecc}_G(v)=k$ for any vertex $v\in V(G)$. For a graph G, we denote by $c(G)$ the cyclomatic number of G, that is, $c(G)=|E(G)|-|V(G)|+\omega(G)$ where $\omega(G)$ is the number of connected components of G. In particular, $c(G)=|E(G)|-|V(G)|+1$ if G is connected. For a connected graph G with $c(G)=2$, there are at least two cycles in G, and G contains one of the following graphs:

– A glasses-type graph, which is the union of two vertex-disjoint cycles;

– An ∞-type graph $C_{p,q}$, which consists of two cycles C_p and C_q sharing one common vertex, being this vertex called the *center* of $C_{p,q}$;
– A Θ-type graph Θ_{k_1,k_2,k_3} with $2 \le k_1 \le k_2 \le k_3$, which consists of three cycles formed by three internally-disjoint paths of lengths k_1, k_2 and k_3 with two common terminals.

Lemma 3.1 ([4]). *If G is a 2-self-centered graph with n vertices and m edges, then $m \ge 2n-5$.*

Lemma 3.2. *Let G be a minimum $(P_k \cup K_3)$-saturated (not necessarily connected) graph on $n > \frac{3k+4}{2}$ vertices. Then there is a triangle in G for $k \ge 4$.*

Proof. Firstly, we show that there is a cycle in G for $k \ge 2$. Otherwise, we suppose that there is no cycle in G, which means that G is a forest. First we claim that G is a tree. Otherwise, G contains at least two distinct trees, say T_1 and T_2. Now $G + v_1v_2$ contains no K_3 as a subgraph where $v_1 \in V(T_1)$ and $v_2 \in V(T_2)$, which is a contradiction. For any edge $e = xy$ missing from G, $G + xy$ contains $P_k \cup K_3$. Then xy must belong to K_3 in $G + xy$ since G is a tree. If $d_G(x, y) \ge 3$, then there is no $P_k \cup K_3$ in $G + xy$ since the unique cycle in it has length at least 4. This is a contradiction. Thus $d_G(x, y) = 2$ for any two non-adjacent vertices x and y in G, which implies that $G \cong S_n$. But $G \cong S_n$ is not $(P_k \cup K_3)$-saturated as a clear contradiction again. Therefore, G must contain a cycle.

We next prove that there is a triangle in G for $k \ge 4$. If not, we have $|C| \ge 4$ for every cycle C in G. Furthermore, $e \in K_3$ in $G + e$ for any edge e missing from G. Then any two non-adjacent vertices in G have at least one common neighbor with $\Delta(G) < n-1$ (if $\Delta(G) = n-1$, then G contains a triangle, contradicting the above assumption). Therefore G is 2-self-centered. Then $e(G) \ge 2n-5$ by Lemma 3.1. However, $e(G) \ge 2n-5 > n + \frac{3k-6}{2}$ for $n > \frac{3k+4}{2}$, which contradicts Theorem 2.1. This completes the proof. ☐

We give a lower bound on the saturation number $\mathrm{sat}'(n, P_k \cup K_3)$ for $k \ge 4$ in the following result.

Theorem 3.1. *If $n > \frac{3k+4}{2}$ with $k \ge 4$, then* $\mathrm{sat}'(n, P_k \cup K_3) \ge n+2$.

Proof. Let G be a minimum $(P_k \cup K_3)$-saturated connected graph on n vertices. It follows from Lemma 3.2 that $c(G) \ge 1$. Next we divide into the following two cases to show that $c(G) \ge 3$.

Case 1. $c(G) = 1$.

In this case there is exactly one cycle C in G. By Lemma 3.2, We have $|C| = 3$ with $C = v_1v_2v_3v_1$. Now we claim that any vertex $y \in V(G)\backslash V(C)$ is adjacent to exactly one vertex in C. Clearly, y has at most one neighbor in $V(C)$. If there is a vertex $x \in V(G)\backslash V(C)$ such that x is not adjacent to any vertex in C, we assume, without loss of generality, that $\min\{d_G(x,u) : u \in V(C)\} = d_G(x, v_1)$. In the graph $G + xv_2$, we have $xv_2 \in K_3$. Otherwise, there is no $P_k \cup K_3$ in $G + xv_2$, contradicting the fact that G is $(P_k \cup K_3)$-saturated. Then there exists a vertex $x_1 \in V(G) \setminus V(C)$ such that $K_3 = xx_1v_2x$. Thus $d_G(x, v_1) = 2$ since $d_G(x, v_1) \le d_G(x, v_2) = 2$ and x is not adjacent to v_1. This creates a second cycle in G as a contradiction to $c(G) = 1$. Therefore, G is a graph of order n obtained by attaching pendant vertices to some or all vertices in C. But G turns out to be not $(P_k \cup K_3)$-saturated from a routine check as a clear contradiction again.

Case 2. $c(G) = 2$.

In this case, by Lemma 3.2, there are at least two cycles C^1 and C^2 in G with $C^1 = v_1v_2v_3v_1$. Now we claim that $V(C^1) \cap V(C^2) \ne \varnothing$. If not, then $V(C^1) \cap V(C^2) = \varnothing$ and set $C^2 = v_1'v_2'v_3' \cdots v_1'$. Thus there are exactly two cycles C^1 and C^2 in G. Assume that $d_G(v_1, v_1') = \min\{d_G(x,y) : x \in V(C^1), y \in V(C^2)\}$. Now we have $v_1'v_2 \in K_3$ in $G + v_1'v_2$. Similarly as above, there is a vertex $u \in V(G) \setminus (V(C^1) \cup V(C^2))$ such that $K_3 = uv_2v_1'u$ and $d_G(v_1, v_1') = 2$, creating a third cycle in G, a contradiction. Thus $V(C^1) \cap V(C^2) \ne \varnothing$.

If $|V(C^1) \cap V(C^2)| = 1$, then there are exactly two cycles C^1 and C^2 in G. Without loss of generality, we assume that $\{v_1\} = V(C^1) \cap V(C^2)$. If $|C^2| \ge 4$ with $C^2 = v_1v_2'v_3'v_4' \cdots v_1$, we consider $G + v_2v_3'$, then $v_2v_3' \in K_3$. Otherwise $v_2v_3' \in P_k$ and there is no $P_k \cup K_3$ in $G + v_2v_3'$, contradicting the fact that

G is $(P_k \cup K_3)$-saturated. Thus there is a $y \in V(G)\backslash\big(V(C^1) \cup V(C^2)\big)$ such that $K_3 = v_2v_3'yv_2$. However, a third cycle occurs in G as a contradiction. Therefore $|C^2| = 3$ with $C^2 = v_1v_2'v_3'v_1$. Any vertex $y \in V(G)\backslash(V(C^1) \cup V(C^2))$ has at most one neighbor in $V(C^1) \cup V(C^2)$ since there are exactly two cycles in G. If there exists a y with no neighbors in $V(C^1) \cup V(C^2)$, we can consider $G + v_1y$, then $v_1y \in K_3$ and there is a $y_1 \in V(G)\backslash(V(C^1) \cup V(C^2))$ such that $K_3 = v_1yy_1v_1$. We claim that $N_G(y) = \{y_1\}$. If not, we assume $y_2 \in N_G(y)\backslash\{y_1\}$ and consider $G + v_1y_2$, then $v_1y_2 \in K_3$ and there is a $y_3 \in V(G)\backslash\big(V(C^1) \cup V(C^2)\big)$ such that $K_3 = v_1y_2y_3v_1$ (possibly $y_1 = y_3$ holds). However, this creates a third cycle in G, a contradiction. Therefore any vertex $y \in V(G)\backslash(V(C^1) \cup V(C^2))$ either has exactly one neighbor in $V(C^1) \cup V(C^2)$ or lies on 2-length pendant path from $v_1 \in V(C^1) \cap V(C^2)$ with $d_G(y, v_1) = 2$. By considering the non-edge v_2v_2' or v_3v_3', we can verify that G with this structure is not a $(P_k \cup K_3)$-saturated graph as a contradiction.

Next it suffices to consider the subcase when G contains a Θ-type graph. In this subcase we consider two cycles C^1 and C^2 in G of less lengths with $|C^1| \le |C^2|$. Then $|V(C^1) \cap V(C^2)| = 2$. By Lemma 3.2 we assume that $\{v_1, v_2\} = V(C^1) \cap V(C^2)$ with $C^1 = v_1v_2v_3v_1$ and $C^2 = v_1v_2v_3' \cdots v_1$. If $|C^2| \ge 5$ with $C^2 = v_1v_2v_3'v_4'v_5' \cdots v_1$, we can consider $G + v_3v_4'$. Then $v_3v_4' \in K_3$ and there is a vertex $y \in V(G)\backslash(V(C^1) \cup V(C^2))$ such that $K_3 = v_3v_4'yv_3$. This creates a new cycle in G as a contradiction. Then $3 \le |C^2| \le 4$. Denote by G_0 the induced subgraph of G by $V(C^1) \cup V(C^2)$. Now we claim that any vertex $y \in V(G)\backslash V(G_0)$ has exactly one neighbor in G_0. Clearly, y has at most one neighbor in G_0 since $c(G) = 2$. If there exists a y without neighbor in G_0, without loss of generality, we assume that $\min\{d_G(y, u) : u \in V(G_0)\} = d_G(y, x)$. Selecting $w \in \{v_1, v_2\} \setminus \{x\}$ and considering the graph $G + wy$, we have $wy \in K_3$. Then there exists another vertex $y_1 \in V(G) \setminus V(G_0)$ such that $K_3 = wyy_1w$ and $d_G(y, x) = 2$. Then a new cycle occurs in G as a clear contradiction. Therefore G is a graph obtained by attaching some pendant vertices to the vertices in G_0. It is routine to verify that G with this structure is not $(P_k \cup K_3)$-saturated as a contradiction.

Therefore we have $c(G) \ge 3$, which implies $\text{sat}'(n, P_k \cup K_3) \ge n + 2$. □

Combining Theorems 2.1 and 3.1, we can get the following result immediately.

Corollary 3.1. *Let G be a minimum $(P_k \cup K_3)$-saturated connected graph on $n > \frac{3k+4}{2}$ vertices with $k \ge 4$. Then $n + 2 \le m(G) \le n + \frac{3k-6}{2}$.*

Remark 3.1. Similar to the proof (paragraphs 2 and 3 of Case 2) of Theorem 3.1, it can be checked that G is not $(P_k \cup K_3)$-saturated for any graph G obtained by attaching some pendant vertices to vertices in $C_{3,3}$ or possibly identifying a leaf of one or more stars with its center, or obtained by attaching some pendant vertices to vertices in G_0 with $G_0 \in \{\Theta_{2,3,3}, \Theta_{2,3,4}\}$.

Next we construct the $(P_k \cup K_3)$-saturated disconnected graph G with specific order such that $m(G) < \text{sat}'(n, P_k \cup K_3)$, which means $\text{sat}''(n, P_k \cup K_3) < \text{sat}'(n, P_k \cup K_3)$ for some n. If $k \ge 4$, the structure of P_k-saturated tree can be described in [20]. Let T_k be a rooted (or double rooted) tree with $\lfloor \frac{k}{2} \rfloor$ levels in which every vertex has degree 3, except for the lowest level, and the highest level contains $k + 1 - 2\lfloor \frac{k}{2} \rfloor$ vertices (see Fig. 3). Then

$$|T_k| = \begin{cases} 3 \cdot 2^{m-1} - 2, & \text{if } k = 2m, \\ 2^{m+1} - 2, & \text{if } k = 2m + 1. \end{cases}$$

Theorem 3.2 ([20])**.** *If T is a P_k-saturated tree, then $T_k \subset T$.*

In addition, any graph obtained from T_k by adding more pendant vertices to the neighbors of leaves maintains the P_k-saturated property. We denote by $T_{k,r}$ the graph obtained from T_k by adding r pendant vertices to the neighbors of leaves.

Let $G = G^6_{\text{even}} \cup 4T_6 \cup T_{6,r}$ with $n(G^6_{\text{even}}) = 9$ and $0 \le r \le 9$. It is not difficult to verify that G is a $(P_6 \cup K_3)$-saturated disconnected graph of order $n = 59 + r$. Note that $m(G^6_{\text{even}}) = 15$. We can know that $m(G) = 15 + 4 \times 9 + 9 + r = 60 + r = n + 1$. Thus $\text{sat}''(n, P_6 \cup K_3) \le m(G) = n(G) + 1$, meaning $\text{sat}''(n, P_6 \cup K_3) < \text{sat}'(n, P_6 \cup K_3)$ for $n = 59 + r$ by Theorem 3.1. Thus we get the following result.

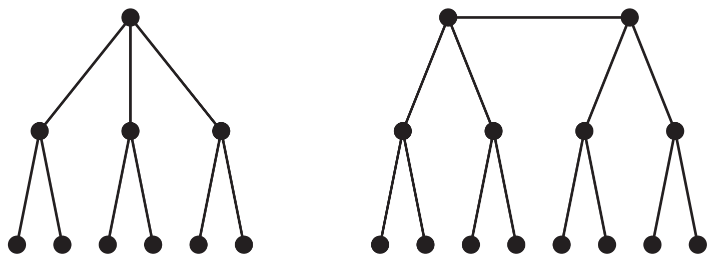

FIGURE 3. T_6 and T_7.

Remark 3.2. The result of Theorem 3.1 cannot be extended to $\mathrm{sat}(n, P_k \cup K_3)$.

4. $\mathrm{sat}'(n, P_k \cup K_3)$ WITH $k \in \{2, 3, 4\}$

In this section we will determine the exact values of $\mathrm{sat}'(n, P_k \cup K_3)$ when k is small. First we deal with the case when $k = 2$.

Theorem 4.1. $\mathrm{sat}'(n, P_2 \cup K_3) = n$ *with* $n \geq 6$.

Proof. Let G be a graph of order n obtained by attaching at least one pendant vertex to each vertex of a triangle. We can easily verify that G is a $(P_2 \cup K_3)$-saturated connected graph, implying that $\mathrm{sat}'(n, P_2 \cup K_3) \leq n$. On the other hand, we have $\mathrm{sat}'(n, P_2 \cup K_3) \geq \mathrm{sat}(n, P_2 \cup K_3) \geq n$ by the proof of Lemma 3.2. Our result follows immediately. □

Note that $\mathrm{sat}(n, P_2 \cup K_3) = \min\{\mathrm{sat}'(n, P_2 \cup K_3), \mathrm{sat}''(n, P_2 \cup K_3)\}$ and $\mathrm{sat}(n, P_2 \cup K_3) \geq n$. Then we have the following result.

Corollary 4.1. $\mathrm{sat}(n, P_2 \cup K_3) = n$ *with* $n \geq 6$.

Next we turn to determine the value of $\mathrm{sat}'(n, P_3 \cup K_3)$.

Theorem 4.2. $\mathrm{sat}'(n, P_3 \cup K_3) = n + 2$.

Proof. Let G be a minimum $(P_3 \cup K_3)$-saturated connected graph on n vertices. By Remark 2.1, we have $m(G) \leq n + 2$. Then it suffices to show $c(G) \geq 3$. From the proof of Lemma 3.2, we have $c(G) \geq 1$. Next we divide into the following two cases.

Case 1. $c(G) = 1$.

Now there is exactly one cycle C in G with $C = v_1 v_2 v_3 \cdots v_1$. We claim that any vertex $y \in V(G) \backslash V(C)$ has exactly one neighbor in $V(C)$. Clearly, y has at most one neighbor in C since $c(G) = 1$. If there is a vertex y without neighbor in C, we assume that $\min\{d_G(y, u) : u \in V(C)\} = d_G(y, v_1)$. Consider the graph $G + v_2 y$, we have $v_2 y \in K_3$ since G is $(P_3 \cup K_3)$-saturated. Then there is a vertex $u \in V(G) \setminus V(C)$ such that $K_3 = v_2 y u v_2$ in $G + v_2 y$ and $d_G(y, v_1) = 2$, which creates a second cycle in G and contradicts the fact that $c(G) = 1$. Therefore G is a graph obtained by attaching one or more pendant vertices to some vertices of $V(C)$. But now G is not $(P_3 \cup K_3)$-saturated, which can be verified similarly as that in the proof (Case 1) of Theorem 3.1 for $|C| = 3$ and can be done by adding a non-edge yx into G for $|C| \geq 4$ where $y \in V(G) \setminus V(C)$ and x lies on C with $d_G(x, y) = 3$. This is a contradiction.

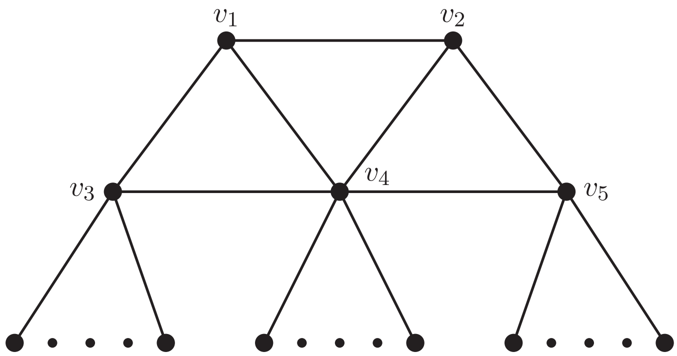

FIGURE 4. Graph $\widehat{G}$.

Case 2. $c(G) = 2$.

In this case we consider two cycles C^1 and C^2 in G of less lengths. By a similar reasoning as that in the proof (paragraph 1 of Case 2) of Theorem 3.1, we have $V(C^1) \cap V(C^2) \neq \varnothing$. Now we divide into three cases base on the value of $\min\{|C^1|, |C^2|\}$. If $\min\{|C^1|, |C^2|\} \geq 5$, there must exist two vertices $x \in V(C^1)$ and $y \in V(C^2)$ such that $d_G(x, y) \geq 3$. In the graph $G + xy$, we have $xy \in K_3 = xyzx$ where $z \in V(G) \setminus (V(C^1) \cup V(C^2))$. This creates a new cycle, contradicting the fact $c(G) = 2$. If $\min\{|C^1|, |C^2|\} = 3$ with $C^1 = v_1v_2v_3v_1$, by a similar reasoning to the proof (paragraphs 2 and 3 of Case 2) of Theorem 3.1 and Remark 3.1, we deduce that G is also not $(P_3 \cup K_3)$-saturated as a contradiction.

If $\min\{|C^1|, |C^2|\} = 4$ with $C^1 = v_1v_2v_3v_4v_1$, we claim $|V(C^1) \cap V(C^2)| = 3$. Otherwise, there must exist two vertices $x \in V(C^1)$ and $y \in V(C^2)$ such that $d_G(x, y) \geq 3$. In the graph $G+xy$, we have $xy \in K_3 = xyzx$ where $z \in V(G) \setminus (V(C^1) \cup V(C^2))$. This creates a new cycle, contradicting the fact $c(G) = 2$ again. Thus $|V(C^1) \cap V(C^2)| = 3$ with $\{v_1, v_2, v_3\} = V(C^1) \cap V(C^2)$. If $C^2 \geq 6$ with $C^2 = v_1v_2v_3v_4'v_5'v_6' \cdots v_1$, we have $v_2v_5' \in K_3$ in $G + v_2v_5'$. This creates a new cycle in G, a contradiction. Therefore $4 = |C^1| \leq |C^2| \leq 5$ with $\{v_1, v_2, v_3\} = V(C^1) \cap V(C^2)$. By a similar reasoning as that in the proof (paragraph 3 of Case 2) of Theorem 3.1, G is a graph obtained by attaching some pendant vertices to the vertices in $G[V(C^1) \cup V(C^2)]$. It is routine to verify that G with this structure is not $(P_3 \cup K_3)$-saturated as a contradiction.

It follows from the above two cases that $c(G) \geq 3$. Our result follows immediately. □

In the following result we determine $\text{sat}'(n, P_4 \cup K_3)$. Let $\widehat{G}$ be the graph on $n > 8$ vertices as shown in Figure 4. A, B and C are the vertex sets of at least one pendant vertex attaching to v_3, v_4 and v_5, respectively. Let $X = \{v_1, v_2, v_3, v_4, v_5\}$ with $Y = V(G) \setminus X$. Then $Y = A \cup B \cup C$.

Theorem 4.3. *$\text{sat}'(n, P_4 \cup K_3) = n + 2$ with $n > 8$.*

Proof. By Theorem 3.1, we have $\text{sat}'(n, P_4 \cup K_3) \geq n+2$. Now we show that $\widehat{G}$ is a $(P_4 \cup K_3)$-saturated connected graph with $m(\widehat{G}) = n + 2$. We can easily check that $\widehat{G}$ is $(P_4 \cup K_3)$-free. It suffices to prove that there exists a copy of $P_4 \cup K_3$ as a subgraph of $\widehat{G} + e$ by adding any non-edge e. We divide into three cases based on the position of non-edge e.

Case 1. $e = v_iv_j$ with $v_i, v_j \in X$.

Note that there are three non-edges v_1v_5, v_2v_3 and v_3v_5 in $\widehat{G}[X]$. From symmetry, we only need to consider the non-edges v_2v_3 and v_3v_5. For any $e \in \{v_2v_3, v_3v_5\}$, it is routine to verify that $\widehat{G} + e$ contains a copy of $P_4 \cup K_3$ as a subgraph by selecting $v_2v_3 \in K_3 = v_2v_3v_1v_2$ and $v_3v_5 \in P_4 = av_3v_5b$ with any $a \in A$, any $b \in B$ in $\widehat{G} + v_2v_3$ and $\widehat{G} + v_3v_5$, respectively.

Case 2. $e = uv$ with $u, v \in Y$.

If u and v are adjacent to a same vertex v_i for $i \in \{3, 4, 5\}$, it is clear that $\widehat{G} + e$ contains a copy of $P_4 \cup K_3$ as a subgraph with $uv \in K_3 = uvv_iu$ in $\widehat{G} + uv$. If u and v are from two distinct vertex sets of A, B and C, it is routine to verify that $\widehat{G} + e$ contains a copy of $P_4 \cup K_3$ as a subgraph with $uv \in P_4 = v_1v_3uv$, $uv \in P_4 = v_3uvv_5$ and $uv \in P_4 = v_2v_5vu$ in the graphs $\widehat{G} + uv$ with $uv \in E(A, B)$, $uv \in E(A, C)$ and $uv \in E(B, C)$, respectively.

Case 3. $e \in E(X, Y)$.

For any $a \in A$, any $b \in B$ and any $c \in C$, if $e \in E(X, A)$, it is clear that $\widehat{G} + e$ contains a copy of $P_4 \cup K_3$ as a subgraph with $av_i \in K_3 = av_iv_3a$, $av_2 \in P_4 = av_2v_5c$ and $av_5 \in P_4 = v_3av_5c$ in the graphs $\widehat{G} + av_i$ with $i \in \{1, 4\}$, $\widehat{G} + av_2$ and $\widehat{G} + av_5$, respectively. If $e \in E(X, B)$, it is clear that $\widehat{G} + e$ contains a copy of $P_4 \cup K_3$ as a subgraph with $bv_i \in K_3 = bv_iv_4b$, $bv_1 \in P_4 = bv_1v_3a$ and $bv_2 \in P_4 = bv_2v_5c$ in the graphs $\widehat{G} + bv_i$ with $i \in \{3, 5\}$, $\widehat{G} + bv_1$ and $\widehat{G} + bv_2$, respectively. If $e \in E(X, C)$, it is clear that $\widehat{G} + e$ contains a copy of $P_4 \cup K_3$ as a subgraph with $cv_i \in K_3 = cv_iv_5c$, $cv_1 \in P_4 = cv_1v_3a$ and $cv_3 \in P_4 = v_5cv_3a$ in the graphs $\widehat{G} + cv_i$ with $i \in \{2, 4\}$, $\widehat{G} + cv_1$ and $\widehat{G} + cv_3$, respectively.

Therefore $\widehat{G}$ is a $(P_4 \cup K_3)$-saturated connected graph with $m(\widehat{G}) = n+2$, implying that $\mathrm{sat}'(n, P_4 \cup K_3) \leq n+2$. Our result follows immediately by Theorem 3.1. □

5. Concluding remarks

In this paper we obtain the upper and lower bounds on $\mathrm{sat}'(n, P_k \cup K_3)$ for $n > \frac{3k+4}{2}$ with $k \geq 4$. Also the exact values of $\mathrm{sat}'(n, P_k \cup K_3)$ are determined for $k \in \{2, 3, 4\}$. Note that $\mathrm{sat}'(n, P_4 \cup K_3) = n + 2$ attains the lower bound in Theorem 3.1, but does not attain the upper bound in Theorem 2.1. Naturally we pose the following problem.

Problem 5.1. Does there exist some $k \geq 5$ such that the lower bound in Theorem 3.1 is sharp for $\mathrm{sat}'(n, P_k \cup K_3)$, that is, $\mathrm{sat}'(n, P_k \cup K_3) = n + 2$, with $n > \frac{3k+4}{2}$? Is there some $k \geq 5$ such that the upper bound in Theorem 2.1 is attained for $\mathrm{sat}'(n, P_k \cup K_3)$?

Recall that a $(P_k \cup K_3)$-saturated (connected) graph G with minimum number of edges is a minimum $(P_k \cup K_3)$-saturated (connected) graph, that is to say, $m(G) = \mathrm{sat}(n, P_k \cup K_3)$ $(m(G) = \mathrm{sat}'(n, P_k \cup K_3))$ with $n(G) = n$. The minimum $(P_k \cup K_3)$-saturated (connected) graph can be viewed as a realization graph for $\mathrm{sat}(n, P_k \cup K_3)$ $(\mathrm{sat}'(n, P_k \cup K_3))$. In $\widehat{G}$ of Figure 4, we have $|A| + |B| + |C| = n - 5 > 3$. Therefore the realization graph for $\mathrm{sat}'(n, P_4 \cup K_3)$ is not unique. Moreover, the realization graph for $\mathrm{sat}(n, P_2 \cup K_3)$ is also not unique. This results in the following problem.

Problem 5.2. How can we determine the number of realization graphs for $\mathrm{sat}(n, P_k \cup K_3)$ and $\mathrm{sat}'(n, P_k \cup K_3)$ for given k? In particular, can we determine some k such that the realization graph for $\mathrm{sat}(n, P_k \cup K_3)$ or $\mathrm{sat}'(n, P_k \cup K_3)$ is unique?

Considering the results in Theorem 4.1 and Corollary 4.1, we propose the following general problem.

Problem 5.3. Under what conditions, do we have $\mathrm{sat}(n, H) = \mathrm{sat}'(n, H)$ $(\mathrm{sat}(n, H) = \mathrm{sat}''(n, H)$, resp.)?

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