COMPUTING ROLE ASSIGNMENTS OF CARTESIAN PRODUCT OF GRAPHS

DIANE CASTONGUAY, ELISÂNGELA SILVA DIAS*, FERNANDA NEIVA MESQUITA* and JULIANO ROSA NASCIMENTO

Abstract. Network science is a growing field of study using Graph Theory as a modeling tool. In social networks, a role assignment is such that individuals play the same role, if they relate in the same way to other individuals playing counterpart roles. In this sense, a role assignment permit to represent the network through a smaller graph modeling its roles. This leads to a problem called \( r \)-Role Assignment whose goal is deciding whether it exists such an assignment of \( r \) distinct roles. This problem is known to be \( \text{NP} \)-complete for any fixed \( r \geq 2 \). The Cartesian product of graphs is a well studied graph operation, often used for modeling interconnection networks. Formally, the Cartesian product of \( G \) and \( H \) is a graph, denoted as \( G \Box H \), whose vertex set is \( V(G) \times V(H) \) and two vertices \((u, v) \) and \((x, y) \) are adjacent precisely if \( u = x \) and \( vy \in E(H) \), or \( ux \in E(G) \) and \( v = y \). In a previous work, we showed that Cartesian product of graphs are always 2-role assignable, however the 3-Role Assignment problem is \( \text{NP} \)-complete on this class. In this paper, we prove that \( r \)-Role Assignment restricted to Cartesian product graphs is still \( \text{NP} \)-complete, for any fixed \( r \geq 4 \).

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1. Introduction

Nowadays, social networks are a part of everybody’s life. Up to study their behavior, a social network is conceptualized as a graph where vertices represent individuals and edges the relationship between them. In 1980, Augluin introduced the concept of covering from which role assignment arise, as a tool for networks of processors [1]. A decade later, based on graph models for social networks, Everett and Borgatti [8] formalized role assignment under the name of role coloring.

Indeed, a \( r \)-role assignment of a simple graph \( G \) is an assignment of \( r \) distinct roles to the vertices of \( G \) such that if two vertices have the same role, then the sets of roles of their neighbors are the same. Moreover, from such an assignment, we obtain a role graph where vertices are the \( r \) distinct roles and there is an edge between two roles whenever there are two neighbors in the graph \( G \) that correspond to those roles. Note that, the role graph has no multiple edges, but permit loops since two related vertices in \( G \) can have the same role. Observe that, while a social network usually give rise to a large graph, a role assignment allows to represent the same network through a smaller graph. On the other hand, when the role graph is a graph without loops, where every pair of vertices are neighbors, role assignment coincides with \( k \)-fall coloring, introduced by Dunbar et al. [7]

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as a variant of vertex coloring. More recently, Kaul and Mitillos [14] studied relations between the parameters associated to fall coloring and vertex coloring.

We define the r-ROLE ASSIGNMENT problem as follows:

<table>
<thead>
<tr>
<th>r-ROLE ASSIGNMENT</th>
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<tr>
<td><strong>Instance:</strong> A simple graph G.</td>
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<tr>
<td><strong>Question:</strong> Does G admit a r-role assignment?</td>
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</table>

Applications of role assignment are highlighted in several contexts such as social networks [8, 18] and distributed computing [3, 4]. Furthermore, Pekeč and Roberts [17] showed that any network represented by a graph, with minimum degree bounded by a suitable bound that depends on r, has a r-role assignment.

The first study devoted to determine the computational complexity of the problem has appeared in 2001, by Roberts and Sheng [18]. They proved the NP-completeness of 2-ROLE ASSIGNMENT. Such a result was strengthened in 2005, by Fiala and Paulusma [9], who showed that r-ROLE ASSIGNMENT is NP-complete for any fixed r ≥ 3. On the positive side, r-ROLE ASSIGNMENT can be solved in polynomial time for trees [10] and for proper interval graphs [13] for any fixed r ≥ 1. Considering chordal and split graphs, a dichotomy for the complexity of r-ROLE ASSIGNMENT arises. While for chordal graphs, the problem is solvable in linear time for r = 2 and NP-complete for r ≥ 3 [20]; for split graphs, the problem is trivial, with true answer, for r = 2, solvable in polynomial time for r = 3, and NP-complete for any fixed r ≥ 4 [6].

Cartesian product of graphs was introduced by Sabidussi [19] in 1959. Since then, it has been applied in many areas, such as space structures [15] and interconnection networks [5]. Some properties of role assignments has also been studied for Cartesian and categorical product of graphs to admit a fall coloring. After that, Laskar and Lyle [16] showed that the k-fall coloring for k ≥ 3 is NP-complete for bipartite graphs and they also determine some fall colourable Cartesian product of graphs.

In 2021, we initiated a study of the r-ROLE ASSIGNMENT problem for Cartesian product of graphs [2]. In that work, we show the NP-completeness of the problem when r = 3 and we prove that it has a true answer for r = 2. In this paper, we answer positively a conjecture opened by this previous work, by proving that the r-role assignment problem for Cartesian product of graphs is NP-complete for any fixed r ≥ 4.

This paper is organized in more two sections. In Section 2, we set notations and terminology. Our complexity results concerning the r-ROLE ASSIGNMENT problem, r ≥ 4, for Cartesian product of graphs follow in Section 3.

2. Preliminaries

All graphs considered are undirected, finite, non-trivial and have no multiple edges. A graph G is a pair (V(G), E(G)), where V(G) is the set of vertices and E(G) is the set of edges. The vertices u and v are adjacent or neighbors if they are joined by an edge e, also denoted by uv. In this case, u and v are incident to e and e is incident to u and v. A loop is an edge incident to only one vertex. The neighborhood of a vertex v, denoted by N_G(v), is the set of all neighbors of v in G. When the graph G is clear from the context, we simply write N(v).

A simple graph is a graph without loops. In a simple graph G, the degree of a vertex v is the cardinality of N_G(v). The neighborhood of a subset U of V(G), denoted as N_G(U), is the union of the neighborhoods of the vertices of U. If U is any set of vertices in G, we denote by G – U the graph obtained by deleting the vertices in U and all edges incident with any of them.

A path is a sequence of distinct vertices with an edge between each pair of consecutive vertices. For n ≥ 2, we denote a path on n vertices by P_n or by the sequence of vertices v_1 . . . v_n. A clique is a subset of vertices
that are adjacent to each other. A graph $G$ is called bipartite graph if we can partition $V(G) = A \cup B$ so that if there is an edge $uv \in E(G)$, then $u \in A$ and $v \in B$, or vice versa. Hence, a bipartite graph with loops have edge that connects a vertex to itself same partition.

Given a simple graph $G$ and a graph $R$, possibly with loops. A $R$-role assignment of $G$ is a surjective vertex mapping $p : V(G) \to V(R)$ such that $p(N_G(v)) = N_R(p(v))$ for all $v \in V(G)$. A graph $G$ has a $r$-role assignment if it admits a $R$-role assignment for some graph $R$, called the role graph, with $|V(R)| = r$. We set $1, \ldots , r$ the vertices of $R$, also called roles. For now on, all graphs (except maybe the role graph) are simple. Observe that, if the graph $G$ is connected, then the role graph $R$ of any role assignment of $G$ is also connected. Also, if role graph $R$ is bipartite, then so $G$ is.

For Cartesian product, we follow the terminology of Hammack et al. [12]. Let $G$ and $H$ be two graphs. The Cartesian product of $G$ and $H$ is a graph, denoted as $G \square H$, whose vertex set is $V(G) \times V(H)$ and two vertices $(u, v)$ and $(x, y)$ are adjacent precisely if $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$.

Remark that, if $H$ is a trivial graph, then $G \square H \cong G$ and the complexity of the problem $r$-Role Assignment [18] is already known. Thus, we always consider $G$ and $H$ to be non-trivial graphs.

## 3. Results

Motivated by the constructions given by van ’t Hof et al. [20], Dourado [6] and Castonguay et al. [2], we propose a new construction to show that the decision problem related to finding a $R$-role assignment, with $r \geq 4$, for Cartesian product of two graphs remains NP-complete. As done in [2], we perform a polynomial time reduction from the NP-complete problem Hypergraph 2-Coloring, see [11]. A hypergraph $\mathcal{H}$ is a pair $\mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{S}(\mathcal{H}))$, where $\mathcal{V}(\mathcal{H})$ is a set of vertices, and $\mathcal{S}(\mathcal{H})$ is a set of non-empty subsets of $\mathcal{V}(\mathcal{H})$ called hyperedges. We consider hypergraphs with at least one hyperedge and hyperedges with at least two elements.

A surjective mapping $c : \mathcal{V}(\mathcal{H}) \to \{1, 2\}$ is a 2-coloring of $\mathcal{H}$, if every hyperedge in $\mathcal{S}(\mathcal{H})$ contains at least two vertices $u$ and $v$ with $c(u) \neq c(v)$. The Hypergraph 2-Coloring problem asks whether a given hypergraph has a 2-coloring. Notice that this problem is NP-complete even if the hyperedges has size at most 3 [11].

In Subsection 3.1, we present our construction. We show the unicity of the role graph in Subsection 3.2. Finally, we conclude our NP-completeness result in Subsection 3.3.

### 3.1. Polynomial time reduction and sufficiency

Given a hypergraph $\mathcal{H}$, we construct a Cartesian product of two graphs that will serve as an instance of $r$-Role Assignment. For this, we will consider $G_r(\mathcal{H}) \sqcup P_2$. The construction of $G_r(\mathcal{H})$, defined in the sequel, is based on the incidence graph. The incidence graph $I_{\mathcal{H}} = (\mathcal{V}(I_{\mathcal{H}}), E(I_{\mathcal{H}}))$ of a hypergraph $\mathcal{H}$ is a bipartite graph whose vertex set is $V(I_{\mathcal{H}}) = \mathcal{V}(\mathcal{H}) \cup \mathcal{S}(\mathcal{H})$, and edge set $E(I_{\mathcal{H}}) = \{vS \mid v \in \mathcal{V}(\mathcal{H}), S \in \mathcal{S}(\mathcal{H}) \text{ with } v \in S\}$.

**Construction 1.** Given a hypergraph $\mathcal{H}$, with $\mathcal{V}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H}) = \{S_1, \ldots , S_m\}$, with $m \geq 1$, we construct a graph $G_r(\mathcal{H})$, arising from the incidence graph $I_{\mathcal{H}}$. We remind that $V(I_{\mathcal{H}}) = \mathcal{V}(\mathcal{H}) \cup \mathcal{S}(\mathcal{H})$.

- We add edges to make $\mathcal{V}(\mathcal{H})$ a clique;
- We add a path $C$ denoted by $c_1 \ldots c_{r-2}$ and the edges $vc_{r-2}$ for every $v \in \mathcal{V}(\mathcal{H})$;
- We add the subgraph $F_r$, illustrated in Figure 1, and the edges $b_{r-1}c_{r-2}$ and $b_rc_{r-2}$. We denote by $A$ the path $a_1 \ldots a_{r-2}$ and by $B$ the path $b_1 \ldots b_{r-2}$;
- For $j = 1, \ldots , m$, we add a path $W_j$, denoted by $w_{j,1} \ldots w_{j,r-3}$ connected to $S_j$ by the vertex $w_{j,r-3}$, that is, with the edge $S_jw_{j,r-3}$.

Figure 1 contains an example of the graph $G_r(\mathcal{H})$, illustrating the Construction 1. We recall that $\mathcal{H}$ has at least one hyperedge and therefore we have a clique of cardinality 3 in $G_r(\mathcal{H})$ and thus in $G_r(\mathcal{H}) \sqcup P_2$, which ensures that both graphs are not bipartite. Therefore, if there exist a $R$-role assignment of $G_r(\mathcal{H}) \sqcup P_2$, then $R$ is not bipartite.
Figure 1. The graph $G_r(\mathcal{H})$ constructed from $\mathcal{H}$ with $V(\mathcal{H}) = \{v_1, v_2, v_3, v_4\}$ and $S(\mathcal{H}) = \{S_1, S_2, S_3\}$, where $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_1, v_4\}$ and $S_3 = \{v_3, v_4\}$ and part of a $r$-role assignment of $G_r(\mathcal{H}) \square P_2$, with $r \geq 4$.

For short, we use the following simplified notations on $G_r(\mathcal{H}) \square P_2$. Considering $V(P_2) = \{v_1, v_2\}$ for each $u \in V(G_r(\mathcal{H}))$, we identify $u$ with the vertex $(u, v_1)$ and denote by $u'$ the vertex $(u, v_2)$. We say that $u'$ is the corresponding vertex of $u$. Let $U \subseteq V(G)$, we identify $U$ with the set $\{(u, v_1) \mid u \in U\}$ and denote by $U'$ the set $\{(u, v_2) \mid u \in U\}$. For short, we denote $N(u) = N_{G_r(\mathcal{H}) \square P_2}(u, v_1)$ and $N(u') = N_{G_r(\mathcal{H}) \square P_2}(u, v_2)$ for any $u \in V(G_r(\mathcal{H}))$.

Let $R_r$ be the graph presented in Figure 2. Given a hypergraph $\mathcal{H}$ and the Construction 1 graph $G_r(\mathcal{H})$, $r \geq 4$, we show that $R_r$ is the unique role graph for a $r$-role assignment of $G_r(\mathcal{H}) \square P_2$. Observe that all vertices of $R_r$ have loops.

First, Theorem 1 presents a $R_r$-role assignment of $G_r(\mathcal{H}) \square P_2$ with $r \geq 4$, when a 2-coloring of $\mathcal{H}$ is known.

**Theorem 1.** Let $\mathcal{H}$ be a hypergraph and $G_r(\mathcal{H})$, with $r \geq 4$, the graph obtained from Construction 1. If $\mathcal{H}$ has a 2-coloring, then $G_r(\mathcal{H}) \square P_2$ has a $R_r$-role assignment.
Proof. Let \( c: V(\mathcal{H}) \rightarrow \{r - 1, r\} \) be a 2-coloring of \( \mathcal{H} \). To simplify the definition of a role assignment, we introduce the following subsets of \( V(G_r(\mathcal{H})) \):

- For \( i = 1, \ldots, r - 3 \), \( A_i = \{a_i, b_i, c_i\} \cup \{w_{j,i} \mid j = 1, \ldots, m\} \);
- \( A_{r-2} = \{a_{r-2}, b_{r-2}, c_{r-2}\} \);
- For \( i = r - 1, r \), \( A_i = \{a_i, b_i\} \).

We define a role assignment \( p: V(G_r(\mathcal{H}) \square P_2) \rightarrow \{1, 2, \ldots, r\} \), as follows. For every \( x \in V(G_r(\mathcal{H})) \):

\[
p(x) = \begin{cases} 
  i, & \text{if } x \in A_i, \text{ with } i \in \{1, \ldots, r\}; \\
  r - 2, & \text{if } x \in S(\mathcal{H}); \\
  c(i), & \text{if } x \in V(\mathcal{H}).
\end{cases}
\]

For all \( x' \), with \( x \in V(G_r(\mathcal{H})) \), we define \( p(x') = p(x) \).

See an example in Figure 1, where the labels inside the vertices indicate the role assigned to that vertex. It is easy to see that \( p \) is a \( r \)-role assignment of \( G_r(\mathcal{H}) \square P_2 \), but to make it clearer let us detail for \( x \in S(\mathcal{H}) \cup V(\mathcal{H}) \).

For \( x \in S(\mathcal{H}) \), we have that \( p(x) = r - 2 \) and \( x = S_j \), for some \( j \in \{1, \ldots, m\} \). Since \( c \) is a 2-coloring of \( \mathcal{H} \), there are \( v_s, v_t \in S_j \) (in \( \mathcal{H} \)), with \( s, t \in \{1, \ldots, n\} \) and \( c(v_s) \neq c(v_t) \). Suppose, without loss of generality, that \( c(v_s) = r - 1 \) and \( c(v_t) = r \). Thus, \( p(N(S_j)) = \{r - 3, r - 2, r - 1, r\} \), because \( p(S'_j) = r - 2 \) and \( p(w_{j,r-3}) = r - 3 \).

For \( x \in V(\mathcal{H}) \), by definition of \( p \), \( p(x) \in \{r - 1, r\} \) and \( p(N(x)) \subseteq \{r - 2, r - 1, r\} \). We observe that there exists \( y \in V(\mathcal{H}) \), such that \( p(y) \neq p(x) \) and \( \{x', c_{r-2}, y\} \subseteq N(x) \). Since \( p(x') = p(x) \) and \( p(c_{r-2}) = r - 2 \), we have that \( p(N(x)) = \{r - 2, r - 1, r\} \).

\[\square\]

3.2. Uniqueness of the role graph

In the next lemmas we show some settings where the Cartesian product of the Construction 1 graph with \( P_2 \) has no \( r \)-role assignment. In Lemma 1, we show that this is the case when \( a_2 \) and \( a'_2 \) have the same role.

Lemma 1. Let \( \mathcal{H} \) be a hypergraph and \( G_r(\mathcal{H}) \), \( r \geq 4 \), the graph obtained from Construction 1. There is no \( r \)-role assignment \( p \) of \( G_r(\mathcal{H}) \square P_2 \) with \( p(a_2) = p(a'_2) \).

Proof. By contradiction, we suppose that \( p \) is a \( r \)-role assignment of \( G_r(\mathcal{H}) \square P_2 \) with \( p(a_2) = p(a'_2) = 2 \). Let \( R \) be the role graph of \( p \). It follows that \( N_R(1) = \{2\} \). Observe that, if the graph \( G_r(\mathcal{H}) \square P_2 \) is connected, then the role graph \( R \) of any role assignment of \( G_r(\mathcal{H}) \square P_2 \) is also connected. Hence, we may assume that \( k + 1 \in N_R(\{1, \ldots, k\}) \), for \( k = 2, \ldots, r - 1 \).

First, we show by induction that \( p(a_i) = i \) and \( p(a'_i) = i + 1 \), for \( i = 1, \ldots, r - 2 \). Clearly, \( p(a_1) = 1 \) and \( p(a'_1) = 2 \). Since \( N_R(1) = \{2\} \) and \( 3 \in N_R(2) \), we have that \( p(a'_2) = 3 \). Thus, \( p(a_2) = 2 \) and \( p(a'_2) = 3 \). Suppose, by induction hypothesis, that \( p(a_i) = i \) and \( p(a'_i) = i + 1 \), for \( i = 1, \ldots, k \), with \( k \in \{2, \ldots, r - 3\} \).

We observe that \( N_R(k) = N_R(p(a'_{k-1})) = \{k - 1, k + 1\} \) and \( k + 2 \in N_R(k + 1) \), thus \( p(a'_{k+1}) = k + 2 \). Since \( p(a_{k+1}) \in N_R(k) \cap N_R(k + 2) \), we have that \( p(a_{k+1}) = k + 1 \) and this proves the induction step.
Observe that $R - \{r - 1, r\}$ is isomorphic to the path $P_{r-2}$ and $N_R(r - 2) = \{r - 3, r - 1\}$. As $r \notin N_R(\{1, \ldots, r-2\})$, we have $r \in N_R(r - 1)$. We may assume that $p(a'_{r-1}) = r$. Since $p(a_{r-1}) \in N_R(r - 2) \cap N_R(r)$, we have that $p(a_{r-1}) = r - 1$. On the other hand, $R$ is not bipartite. Consequently, at least one of the roles $r - 1$ or $r$ has a loop. If there is a loop on $r - 1$, then $p(a'_r) = r - 1$. However, $a'_r$ has no neighbor with role $r - 2$, since $r - 2 \notin N_R(\{r - 2, r\})$. We conclude that $R$ is isomorphic to the path $P_r$ with loop on role $r$ and we can assume that $p(b'_r) = r$.

We remind that $p(a_{r-2}) = r - 2$, $p(a_{r-1}) = p(a'_r - 2) = r - 1$ and $p(a'_r - 1) = p(b'_r - 1) = r$. Since $N_R(r) = \{r - 1, r\}$ and there is no loop on role $r - 1$, we have that $p(b_{r-1}) = p(a'_r) = r$. We deduce that $p(a_r) = r - 1$. On the other hand, $r - 1 \in \{b'_r - 2, p(c'_{r-2})\}$. Therefore, $p(b_r) = p(b'_r) = r$. We look at the possible roles for $b'_r - 2$. Recall that $p(b'_r - 2) \in \{r - 1, r\}$.

If $p(b'_r - 2) = r - 1$, then $p(b_{r-2}) = r$ and $p(b'_{r-3}) = r - 2$. We show by induction that $p(b_i) = i + 1$ and $p(b'_i) = i + 1$, for $i = 1, \ldots, r - 2$. We observe that this induction is decreasing. Clearly, the roles hold for $i = r - 2$. Suppose that $p(b_i) = i + 2$ and $p(b'_i) = i + 1$ for $i = k, \ldots, r - 2$, for some $k \in \{2, \ldots, r - 2\}$. We show that $p(b_{k-1}) = k + 1$ and $p(b'_{k-1}) = k$. We have that $p(b'_k) = k + 1$ and $N_R(k + 1) = \{k, k + 1\}$, since $k + 1 \in \{3, \ldots, r - 1\}$. On the other hand, $p(N(b'_k)) = \{p(b'_{k-1}), k + 2\}$. Hence, $p(b'_{k-1}) = k$. Since $p(b_{k-1}) \in N_R(k) \cap N_R(k + 2)$ and $N_R(k) \cap N_R(k + 2) = \{k + 1\}$, then $p(b_{k-1}) = k + 1$. Therefore, we conclude that $p(b_1) = 3$, $p(b'_1) = 2$ and $b'_1$ has no neighbor with role 1, a contradiction.

If $p(b'_r - 2) = r$, then we have two possibilities for the role of $b_{r-2}$: $r - 1$ or $r$. In case $p(b_{r-2}) = r - 1$, with a similar argument of the previous paragraph we get a contradiction. If $p(b_{r-2}) = r$, we have that $p(b_{r-3}) = p(b'_{r-3}) = r - 1$, a contradiction, since there is no loop on $r - 1$. \qed

In the following lemmas, we show that when the roles of $a_1$, $a_2$ and $a'_1$ are distinct, the Cartesian product of the Construction 1 graph with $P_2$, has no $r$-role assignment. We prove specific cases for $r = 4$, 5 and 6 in Lemmas 2–4, respectively, and the general case in Lemma 5. First, we consider, in the next lemma, the case $r = 4$.

**Lemma 2.** Let $\mathcal{H}$ be a hypergraph and $G_4(\mathcal{H})$ the graph obtained from Construction 1. There is no 4-role assignment $p$ of $G_4(\mathcal{H}) \boxtimes P_2$, when $p(a_1)$, $p(a_2)$, and $p(a'_1)$ are all distinct.

**Proof.** By contradiction, we suppose that $p$ is a $r$-role assignment of $G_4(\mathcal{H}) \boxtimes P_2$ with $p(a_1) = 1$, $p(a'_1) = 2$ and $p(a_2) = 3$. Let $R$ be the role graph of $p$. It follows that $N_R(1) = \{2, 3\}$. Since, $G_4(\mathcal{H}) \boxtimes P_2$ is connected, so $R$ is. Therefore, we can assume that $4 \in N_R(\{2, 3\})$. We consider the possible roles for $a'_2$. By Lemma 1, $p(a'_2) \neq 1$.

If $p(a'_2) = 2$, then $N_R(2) = N_R(p(a'_2)) = \{1, 2\}$. This causes a contradiction, because $p(a_2) = 3$ and $p(a_2) \in N_R(2)$.

If $p(a'_2) = 3$, then $N_R(2) = \{1, 3\}$ and $N_R(3) = \{1, 2, 3, 4\}$. We may assume that $p(a_3) = 4$ and $p(a'_3) = 2$. Since $1 \in N_R(p(a_3))$ and $1 \notin N_R(4)$, we have that $p(a'_3) = 1$. Given that $N_R(2) = \{1, 3\}$, we have $p(b_3) = p(a'_3) = 3$. Since role 1 has no loop on and $1 \in N_R(3)$, we have that $1 \in p(\{b_2, c_2\})$. We remark that $N_R(1) = \{2, 3\}$ and $N_R(3) = \{1, 2, 3, 4\}$, we obtain that no vertices of degree two or three has role 3, and $2 \in p(\{b_2, c_1\})$. In both cases, this leads to a contradiction, since there is no possibility of vertices of role 3 in the neighborhood of neither $b_1$, nor $c_1$.

If $p(a'_2) = 4$, then $N_R(2) = \{1, 4\}$. Since $R$ is not bipartite, we may suppose that role 3 has a loop on, that means $N_R(3) = \{1, 3, 4\}$. Therefore, we can assume that $p(a_3) = 3$. We now consider whether there is a loop on role 4 or not.

If there is no loop on role 4, we have that $N_R(4) = \{2, 3\}$. On the other hand, $p(a'_3) \in N_R(3) \cap N_R(4)$, hence $p(a'_3) = 3$. Looking at the neighborhood of $a_3$, whose role is 3, we may assume that $p(b_3) = 4$. Since $N_R(3) \cap N_R(4) = \{3\}$, we have $p(b'_3) = 3$. We consider the possible roles for $b_2$. We recall that $p(b_2) \in N_R(4) = \{2, 3\}$. If $p(b_2) = 2$, then $p(b_1) \in \{1, 4\}$. Consequently $p(b'_1) = 3$, a contradiction by the degree of $b_1$ which is two. Otherwise, $p(b_2) = 3$ and in the same way, $p(c_2) = 3$. Since $p(b_3) = 4$ and $2 \in N_R(3)$, therefore $p(a_4) = 2$. That is a contradiction, since $2 \notin N_R(3)$.
If there is a loop on role 4, we have that \( N_R(4) = \{2, 3, 4\} \) and by Construction 1 \( p(\{b_1, b'_1\}) = p(\{c_1, c'_1\}) = \{1, 2\} \) and \( p(\{b_2, b'_2\}) = p(\{c_2, c'_2\}) = \{3, 4\} \). Since \( p(a'_3) \in N_R(3) \cap N_R(4) \), we have that \( p(a'_3) \neq 1 \) and we may assume that \( p(b_3) = 1 \). Hence, \( p(b_2) = p(c_2) = 3 \) and \( p(b'_2) = p(c'_2) = 4 \). Therefore, \( p(b_1) = p(c_1) = 1 \) and \( p(b'_1) = p(c'_1) = 2 \). It follows from the fact that \( 3 \in N_R(3) \), that \( p(b_4) = 3 \). Since \( 1 \notin N_R(4) \) and there is no loop on the role 1, we have that \( 1 \notin p(N_R(b_4)) \) is a contradiction. We remind that \( N_R(1) = \{2, 3\} \), \( N_R(3) = \{1, 3, 4\} \), and \( p(a_2) = 3 \), then \( p(a_4) = 3 \). □

Now, in Lemma 3, we consider the case \( r = 5 \).

**Lemma 3.** Let \( \mathcal{H} \) be a hypergraph and \( G_5(\mathcal{H}) \) the graph obtained from Construction 1. There is no 5-role assignment \( p \) of \( G_5(\mathcal{H}) \square P_2 \), when \( p(a_1), p(a_2), \) and \( p(a'_1) \) are all distinct.

**Proof.** By contradiction, we suppose that \( p \) is a \( r \)-role assignment of \( G_r(\mathcal{H}) \square P_2 \) \( p(a_1) = 1, p(a_2) = p(a'_1) = 2 \). Let \( R \) be the role graph of \( p \). It follows that \( N_R(1) = \{2, 3\} \). Since \( G_5(\mathcal{H}) \square P_2 \) is connected, so \( R \) is. Hence, we may assume that \( 4 \in N_R(\{2, 3\}) \) and \( 5 \in N_R(\{2, 3, 4\}) \). We consider the possible roles for \( a'_2 \). As before, if \( p(a'_2) = 2 \), then \( N_R(2) = \{1, 2\} \) and a contradiction, since \( 3 \in p(N(a'_2)) \). By Lemma 1, \( p(a'_2) \neq 1 \).

If \( p(a'_2) = 3 \), then \( N_R(2) = \{1, 3\} \) and we conclude that \( \{1, 2, 3, 4\} \subseteq N_R(3) \). However, \( |N(a'_2)| = 3 \), so we do not have enough neighbors, a contradiction. As \( p(a'_2) \notin \{1, 2, 3\} \), we may assume that \( p(a'_2) = 4 \) and \( N_R(2) = \{1, 4\} \).

Given the configuration of the roles so far and the fact that \( 5 \in N_R(\{3, 4\}) \), we may assume that \( 5 \in N_R(3) \). Hence, \( p(a_3) = 5 \) and \( N_R(3) = \{1, 4, 5\} \). We consider the potential roles for \( a'_3 \). We observe that \( p(a'_3) \in N_R(5) \subseteq \{3, 4, 5\} \). If \( p(a'_3) = 3 \), then \( N_R(4) = \{2, 3\} \) and we have a loop on role 5, since \( R \) is not bipartite. Hence, \( N_R(5) = \{3, 5\} \) and we may assume that \( p(a_4) = 5 \). We have that \( p(a'_4) \in N_R(3) \cap N_R(5) \), that is, \( p(a'_4) = 5 \). Since \( 1 \in N_R(3) \), we have that \( p(a'_3) = 1 \). Therefore, \( p(a_5) \in N_R(1) \cap N_R(5) \), that is, \( p(a_5) = 3 \). Hence, \( 4 \notin p(N(a_5)) \), because \( p(\{b_1, b'_1\}) \subseteq N_R(5) \) is a contradiction.

If \( p(a'_3) = 4 \), then \( N_R(4) = \{2, 3, 4\} \), a contradiction, because \( p(a_3) = 5 \).

If \( p(a'_3) = 5 \), then \( N_R(4) = \{2, 3, 5\} \) and \( N_R(5) = \{3, 4, 5\} \). We may assume that \( p(a_4) = 4 \). As \( p(a'_3) \in N_R(4) \cap N_R(5) \), we have that \( p(a'_4) \notin 2 \) and we may assume that \( p(b_2) = 2 \). Since \( |N(b_1)| = |N(b'_1)| = 2 \), we have \( p(\{b_1, b'_1\}) \subseteq \{1, 2\} \). Given that there is no loop on roles 1 and 2, we conclude that \( p(\{b_1, b'_1\}) = \{1, 2\} \). Therefore, \( p(\{b_2, b'_2\}) = \{3, 4\} \) and \( p(b_3) = p(b'_3) = 5 \), a contradiction, because \( p(b_4) = 2 \) and \( 2 \notin N_R(5) \). □

Next, in Lemma 4 we consider the case \( r = 6 \).

**Lemma 4.** Let \( \mathcal{H} \) be a hypergraph and \( G_6(\mathcal{H}) \) the graph obtained from Construction 1. There is no 6-role assignment \( p \) of \( G_6(\mathcal{H}) \square P_2 \), when \( p(a_1), p(a_2), \) and \( p(a'_1) \) are all distinct.

**Proof.** By contradiction, we suppose that \( p \) is a \( r \)-role assignment of \( G_r(\mathcal{H}) \square P_2 \) \( p(a_1) = 1, p(a_2) = p(a'_1) = 2 \). Let \( R \) be the role graph of \( p \). It follows that \( N_R(1) = \{2, 3\} \). Since \( G_6(\mathcal{H}) \square P_2 \) is connected, so \( R \) is. We consider \( 4 \in N_R(\{2, 3\}), 5 \in N_R(\{2, 3, 4\}) \) and \( 6 \in N_R(\{2, 3, 4, 5\}) \). We consider the possible roles for \( a'_2 \). As before, if \( p(a'_2) = 2 \), then \( N_R(2) = \{1, 2\} \) and a contradiction, since \( 3 \in p(N(a'_2)) \). By Lemma 1, \( p(a'_2) \neq 1 \). Similar as the proof of Lemma 3, we may assume that \( p(a'_2) = 4 \) and \( N_R(2) = \{1, 4\} \).

Given the configuration of the roles so far and the fact that \( 5 \in N_R(\{3, 4\}) \) and \( N_R(3) = \{1, 4, 5\} \). We may assume that \( p(a_3) = 5 \). And we consider the feasible roles for \( a'_3 \). Since \( p(a'_3) \in N_R(4) \cap N_R(5) \), we have \( p(a'_3) \in \{3, 4, 5, 6\} \).

If \( p(a'_3) = 3 \), then \( N_R(4) = \{2, 3\} \), \( 6 \in N_R(5) \) and \( p(a_4) = 6 \). Since \( 1 \in N_R(3) \), we have that \( p(a'_3) = 1 \), a contradiction, since \( 1 \notin N_R(6) \). If \( p(a'_4) = 4 \), then \( N_R(4) = \{2, 3, 4\} \), but \( p(a'_3) = 5 \), a contradiction. If \( p(a'_3) = 5 \), then \( N_R(4) = \{2, 3, 5\} \). Accordingly, \( N_R(5) = \{3, 4, 5, 6\} \) is a contradiction, since \( |N(a'_3)| = 3 \). So, we have that \( p(a'_3) = 6 \) and \( N_R(4) = \{2, 3, 6\} \).

Since, \( R \) is not bipartite and we may assume that \( R \) has a loop on role 6. Consequently, \( p(a'_4) = 6 \) and \( N_R(6) = \{4, 5, 6\} \). As \( p(a_4) \in N_R(5) \cap N_R(6) \), we consider the potential for \( p(a_4) \) are roles 5 and 6.
If \( p(a_4) = 5 \), then \( N_R(5) = \{3,5,6\} \) and we may assume that \( p(a_5) = 3 \). Since \( p(a'_4) \in N_R(3) \cap N_R(6) \), we have that \( p(a'_4) \neq 1 \) and we may assume that \( p(b_3) = 1 \). On the other hand \( |N(b_3)| = |N(b'_3)| = 2 \), since there are no loop on roles 1 and 2, we have that \( p(\{b_1, b'_1\}) = \{1,2\} \). Therefore, \( p(\{b_2, b'_2\}) = \{3,4\} \) and \( p(\{b_3, b'_3\}) = p(\{b_4, b'_4\}) = \{5,6\} \). Note that \( 1 \notin N_R(\{5,6\}) \) and we have a contradiction.

If \( p(a_4) = 6 \), then \( N_R(5) = \{3,6\} \). On the other hand, since \( N_R(6) = \{4,5,6\} \), we may assume that \( p(a'_4) = 4 \) and \( p(a'_5) = 5 \). We remind that \( 2 \in N_R(4) \), but all neighbors of \( a'_5 \) are adjacent to a vertex role 5 or 6, a contradiction, given that \( 2 \notin N_R(\{5,6\}) \). \( \square \)

After considered the particular cases, we conclude in Lemma 5 that, for \( r \geq 4 \), the Cartesian product of the Construction 1 graph with \( P_2 \) has no \( r \)-role assignment, when the roles of \( a_1 \), \( a_2 \) and \( a'_1 \) are distinct.

**Lemma 5.** Let \( \mathcal{H} \) be a hypergraph and \( G_r(\mathcal{H}) \), with \( r \geq 4 \), the graph obtained from Construction 1. There is no \( r \)-role assignment \( p \) of \( G_r(\mathcal{H}) \square P_2 \), such that \( p(a_1), p(a_2) \), and \( p(a'_1) \) are all distinct.

**Proof.** By the Lemmas 2, 3 and 4, the result follows, for \( r = 4, 5 \) and \( 6 \), respectively. From now on, we consider, \( r \geq 7 \). By contradiction, we suppose that \( p \) is a \( r \)-role assignment of \( G_r(\mathcal{H}) \square P_2 \) and \( p(a_1) = 1 \), \( p(a'_1) = 2 \) and \( p(a_2) = 3 \). Let \( R \) be the role graph of \( p \). It follows that \( N_R(1) = \{2,3\} \). Since \( G_r(\mathcal{H}) \square P_2 \) is connected, so \( R \) is.

Hence, we may assume that \( k+1 \notin N_R(\{1,\ldots,k\}) \), for \( k = 3,\ldots, r-1 \). We consider the possible roles for \( a'_2 \).

As before, if \( p(a'_2) = 2 \), then \( N_R(2) = \{1,2\} \) is a contradiction, since \( 3 \notin p(N(a'_2)) \). By Lemma 1, \( p(a'_2) \neq 1 \).

If \( p(a'_2) = 3 \), then \( \{1,2,3,4\} \subseteq N_R(3) \), but \( |N_R(a'_2)| = 3 \) and \( a'_2 \) does not have enough neighbors, a contradiction. Therefore, we can assume \( p(a'_2) = 4 \).

Let \( t \) be the largest integer, such that, \( |\{p(a_i), p(a'_i) \mid i = 1,\ldots, t\}| = 2t \). Clearly, \( t \geq 2 \) and \( 2t \leq r \). For sake of simplicity, let \( p(a_i) = 2t-1 \) and \( p(a'_i) = 2t \), for \( i = 1,\ldots, t \). We observe that \( N_R(1) = \{2,3\} \), \( N_R(2) = \{1,4\} \) and for \( i = 2,\ldots, t-1 \), \( N_R(2i-1) = \{2i-3,2i,2i+1\} \), \( N_R(2i) = \{2i-2,2i-1,2i+2\} \). We analyze three cases: \( 2t \leq r-2 \), \( 2t = r-1 \) and \( 2t = r \).

**Case 2t ≤ r − 2.**

In this case, \( 2t+1 \in N(\{2t-1,2t\}) \) and since \( t \geq 2 \), \( t \leq r-t-2 \leq r-4 \). We may assume that \( p(a_{t+1}) = 2t+1 \). Hence, \( N_R(2t-1) = \{2t-3,2t,2t+1\} \). By the hypotheses about \( t \), \( p(a'_t+1) \neq 2t+2 \) and as we know that \( 2t+2 \leq r \), \( 2t+2 \in N_R(2t+1) \). On the other hand, \( t+1 \leq r-3 \) and \( |N(a_{t+1})| = 3 \), which implies that \( p(a_{t+2}) = 2t+2 \).

We show a contradiction for every possible role of \( a'_t+1 \). We observe that \( p(a'_t+1) \in \{2t-1,2t,2t+1\} \). If \( p(a'_t+1) = 2t-1 \), then \( p(a'_t+2) = 2t-3 \) is a contradiction, since \( 2t+2 \notin N_R(2t-3) \).

If \( p(a'_t+1) = 2t \), then \( N_R(2t) = \{2t-2,2t-1,2t,2t+1\} \) is a contradiction, because \( |N(a'_t+1)| = 3 \).

If \( p(a'_t+1) = 2t+1 \), then \( N_R(2t+1) = \{2t-1,2t,2t+1,2t+2\} \), which again leads to a contradiction.

**Case 2t = r − 1, that is, r = 2t + 1.**

As \( r \geq 7 \), we have \( t \geq 3 \) and \( t \leq r-4 \). Analogously to the previous case, we may assume that \( p(a_{t+1}) = 2t+1 \) and \( N_R(2t-1) = \{2t-3,2t,2t+1\} \). We observe that \( p(a'_t+1) \in \{2t-2,2t-1,2t,2t+1\} \) and we reach a contradiction for each of the potential roles of \( a'_t+1 \).

If \( p(a'_t+1) = 2t-2 \), then we have a contradiction, because \( 2t+1 \in N_R(2t+2) \). If \( p(a'_t+1) = 2t-1 \), then \( N_R(2t) = \{2t-2,2t-1\} \). As, \( t+1 \leq r-3 \), \( p(a'_t+2) = 2t-3 \). Thus, \( p(a_{t+2}) \in N_R(2t-3) \cap N_R(2t+1) \), that is \( p(a_{t+2}) = 2t-1 \). Therefore, \( N_R(2t+1) = \{2t-1\} \). Hence, \( R \) is bipartite, a contradiction. If \( p(a'_t+1) = 2t \), then \( N_R(2t) = \{2t-2,2t-1,2t,2t+1\} \), a contradiction to the fact that \( |N_R(a'_t+1)| = 3 \). We conclude that, \( p(a'_t+1) = 2t+1 \), \( N_R(2t) = \{2t-2,2t-1,2t+1\} \) and \( N_R(2t+1) = \{2t-1,2t,2t+1\} \). We observe that all neighborhood of roles of \( R \) are defined, since \( r = 2t+1 \).

Recall that \( t \leq r-4 \), that is, \( t+1 \leq r-3 \). Therefore, \( p(a_{t+2}) = 2k \) and \( p(a'_t+2) = 2t-1 \). It is easy to see, using induction, that \( p(a_i) = 2(r-i) + 2 \) and \( p(a'_i) = 2(r-i) + 1 \) for \( i = t+2, \ldots, r-2 \).

We deduce that \( p(a_{r-2}) = 2(r-(r-2)) + 2 = 6 \) and \( p(a'_{r-2}) = 5 \). Hence, \( 3 \in p(\{a'_{r-1}, a'_r\}) \). We may assume that \( p(a'_{r-1}) = 3 \). Since \( 1 \notin N_R(6) \), we can suppose that \( p(b'_{r-1}) = 1 \).
Clearly, $p(\{b_i, b'_i\}) = p(\{a_i, a'_i\})$, for $i = \{1, \ldots, r - 2\}$. We conclude that $p(b'_{r-2}) \in \{5, 6\}$, a contradiction, since $p(b'_{r-1}) = 1$.

**Case 2t = r.**

We observe that, in this case, $r$ is even and therefore $r \geq 8$. We remind that $N_R(1) = \{2, 3\}$, $N_R(2) = \{1, 4\}$ and for $i = 2, \ldots, t - 1$, with $N_R(2i - 1) = \{2i - 3, 2i, 2i + 1\}$ and $N_R(2i) = \{2i - 2, 2i - 1, 2i + 2\}$. On the other hand, $R$ is not bipartite and we may assume that there is a loop on the role $r$. As $r \geq 8$, $4 \leq t \leq r - 4$, we have that $p(a'_{t+1}) = 2t = r$ and $N_R(2t) = \{2t - 2, 2t - 1, 2t\}$.

We deduce that $p(a_{t+1}) \in N_R(2t - 1) \cap N_R(2t)$. Clearly, $p(a_{t+1}) \in \{2t - 1, 2t\}$.

If $p(a_{t+1}) = 2t$, then $N_R(2t - 1) = \{2t - 3, 2t\}$. We remark that $N_R(2t) = \{2t - 2, 2t - 1, 2t\}$ and $t + 1 \leq r - 3$. However, $p(N(a_{t+1})) = \{2t, p(a_{t+2})\}$, a contradiction. Therefore, $p(a_{t+1}) = 2t - 1$ and $N_R(2t - 1) = \{2t - 3, 2t - 1, 2t\}$. Clearly, $p(a_{t+2}) = 2t - 3$ and $p(a'_{t+2}) = 2t - 2$. It is easy to see, using induction, that $p(a_i) = 2(r - i) + 1$ and $p(a'_i) = 2(r - i) + 2$ for $i = t + 1, \ldots, r - 2$. We conclude that $p(a_{r-2}) = 5$ and $p(a'_{r-2}) = 6$. Hence, $3 \in p(\{a_{r-1}, a_r\})$. We can suppose that $p(a_{r-1}) = 3$. Since $1 \notin N_R(6)$, we can suppose that $p(b_{r-1}) = 1$.

Similarly, using induction, it is possible to show that $p(\{b_i, b'_i\}) = p(\{a_i, a'_i\})$, for $i = 1, \ldots, r - 2$. We conclude that $p(b'_{r-2}) \in \{5, 6\}$ which leads to a contradiction, since $p(b'_{r-1}) = 1$.

Now, we show that the role graph of a $r$-role assignment of the Cartesian product of $G_r(\mathcal{H}) \square P_2$ is unique. For this purpose, we will use the graph $R_r$ of Figure 2. For a better use, let recall this graph in Figure 3.

The proof is separated in two lemmas: Lemma 6 for the specific case when $r = 4$ and Lemma 7 for the general case.

**Lemma 6.** Let $\mathcal{H}$ be a hypergraph and $G_4(\mathcal{H})$ the graph obtained from Construction 1. If $G_4(\mathcal{H}) \square P_2$ has a $R$-role assignment with $|V(R)| = 4$, then $R \simeq R_4$.

**Proof.** Let $p$ be a $R$-role assignment of $G_4(\mathcal{H}) \square P_2$, with $|V(R)| = 4$. Suppose that $p(a_1) = 1$. Since, $G_4(\mathcal{H}) \square P_2$ is connected, so $R$ is. Hence, we may assume that $2 \in N_R(1)$, $3 \in N_R(\{1, 2\})$ and $4 \in N_R(\{1, 2, 3\})$. We observe that $2 \in p(N(a_1))$ and $N(a_1) = \{a'_1, a_2\}$. By Lemma 1, $p(a_2) \neq p(a'_1)$. It follows from the Lemma 5 that $p(a_1), p(a_2)$ and $p(a'_1)$ are not all distinct and we conclude that $p(\{a'_1, a_2\}) = \{1, 2\}$. If $p(a'_1) = 2$ and $p(a_2) = 1$, then $N_R(1) = \{1, 2\}$. Hence, $3 \in N_R(2)$ and $p(a'_2) = 3$, a contradiction. Therefore, $p(a'_1) = 1$ and $p(a_2) = 2$.

We show that $R \simeq R_4$. We remark that $p(a_1) = 1$, $p(a'_1) = 1$, $p(a_2) = 2$ and $N_R(1) = \{1, 2\}$. Hence, $p(a'_2) = 2$ and $\{1, 2, 3\} \subseteq N_R(2)$. Since $|N(b_1)| = |N(b'_1)| = 2$, we have $2 \notin p(\{b_1, b'_1\})$. In the same way, $2 \notin p(\{c_1, c'_1\})$. We may assume that $p(a_3) = 3$. We divide in two cases, depending whether role 4 belongs or not to the neighborhood of the role 2.

If $4 \notin N_R(2)$, then $4 \in N_R(3)$, $p(a'_3) \in \{2, 3\}$ and we may assume that $p(b_3) = 4$. If $p(a'_3) = 2$, then $p(a'_3) = 3$. Therefore, all neighbors of $a'_3$ are adjacent to the vertices with role 3 or 4, that is, $1 \notin p(N(a'_3))$ a contradiction. Thus, $p(a'_3) = 3$ and $|N_R(3)| = \{2, 3, 4\} = 3$. Again, we have that $3 \notin p(\{b_1, b'_1\})$, that is, $|p(\{b_1, b'_1\})| \subseteq \{1, 4\}$. Since there is no edge between roles 1 and 4, we have that $p(b_1) = p(b'_1)$. On the other hand, $p(b_2) \neq 2$, since $4 \notin N_R(2)$. We conclude that $p(b_1) = p(b'_1) = 4$ and $p(b_2) = p(b'_2) = 3$. In the same way, we get $p(c_2) = p(c'_2) = 3$. Observing the neighborhood of $b_2$, which has role 3, we obtain that $p(b_4) = 2$. 

![Figure 3. Role graph $R_r$, with $r \geq 4$.](image)
However, all neighbors of $b_4$ have role 3, except for $a_4$. Nonetheless, $p(a_4) \neq 1$, since $p(b_3) = 4$ and $1 \not\in N_R(4)$. Therefore, $1 \not\in p(N(b_2))$, a contradiction.

If $4 \in N_R(2)$, then $N_R(2) = \{1,2,3,4\}$ and $p(a_4) = 4$. We remind that $p(b_1), p(b'_1) \in \{1,3,4\}$. If $p(b_1) = 3$, then $|N_R(3)| = 2$ and $p(b'_1) \in \{3,4\}$. In both cases, $p(b_2) = p(b'_2) = 2$. However, all neighbors of $b_2$ has adjacent vertices with role 3 or 4. Therefore, $1 \not\in p(N(b_2))$. In the same way, we get a contradiction when $p(b_1) = 4$. We conclude that, $p(b_1) = 1$ and thus $p(b'_1) = 1$. Hence, $p(b_2) = p(b'_2) = 2$. We conclude that $p(\{b_3,b_4\}) = \{3,4\}$ and $R \simeq R_3$.

In the following lemma, we show the uniqueness of the role graph for the general case.

**Lemma 7.** Let $\mathcal{H}$ be a hypergraph, $r \geq 4$ and $G_r(\mathcal{H})$ the graph obtained from Construction 1. If $G_r(\mathcal{H}) \square P_2$ has a $R$-role assignment with $|V(R)| = r$, then $R \simeq R_r$.

**Proof.** Let $p$ be a $R$-role assignment of $G_r(\mathcal{H}) \square P_2$ and $|V(R)| = r$. By Lemma 6, we can assume that $r \geq 5$. Suppose that $p(a_1) = 1$. Since, $G_r(\mathcal{H}) \square P_2$ is connected, so $R$ is. Hence, we may assume that, $V(R) = \{1,\ldots,r\}$ and $k + 1 \in N_R(\{1,\ldots,k\})$, for $k = 1,\ldots,r - 1$. We observe that $2 \in p(N(a_1))$ and $N(a_1) = \{a'_1, a_2\}$. By Lemma 1, $p(a_2) \neq p(a'_1)$. It follows from the Lemma 5 that $p(a_1), p(a_2)$ and $p(a'_1)$ are not all distinct and we conclude that $p(\{a'_1, a_2\}) = \{1,2\}$. If $p(a'_1) = 2$ and $p(a_2) = 1$, then $N_R(1) = \{1,2\}$. Hence, $3 \in N_R(2)$ and $p(a'_2) = 3$, a contradiction. Therefore, $p(a'_1) = 1$ and $p(a_2) = 2$.

We remark that $p(a_1) = p(a'_1) = 1$ and $p(a_2) = 2$. Hence, $N_R(1) = \{1,2\}$ and $p(a'_2) = 2$. By induction, we have that $p(a_i) = p(a'_i) = i$ for $i = 1,\ldots,r - 2$. We note that in this case, except for $i = 2, N_R(i - 1) = \{i - 2, i - 1, i\}$.

We consider the path $B = b_1 \ldots b_{r-2}$ of $G_r(\mathcal{H})$. We show that $p(b_i) = p(b'_i) = i$, for $i = 1,\ldots,r - 2$. Similarly, we get the roles for the vertices of the paths $C$ and $W_j$, with $j = 1,\ldots,m$.

We observe that the graph $R - \{r - 1, r\}$ is isomorphic to the path $P_{r-2}$ with loop on all roles and $r - 1 \in N_R(r - 2)$. Hence, $\{r - 3, r - 2, r - 1\} \subseteq N_R(r - 2)$. We may assume that $p(a_{r-1}) = r - 1$. Next, we consider the potential roles for $p(b_1)$. We remark that $|N(b_1)| = 2$ and therefore $p(b_1) \in \{1, r - 1, r\}$. In the same way, $p(b'_1) \in \{1, r - 1, r\}$.

If $p(b_1) = 1$, then similarly to the path $A$, we have that $p(b_i) = p(b'_i) = i$, for $i = 1,\ldots,r - 2$. Hence, we may assume that, $p(b_{r-1}) = r - 1$. We remark that, $p(a_{r-1}) = r - 1$ and $r \in N_R(\{r - 2, r - 1\})$. Since all neighbors of $a_{r-1}$ are neighbors of vertices of role $r - 2$, we have that, $r \in N_R(r - 2)$ and $N_R(r - 2) = \{r - 3, r - 2, r - 1, r\}$. Therefore, $p(a_r) = r$. In the same way, $p(b_r) = r$ and we have that, $R \simeq R_r$.

Next, we reach contradictions for the remaining cases. If $p(b_1) = r - 1$, then as $p(b'_1) \in \{1, r - 1, r\}$ we have that $p(b_2) = r - 2$. Since $|N(b_2)| = 3$ and $r \geq 5$, we have that, $N_R(r - 2) = \{r - 3, r - 2, r - 1\}$. We get that, $r \in N_R(r - 1)$ and therefore $p(b'_2) = r$. Hence, $p(b'_2) \in N_R(r - 2) \cap N_R(r)$, that is, $p(b'_2) = r - 1$, a contradiction, because $p(N(b_2)) = \{r - 1, 1, p(b_1)\}$.

If $p(b_1) = r$, then from the previous cases, we may assume that, $p(b'_1) = r$ and $r \not\in N_R(r - 2)$. We find that, $N_R(r - 2) = \{r - 3, r - 2, r - 1\}$ and $N_R(r) = \{r, r - 1\}$. Therefore, $p(b_2) = p(b'_2) = r - 1$ and the graph $R$ is isomorphic to the path $P_r$ with loops on all roles. Clearly, by induction, $p(b_i) = p(b'_i) = r - i + 1$, for $i = 2,\ldots,r - 2$. We conclude that, $p(b_{r-2}) = p(b'_{r-2}) = r - (r - 2) + 1 = 3$. Since $p(b_{r-3}) = 4$, we may assume that, $p(b_{r-1}) = 2$. We remark that $p(a_{r-1}) = r - 1$, a contradiction, once $r \geq 5$. □

### 3.3. NP-completeness

We show that one can obtain a 2-coloring of $\mathcal{H}$ when it is given a $r$-role assignment of $G_r(\mathcal{H}) \square P_2$, $r \geq 4$.

**Theorem 2.** Let $\mathcal{H}$ be a hypergraph and $G_r(\mathcal{H})$ with $r \geq 4$, the graph obtained from Construction 1. If $G_r(\mathcal{H}) \square P_2$ has a $R$-role assignment, then $\mathcal{H}$ has a 2-coloring.

**Proof.** Let $p$ be a $R$-role assignment of $G_r(\mathcal{H}) \square P_2$, with $|V(R)| = r \geq 4$. By Lemma 7, we may assume that, $R = R_r$.

Consider the path $A = a_1 \ldots a_{r-2}$ in $G_r(\mathcal{H})$. Similarly, we assign the roles for the paths $B, C$ and $W_j S_j$ with $j = 1,\ldots,m$. 

We show, by induction, that \( p(a_i) = p(a'_i) = i \), for \( i = 1, \ldots, r - 2 \). Since \(|N(a_1)| = |N(a'_1)| = 2\), we have that, \( P(a_i) = P(a'_i) = 1 \). Suppose that, \( p(a_i) = p(a'_i) = i \), for \( i = 1, \ldots, k \) with \( k \in \{1, \ldots, r - 3\} \). We remind that, \( N_R(1) = \{1, 2\} \) and \( N_R(i) = \{i - 1, i, i + 1\} \), for \( i = 2, \ldots, r - 3 \). Observing the neighborhood of \( a_k \), we get that \( P(a_{k+1}) = k + 1 \). In the same way, \( P(a'_{k+1}) = k + 1 \).

We want to define a 2-coloring of \( \mathcal{H} \), \( c : \mathcal{V}(\mathcal{H}) \rightarrow \{r - 1, r\} \), using \( p \). First, we show that, \( P(\mathcal{V}(\mathcal{H})) = \{r - 1, r\} \). Observe that, for \( j = 1, \ldots, m \), \( P(w_{j, r - 3}) = r - 3 \) and \( P(S_j) = P(S'_j) = r - 2 \). Since, \( m \geq 1 \) and \( N_R(r - 2) = \{r - 3, r - 2, r - 1, r\} \), we have that, \( \{r - 1, r\} \subseteq P(\mathcal{V}(\mathcal{H})) \). In the same way, \( \{r - 1, r\} \subseteq P(\mathcal{V}(\mathcal{V})) \).

Since \( \mathcal{V}(\mathcal{H}) \) induces a clique on \( G_r(\mathcal{H}) \), we have that \( P(\mathcal{V}(\mathcal{H})) \subseteq \{r - 2, r - 1, r\} \). However, if a vertex has role \( r - 2 \), then it will have no neighbor with role \( r - 3 \), a contradiction. Therefore, \( P(\mathcal{V}(\mathcal{H})) = \{r - 1, r\} \).

We define \( c(v) = p(v) \) for every \( v \in \mathcal{V}(\mathcal{H}) \). We show that \( c \) is a 2-coloring of \( \mathcal{H} \). Let \( S_j \in S(\mathcal{H}) \), for some \( j \in \{1, \ldots, m\} \). We remind that, \( P(S_j) = r - 2 \) and \( P(N(S_j)) = \{r - 2, r - 3\} \cup \{p(v) | v \in S_j \text{ in } \mathcal{H}\} \). Since \( N_R(r - 2) = \{r - 3, r - 2, r - 1, r\} \), there exists \( s, t \in \{1, \ldots, n\} \), such that, \( v_s, v_t \in S_j \) in \( \mathcal{H} \), \( p(v_s) = r - 1 \) and \( p(v_t) = r \). Therefore, \( c \) is a 2-coloring of \( \mathcal{H} \).

The above results imply directly in the \( \text{NP} \)-completeness of \( r \)-Role Assignment, \( r \geq 4 \).

**Theorem 3.** The problem \( r \)-Role Assignment, for any fixed \( r \geq 4 \), is \( \text{NP} \)-complete even when restricted to the Cartesian product of two non-trivial graphs.

**Proof.** The problem is clearly in \( \text{NP} \) (cf. Roberts and Sheng [18]). To show the \( \text{NP} \)-hardness, we use a reduction from the \( \text{NP} \)-complete problem HYPERGRAPH 2-COLORING [11]. Given a hypergraph \( \mathcal{H} \), we construct the graph \( G_r(\mathcal{H}) \) according to Construction 1, which is used to compute \( G_r(\mathcal{H}) \square P_2 \). It is easy to see that \( G_r(\mathcal{H}) \square P_2 \) may be obtained in polynomial time. By Theorems 1 and 2, we obtain that, \( \mathcal{H} \) has 2-coloring if and only if \( G_r(\mathcal{H}) \square P_2 \) has a \( r \)-role assignment for any fixed \( r \geq 4 \), and the proof is complete. \( \square \)

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**REFERENCES**


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