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COMPUTING ROLE ASSIGNMENTS OF CARTESIAN PRODUCT OF GRAPHS

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Abstract. Network science is a growing field of study using Graph Theory as a modeling tool. In social networks, a role assignment is such that individuals play the same role, if they relate in the same way to other individuals playing counterpart roles. In this sense, a role assignment permit to represent the network through a smaller graph modeling its roles. This leads to a problem called r-Role Assignment whose goal is deciding whether it exists such an assignment of r distinct roles. This problem is known to be NP-complete for any fixed $r \geq 2$. The Cartesian product of graphs is a well studied graph operation, often used for modeling interconnection networks. Formally, the *Cartesian product* of G and H is a graph, denoted as $G\square H$, whose vertex set is $V(G) \times V(H)$ and two vertices (u, v) and (x, y) are adjacent precisely if $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$. In a previous work, we showed that Cartesian product of graphs are always 2-role assignable, however the 3-Role Assignment problem is NP-complete on this class. In this paper, we prove that r-Role Assignment restricted to Cartesian product graphs is still NP-complete, for any fixed $r \geq 4$.

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1. Introduction

Nowadays, social networks are a part of everybody's life. Up to study their behavior, a social network is conceptualized as a graph where vertices represent individuals and edges the relationship between them. In 1980, Augluin introduced the concept of *covering* from which role assignment arise, as a tool for networks of processors [1]. A decade later, based on graph models for social networks, Everett and Borgatti [8] formalized role assignment under the name of *role coloring*.

Indeed, a r-*role assignment* of a simple graph G is an assignment of r distinct roles to the vertices of G such that if two vertices have the same role, then the sets of roles of their neighbors are the same. Moreover, from such an assignment, we obtain a *role graph* where vertices are the r distinct roles and there is an edge between two roles whenever there are two neighbors in the graph G that correspond to those roles. Note that, the role graph has no multiple edges, but permit loops since two related vertices in G can have the same role. Observe that, while a social network usually give rise to a large graph, a role assignment allows to represent the same network through a smaller graph. On the other hand, when the role graph is a graph without loops, where every pair of vertices are neighbors, role assignment coincides with *k-fall coloring*, introduced by Dunbar *et al.* [7]

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as a variant of *vertex coloring*. More recently, Kaul and Mitillos [14] studied relations between the parameters associated to fall coloring and vertex coloring.

We define the r-Role Assignment problem as follows:

r-Role Assignment
Instance: A simple graph G.
Question: Does G admit a r-role assignment?

Applications of role assignment are highlighted in several contexts such as social networks [8, 18] and distributed computing [3,4]. Furthermore, Pekeč and Roberts [17] showed that any network represented by a graph, with minimum degree bounded by a suitable bound that depends on r, has a r-role assignment.

The first study devoted to determine the computational complexity of the problem has appeared in 2001, by Roberts and Sheng [18]. They proved the NP-completeness of 2-Role Assignment. Such a result was strengthened in 2005, by Fiala and Paulusma [9], who showed that r-Role Assignment is NP-complete for any fixed $r \geq 3$. On the positive side, r-Role Assignment can be solved in polynomial time for trees [10] and for proper interval graphs [13] for any fixed $r \geq 1$. Considering chordal and split graphs, a dichotomy for the complexity of r-Role Assignment arises. While for chordal graphs, the problem is solvable in linear time for $r = 2$ and NP-complete for $r \geq 3$ [20]; for split graphs, the problem is trivial, with true answer, for $r = 2$, solvable in polynomial time for $r = 3$, and NP-complete for any fixed $r \geq 4$ [6].

Cartesian product of graphs was introduced by Sabidussi [19] in 1959. Since then, it has been applied in many areas, such as space structures [15] and interconnection networks [5]. Some properties of role assignments has also been studied for Cartesian product and other graph operations [21]. However, the only operation that study the complexity of r-Role Assignment problem is the complementary prism, which arise from complementary product, a generalization of Cartesian product. In fact, 2-Role Assignment has a true answer for all complementary prisms, except the one of a path with three vertices [2].

Related to fall coloring, Dunbar *et al.* [7] proved that, given a graph G, determining whether G has a k-fall coloring is NP-complete, for each fixed $k \geq 3$. In the same work, they also showed some necessary conditions for Cartesian and categorical product of graphs to admit a fall coloring. After that, Laskar and Lyle [16] showed that the k-fall coloring for $k \geq 3$ is NP-complete for bipartite graphs and they also determine some fall colourable Cartesian product of graphs.

In 2021, we initiated a study of the r-Role Assignment problem for Cartesian product of graphs [2]. In that work, we show the NP-completeness of the problem when $r = 3$ and we prove that it has a true answer for $r = 2$. In this paper, we answer positively a conjecture opened by this previous work, by proving that the r-role assignment problem for Cartesian product of graphs is NP-complete for any fixed $r \geq 4$.

This paper is organized in more two sections. In Section 2, we set notations and terminology. Our complexity results concerning the r-Role Assignment problem, $r \geq 4$, for Cartesian product of graphs follow in Section 3.

2. Preliminaries

All graphs considered are undirected, finite, non-trivial and have no multiple edges. A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. The vertices u and v are *adjacent* or *neighbors* if they are joined by an edge e, also denoted by uv. In this case, u and v are *incident* to e and e is *incident* to u and v. A *loop* is an edge incident to only one vertex. The *neighborhood* of a vertex v, denoted by $N_G(v)$, is the set of all neighbors of v in G. When the graph G is clear from the context, we simply write $N(v)$. A *simple graph* is a graph without loops. In a simple graph G, the *degree* of a vertex v is the cardinality of $N_G(v)$. The neighborhood of a subset U of $V(G)$, denoted as $N_G(U)$, is the union of the neighborhoods of the vertices of U. If U is any set of vertices in G, we denote by $G - U$ the graph obtained by deleting the vertices in U and all edges incident with any of them.

A *path* is a sequence of distinct vertices with an edge between each pair of consecutive vertices. For $n \geq 2$, we denote a path on n vertices by P_n or by the sequence of vertices $v_1 \dots v_n$. A *clique* is a subset of vertices that are adjacent to each other. A graph G is called *bipartite graph* if we can partition $V(G) = A \cup B$ so that if there is an edge $uv \in E(G)$, then $u \in A$ and $v \in B$, or *vice versa*. Hence, a *bipartite graph with loops* have edge that connects a vertex to itself same partition.

Given a simple graph G and a graph R, possibly with loops. A *R-role assignment* of G is a surjective vertex mapping $p : V(G) \to V(R)$ such that $p(N_G(v)) = N_R(p(v))$ for all $v \in V(G)$. A graph G has a *r-role assignment* if it admits a R-role assignment for some graph R, called the *role graph*, with $|V(R)| = r$. We set $1, \dots, r$ the vertices of R, also called *roles*. For now on, all graphs (except maybe the role graph) are simple. Observe that, if the graph G is connected, then the role graph R of any role assignment of G is also connected. Also, if role graph R is bipartite, then so G is.

For Cartesian product, we follow the terminology of Hammack *et al.* [12]. Let G and H be two graphs. The *Cartesian product* of G and H is a graph, denoted as $G \square H$, whose vertex set is $V(G) \times V(H)$ and two vertices (u, v) and (x, y) are adjacent precisely if $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$.

Remark that, if H is a *trivial graph*, then $G \square H \simeq G$ and the complexity of the problem r-Role Assignment [18] is already known. Thus, we always consider G and H to be non-trivial graphs.

3. Results

Motivated by the constructions given by van 't Hof *et al.* [20], Dourado [6] and Castonguay *et al.* [2], we propose a new construction to show that the decision problem related to finding a r-role assignment, with $r \geq 4$, for Cartesian product of two graphs remains NP-complete. As done in [2], we perform a polynomial time reduction from the NP-complete problem Hypergraph 2-Coloring, see [11]. A *hypergraph* $\mathcal{H}$ is a pair $\mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{S}(\mathcal{H}))$, where $\mathcal{V}(\mathcal{H})$ is a set of vertices, and $\mathcal{S}(\mathcal{H})$ is a set of non-empty subsets of $\mathcal{V}(\mathcal{H})$ called *hyperedges*. We consider hypergraphs with at least one hyperedge and hyperedges with at least two elements.

A surjective mapping $c : \mathcal{V}(\mathcal{H}) \to \{1, 2\}$ is a *2-coloring* of $\mathcal{H}$, if every hyperedge in $\mathcal{S}(\mathcal{H})$ contains at least two vertices u and v with $c(u) \neq c(v)$. The Hypergraph 2-Coloring problem asks whether a given hypergraph has a 2-coloring. Notice that this problem is NP-complete even if the hyperedges has size at most 3 [11].

In Subsection 3.1, we present our construction. We show the unicity of the role graph in Subsection 3.2. Finally, we conclude our NP-completeness result in Subsection 3.3.

3.1. Polynomial time reduction and sufficiency

Given a hypergraph $\mathcal{H}$, we construct a Cartesian product of two graphs that will serve as an instance of r-Role Assignment. For this, we will consider $G_r(\mathcal{H}) \square P_2$. The construction of $G_r(\mathcal{H})$, defined in the sequel, is based on the incidence graph. The *incidence graph* $I_{\mathcal{H}} = (V(I_{\mathcal{H}}), E(I_{\mathcal{H}}))$ of a hypergraph $\mathcal{H}$ is a bipartite graph whose vertex set is $V(I_{\mathcal{H}}) = \mathcal{V}(\mathcal{H}) \cup \mathcal{S}(\mathcal{H})$, and edge set $E(I_{\mathcal{H}}) = \{vS \mid v \in \mathcal{V}(\mathcal{H}),\, S \in \mathcal{S}(\mathcal{H}) \text{ with } v \in S\}$.

Construction 1. Given a hypergraph $\mathcal{H}$, with $\mathcal{V}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H}) = \{S_1, \dots, S_m\}$, with $m \geq 1$, we construct a graph $G_r(\mathcal{H})$, arising from the incidence graph $I_{\mathcal{H}}$. We remind that $V(I_{\mathcal{H}}) = \mathcal{V}(\mathcal{H}) \cup \mathcal{S}(\mathcal{H})$.

- We add edges to make $\mathcal{V}(\mathcal{H})$ a clique;
- We add a path C denoted by $c_1 \dots c_{r-2}$ and the edges vc_{r-2} for every $v \in \mathcal{V}(\mathcal{H})$;
- We add the subgraph F_r, illustrated in Figure 1, and the edges $b_{r-1}c_{r-2}$ and $b_r c_{r-2}$. We denote by A the path $a_1 \dots a_{r-2}$ and by B the path $b_1 \dots b_{r-2}$;
- For $j = 1, \dots, m$, we add a path W_j, denoted by $w_{j,1} \dots w_{j,r-3}$ connected to S_j by the vertex $w_{j,r-3}$, that is, with the edge $S_j w_{j,r-3}$.

Figure 1 contains an example of the graph $G_r(\mathcal{H})$, illustrating the Construction 1. We recall that $\mathcal{H}$ has at least one hyperedge and therefore we have a clique of cardinality 3 in $G_r(\mathcal{H})$ and thus in $G_r(\mathcal{H}) \square P_2$, which ensures that both graphs are not bipartite. Therefore, if there exist a R-role assignment of $G_r(\mathcal{H}) \square P_2$, then R is not bipartite.

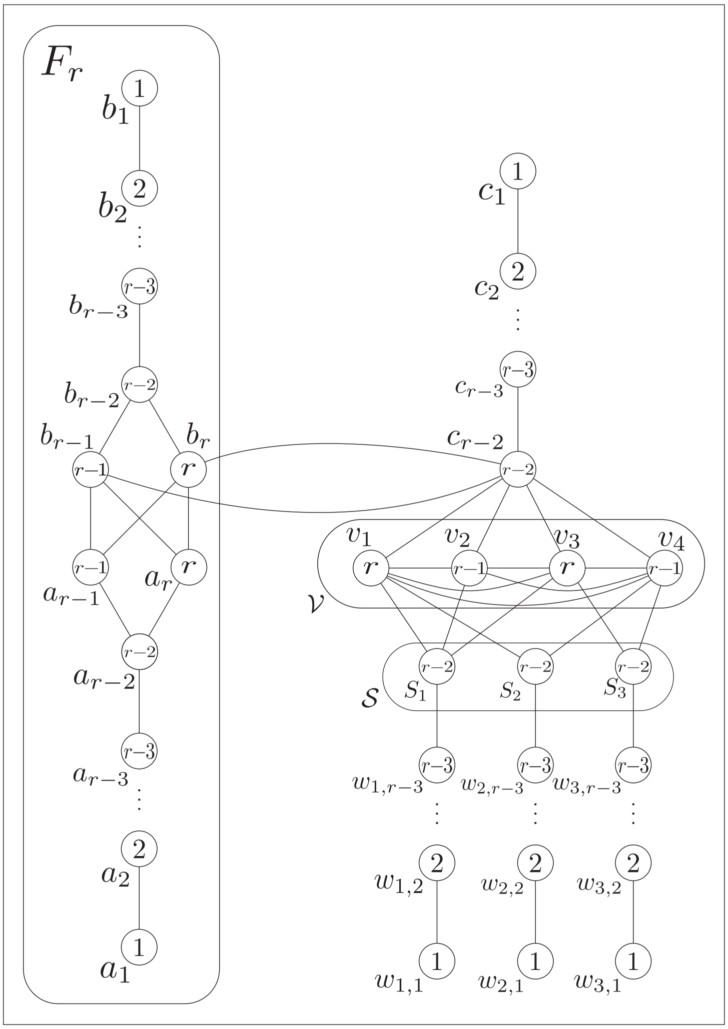

FIGURE 1. The graph $G_r(\mathcal{H})$ constructed from $\mathcal{H}$ with $V(\mathcal{H}) = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{S}(\mathcal{H}) = \{S_1, S_2, S_3\}$, where $S_1 = \{v_1, v_2, v_3\}$, $S_1 = \{v_1, v_4\}$ and $S_3 = \{v_3, v_4\}$ and part of a r-role assignment of $G_r(\mathcal{H})\Box P_2$, with $r \geq 4$.

For short, we use the following simplified notations on $G_r(\mathcal{H})\Box P_2$. Considering $V(P_2) = \{v_1, v_2\}$ for each $u \in V(G_r(\mathcal{H}))$, we identify u with the vertex (u, v_1) and denote by u' the vertex (u, v_2). We say that u' is the *corresponding vertex* of u. Let $U \subseteq V(G)$, we identify U with the set $\{(u, v_1) \mid u \in U\}$ and denote by U' the set $\{(u, v_2) \mid u \in U\}$. For short, we denote $N(u) = N_{G_r(\mathcal{H})\Box P_2}(u, v_1)$ and $N(u') = N_{G_r(\mathcal{H})\Box P_2}(u, v_2)$ for any $u \in V(G_r(\mathcal{H}))$.

Let R_r be the graph presented in Figure 2. Given a hypergraph $\mathcal{H}$ and the Construction 1 graph $G_r(\mathcal{H})$, $r \geq 4$, we show that R_r is the unique role graph for a r-role assignment of $G_r(\mathcal{H})\Box P_2$. Observe that all vertices of R_r have loops.

First, Theorem 1 presents a R_r-role assignment of $G_r(\mathcal{H})\Box P_2$ with $r \geq 4$, when a 2-coloring of $\mathcal{H}$ is known.

Theorem 1. *Let $\mathcal{H}$ be a hypergraph and $G_r(\mathcal{H})$, with $r \geq 4$, the graph obtained from Construction 1. If $\mathcal{H}$ has a 2-coloring, then $G_r(\mathcal{H})\Box P_2$ has a R_r-role assignment.*

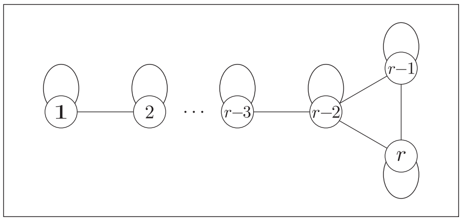

FIGURE 2. Role graph R_r with $r \geq 4$.

Proof. Let $c : V(\mathcal{H}) \to \{r-1, r\}$ be a 2-coloring of $\mathcal{H}$. To simplify the definition of a role assignment, we introduce the following subsets of $V(G_r(\mathcal{H}))$.

– For $i = 1, \ldots, r-3$, $A_i = \{a_i, b_i, c_i\} \cup \{w_{j,i} \mid j = 1, \ldots, m\}$;
– $A_{r-2} = \{a_{r-2}, b_{r-2}, c_{r-2}\}$;
– For $i = r-1, r$, $A_i = \{a_i, b_i\}$.

We define a role assignment $p : V(G_r(\mathcal{H}) \Box P_2) \to \{1, 2, \ldots, r\}$, as follows. For every $x \in V(G_r(\mathcal{H}))$:

$$p(x) = \begin{cases} i, & \text{if } x \in A_i, \text{ with } i \in \{1 \ldots, r\}; \\ r-2, & \text{if } x \in \mathcal{S}(\mathcal{H}); \\ c(i), & \text{if } x \in \mathcal{V}(\mathcal{H}). \end{cases}$$

For all x', with $x \in V(G_r(\mathcal{H}))$, we define $p(x') = p(x)$.

See an example in Figure 1, where the labels inside the vertices indicate the role assigned to that vertex. It is easy to see that p is a R_r-role assignment of $G_r(\mathcal{H}) \Box P_2$, but to make it clearer let us detail for $x \in \mathcal{S}(\mathcal{H}) \cup \mathcal{V}(\mathcal{H})$.

For $x \in \mathcal{S}(\mathcal{H})$, we have that $p(x) = r-2$ and $x = S_j$, for some $j \in \{1, \ldots, m\}$. Since c is a 2-coloring of $\mathcal{H}$, there are $v_s, v_t \in S_j$ (in $\mathcal{H}$), with $s, t \in \{1, \ldots, n\}$ and $c(v_s) \neq c(v_t)$. Suppose, without loss of generality, that $c(v_s) = r-1$ and $c(v_t) = r$. Thus, $p(N(S_j)) = \{r-3, r-2, r-1, r\}$, because $p(S'_j) = r-2$ and $p(w_{j,r-3}) = r-3$.

For $x \in \mathcal{V}(\mathcal{H})$, by definition of p, $p(x) \in \{r-1, r\}$ and $p(N(x)) \subseteq \{r-2, r-1, r\}$. We observe that there exists $y \in \mathcal{V}(\mathcal{H})$, such that $p(y) \neq p(x)$ and $\{x', c_{r-2}, y\} \subseteq N(x)$. Since $p(x') = p(x)$ and $p(c_{r-2}) = r-2$, we have that $p(N(x)) = \{r-2, r-1, r\}$. □

3.2. Uniqueness of the role graph

In the next lemmas we show some settings where the Cartesian product of the Construction 1 graph with P_2 has no r-role assignment. In Lemma 1, we show that this is the case when a_2 and a'_1 have the same role.

Lemma 1. *Let $\mathcal{H}$ be a hypergraph and $G_r(\mathcal{H})$, $r \geq 4$, the graph obtained from Construction 1. There is no r-role assignment p of $G_r(\mathcal{H}) \Box P_2$ with $p(a_2) = p(a'_1)$.*

Proof. By contradiction, we suppose that p is a r-role assignment of $G_r(\mathcal{H}) \Box P_2$ $p(a_1) = 1$, $p(a_2) = p(a'_1) = 2$. Let R be the role graph of p. It follows that $N_R(1) = \{2\}$. Observe that, if the graph $G_r(\mathcal{H}) \Box P_2$ is connected, then the role graph R of any role assignment of $G_r(\mathcal{H}) \Box P_2$ is also connected. Hence, we may assume that $k+1 \in N_R(\{1, \ldots, k\})$, for $k = 2, \ldots, r-1$.

First, we show by induction that $p(a_i) = i$ and $p(a'_i) = i+1$, for $i = 1, \ldots, r-2$. Clearly, $p(a_1) = 1$ and $p(a'_1) = 2$. Since $N_R(1) = \{2\}$ and $3 \in N_R(2)$, we have that $p(a'_2) = 3$. Thus, $p(a_2) = 2$ and $p(a'_2) = 3$. Suppose, by induction hypothesis, that $p(a_i) = i$ and $p(a'_i) = i+1$, for $i = 1, \ldots, k$, with $k \in \{2, \ldots r-3\}$. We observe that $N_R(k) = N_R(p(a'_{k-1})) = \{k-1, k+1\}$ and $k+2 \in N_R(k+1)$, thus $p(a'_{k+1}) = k+2$. Since $p(a_{k+1}) \in N_R(k) \cap N_R(k+2)$, we have that $p(a_{k+1}) = k+1$ and this proves the induction step.

Observe that $R - \{r-1, r\}$ is isomorphic to the path P_{r-2} and $N_R(r-2) = \{r-3, r-1\}$. As $r \notin N_R(\{1, \ldots, r-2\})$, we have $r \in N_R(r-1)$. We may assume that $p(a'_{r-1}) = r$. Since $p(a_{r-1}) \in N_R(r-2) \cap N_R(r)$, we have that $p(a_{r-1}) = r-1$. On the other hand, R is not bipartite. Consequently, at least one of the roles $r-1$ or r has a loop. If there is a loop on $r-1$, then $p(a'_r) = r-1$. However, a'_r has no neighbor with role $r-2$, since $r-2 \notin N_R(\{r-2, r\})$. We conclude that R is isomorphic to the path P_r with loop on role r and we can assume that $p(b'_{r-1}) = r$.

We remind that $p(a_{r-2}) = r-2$, $p(a_{r-1}) = p(a'_{r-2}) = r-1$ and $p(a'_{r-1}) = p(b'_{r-1}) = r$. Since $N_R(r) = \{r-1, r\}$ and there is no loop on $r-1$, we have that $p(b_{r-1}) = p(a'_r) = r$. We deduce that $p(a_r) = r-1$. On the other hand, $r-1 \in \{p(b'_{r-2}), p(c'_{r-2})\}$. Therefore, $p(b_r) = p(b'_r) = r$. We look at the possible roles for b'_{r-2}. Recall that $p(b'_{r-2}) \in \{r-1, r\}$.

If $p(b'_{r-2}) = r-1$, then $p(b_{r-2}) = r$ and $p(b'_{r-3}) = r-2$. We show by induction that $p(b_i) = i+2$ and $p(b'_i) = i+1$, for $i = 1, \ldots, r-2$. We observe that this induction is decreasing. Clearly, the roles hold for $i = r-2$. Suppose that $p(b_i) = i+2$ and $p(b'_i) = i+1$ for $i = k, \ldots, r-2$, for some $k \in \{2, \ldots, r-2\}$. We show that $p(b_{k-1}) = k+1$ and $p(b'_{k-1}) = k$. We have that $p(b'_k) = k+1$ and $N_R(k+1) = \{k, k+2\}$, since $k+1 \in \{3, \ldots, r-1\}$. On the other hand, $p(N(b'_k)) = \{p(b'_{k-1}), k+2\}$. Hence, $p(b'_{k-1}) = k$. Since $p(b_{k-1}) \in N_R(k) \cap N_R(k+2)$ and $N_R(k) \cap N_R(k+2) = \{k+1\}$, then $p(b_{k-1}) = k+1$. Therefore, we conclude that $p(b_1) = 3$, $p(b'_1) = 2$ and b'_1 has no neighbor with role 1, a contradiction.

If $p(b'_{r-2}) = r$, then we have two possibilities for the role of b_{r-2}: $r-1$ or r. In case $p(b_{r-2}) = r-1$, with a similar argument of the previous paragraph we get a contradiction. If $p(b_{r-2}) = r$, we have that $p(b_{r-3}) = p(b'_{r-3}) = r-1$, a contradiction, since there is no loop on $r-1$. □

In the following lemmas, we show that when the roles of a_1, a_2 and a'_1 are distinct, the Cartesian product of the Construction 1 graph with P_2, has no r-role assignment. We prove specific cases for $r = 4$, 5 and 6 in Lemmas 2–4, respectively, and the general case in Lemma 5. First, we consider, in the next lemma, the case $r = 4$.

Lemma 2. *Let $\mathcal{H}$ be a hypergraph and $G_4(\mathcal{H})$ the graph obtained from Construction 1. There is no 4-role assignment p of $G_4(\mathcal{H})\square P_2$, when $p(a_1)$, $p(a_2)$, and $p(a'_1)$ are all distinct.*

Proof. By contradiction, we suppose that p is a r-role assignment of $G_r(\mathcal{H})\square P_2$ $p(a_1) = 1$, $p(a'_1) = 2$ and $p(a_2) = 3$. Let R be the role graph of p. It follows that $N_R(1) = \{2, 3\}$. Since, $G_r(\mathcal{H})\square P_2$ is connected, so R is. Therefore, we can assume that $4 \in N_R(\{2, 3\})$. We consider the possible roles for a'_2. By Lemma 1, $p(a'_2) \neq 1$.

If $p(a'_2) = 2$, then $N_R(2) = N_R(p(a'_1)) = \{1, 2\}$. This causes a contradiction, because $p(a_2) = 3$ and $p(a_2) \in N_R(2)$.

If $p(a'_2) = 3$, then $N_R(2) = \{1, 3\}$ and $N_R(3) = \{1, 2, 3, 4\}$. We may assume that $p(a_3) = 4$ and $p(a_4) = 2$. Since $1 \in N_R(p(a_4))$ and $1 \notin N_R(4)$, we have that $p(a'_4) = 1$. Given that $N_R(2) = \{1, 3\}$, we have $p(b_3) = p(b_4) = 3$. Since role 1 has no loop on and $1 \in N_R(3)$, we have that $1 \in p(\{b_2, c_2\})$. We remind that $N_R(1) = \{2, 3\}$ and $N_R(3) = \{1, 2, 3, 4\}$, we obtain that no vertices of degree two or three has role 3, and $2 \in p(\{b_1, c_1\})$. In both cases, this leads to a contradiction, since there is no possibility of vertices of role 3 in the neighborhood of neither b_1, nor c_1.

If $p(a'_2) = 4$, then $N_R(2) = \{1, 4\}$. Since R is not bipartite, we may suppose that role 3 has a loop on, that means $N_R(3) = \{1, 3, 4\}$. Therefore, we can assume that $p(a_3) = 3$. We now consider whether there is a loop on role 4 or not.

If there is no loop on role 4, we have that $N_R(4) = \{2, 3\}$. On the other hand, $p(a'_3) \in N_R(3) \cap N_R(4)$, hence $p(a'_3) = 3$. Looking at the neighborhood of a_3, whose role is 3, we may assume that $p(b_3) = 4$. Since $N_R(3) \cap N_R(4) = \{3\}$, we have $p(b'_3) = 3$. We consider the possible roles for b_2. We recall that $p(b_2) \in N_R(4) = \{2, 3\}$. If $p(b_2) = 2$, then $p(b_1) \in \{1, 4\}$. Consequently $p(b'_1) = 3$, a contradiction by the degree of b_1 which is two. Otherwise, $p(b_2) = 3$ and in the same way, $p(c_2) = 3$. Since $p(b_3) = 4$ and $2 \in N_R(3)$, therefore $p(a_4) = 2$. That is a contradiction, since $2 \notin N_R(3)$.

If there is a loop on role 4, we have that $N_R(4) = \{2, 3, 4\}$ and by Construction 1 $p(\{b_1, b_1'\}) = p(\{c_1, c_1'\}) = \{1, 2\}$ and $p(\{b_2, b_2'\}) = p(\{c_2, c_2'\}) = \{3, 4\}$. Since $p(a_3') \in N_R(3) \cap N_R(4)$, we have that $p(a_3') \neq 1$ and we may assume that $p(b_3) = 1$. Hence, $p(b_2) = p(c_2) = 3$ and $p(b_2') = p(c_2') = 4$. Therefore, $p(b_1) = p(c_1) = 1$ and $p(b_1') = p(c_1') = 2$. It follows from the fact that $3 \in N_R(3)$, that $p(b_4) = 3$. Since $1 \notin N_R(4)$ and there is no loop on the role 1, we have that $1 \notin p(N_R(b_4))$ is a contradiction. We remind that $N_R(1) = \{2, 3\}$, $N_R(3) = \{1, 3, 4\}$, and $p(a_2) = 3$, then $p(a_4) = 3$. ☐

Now, in Lemma 3, we consider the case $r = 5$.

Lemma 3. *Let $\mathcal{H}$ be a hypergraph and $G_5(\mathcal{H})$ the graph obtained from Construction 1. There is no 5-role assignment p of $G_5(\mathcal{H})\square P_2$, when $p(a_1)$, $p(a_2)$, and $p(a_1')$ are all distinct.*

Proof. By contradiction, we suppose that p is a r-role assignment of $G_r(\mathcal{H})\square P_2$ $p(a_1) = 1$, $p(a_2) = p(a_1') = 2$. Let R be the role graph of p. It follows that $N_R(1) = \{2, 3\}$. Since, $G_5(\mathcal{H})\square P_2$ is connected, so R is. Hence, we may assume that $4 \in N_R(\{2, 3\})$ and $5 \in N_R(\{2, 3, 4\})$. We consider the possible roles for a_2'. As before, if $p(a_2') = 2$, then $N_R(2) = \{1, 2\}$ and a contradiction, since $3 \in p(N(a_2'))$. By Lemma 1, $p(a_2') \neq 1$.

If $p(a_2') = 3$, then $N_R(2) = \{1, 3\}$ and we conclude that $\{1, 2, 3, 4\} \subseteq N_R(3)$. However, $|N(a_2')| = 3$, so we do not have enough neighbors, a contradiction. As $p(a_2') \notin \{1, 2, 3\}$, we may assume that $p(a_2') = 4$ and $N_R(2) = \{1, 4\}$

Given the configuration of the roles so far and the fact that $5 \in N_R(\{3, 4\})$, we may assume that $5 \in N_R(3)$. Hence, $p(a_3) = 5$ and $N_R(3) = \{1, 4, 5\}$. We consider the potential roles for a_3'. We observe that $p(a_3') \in N_R(5) \subseteq \{3, 4, 5\}$.

If $p(a_3') = 3$, then $N_R(4) = \{2, 3\}$ and we have a loop on role 5, since R is not bipartite. Hence, $N_R(5) = \{3, 5\}$ and we may assume that $p(a_4) = 5$. We have that $p(a_4') \in N_R(3) \cap N_R(5)$, that is, $p(a_4') = 5$. Since $1 \in N_R(3)$, we have that $p(a_5') = 1$. Therefore, $p(a_5) \in N_R(1) \cap N_R(5)$, that is, $p(a_5) = 3$. Hence, $4 \notin p(N(a_5))$, because $p(\{b_4, b_5\}) \subseteq N_R(5)$ is a contradiction.

If $p(a_3') = 4$, then $N_R(4) = \{2, 3, 4\}$, a contradiction, because $p(a_3) = 5$.

If $p(a_3') = 5$, then $N_R(4) = \{2, 3, 5\}$ and $N_R(5) = \{3, 4, 5\}$. We may assume that $p(a_4) = 4$. As $p(a_4') \in N_R(4) \cap N_R(5)$, we have that $p(a_4') \neq 2$ and we may assume that $p(b_4) = 2$. Since $|N(b_1)| = |N(b_1')| = 2$, we have $p(\{b_1, b_1'\}) \subseteq \{1, 2\}$. Given that there is no loop on roles 1 and 2, we conclude that $p(\{b_1, b_1'\}) = \{1, 2\}$. Therefore, $p(\{b_2, b_2'\}) = \{3, 4\}$ and $p(b_3) = p(b_3') = 5$, a contradiction, because $p(b_4) = 2$ and $2 \notin N_R(5)$. ☐

Next, in Lemma 4 we consider the case $r = 6$.

Lemma 4. *Let $\mathcal{H}$ be a hypergraph and $G_6(\mathcal{H})$ the graph obtained from Construction 1. There is no 6-role assignment p of $G_6(\mathcal{H})\square P_2$, when $p(a_1)$, $p(a_2)$, and $p(a_1')$ are all distinct.*

Proof. By contradiction, we suppose that p is a r-role assignment of $G_r(\mathcal{H})\square P_2$ $p(a_1) = 1$, $p(a_2) = p(a_1') = 2$. Let R be the role graph of p. It follows that $N_R(1) = \{2, 3\}$. Since, $G_6(\mathcal{H})\square P_2$ is connected, so R is. We consider $4 \in N_R(\{2, 3\})$, $5 \in N_R(\{2, 3, 4\})$ and $6 \in N_R(\{2, 3, 4, 5\})$. We consider the possible roles for a_2'. As before, if $p(a_2') = 2$, then $N_R(2) = \{1, 2\}$ and a contradiction, since $3 \in p(N(a_2'))$. By Lemma 1, $p(a_2') \neq 1$. Similar as the proof of Lemma 3, we may assume that $p(a_2') = 4$ and $N_R(2) = \{1, 4\}$.

Given the configuration of the roles so far and the fact that $5 \in N_R(\{3, 4\})$ and $N_R(3) = \{1, 4, 5\}$. We may assume that $p(a_3) = 5$. And we consider the feasible roles for a_3'. Since $p(a_3') \in N_R(4) \cap N_R(5)$, we have $p(a_3') \in \{3, 4, 5, 6\}$.

If $p(a_3') = 3$, then $N_R(4) = \{2, 3\}$, $6 \in N_R(5)$ and $p(a_4) = 6$. Since $1 \in N_R(3)$, we have that $p(a_4') = 1$, a contradiction, since $1 \notin N_R(6)$. If $p(a_3') = 4$, then $N_R(4) = \{2, 3, 4\}$, but $p(a_3') = 5$, a contradiction. If $p(a_3') = 5$, then $N_R(4) = \{2, 3, 5\}$. Accordingly, $N_R(5) = \{3, 4, 5, 6\}$ is a contradiction, since $|N(a_3')| = 3$. So, we have that $p(a_3') = 6$ and $N_R(4) = \{2, 3, 6\}$.

Since, R is not bipartite and we may assume that R has a loop on role 6. Consequently, $p(a_4') = 6$ and $N_R(6) = \{4, 5, 6\}$. As $p(a_4) \in N_R(5) \cap N_R(6)$, we consider the potential for $p(a_4)$ are roles 5 and 6.

If $p(a_4) = 5$, then $N_R(5) = \{3, 5, 6\}$ and we may assume that $p(a_5) = 3$. Since $p(a'_5) \in N_R(3) \cap N_R(6)$, we have that $p(a'_5) \neq 1$ and we way assume that $p(b_5) =1$. On the other hand $|N(b_1)| = |N(b'_1)| = 2$, since there are no loop on roles 1 and 2, we have that $p(\{b_1, b'_1\}) = \{1, 2\}$. Therefore, $p(\{b_2, b'_2\}) = \{3, 4\}$ and $p(\{b_3, b'_3\}) = p(\{b_4, b'_4\}) = \{5, 6\}$. Note that $1 \notin N_R(\{5, 6\})$ and we have a contradiction.

If $p(a_4) = 6$, then $N_R(5) = \{3, 6\}$. On the other hand, since $N_R(6) = \{4, 5, 6\}$, we may assume that $p(a'_5) = 4$ and $p(a'_6) = 5$. We remind that $2 \in N_R(4)$, but all neighbors of a'_5 are adjacent to a vertex role 5 or 6, a contradiction, given that $2 \notin N_R(\{5, 6\})$. ☐

After considered the particular cases, we conclude in Lemma 5 that, for $r \geq 4$, the Cartesian product of the Construction 1 graph with P_2 has no r-role assignment, when the roles of a_1, a_2 and a'_1 are distinct.

Lemma 5. *Let $\mathcal{H}$ be a hypergraph and $G_r(\mathcal{H})$, with $r \geq 4$, the graph obtained from Construction 1. There is no r-role assignment p of $G_r(\mathcal{H})\square P_2$, such that $p(a_1)$, $p(a_2)$, and $p(a'_1)$ are all distinct.*

Proof. By the Lemmas 2, 3 and 4, the result follows, for $r = 4, 5$ and 6, respectively. From now on, we consider, $r \geq 7$. By contradiction, we suppose that p is a r-role assignment of $G_r(\mathcal{H})\square P_2$ and $p(a_1) = 1$, $p(a'_1) = 2$ and $p(a_2) = 3$. Let R be the role graph of p. It follows that $N_R(1) = \{2, 3\}$. Since $G_r(\mathcal{H})\square P_2$ is connected, so R is. Hence, we may assume that $k + 1 \in N_R(\{1, \ldots, k\})$, for $k = 3, \ldots, r - 1$. We consider the possible roles for a'_2. As before, if $p(a'_2) = 2$, then $N_R(2) = \{1, 2\}$ is a contradiction, since $3 \in p(N(a'_2))$. By Lemma 1, $p(a'_2) \neq 1$.

If $p(a'_2) = 3$, then $\{1, 2, 3, 4\} \subseteq N_R(3)$, but $|N_R(a'_2)| = 3$ and a'_2 does not have enough neighbors, a contradiction. Therefore, we can assume $p(a'_2) = 4$.

Let t be the largest integer, such that, $|\{p(a_i), p(a'_i) \mid i = 1, \ldots, t\}| = 2t$. Clearly, $t \geq 2$ and $2t \leq r$. For sake of simplicity, let $p(a_i) = 2i - 1$ and $p(a'_i) = 2i$, for $i \in \{1, \ldots, t\}$. We observe that $N_R(1) = \{2, 3\}$, $N_R(2) = \{1, 4\}$ and for $i = 2, \ldots, t - 1$, $N_R(2i - 1) = \{2i - 3, 2i, 2i + 1\}$, $N_R(2i) = \{2i - 2, 2i - 1, 2i + 2\}$. We analyze three cases: $2t \leq r - 2$, $2t = r - 1$ and $2t = r$.

Case $2t \leq r - 2$.

In this case, $2t+1 \in N(\{2t-1, 2t\})$ and since $t \geq 2$, $t \leq r-t-2 \leq r-4$. We may assume that $p(a_{t+1}) = 2t+1$. Hence, $N_R(2t - 1) = \{2t - 3, 2t, 2t + 1\}$. By the hypotheses about t, $p(a'_{t+1}) \neq 2t + 2$ and as we know that $2t + 2 \leq r$, $2t + 2 \in N_R(2t + 1)$. On the other hand, $t + 1 \leq r - 3$ and $|N(a_{t+1})| = 3$, which implies that $p(a_{t+2}) = 2t{+}2$.

We show a contradiction for every possible role of $p(a'_{t+1})$. We observe that $p(a'_{t+1}) \in \{2t - 1, 2t, 2t + 1\}$. If $p(a'_{t+1}) = 2t - 1$, then $p(a'_{t+2}) = 2t - 3$ is a contradiction, since $2t + 2 \notin N_R(2t - 3)$.

If $p(a'_{t+1}) = 2t$, then $N_R(2t) = \{2t - 2, 2t - 1, 2t, 2t + 1\}$ is a contradiction, because $|N(a'_{t+1})| = 3$.

If $p(a'_{t+1}) = 2t + 1$, then $N_R(2t + 1) = \{2t - 1, 2t, 2t + 1, 2t + 2\}$, which again leads to a contradiction.

Case $2t = r - 1$, that is, $r = 2t + 1$.

As $r \geq 7$, we have $t \geq 3$ and $t \leq r - 4$. Analogue to the previous case, we may assume that $p(a_{t+1}) = 2t + 1$ and $N_R(2t - 1) = \{2t - 3, 2t, 2t + 1\}$. We observe that $p(a'_{t+1}) \in \{2t - 2, 2t - 1, 2t, 2t + 1\}$ and we reach a contradiction for each of the potential roles of $p(a'_{t+1})$.

If $p(a'_{t+1}) = 2t - 2$, then we have a contradiction, because $2t + 1 \in N_R(2t + 2)$. If $p(a'_{t+1}) = 2t - 1$, then $N_R(2t) = \{2t - 2, 2t - 1\}$. As, $t + 1 \leq r - 3$, $p(a'_{t+2}) = 2t - 3$. Thus, $p(a_{t+2}) \in N_R(2t - 3) \cap N_R(2t + 1)$, that is $p(a_{t+2}) = 2t - 1$. Therefore, $N_R(2t + 1) = \{2t - 1\}$. Hence, R is bipartite, a contradiction. If $p(a'_{t+1}) = 2t$, then $N_R(2t) = \{2t - 2, 2t - 1, 2t, 2t + 1\}$, a contradiction to the fact that $|N_R(a'_{t+1})| = 3$. We conclude that, $p(a'_{t+1}) = 2t + 1$, $N_R(2t) = \{2t - 2, 2t - 1, 2t + 1\}$ and $N_R(2t + 1) = \{2t - 1, 2t, 2t + 1\}$. We observe that all neighborhood of roles of R are defined, since $r = 2t + 1$.

Recall that $t \leq r - 4$, that is, $t + 1 \leq r - 3$. Therefore, $p(a_{t+2}) = 2k$ and $p(a'_{t+2}) = 2t - 1$. It is easy to see, using induction, that $p(a_i) = 2(r - i) + 2$ and $p(a'_i) = 2(r - i) + 1$ for $i = t + 2, \ldots, r - 2$.

We deduce that $p(a_{r-2}) = 2(r - (r - 2)) + 2 = 6$ and $p(a'_{r-2}) = 5$. Hence, $3 \in p(\{a'_{r-1}, a'_r\})$. We may assume that $p(a'_{r-1}) = 3$. Since $1 \notin N_R(6)$, we can suppose that $p(b'_{r-1}) = 1$.

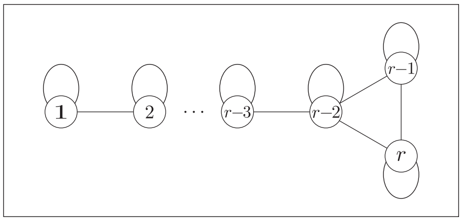

FIGURE 3. Role graph R_r, with $r \geq 4$.

Clearly, $p(\{b_i, b'_i\}) = p(\{a_i, a'_i\})$, for $i = \{1, \ldots, r-2\}$. We conclude that $p(b'_{r-2}) \in \{5, 6\}$, a contradiction, since $p(b'_{r-1}) = 1$.

Case $2t = r$.

We observe that, in this case, r is even and therefore $r \geq 8$. We remind that $N_R(1) = \{2, 3\}$, $N_R(2) = \{1, 4\}$ and for $i = 2, \ldots, t-1$, with $N_R(2i-1) = \{2i-3, 2i, 2i+1\}$ and $N_R(2i) = \{2i-2, 2i-1, 2i+2\}$. On the other hand, R is not bipartite and we may assume that there is a loop on the role r. As $r \geq 8$, $4 \leq t \leq r-4$, we have that $p(a'_{t+1}) = 2t = r$ and $N_R(2t) = \{2t-2, 2t-1, 2t\}$.

We deduce that $p(a_{t+1}) \in N_R(2t-1) \cap N_R(2t)$. Clearly, $p(a_{t+1}) \in \{2t-1, 2t\}$.

If $p(a_{t+1}) = 2t$, then $N_R(2t-1) = \{2t-3, 2t\}$. We remind that $N_R(2t) = \{2t-2, 2t-1, 2t\}$ and $t+1 \leq r-3$. However, $p(N(a'_{t+1})) = \{2t, p(a_{t+2})\}$, a contradiction. Therefore, $p(a_{t+1}) = 2t-1$ and $N_R(2t-1) = \{2t-3, 2t-1, 2t\}$. Clearly, $p(a_{t+2}) = 2t-3$ and $p(a'_{t+2}) = 2t-2$. It is easy to see, using induction, that $p(a_i) = 2(r-i)+1$ and $p(a'_i) = 2(r-i)+2$ for $i = t+1, \ldots, r-2$. We conclude that, $p(a_{r-2}) = 5$ and $p(a'_{r-2}) = 6$. Hence, $3 \in p(\{a_{r-1}, a_r\})$. We can suppose that $p(a_{r-1}) = 3$. Since $1 \notin N_R(6)$, we can suppose that $p(b_{r-1}) = 1$.

Similarly, using induction, it is possible to show that $p(\{b_i, b'_i\}) = p(\{a_i, a'_i\})$, for $i = 1, \ldots, r-2$. We conclude that $p(b'_{r-2}) \in \{5, 6\}$ which leads to a contradiction, since $p(b_{r-1}) = 1$. □

Now, we show that the role graph of a r-role assignment of the Cartesian product of $G_r(\mathcal{H})\square P_2$ is unique. For this purpose, we will use the graph R_r of Figure 2. For a better use, let recall this graph in Figure 3.

The proof is separated in two lemmas: Lemma 6 for the specific case when $r = 4$ and Lemma 7 for the general case.

Lemma 6. *Let $\mathcal{H}$ be a hypergraph and $G_4(\mathcal{H})$ the graph obtained from Construction 1. If $G_4(\mathcal{H})\square P_2$ has a R-role assignment with $|V(R)| = 4$, then $R \simeq R_4$.*

Proof. Let p be a R-role assignment of $G_4(\mathcal{H})\square P_2$, with $|V(R)| = 4$. Suppose that $p(a_1) = 1$. Since, $G_4(\mathcal{H})\square P_2$ is connected, so R is. Hence, we may assume that $2 \in N_R(1)$, $3 \in N_R(\{1, 2\})$ and $4 \in N_R(\{1, 2, 3\})$. We observe that $2 \in p(N(a_1))$ and $N(a_1) = \{a'_1, a_2\}$. By Lemma 1, $p(a_2) \neq p(a'_1)$. It follows from the Lemma 5 that $p(a_1), p(a_2)$ and $p(a'_1)$ are not all distinct and we conclude that $p(\{a'_1, a_2\}) = \{1, 2\}$. If $p(a'_1) = 2$ and $p(a_2) = 1$, then $N_R(1) = \{1, 2\}$. Hence, $3 \in N_R(2)$ and $p(a'_2) = 3$, a contradiction. Therefore, $p(a'_1) = 1$ and $p(a_2) = 2$.

We show that $R \simeq R_4$. We remind that $p(a_1) = 1$, $p(a'_1) = 1$, $p(a_2) = 2$ and $N_R(1) = \{1, 2\}$. Hence, $p(a'_2) = 2$ and $\{1, 2, 3\} \subseteq N_R(2)$. Since $|N(b_1)| = |N(b'_1)| = 2$, we have $2 \notin p(\{b_1, b'_1\})$. In the same way, $2 \notin p(\{c_1, c'_1\})$. We may assume that $p(a_3) = 3$. We divide in two cases, depending whether role 4 belongs or not to the neighborhood of the role 2.

If $4 \notin N_R(2)$, then $4 \in N_R(3)$, $p(a'_3) \in \{2, 3\}$ and we may assume that $p(b_3) = 4$. If $p(a'_3) = 2$, then $p(a'_4) = 3$. Therefore, all neighbors of a'_3 are adjacent to the vertices with role 3 or 4, that is, $1 \notin p(N(a'_3))$ a contradiction. Thus, $p(a'_3) = 3$ and $|N_R(3)| = |\{2, 3, 4\}| = 3$. Again, we have that $3 \notin p(\{b_1, b'_1\})$, that is, $p(\{b_1, b'_1\}) \subseteq \{1, 4\}$. Since there is no edge between roles 1 and 4, we have that $p(b_1) = p(b'_1)$. On the other hand, $p(b_2) \neq 2$, since $4 \notin N_R(2)$. We conclude that $p(b_1) = p(b'_1) = 4$ and $p(b_2) = p(b'_2) = 3$. in the same way, we get $p(c_2) = p(c'_2) = 3$. Observing the neighborhood of b_2, which has role 3, we obtain that $p(b_4) = 2$.

However, all neighbors of b_4 have role 3, except for a_4. Nonetheless, $p(a_4) \neq 1$, since $p(b_3) = 4$ and $1 \notin N_R(4)$. Therefore, $1 \notin p(N(b_4))$, a contradiction.

If $4 \in N_R(2)$, then $N_R(2) = \{1, 2, 3, 4\}$ and $p(a_4) = 4$. We remind that $p(b_1), p(b'_1) \in \{1, 3, 4\}$. If $p(b_1) = 3$, then $|N_R(3)| = 2$ and $p(b'_1) \in \{3, 4\}$. In both cases, $p(b_2) = p(b'_2) = 2$. However, all neighbors of b_2 has adjacent vertices with role 3 or 4. Therefore, $1 \notin p(N(b_2))$. In the same way, we get a contradiction when $p(b_1) = 4$. We conclude that, $p(b_1) = 1$ and thus $p(b'_1) = 1$. Hence, $p(b_2) = p(b'_2) = 2$. We conclude that $p(\{b_3, b_4\}) = \{3, 4\}$ and $R \simeq R_4$. □

In the following lemma, we show the uniqueness of the role graph for the general case.

Lemma 7. *Let $\mathcal{H}$ be a hypergraph, $r \geq 4$ and $G_r(\mathcal{H})$ the graph obtained from Construction 1. If $G_r(\mathcal{H})\Box P_2$ has a R-role assignment with $|V(R)| = r$, then $R \simeq R_r$.*

Proof. Let p be a R-role assignment of $G_r(\mathcal{H})\Box P_2$ and $|V(R)| = r$. By Lemma 6, we can assume that $r \geq 5$. Suppose that $p(a_1) = 1$. Since, $G_r(\mathcal{H})\Box P_2$ is connected, so R is. Hence, we may assume that, $V(R) = \{1, \ldots, r\}$ and $k + 1 \in N_R(\{1, \ldots, k\})$, for $k = 1, \ldots, r - 1$. We observe that $2 \in p(N(a_1))$ and $N(a_1) = \{a'_1, a_2\}$. By Lemma 1, $p(a_2) \neq p(a'_1)$. It follows from the Lemma 5 that $p(a_1), p(a_2)$ and $p(a'_1)$ are not all distinct and we conclude that $p(\{a'_1, a_2\}) = \{1, 2\}$. If $p(a'_1) = 2$ and $p(a_2) = 1$, then $N_R(1) = \{1, 2\}$. Hence, $3 \in N_R(2)$ and $p(a'_2) = 3$, a contradiction. Therefore, $p(a'_1) = 1$ and $p(a_2) = 2$.

We remind that $p(a_1) = p(a'_1) = 1$ and $p(a_2) = 2$. Hence, $N_R(1) = \{1, 2\}$ and $p(a'_2) = 2$. By induction, we have that $p(a_i) = p(a'_i) = i$ for $i = 1, \ldots, r-2$. We note that in this case, except for $i = 2$, $N_R(i-1) = \{i-2, i-1, i\}$.

We consider the path $B = b_1 \ldots b_{r-2}$ of $G_r(\mathcal{H})$. We show that $p(b_i) = p(b'_i) = i$, for $i = 1, \ldots, r-2$. Similarly, we get the roles for the vertices of the paths C and W_j, with $j = 1, \ldots, m$.

We observe that the graph $R - \{r-1, r\}$ is isomorphic to the path P_{r-2} with loop on all roles and $r - 1 \in N_R(r-2)$. Hence, $\{r-3, r-2, r-1\} \subseteq N_R(r-2)$. We may assume that $p(a_{r-1}) = r - 1$. Next, we consider the potential roles for $p(b_1)$. We remind that $|N(b_1)| = 2$ and therefore $p(b_1) \in \{1, r-1, r\}$. In the same way, $p(b'_1) \in \{1, r-1, r\}$.

If $p(b_1) = 1$, then similarly to the path A, we have that $p(b_i) = p(b'_i) = i$, for $i = 1, \ldots, r-2$. Hence, we may assume that, $p(b_{r-1}) = r-1$. We remind that, $p(a_{r-1}) = r-1$ and $r \in N_R(\{r-2, r-1\})$. Since all neighbors of a_{r-1} are neighbors of vertices of role $r-2$, we have that, $r \in N_R(r-2)$ and $N_R(r-2) = \{r-3, r-2, r-1, r\}$. Therefore, $p(a_r) = r$. In the same way, $p(b_r) = r$ and we have that, $R \simeq R_r$.

Next, we reach contradictions for the remaining cases. If $p(b_1) = r - 1$, then as $p(b'_1) \in \{1, r-1, r\}$ we have that $p(b_2) = r - 2$. Since $|N(b_2)| = 3$ and $r \geq 5$, we have that, $N_R(r-2) = \{r-3, r-2, r-1\}$. We get that, $r \in N_R(r-1)$ and therefore $p(b'_1) = r$. Hence, $p(b'_2) \in N_R(r-2) \cap N_R(r)$, that is, $p(b'_2) = r-1$, a contradiction, because $p(N(b_2)) = \{r-1, p(b_3)\}$.

If $p(b_1) = r$, then from the previous cases, we may assume that, $p(b'_1) = r$ and $r \notin N_R(r-2)$. We find that, $N_R(r-2) = \{r-3, r-2, r-1\}$ and $N_R(r) = \{r, r-1\}$. Therefore, $p(b_2) = p(b'_2) = r - 1$ and the graph R is isomorphic to the path P_r with loops on all roles. Clearly, by induction, $p(b_i) = p(b'_i) = r - i + 1$, for $i = 2, \ldots, r-2$. We conclude that, $p(b_{r-2}) = p(b'_{r-2}) = r - (r-2) + 1 = 3$. Since $p(b_{r-3}) = 4$, we may assume that, $p(b_{r-1}) = 2$. We remind that $p(a_{r-1}) = r - 1$, a contradiction, once $r \geq 5$. □

3.3. NP-completeness

We show that one can obtain a 2-coloring of $\mathcal{H}$ when it is given a r-role assignment of $G_r(\mathcal{H})\Box P_2$, $r \geq 4$.

Theorem 2. *Let $\mathcal{H}$ be a hypergraph and $G_r(\mathcal{H})$ with $r \geq 4$, the graph obtained from Construction 1. If $G_r(\mathcal{H})\Box P_2$ has a r-role assignment, then $\mathcal{H}$ has a 2-coloring.*

Proof. Let p be a R-role assignment of $G_r(\mathcal{H})\Box P_2$, with $|V(R)| = r \geq 4$. By Lemma 7, we may assume that, $R = R_r$.

Consider the path $A = a_1 \ldots a_{r-2}$ in $G_r(\mathcal{H})$. Similarly, we assign the roles for the paths B, C and $W_j S_j$ with $j = 1 \ldots, m$.

We show, by induction, that $p(a_i) = p(a'_i) = i$, for $i = 1, \ldots, r-2$. Since $|N(a_1)| = |N(a'_1)| = 2$, we have that, $p(a_1) = p(a'_1) = 1$. Suppose that, $p(a_i) = p(a'_i) = i$, for $i = 1, \ldots, k$ with $k \in \{1, \ldots, r-3\}$. We remind that, $N_R(1) = \{1, 2\}$ and $N_R(i) = \{i-1, i, i+1\}$, for $i = 2, \ldots, r-3$. Observing the neighborhood of a_k, we get that $p(a_{k+1}) = k+1$. In the same way, $p(a'_{k+1}) = k+1$.

We want to define a 2-coloring of $\mathcal{H}$, $c : \mathcal{V}(\mathcal{H}) \to \{r-1, r\}$, using p. First, we show that, $p(\mathcal{V}(\mathcal{H})) = \{r-1, r\}$.Observe that, for $j = 1, \ldots, m$, $p(w_{j,r-3}) = r-3$ and $p(S_j) = p(S'_j) = r-2$. Since, $m \geq 1$ and $N_R(r-2) = \{r-3, r-2, r-1, r\}$, we have that, $\{r-1, r\} \subseteq p(\mathcal{V}(\mathcal{H}))$. In the same way, $\{r-1, r\} \subseteq p(\mathcal{V}'(\mathcal{V})$. Since $\mathcal{V}(\mathcal{H})$ induces a clique on $G_r(\mathcal{H})$, we have that $p(\mathcal{V}(\mathcal{H})) \subseteq \{r-2, r-1, r\}$. However, if a vertex has role $r-2$, then it will have no neighbor with role $r-3$, a contradiction. Therefore, $p(\mathcal{V}(\mathcal{H})) = \{r-1, r\}$. We define $c(v) = p(v)$ for every $v \in \mathcal{V}(\mathcal{H})$. We show that c is a 2-coloring of $\mathcal{H}$. Let $S_j \in \mathcal{S}(\mathcal{H})$, for some $j \in \{1, \ldots, m\}$. We remind that, $p(S_j) = r-2$ and $p(N(\mathcal{S}_j)) = \{r-2, r-3\} \cup \{p(v) \mid v \in \mathcal{S}_j \text{ (in } \mathcal{H})\}$. Since $N_R(r-2) = \{r-3, r-2, r-1, r\}$, there exists $s, t \in \{1, \ldots, n\}$, such that, $v_s, v_t \in \mathcal{S}_j$ (in $\mathcal{H}$), $p(v_s) = r-1$ and $p(v_t) = r$. Therefore, c is a 2-coloring of $\mathcal{H}$. ☐

The above results imply directly in the NP-completeness of r-Role Assignment, $r \geq 4$.

Theorem 3. *The problem r-Role Assignment, for any fixed $r \geq 4$, is NP-complete even when restricted to the Cartesian product of two non-trivial graphs.*

Proof. The problem is clearly in NP (cf. Roberts and Sheng [18]). To show the NP-hardness, we use a reduction from the NP-complete problem Hypergraph 2-Coloring [11]. Given a hypergraph $\mathcal{H}$, we construct the graph $G_r(\mathcal{H})$ according to Construction 1, which is used to compute $G_r(\mathcal{H}) \Box P_2$. It is easy to see that $G_r(\mathcal{H}) \Box P_2$ may be obtained in polynomial time. By Theorems 1 and 2, we obtain that, $\mathcal{H}$ has 2-coloring if and only if $G_r(\mathcal{H}) \Box P_2$ has a r-role assignment for any fixed $r \geq 4$, and the proof is complete. ☐

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