RAIRO-Oper. Res. 57 (2023) 927–938
https://doi.org/10.1051/ro/2023041

RAIRO Operations Research
www.rairo-ro.org

ON CONVERGENCE OF EXPONENTIAL PENALTY FOR THE MULTI-DIMENSIONAL VARIATIONAL PROBLEMS

Anurag Jayswal and Ayushi Baranwal*

Abstract. In this article, we describe a method to deal with a multi-dimensional variational problem with inequality constraints using an exponential penalty function. We formulate an unconstrained multi-dimensional variational problem and examine the relationships between the optimal solution to the considered multi-dimensional variational problem and the sequence of minimizers of the unconstrained multi-dimensional variational problem. The convergence of the proposed exponential penalty approach is also investigated, which shows that a convergent subsequence of the sequence of minimizers of the unconstrained multi-dimensional variational problem approaches an optimal solution to the multi-dimensional variational problem. Further, an illustrative application (to minimize a manufacturing cost functional of a production firm) is also presented to confirm the effectiveness of the proposed outcomes.

Mathematics Subject Classification. 28A20, 58E15, 90C30.

Received January 30, 2023. Accepted March 28, 2023.

1. Introduction

Variational problems arising from the calculus of variations have received much attention in various fields, including science, engineering, pure and applied mathematics, decision-making, economics, etc. and are defined as the task of determining a function that minimizes a given functional. The mathematical association between the optimization problems and the calculus of variation was first investigated and developed by Hanson [10]. Afterward, several researchers have made their contributions to this field of study. Mond and Husain [19] have given sufficient optimality and duality results for the variational problem under generalized invexity. Jiang [17] proved the existence and uniqueness of the solution to a variational problem with inequality constraints by applying the saddle-point theory. Jiménez *et al.* [4] studied optimality conditions and duality results for a class of variational problems under generalized invexity. Recently, Jayswal *et al.* [15] have investigated an equivalence relation between the solution of the variational problem and its associated modified variational problem using the notion of convexity. For more insights readers are referred to see [3, 12, 22, 23] and references therein.

The theoretical and practical views of variational problems encouraged authors to investigate this field intensively. In 2012, studied the multi-dimensional variational problem with curvilinear integral functional and presented the Mond–Weir type duality results under some generalized convexity. Treanţă [25] investigated the

Keywords. Convergence, exponential penalty function method, optimal solution, multi-dimensional variational problem.

Department of Mathematics and Computing, Indian Institute of Technology (Indian School of Mines), Dhanbad 826004, India.
*Corresponding author ayushibaranwal06@gmail.com; ayushibaranwal.maths@gmail.com

well-posedness of variational problems with inequality constraints by defining the set of approximating solutions. Later on, Jayswal *et al.* [11] established the equivalence relatiuon between an optimal solution to a class of multi-dimensional variational problem and its associated variational inequality under the hypotheses of convexity. For more insights in this field, readers are referred to see [6, 7, 21] and references therein.

On the other hand, the penalty function method is a systematic technique for solving constrained optimization problems. In this method, the constrained problem is transformed into a sequence of unconstrained problems using a penalty function. If the sequence of solutions to unconstrained problems converges towards the solution of the constrained optimization problem, then the technique is said to be effective. Many authors have explored this method to solve different types of constrained optimization problems; see, for example, [1, 14, 16, 24] and references therein.

The exponential penalty function method is one of an efficient approach to solve the constrained optimization problem and in this method no need to restrict the involved functions to have convexity or some generalized convexity properties. Murphy [20] first introduced to a class of exponential penalty functions to solve constrained optimization problems. In 2011, Antczak [2] solved nonconvex mathematical programming problems using the l_1 exact exponential penalty function method. Later on, Mandal *et al.* [18] transferred a constrained variational problem into an unconstrained one by using the exponential penalty function and presented optimality conditions and duality results. Recently, Jayswal and Choudhury [13] utilize the exponential penalty function method to address a constrained fractional problem and investigate the convergence of the utilized method.

Motivated by the works mentioned earlier, in this paper we consider a multi-dimensional variational problem with inequality constraints and obtain the optimal solution of the problem using the concept of exponential penalty function method. The method presented in this paper does not require the convexity or differentiability of the involved functionals to establish an association between minimizers and optimal solution of the unconstrained and constrained problem, respectively. The management of this paper is as follows: Section 2 consists of some notations and preliminaries which are helpful in the presentation of the main results. In Section 3, we formulate an unconstrained multi-dimensional variational problem $(\text{MVP})_k$ with the help of exponential penalty function associated with the variational problem with inequality constraints (MVP). In Section 4, we show that the sequence of minimizer of problem $(\text{MVP})_k$ converges towards an optimal solution to the problem (MVP). Furthermore, a suitable example has been fabricated to validate the main results. Finally, Section 5 concludes the paper.

2. Notations and preliminaries

This section contains some notations that will assist in framing the problem and presenting the significant results.

- Consider the finite dimensional Euclidean spaces R^m and R^n. Let $t = (t^\gamma),\ \gamma = \overline{1, m}$ and $a = (a^i),\ i = \overline{1, n}$ be the local coordinates of R^m and R^n respectively.
- Let $\Omega = \Omega_{t_0,t_1} \subset R^m$ be a hyperparallelepiped, fixed by the diagonally opposite points $t_0 = (t_0^\gamma)$ and $t_1 = (t_1^\gamma),\ \gamma = \overline{1, m}$ in R^m, as well as $\mathrm{d}\omega = \mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^m$ denotes the volume element in $R^m \supset \Omega$.
- Let $a(t)$ be the bounded uniformly equicontinuous piecewise smooth state functions from Ω to R^n and X be the collection of all such piecewise smooth state functions $a(t)$ endowed with norm

$$\|a(t)\| = \|a(t)\|_\infty + \sum_{\gamma=1}^{m} \|a_\gamma(t)\|_\infty,$$

  where $\|a(t)\|_\infty = \max(|a^1(t)|,\ |a^2(t)|, \ldots,\ |a^n(t)|)$ is a uniform norm and $a_\gamma(t) = \frac{\partial a(t)}{\partial t^\gamma}$.

Definition 2.1 ([9]). A family of functions is said to be uniformly equicontinuous if all the functions are continuous and they have equal variation over a given interval. Mathematically, the family of piecewise smooth (that implies continuity) functions $a(t) : \Omega \mapsto X$ is said to be uniformly equicontinuous, if for each $\epsilon > 0,\ \epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ there exists a $\delta > 0,\ \delta = (\delta_1, \delta_2, \ldots, \delta_m)$, such that $||a(t_1) - a(t_2)|| < \epsilon$ where $|t_1 - t_2| < \delta$.

– For any two points, $x = (x^\ell)$ and $y = (y^\ell)$, $\ell = \overline{1,l}$ in R^l, the following convention will be used through paper

$$x = y \Leftrightarrow x^\ell = y^\ell,\ x \le y \Leftrightarrow x^\ell \le y^\ell,\ x < y \Leftrightarrow x^\ell < y^\ell,\quad \ell = \overline{1,l}.$$

– Let A and B are two subsets in R^ℓ. Then $x \in A \setminus B$ describes that $x \in A$ but $x \notin B$.
– Let $A_k \subset R^\ell,\ k \in N = \{1, 2, ...\}$. Then we denote

$$\overline{\lim}_{k\to\infty} A_k = \left\{x \in R^\ell : x \in A_k \text{ for infinitely many } k \in N\right\};$$
$$\underline{\lim}_{k\to\infty} A_k = \left\{x \in R^\ell : x \in A_k \text{ for all but finitely many } k \in N\right\};$$

Therefore, $\underline{\lim}_{k\to\infty} A_k \subseteq \overline{\lim}_{k\to\infty} A_k$. And, if $\overline{\lim}_{k\to\infty} A_k \subseteq \underline{\lim}_{k\to\infty} A_k$, then

$$\lim_{k\to\infty} A_k = \underline{\lim}_{k\to\infty} A_k = \overline{\lim}_{k\to\infty} A_k.$$

Considering the above mathematical tools, we formulate the following multi-dimensional variational problem with inequality constraints (MVP) as:

$$\text{(MVP)} \qquad \min \int_\Omega f(t, a(t), a_\gamma(t))\, \mathrm{d}\omega$$

subject to

$$g_\beta(t, a(t), a_\gamma(t)) \le 0, \quad \beta = \overline{1,q}, \tag{1}$$
$$a(t_0) = a_0,\ a(t_1) = a_1, \tag{2}$$

where $t \in \Omega,\ f : \Omega \times X \times X \mapsto R,\ g_\beta : \Omega \times X \times X \mapsto R,\ \beta = \overline{1,q},$ are continuously differentiable functionals.

Remark 2.1. (i) If we take the variable t of single dimension in the problem (MVP), then it reduces to the problems (VP) and (CVP) given Jimenez *et al.* [4,5], respectively.
(ii) If we take the variable t of single dimension and assume the interval-valued objective functional in the problem (MVP), then it becomes the problem (IVP) given by Debnath and Pokharna [8].

Let $D = \{a(t) \in X \mid g_\beta(t, a(t), a_\gamma(t)) \le 0,\ a(t_0) = a_0,\ a(t_1) = a_1,\ t \in \Omega,\ \beta = \overline{1,q}\}$, be the set of all feasible solutions to (MVP).

Definition 2.2. A point $\hat{a}(t) \in D$ is said to be an optimal solution to the multi-dimensional variational problem with inequality constraints (MVP), if

$$\int_\Omega f(t, a(t), a_\gamma(t))\, \mathrm{d}\omega \ge \int_\Omega f(t, \hat{a}(t), \hat{a}_\gamma(t))\, \mathrm{d}\omega, \quad \forall\ a(t) \in D.$$

Let all across the article $\hat{D}$ indicate the set of all optimal solutions to (MVP).

3. Exponential penalty function method

Now, we introduce the unconstrained multi-dimensional variational problem $(\text{MVP})_k$ with exponential penalty function $P(a(t), \sigma_k)$ associated to the above mentioned variational problem (MVP) as follows:

$$(\text{MVP})_k \quad \min P(a(t), \sigma_k) = \int_\Omega f(t, a(t), a_\gamma(t))\, \mathrm{d}\omega + \frac{1}{\sigma_k} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_k \int_\Omega g_\beta(t, a(t), a_\gamma(t))\, \mathrm{d}\omega\right)\right],$$

where the penalty parameter $\sigma_k > 0$ and $\lim_{k\to+\infty} \sigma_k = +\infty$, $\Phi(x) = \exp(x) - 1$ is exponential penalty.

Definition 3.1. A point $\hat{a}(t) \in X$ is said to be a minimizer of $(\mathrm{MVP})_k$, if

$$P(a(t), \sigma_k) \geq P(\hat{a}(t), \sigma_k), \qquad \forall\ a(t) \in X.$$

Also, consider all across the article, $\hat{D}_k$ denote the the set of all minimizer to $(\mathrm{MVP})_k$. From now onwards, we utilize the following notions for the convenience of presentation: $a = a(t), a_\gamma = a_\gamma(t), \varrho = (t, a(t), a_\gamma(t)), \hat{\varrho} = (t, \hat{a}(t), \hat{a}_\gamma(t)), \varrho^0 = (t, a^0(t), a^0_\gamma(t)),\ \varrho_k = (t, a_k(t), a_{k\gamma}(t))$, etc.

The following lemma proves that a feasible solution to the constrained multi-dimensional variational problem predicts the limit point of the exponential penalty function concerning the penalty parameter σ_k. The lemma is essential as it is used to prove some of the significant results of this paper.

Lemma 3.1. *Let D be the set of feasible solutions to (MVP)*

(i) *If $a \in D$, then*

$$\lim_{\sigma \to +\infty} \frac{1}{\sigma} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma \int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega \right) \right] = 0.$$

(ii) *If $a \notin D$, then*

$$\lim_{\sigma \to +\infty} \frac{1}{\sigma} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma \int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega \right) \right] = +\infty.$$

Proof. (i) Let $a \in D$ be a feasible solution to (MVP), and so $\int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega \leq 0,\ \beta \in \overline{1, q}$. If $\int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega = 0,\ \beta \in \overline{1, q}$, then the result is obvious. And, if $\int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega = -\int_\Omega h_\beta(\varrho)\, \mathrm{d}\omega$, where $h_\beta(\varrho) \geq 0,\ \beta \in \overline{1, q}$. Then

$$\begin{aligned}
\lim_{\sigma \to +\infty} \frac{1}{\sigma} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma \int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega \right) \right] &= \lim_{\sigma \to +\infty} \frac{1}{\sigma} \sum_{\beta=1}^{q} \left[\exp\left(\sigma \int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega \right) - 1 \right] \\
&= \lim_{\sigma \to +\infty} \frac{1}{\sigma} \sum_{\beta=1}^{q} \left[\exp\left(-\sigma \int_\Omega h_\beta(\varrho)\, \mathrm{d}\omega \right) - 1 \right] \\
&= \sum_{\beta=1}^{q} \lim_{\sigma \to +\infty} \frac{-\int_\Omega h_\beta(\varrho)\, \mathrm{d}\omega}{1 + \sigma \int_\Omega h_\beta(\varrho)\, \mathrm{d}\omega} \qquad \text{(by L-Hospital rule)} \\
&= 0.
\end{aligned}$$

(ii) Let a is not a feasible solution to (MVP), and so $\int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega > 0$, for some $\beta \in \overline{1, q}$. If $\int_\Omega g_{\beta_1}(\varrho)\, \mathrm{d}\omega = 0$, for $\beta_1 \in \overline{1, q}$, then we get

$$\begin{aligned}
\lim_{\sigma \to +\infty} \frac{1}{\sigma} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma \int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega \right) \right] &= \lim_{\sigma \to +\infty} \frac{1}{\sigma} \sum_{\beta=1}^{q} \left[\exp\left(\sigma \int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega \right) - 1 \right] \\
&= \lim_{\sigma \to +\infty} \frac{1}{\sigma} \sum_{\substack{\beta=1 \\ \beta \neq \beta_1}}^{q} \left[\exp\left(\sigma \int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega \right) - 1 \right] \\
&\quad + \lim_{\sigma \to +\infty} \frac{1}{\sigma} \left[\exp\left(\sigma \int_\Omega g_{\beta_1}(\varrho)\, \mathrm{d}\omega \right) - 1 \right] \\
&= \lim_{\sigma \to +\infty} \frac{1}{\sigma} \sum_{\substack{\beta=1 \\ \beta \neq \beta_1}}^{q} \left[\exp\left(\sigma \int_\Omega g_\beta(\varrho)\, \mathrm{d}\omega \right) - 1 \right]
\end{aligned}$$

$$+\lim_{\sigma\to\infty}\exp\left(\sigma\int_\Omega g_{\beta_1}(\varrho)\,\mathrm{d}\omega\right)\int_\Omega g_{\beta_1}(\varrho)\,\mathrm{d}\omega \quad \text{(by L-Hospital rule)}$$
$$=+\infty.$$

Hence the proof is complete. □

Lemma 3.2. *Let $\sigma_k > 0$ such that $\lim_{k\to\infty}\sigma_k = +\infty$. If $\hat{a} \in \overline{\lim}_{k\to\infty}\hat{D}_k$, then $\hat{a}$ is the feasible solution to (MVP).*

Proof. Since $\hat{a} \in \overline{\lim}_{k\to\infty}\hat{D}_k$, then there exist a subsequence $\{k_n\}$ of $\{k\}$ such that $\hat{a} \in \hat{D}_{k_n},\ n=\{1,2,...\}$. Therefore, by the definition of minimizer of the $(\mathrm{MVP})_{k_n},\ n=\{1,2,...\}$, we can write

$$P(a,\sigma_{k_n}) \geq P(\hat{a},\sigma_{k_n}), \qquad \forall a\in X.$$

Or

$$\int_\Omega f(\varrho)\,\mathrm{d}\omega + \frac{1}{\sigma_{k_n}}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_{k_n}\int_\Omega g_\beta(\varrho)\,\mathrm{d}\omega\right)\right]$$
$$\geq \int_\Omega f(\hat{\varrho})\,\mathrm{d}\omega + \frac{1}{\sigma_{k_n}}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_{k_n}\int_\Omega g_\beta(\hat{\varrho})\,\mathrm{d}\omega\right)\right], \qquad \forall a\in X,\ n=\{1,2,\ldots\}. \tag{3}$$

On contrary we assume that $\hat{a}\notin D$, so from Lemma 3.1, we get

$$\lim_{n\to\infty}\frac{1}{\sigma_{k_n}}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_{k_n}\int_\Omega g_\beta(\hat{\varrho})\,\mathrm{d}\omega\right)\right]=+\infty.$$

Again, from Lemma 3.1, for any $a\in D$, we obtain

$$\lim_{n\to\infty}\frac{1}{\sigma_{k_n}}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_{k_n}\int_\Omega g_\beta(\varrho)\,\mathrm{d}\omega\right)\right]=0.$$

From the above two relations, for sufficiently large n (say $n>n_0$), we get

$$\int_\Omega f(\varrho)\,\mathrm{d}\omega + \frac{1}{\sigma_{k_n}}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_{k_n}\int_\Omega g_\beta(\varrho)\,\mathrm{d}\omega\right)\right] < \int_\Omega f(\hat{\varrho})\,\mathrm{d}\omega + \frac{1}{\sigma_{k_n}}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_{k_n}\int_\Omega g_\beta(\hat{\varrho})\,\mathrm{d}\omega\right)\right],$$

which contradicts the inequality (3). This completes the proof. □

4. Convergence of the exponential penalty function method

In this section, we show the convergence of the exponential penalty function method for the multi-dimensional variational problem with inequality constraints (MVP).

Theorem 4.1. $\underline{\lim}_{k\to\infty}(\hat{D}_k\setminus\hat{D})$ *is an empty set.*

Proof. Assume to contrary that $\underline{\lim}_{k\to\infty}(\hat{D}_k\setminus\hat{D})$ is not an empty set and let $a\in\underline{\lim}_{k\to\infty}(\hat{D}_k\setminus\hat{D})$. Then, there exists a number $k_0>0$ such that $a\in\hat{D}_k\setminus\hat{D}$ for all $k\geq k_0$.

Case 1. Let $a \in D$. Since, $a \notin \hat{D}$, then

$$\int_\Omega f(\hat{\varrho})\,\mathrm{d}\omega < \int_\Omega f(\varrho)\,\mathrm{d}\omega, \qquad \text{for some } \hat{a} \in D. \tag{4}$$

Also, $a \in \hat{D}_k$, yields

$$\int_\Omega f(\varrho)\,\mathrm{d}\omega + \frac{1}{\sigma_k}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_k \int_\Omega g_\beta(\varrho)\,\mathrm{d}\omega\right)\right]$$
$$\leq \int_\Omega f(\hat{\varrho})\,\mathrm{d}\omega + \frac{1}{\sigma_k}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_k \int_\Omega g_\beta(\hat{\varrho})\,\mathrm{d}\omega\right)\right], \qquad \text{for some } \hat{a} \in D. \tag{5}$$

On taking limit $k \to \infty$ and using Lemma 3.1, the above inequality becomes

$$\int_\Omega f(\varrho)\,\mathrm{d}\omega \leq \int_\Omega f(\hat{\varrho})\,\mathrm{d}\omega,$$

which contradicts the inequality (4).

Case 2. Let $a \notin D$ and thus from Lemma 3.1, we get

$$\int_\Omega f(\varrho)\,\mathrm{d}\omega + \frac{1}{\sigma_k}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_k \int_\Omega g_\beta(\varrho)\,\mathrm{d}\omega\right)\right]$$
$$\leq \int_\Omega f(\hat{\varrho})\,\mathrm{d}\omega + \frac{1}{\sigma_k}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_k \int_\Omega g_\beta(\hat{\varrho})\,\mathrm{d}\omega\right)\right], \qquad \text{for some } \hat{a} \in D. \tag{6}$$

Again, from Lemma 3.1, we have

$$\lim_{k\to\infty} \frac{1}{\sigma_k}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_k \int_\Omega g_\beta(\hat{\varrho})\,\mathrm{d}\omega\right)\right] = 0, \qquad \forall \hat{a} \in D.$$

Hence we have come to conclusion that for sufficiently large k

$$\int_\Omega f(\hat{\varrho})\,\mathrm{d}\omega + \frac{1}{\sigma_k}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_k \int_\Omega g_\beta(\hat{\varrho})\,\mathrm{d}\omega\right)\right] < \int_\Omega f(\varrho)\,\mathrm{d}\omega + \frac{1}{\sigma_k}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_k \int_\Omega g_\beta(\varrho)\,\mathrm{d}\omega\right)\right],$$

which contradicts the inequality (6). This completes proof. □

Theorem 4.2. $\overline{\lim}_{k\to\infty}(\hat{D}_k \setminus \hat{D})$ *is an empty set.*

Proof. Assume to the contrary that $\overline{\lim}_{k\to\infty}(\hat{D}_k \setminus \hat{D})$ is not an empty set. Then, there exists a subsequence $\{k_n\}$ of $\{k\}$ such that, $a \in \hat{D}_{k_n} \setminus \hat{D}$.

Case 1. Let $a \in D$. Since, $a \notin \hat{D}$, then

$$\int_\Omega f(\hat{\varrho})\,\mathrm{d}\omega < \int_\Omega f(\varrho)\,\mathrm{d}\omega, \qquad \text{for some } \hat{a} \in D. \tag{7}$$

Now, since $a, \hat{a} \in D$, then using Lemma 3.1, the above inequality becomes

$$\int_\Omega f(\varrho)\,\mathrm{d}\omega + \frac{1}{\sigma_{k_n}}\sum_{\beta=1}^{q}\left[\Phi\left(\sigma_{k_n} \int_\Omega g_\beta(\varrho)\,\mathrm{d}\omega\right)\right]$$

$$> \int_{\Omega} f(\hat{\varrho})\,\mathrm{d}\omega + \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_n} \int_{\Omega} g_{\beta}(\hat{\varrho})\,\mathrm{d}\omega \right) \right], \qquad \text{for some } \hat{a} \in D. \tag{8}$$

Again, since $a \in \hat{D}_{k_n}$, it yields

$$\int_{\Omega} f(\varrho)\,\mathrm{d}\omega + \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_n} \int_{\Omega} g_{\beta}(\varrho)\,\mathrm{d}\omega \right) \right] \leq \int_{\Omega} f(\hat{\varrho})\,\mathrm{d}\omega + \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_n} \int_{\Omega} g_{\beta}(\hat{\varrho})\,\mathrm{d}\omega \right) \right], \tag{9}$$

which contradicts the inequality (8).

Case 2. Let $a \notin D$ and thus from Lemma 3.1, we get

$$\lim_{n\to\infty} \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\left(\sigma_{k_n} \int_{\Omega} g_{\beta}(\varrho)\,\mathrm{d}\omega \right) \right] = +\infty.$$

Again, from Lemma 3.1, we have

$$\lim_{n\to\infty} \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_n} \int_{\Omega} g_{\beta}(\hat{\varrho})\,\mathrm{d}\omega \right) \right] = 0, \qquad \forall \hat{a} \in D.$$

Hence we have come to conclusion that for sufficiently large k_n

$$\int_{\Omega} f(\hat{\varrho})\,\mathrm{d}\omega + \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_n} \int_{\Omega} g_{\beta}(\hat{\varrho})\,\mathrm{d}\omega \right) \right] < \int_{\Omega} f(\varrho)\,\mathrm{d}\omega + \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_n} \int_{\Omega} g_{\beta}(\varrho)\,\mathrm{d}\omega \right) \right], \qquad \forall \hat{a} \in D,$$

which contradicts the inequality (9). This completes the proof. □

Theorem 4.3. $\lim_{k\to\infty}(\hat{D}_k \setminus \hat{D})$ *is an empty set.*

Proof. This proof follows from Theorems 4.1 and 4.2. □

The following theorem shows that the limit point of the penalty function at each point of subsequence will be zero if the subsequence of the minimizers of the unconstrained multi-dimensional variational problem $(\text{MVP})_k$ converges towards a feasible solution to the constrained multi-dimensional variational problem (MVP).

Theorem 4.4. *Let $\hat{a}_k \in X$ be a minimizer to the probem $(MVP)_k$ and $\{\hat{a}_{k_n}\}$ be the convergent subsequence of $\{\hat{a}_k\}$ with $\lim_{n\to\infty} \hat{a}_{k_n} \in D$. Further, if X is measurable and $f_k = f(\varrho_k)$ be a sequence of non-negative measurable functions, then*

$$\lim_{n\to\infty} \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_n} \int_{\Omega} g_{\beta}(\hat{\varrho}_{k_n})\,\mathrm{d}\omega \right) \right] = 0.$$

Proof. We begin with the contradiction and assume that

$$\lim_{n\to\infty} \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_n} \int_{\Omega} g_{\beta}(\hat{\varrho}_{k_n})\,\mathrm{d}\omega \right) \right] \neq 0.$$

Then the sequence $\{\hat{a}_{k_n}\}$ has a subsequence $\{\hat{a}_{k_{n_r}}\}$, $r = \{1, 2, \ldots\}$ such that

$$\lim_{r\to\infty} \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\hat{\varrho}_{k_{n_r}})\,\mathrm{d}\omega \right) \right] > \epsilon, \quad \epsilon > 0. \tag{10}$$

$$\Rightarrow \underline{\lim}_{r\to\infty} \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\hat{\varrho}_{k_{n_r}})\, \mathrm{d}\omega \right) \right] > \epsilon. \tag{11}$$

Since $\hat{a}_{k_{n_r}}$ is a minimizer to the $(\mathrm{MVP})_{k_{n_r}}$, we get

$$\int_{\Omega} f(\hat{\varrho}_{k_{n_r}})\, \mathrm{d}\omega + \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\hat{\varrho}_{k_{n_r}})\, \mathrm{d}\omega \right) \right]$$
$$\leq \int_{\Omega} f(\varrho)\, \mathrm{d}\omega + \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\varrho)\, \mathrm{d}\omega \right) \right], \qquad \forall a \in X.$$

From Arzelà–Ascoli theorem [9], there exist a convergent subsequence $\{\hat{a}_{k_n}\}$ of $\{\hat{a}_k\}$ with $\lim_{n\to\infty} \hat{a}_{k_n} = \hat{a}$, then the above inequality becomes

$$\int_{\Omega} f(\hat{\varrho}_{k_{n_r}})\, \mathrm{d}\omega + \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\hat{\varrho}_{k_{n_r}})\, \mathrm{d}\omega \right) \right] \leq \int_{\Omega} f(\hat{\varrho})\, \mathrm{d}\omega + \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\hat{\varrho})\, \mathrm{d}\omega \right) \right].$$

On taking limit inferior with $r \to \infty$ and using inequality (11), we get

$$\underline{\lim}_{r\to\infty} \int_{\Omega} f(\hat{\varrho}_{k_{n_r}})\, \mathrm{d}\omega + \epsilon \leq \int_{\Omega} f(\hat{\varrho})\, \mathrm{d}\omega + \underline{\lim}_{r\to\infty} \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\hat{\varrho})\, \mathrm{d}\omega \right) \right].$$

Since f_k is a sequence of non-negative measurable functions and $\underline{\lim}_{n\to\infty} \hat{a}_{k_n} = \hat{a}$, It follow from Fatou's lemma

$$\int_{\Omega} \underline{\lim}_{r\to\infty} f(\hat{\varrho}_{k_{n_r}})\, \mathrm{d}\omega + \epsilon \leq \int_{\Omega} f(\hat{\varrho})\, \mathrm{d}\omega + \underline{\lim}_{r\to\infty} \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\hat{\varrho})\, \mathrm{d}\omega \right) \right],$$

Again, since $\lim_{n\to\infty} \hat{a}_{k_n} = \hat{a}$, from the continuity of f, we obtain

$$\int_{\Omega} f(\hat{\varrho})\, \mathrm{d}\omega + \epsilon \leq \int_{\Omega} f(\hat{\varrho})\, \mathrm{d}\omega + \underline{\lim}_{r\to\infty} \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\hat{\varrho})\, \mathrm{d}\omega \right) \right],$$

$$\text{Or } \underline{\lim}_{r\to\infty} \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\hat{\varrho})\, \mathrm{d}\omega \right) \right] \geq \epsilon. \tag{12}$$

On the other hand, since $\hat{a} \in D$, from Lemma 3.1, we have

$$\underline{\lim}_{r\to\infty} \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_{\Omega} g_{\beta}(\hat{\varrho})\, \mathrm{d}\omega \right) \right] = 0. \tag{13}$$

Therefore from the inequality (12) and equation (13), we conclude that $\epsilon \leq 0$, Hence the theorem. □

Theorem 4.5. $\underline{\lim}_{k\to\infty} \hat{D}_k \subseteq \overline{\lim}_{k\to\infty} \hat{D}_k \subseteq D$.

Proof. This proof follows from Lemma 3.2. □

In next theorem, we shall establish that a convergent subsequence of the sequence of minimizers to the unconstrained multi-dimensional variational problem $(\mathrm{MVP})_k$ can approach an optimal solution to the constrained multi-dimensional variational problem (MVP).

Theorem 4.6. *Let $\hat{a}_k \in X$ be a minimizer to the probem (MVP)$_k$ and $\{\hat{a}_{k_n}\}$ is a convergent subsequence of $\{\hat{a}_k\}$ with $\lim_{n\to\infty} \hat{a}_{k_n} = \hat{a} \in D$. Further, if X is measurable and $f_k = f(\varrho_k)$ be a sequence of non-negative measurable functions, then $\hat{a}$ is an optimal solution to (MVP).*

Proof. Since, $\hat{a}_{k_n} \in \hat{D}_{k_n}$, so for all $a^0 \in D$, we get

$$\int_\Omega f(\hat{\varrho}_{k_{n_r}})\,\mathrm{d}\omega + \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_\Omega g_\beta(\hat{\varrho}_{k_{n_r}})\,\mathrm{d}\omega\right)\right] \le \int_\Omega f(\varrho^0)\,\mathrm{d}\omega + \frac{1}{\sigma_{k_{n_r}}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_{n_r}} \int_\Omega g_\beta(\varrho^0)\,\mathrm{d}\omega\right)\right]. \quad (14)$$

From the feasibility of a^0 and Lemma 3.1, we get

$$\lim_{n\to\infty} \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_n} \int_\Omega g_\beta(\varrho^0)\,\mathrm{d}\omega\right) - 1\right] = 0.$$

Also, from Theorem 4.4, we have

$$\lim_{n\to\infty} \frac{1}{\sigma_{k_n}} \sum_{\beta=1}^{q} \left[\Phi\left(\sigma_{k_n} \int_\Omega g_\beta(\hat{\varrho}_{k_n})\,\mathrm{d}\omega\right)\right] = 0.$$

On taking limit inferior as $n \to \infty$ and in agreement of the above two equations, inequality (14) becomes

$$\underline{\lim}_{n\to\infty} \int_\Omega f(\hat{\varrho}_{k_n})\,\mathrm{d}\omega \le \int_\Omega f(\varrho^0)\,\mathrm{d}\omega,$$

which in view of Fatou's lemma gives

$$\int_\Omega \underline{\lim}_{n\to\infty} f(\hat{\varrho}_{k_n})\,\mathrm{d}\omega \le \int_\Omega f(\varrho^0)\,\mathrm{d}\omega.$$

From Arzelà–Ascoli theorem [9], there exist a convergent subsequence$\{\hat{a}_{k_n}\}$ of $\{\hat{a}_k\}$ with $\lim_{n\to\infty} \hat{a}_{k_n} = \hat{a}$, then the above inequality becomes

$$\int_\Omega f(\hat{\varrho})\,\mathrm{d}\omega \le \int_\Omega f(\varrho^0)\,\mathrm{d}\omega.$$

Therefore $\hat{a}$ is an optimal solution to (MVP). This completes the proof. □

Now we'll give an example to demonstrate the effectiveness of the exponential penalty function method in solving the multi-dimensional constrained variational problem.

Example 4.1. A production firm manufactures various merchandise items and the company wants to keep the cost of manufacturing as low as possible. The total production cost of production house is given as:

$$f(\varrho) = (a^1(t) + 9(a^2(t))^2),$$

where $t = (t^1, t^2) \in R^2$, $a(t) = (a^1(t), a^2(t)) \in R^2$ denotes the level of production and $a_\gamma(t) = (a^1_\gamma(t), a^2_\gamma(t)), \gamma = \{1, 2\}$ denotes the rate of change of the level of production. The production cost should be kept as low as possible with respect to the following constraint :

$$g(\varrho) = (a^1(t))^2 - 4 \le 0,$$

with the boundary conditions

$$a(t_0) = (0, -1),\ t_0 = (t^1_0, t^2_0) = (0, 0),$$

$$a(t_1) = (1,1),\ t_1 = (t_1^1, t_1^2) = (1,1).$$

The problem is concern to find an appropriate production level function that reduces manufacturing cost. The above information can be expressed in the form of two-dimensional variational problem as:

$$\text{(MVP1)} \quad \min \int_\Omega \big(a^1(t) + 9(a^2(t))^2\big)\, \mathrm{d}t^1\, \mathrm{d}t^2$$

$$\text{subject to}$$

$$\big(a^1(t)\big)^2 - 4 \le 0,$$
$$a(t_0) = (0,-1) \text{ and } a(t_1) = (1,1).$$

where $t_0 = (t_0^1, t_0^2) = (0,0)$ and $t_1 = (t_1^1, t_1^2) = (1,1)$ are the diagonally opposite points of a square $\Omega \subset R^2$. The set of all feasible solutions of (MVP1) is:

$$D = \big\{a(t) = \big(a^1(t), a^2(t)\big) \in R^2 :\ -2 \le a^1(t) \le 2,\ a(t_0) = (0,-1),\ a(t_1) = (1,1)\big\}.$$

Thus the two-dimensional exponential penalized variational problem associated to the problem (MVP1) is given by

$$\text{(MVP1)}_k \quad \min P(a(t),\sigma_k) = \int_\Omega \Big(a^1(t) + 9\big(a^2(t)\big)^2\Big)\, \mathrm{d}t^1\, \mathrm{d}t^2 + \frac{1}{\sigma_k}\bigg[\exp\bigg(\sigma_k \int_\Omega \Big(\big(a^1(t)\big)^2 - 4\Big)\, \mathrm{d}t^1\, \mathrm{d}t^2\bigg) - 1\bigg].$$

Consider the penalty parameter $\sigma_k = (k+1)$, so $\lim_{k\to\infty} \sigma_k = +\infty$, and the functions $a(t) = (a^1(t), a^2(t))$ as:

$$a^1(t) = \frac{t^1 + t^2}{2},\ a^2(t) = \alpha + t^1 + t^2,\ \alpha \in R,\ t = (t^1, t^2) \in \Omega.$$

Therefore the problem (MVP1)_k becomes

$$\begin{aligned} \min P(a(t),\sigma_k) &= \int_\Omega \bigg\{\bigg(\frac{t^1+t^2}{2}\bigg) + 9(t^1+t^2+\alpha)^2\bigg\}\, \mathrm{d}t^1\, \mathrm{d}t^2 \\ &\quad + \frac{1}{(k+1)}\bigg[\exp\bigg((k+1)\int_\Omega \bigg\{\bigg(\frac{t^1+t^2}{2}\bigg)^2 - 4\bigg\}\, \mathrm{d}t^1\, \mathrm{d}t^2\bigg) - 1\bigg] \\ &= 9(\alpha+1)^2 + 2 + \frac{1}{(k+1)}\bigg[\exp\bigg(\frac{-89(k+1)}{24}\bigg) - 1\bigg]. \end{aligned}$$

By using Matlab minimization function fminunc, we can see that $a(t) = (\frac{t^1+t^2}{2}, t^1 + t^2 - 1)$ with $\alpha = -1,\ t = (t^1, t^2) \in \Omega$ is a minimizer to each of the exponential penalty problem (MVP1)_k, $k \in N$.

So the subsequence $\{a_{k_n}(t)\} = \{(\frac{t^1+t^2}{2}, t^1+t^2-1)\}$ of minimizers to (MVP1)_k is converging to the limit point $(\frac{t^1+t^2}{2}, t^1+t^2-1)$. Also, the point $(\frac{t^1+t^2}{2}, t^1+t^2-1)$ is feasible solution to (MVP1). Further, we can observe that $f_k = f(\varrho_k)$ is a sequence of non-negative measurable functions for $a(t) = (\frac{t^1+t^2}{2}, t^1+t^2+\alpha),\ \alpha \in R,\ t \in \Omega$. Since all the hypotheses of Theorems 4.4 and 4.6 are fulfilled, we have

$$\lim_{n\to\infty} \frac{1}{(k_n+1)}\bigg[\exp\bigg((k_n+1)\int_\Omega \big((a^1(t))^2 - 4\big)\, \mathrm{d}t^1\, \mathrm{d}t^2\bigg)\bigg] = 0,$$

and $a(t) = (\frac{t^1+t^2}{2}, t^1+t^2-1)$ is an optimal solution to the (MVP1). Hence, the best production level function for minimising manufacturing cost is $a(t) = (\frac{t^1+t^2}{2}, t^1+t^2-1)$.

5. Conclusion

In this article, the exponential penalty function method has been used to solve a multi-dimensional variational problem and then demonstrated the convergence of the method. An example has also been included to understand the applicability of the findings. Consequently, we suggest that the exponential penalty function method effectively solves constrained multi-dimensional variational problems. So, the results presented in this work give a powerful technique for solving the problems mentioned above that do not rely on the characterization of convex functions or generalized convex. Therefore, the multi-dimensional variational problems with inequality constraints that naturally emerge in various mathematical models may be handled more quickly and efficiently utilizing this method.

Acknowledgements. The research of the first author is financially supported by the MATRICS, SERB-DST, New Delhi, India (No. MTR/2021/000002).

References

[1] T. Antczak, Exact penalty functions method for mathematical programming problems involving invex functions. *Eur. J. Oper. Res.* **198** (2009) 29–36.

[2] T. Antczak, A new exact exponential penalty function method and nonconvex mathematical programming. *Appl. Math. Comput.* **217** (2011) 6652–6662.

[3] T. Antczak and A. Pitea, Parametric approach to multitime multiobjective fractional variational problems under (f, ρ)-convexity. *Opt. Control App. Methods* **37** (2016) 831–847.

[4] M. Arana-Jiménez, R. Osuna-Gómez, G. Ruiz-Garzón and M. Rojas-Medar, On variational problems: characterization of solutions and duality. *J. Math. Anal. App.* **311** (2005) 1–12.

[5] M. Arana-Jiménez, G. Ruiz-Garzón, A. Rufián-Lizana and R. Osuna-Gómez, A necessary and sufficient condition for duality in multiobjective variational problems. *Eur. J. Oper. Res.* **201** (2010) 672–681.

[6] A. Baranwal, A. Jayswal and Preeti, Robust duality for the uncertain multitime control optimization problems. *Int. J. Robust Nonlinear Control* **32** (2022) 5837–5847.

[7] K. Das and S. Treanţă, Constrained controlled optimization problems involving second-order derivatives. *Quaestiones Math.* (2022) 1–11.

[8] I.P. Debnath and N. Pokharna, On optimality and duality in interval-valued variational problem with $b-(p, r)$-invexity. *RAIRO: Oper. Res.* **55** (2021) 1909–1932.

[9] J. Green and F. Valentine, On the arzela-ascoli theorem. *Math. Mag.* **34** (1961) 199–202.

[10] M. Hanson, Bounds for functionally convex optimal control problems. *J. Math. Anal. App.* **8** (1964) 84–89.

[11] A. Jayswal and A. Baranwal, Relations between multidimensional interval-valued variational problems and variational inequalities. *Kybernetika* **58** (2022) 564–577.

[12] A. Jayswal and A. Baranwal, Robust approach for uncertain multi-dimensional fractional control optimization problems. *Bull. Malaysian Math. Sci. Soc.* **46** (2023) 75.

[13] A. Jayswal and S. Choudhury, Convergence of exponential penalty function method for variational problems. *Proc. Nat. Acad. Sci. India Sect. A: Phys. Sci.* **89** (2019) 517–524.

[14] A. Jayswal and Preeti, An exact minimax penalty function approach to solve multitime variational problems. *RAIRO: Oper. Res.* **54** (2020) 637–652.

[15] A. Jayswal, T. Antczak and S. Jha, On equivalence between a variational problem and its modified variational problem with the η-objective function under invexity. *Int. Trans. Oper. Res.* **26** (2019) 2053–2070.

[16] A. Jayswal, A. Baranwal and M. Arana-Jiménez, G-penalty approach for multi-dimensional control optimization problem with non-linear dynamical system. *Int. J. Control* (2022) 1–22.

[17] C. Jiang, Existence and uniqueness result for a class of mixed variational problems. *J. Inequalities App.* **2018** (2018) 1–16.

[18] P. Mandal, B. Giri and C. Nahak, Variational problems and l_1 exact exponential penalty function with (p, r)-ρ-(η, θ)-invexity, *Adv. Model. Optim.* **16** (2014) 243–259.

[19] B. Mond and I. Husain, Sufficient optimality criteria and duality for variational problems with generalised invexity, *ANZIAM J.* **31** (1989) 108–121.

[20] F.H. Murphy, A class of exponential penalty functions. *SIAM J. Control* **12** (1974) 679–687.

[21] A. Pitea and M. Postolache, Duality theorems for a new class of multitime multiobjective variational problems. *J. Global Optim.* **54** (2012) 47–58.

[22] A. Pitea and M. Postolache, Minimization of vectors of curvilinear functionals on the second order jet bundle: necessary conditions. *Optim. Lett.* **6** (2012) 459–470.

[23] A. Pitea and M. Postolache, Minimization of vectors of curvilinear functionals on the second order jet bundle: sufficient efficiency conditions. *Optim. Lett.* **6** (2012) 1657–1669.

[24] S. Treanţă, PDE-constrained vector variational problems governed by curvilinear integral functionals. *Appl. Anal. Optim.* **3** (2019) 83–101.

[25] S. Treanţă, On well-posedness of some constrained variational problems. *Mathematics* **9** (2021) 2478.