RAIRO-Oper. Res. 57 (2023) 1009–1025
https://doi.org/10.1051/ro/2023044

RAIRO Operations Research
www.rairo-ro.org

ON INTERVAL-VALUED BILEVEL OPTIMIZATION PROBLEMS USING UPPER CONVEXIFICATORS

Stephan Dempe[1], Nazih Abderrazzak Gadhi[1] and Mohamed Ohda[2,*]

Abstract. In this paper, we investigate a bilevel interval valued optimization problem. Reducing the problem into a one-level nonlinear and nonsmooth program, necessary optimality conditions are developed in terms of upper convexificators. Our approach consists of using an Abadie's constraint qualification together with an appropriate optimal value reformulation. Later on, using an upper estimate for upper convexificators of the optimal value function, we give a more detailed result in terms of the initial data. The appearing functions are not necessarily Lipschitz continuous, and neither the objective function nor the constraint functions of the lower-level optimization problem are assumed to be convex. There are additional examples highlighting both our results and the limitations of certain past studies.

Mathematics Subject Classification. 65K10, 90C25, 90C29, 90C46.

Received February 24, 2022. Accepted March 30, 2023.

1. Introduction

Bilevel optimization problems are mathematical programs defined by two optimization problems where one optimization problem contains another optimization problem as a constraint. Indeed, feasible solutions of the bilevel optimization problem are restricted to lower level optimal solutions satisfying the upper level constraints. These problems have been the topic of a number of monographs, *e.g.* [2,8,13,29] and articles, see the annotated bibliographies [9,10]. The upper level optimization problem is commonly referred to as the leader's optimization problem and the lower level optimization problem is known as the follower's optimization problem. The latter one is in fact a parametric optimization problem that is solved with respect to the lower level variables, while some of the upper level variables act as parameters. Recently, extensive work on bilevel optimization problems has been carried out [11,14–16,18,19,22–24]. For instance, Kohli [22] gave the necessary optimality conditions for bilevel programming problems with a convex lower-level problem. Using a scalarization technique together with a generalized Abadie constraint qualification, Lafhim *et al.* [23] derived Karush–Kuhn–Tucker (KKT)-type necessary optimality conditions for bilevel multiobjective optimization problems. Gadhi and Ohda [19] recently developed necessary optimality conditions in terms of tangential subdifferentials for multiobjective bilevel programming problems using the concept of efficiency.

Keywords. Bilevel optimization, Constraint qualification, Upper convexificators, Optimality conditions, Optimal value function.

[1] Department of Mathematics and Computer Science, TU Bergakademie Freiberg, Freiberg, Germany.

[2] LAMA, FSDM, Sidi Mohamed Ben Abdellah University, Fes, Morocco.

[*] Corresponding author: ohda.fsdm2021@gmail.com

© The authors. Published by EDP Sciences, ROADEF, SMAI 2023

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

If we take the objective function as a closed interval in optimization, we get an interval-valued optimization problem. This sort of problem may provide the possibility for a different choice that will address uncertainty in optimization. Many researchers have studied optimality conditions for interval-valued nonlinear programming problems [3, 4, 20, 27].

In this paper, we are concerned with the following interval-valued bilevel optimization problem:

$$
(\mathrm{IVP}):\begin{cases}\min\limits_{x,y} & F(x,y)=[F_L(x,y),\ F_U(x,y)]\\ \text{subject to}: & G_j(x,y)\le 0, \qquad\qquad \forall j\in J,\ y\in\psi(x),\ (x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},\end{cases}
$$

where, for each $x\in\mathbb{R}^{n_1},\ \psi(x)$ is the LU-solution set of the following parametric interval-valued optimization problem:

$$
(\mathrm{IVP}_x):\begin{cases}\min\limits_{y} & f(x,y)=[f_L(x,y),\ f_U(x,y)]\\ \text{subject to}: & g_i(x,y)\le 0, \qquad\qquad \forall i\in I,\ y\in\mathbb{R}^{n_2}.\end{cases}
$$

Here, $I:=\{1,\dots,m_1\},\ J:=\{1,\dots,m_2\},\ m_1,\ m_2,\ n_1\ge 1,\ n_2\ge 1$ are integers and $f_L,\ f_U,\ g_L,\ g_U,\ F_i,\ G_j:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$ are given functions such that $F_L(x,y)\le F_U(x,y)$, for all $(x,y)\in\Omega$, where

$$
\Omega:=\{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}: G_j(x,y)\le 0,\quad \forall j\in J,\ \text{and } y\in\psi(x)\}
$$

is the feasible set of (IVP). We suppose that for each $x\in\mathbb{R}^{n_1}$ and each $y\in\Xi(x)$, the lower level objective interval-function f satisfies the inequality $f_L(x,y)\le f_U(x,y)$, where

$$
\Xi(x):=\{y\in\mathbb{R}^{n_2}: g_i(x,y)\le 0,\ \forall i\in I\}
$$

is the lower-level feasible set of (IVP). In this work, for two intervals $A=[a_L,\ a_U]$ and $B=[b_L,\ b_U]$, the partial order is defined as follows:

$$
A\leqslant_{LU} B \ \text{ if and only if }\ a_L\leqslant b_L \text{ and } a_U\leqslant b_U.
$$

Consequently, a point $\overline{y}\in\psi(\overline{x})$ if and only if

$$
\begin{aligned}
& f(\overline{x},\overline{y})\le_{LU} f(\overline{x},y), && \forall y\in\Xi(\overline{x}).\\
\textit{i.e.}\quad & f_L(\overline{x},y)-f_L(\overline{x},\overline{y})\ge 0 \text{ and } f_U(\overline{x},y)-f_U(\overline{x},\overline{y})\ge 0, && \forall y\in\Xi(\overline{x})
\end{aligned}
$$

A feasible point $(\overline{x},\overline{y})$ is said to be a local LU-optimal solution of (IVP) if there exists a neighborhood U of $(\overline{x},\overline{y})$ such that

$$
F(\overline{x},\overline{y})\le_{LU} F(x,y),\quad \forall (x,y)\in\Omega\cap U.
$$

Notice that $\psi(x)=\psi_L(x)\cap\psi_L(x)$, where $\psi_L(x)$ and $\psi_L(x)$ are the solution sets of the following parametric optimization problems

$$
(\mathrm{IVP}_x^L):\begin{cases}\min\limits_{y} & f_L(x,y)\\ \text{subject to}: & g_i(x,y)\le 0,\quad \forall i\in I.\end{cases}
$$

and

$$
(\mathrm{IVP}_x^U):\begin{cases}\min\limits_{y} & f_U(x,y)\\ \text{subject to}: & g_i(x,y)\le 0,\quad \forall i\in I.\end{cases}
$$

respectively. Moreover, if $F_L(x,y)=F_U(x,y)$ and $f_L(x,y)=f_U(x,y)$ for all $(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, the optimization problem (IVP) under consideration reduces to the well-known usual bilevel optimization problem that we've talked about previously [1, 2, 8, 10, 13, 15, 18, 22, 28].

In this paper, we seek to develop Karush–Kuhn–Tucker type optimality conditions for the interval-valued bilevel optimization problem (IVP). Using an optimal value reformulation adapted to (IVP), we first reformulate (IVP) as a one-level scalar optimization problem. Then, using a suitable Abadie's constraint qualification expressed in terms of upper convexifactors, we provide the necessary optimality conditions of (IVP). Afterwards, using an upper estimate for upper convexificators of the optimal value function, we give a more detailed result in terms of the initial data. Even for ordinary bilevel optimization problems, our approach is novel, and our results are extremely relevant in comparison to those of Kohli [22], because we are not limited to the case where the lower-level objective function and constraints functions must be convex or even locally Lipschitz. Examples that illustrate both our findings and the limitations of those published in [22, Thm. 5.1] (see Examples 15, 18 and 20) are provided. To the best of our knowledge, there is no paper studying the KKT optimality conditions for interval-valued bilevel optimization problems via upper convexificators.

The rest of the paper is organized in this way: Section 2 contains basic definitions and preliminary material. Section 3 addresses the main results (necessary optimality conditions). A conclusion is given in Section 4.

2. Preliminaries

In this section, we give some definitions, notations, and results, that will be used in the sequel. For a subset D of $\mathbb{R}^n$, the sets $cl\ D$, $conv\ D$, $pos\ D$ and D^- stand for the closure of D, the convex hull of D, the convex cone (including the origin) generated by D, and the negative polar cone of D, respectively. Let $x \in cl\ D$, the contingent cone $T_D(x)$ to D at x is defined by

$$T_D(x) = \{v \in \mathbb{R}^n : \exists t_n \searrow 0 \text{ and } \exists v_n \to v \text{ such that } x + t_n v_n \in D,\ \forall\, n\}.$$

The negative polar cone of a set $D \subseteq \mathbb{R}^n$ is defined by,

$$D^- = \{d \in \mathbb{R}^n : \langle d, x\rangle \leq 0,\ \forall\ x \in D\}.$$

A function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to be locally Lipschitz around $\overline{x} \in dom f = \{x \in \mathbb{R}^n \mid\ f(x) \in \mathbb{R}\}$, if there exist a neighbourhood U of $\overline{x}$ and $k \geq 0$ such that

$$|f(x) - f(y)| \leq k\|x - y\| \quad \forall x, y \in U,$$

where $\|.\|$ denotes any norm in $\mathbb{R}^n$. In [5], it was shown that when f is locally Lipschitz, the generalized directional derivative

$$v \longrightarrow f^0(\overline{x},\ v) = \limsup_{x\to\overline{x},\ t\searrow 0} \frac{f(x+tv) - f(x)}{t},$$

is a finite sublinear function; moreover, it is positively homogeneous and thus convex. The following set

$$\partial^c f(\overline{x}) = \left\{x^* \in \mathbb{R}^n : \langle x^*,\ v\rangle \leq f^0(\overline{x},\ v),\ \forall v \in \mathbb{R}^n\right\},$$

called the Clarke subdifferential [5] of f at $\overline{x}$, is then a nonempty convex compact subset of $\mathbb{R}^n$. When f is convex and continuous at $\overline{x}$, then f is locally Lipschitz and $f'(\overline{x},\ v) = f^0(\overline{x},\ v)$ for any $v \in \mathbb{R}^n$, where $v \longrightarrow f'(\overline{x}, v)$ is the usual directional derivative defined by

$$v \longrightarrow f'(\overline{x},\ v) = \limsup_{t\searrow 0} \frac{f(\overline{x}+tv) - f(\overline{x})}{t}\cdot$$

As a result, $\partial^c f(\overline{x})$ is exact the subdifferential of f in the sense of the convex analysis, usually denoted by $\partial f(\bar{x})$.

Now, we recall the definitions related to convexificators given by Jeyakumar and Luc [21] as well as Dutta and Chandra [17]. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function and let $x \in dom f$. The expressions

$$f_d^-(x, v) = \liminf_{t\searrow 0}\ [f(x+tv) - f(x)]/t$$

and

$$f_d^+(x, v) = \limsup_{t\searrow 0}\ [f(x+tv) - f(x)]/t$$

signify, respectively, the lower and upper Dini directional derivatives of f at x in the direction v.

Definition 1. [17] Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a given function.

- f is said to have an upper convexifactor (UCF) $\partial^* f(x)$ at $x \in \mathbb{R}^n$, if $\partial^* f(x) \subseteq \mathbb{R}^n$ is a closed set and

$$f_d^-(x,v) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, v\rangle, \quad \forall v \in \mathbb{R}^n.$$

- f is said to have a lower convexifactor (LCF) $\partial^* f(x)$ at $x \in \mathbb{R}^n$, if $\partial^* f(x) \subseteq \mathbb{R}^n$ is a closed set and

$$f_d^+(x,v) \geq \inf_{x^* \in \partial^* f(x)} \langle x^*, v\rangle, \quad \forall v \in \mathbb{R}^n.$$

A closed set $\partial^* f(x) \subseteq \mathbb{R}^n$ is said to be a convexificator (CF) of f at x if it is both an upper and lower convexificator of f at x.

Definition 2. [17] Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a given function. f is said to have an upper semi-regular convexifactor (USRCF) $\partial^* f(x)$ at $x \in \mathbb{R}^n$, if $\partial^* f(x) \subseteq \mathbb{R}^n$ is a closed set and

$$f_d^+(x,v) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, v\rangle, \quad \forall v \in \mathbb{R}^n.$$

Remark 3. For locally Lipschitz functions, one may find upper (semi-regular) convexifactors which are smaller than the Clarke subdifferential [5] and the Mordukhovich subdifferential [25] as Example 4 shows. In addition, an upper (semi-regular) convexifactor may contain only a finite number of elements.

Example 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = 3|x| - |y|, \; \forall (x,y) \in \mathbb{R}^2.$$

- The function f admits
$$\partial^* f(0,0) := \{(3,-1),(-3,1)\}$$
as an upper semi-regular convexificator at $(0,0)$; whereas the Mordukhovich subdifferential of f at $(0,0)$ and the Clarke subdifferential of f at $(0,0)$ are respectively the sets
$$\partial^m f(0,0) = \big\{(t,1) \in \mathbb{R}^2 : -3 \leq t \leq 3\big\} \cup \big\{(t,-1) \in \mathbb{R}^2 : -3 \leq t \leq 3\big\}$$
and
$$\partial^c f(0,0) = conv\ \{(3,-1),(-3,1),(3,1),(-3,-1)\}.$$
- Observe that the upper semi-regular convexificator $\partial^* f(0,0)$ is strictly included in the Mordukhovich subdifferential $\partial^m f(0,0)$. More than that, the convex hull of $\partial^* f(0,0)$ is a proper subset of both $\partial^c f(0,0)$ and $conv\ \partial^m f(0,0)$.

Remark 5. Example 4 shows that necessary optimality conditions that are expressed in terms of UCFs or USRCFs may provide sharp conditions even for locally Lipschitz functions.

Proposition 6. *Let $\mathcal{F} := \{1,2\}$. Suppose that the functions $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ are continuous at $\overline{x}$ and admit USRCFs $\partial^* f_1(\overline{x})$ and $\partial^* f_2(\overline{x})$ at $\overline{x}$. Then $\bigcup_{i \in \mathcal{F}(\overline{x})} \partial^* f_i(\overline{x})$ is an USRCF at $\overline{x}$ of the min-function*

$$h(x) := \min\{f_1(x), \; f_2(x)\},$$

where $\mathcal{F}(\overline{x}) := \{i \in \mathcal{F} : f_i(\overline{x}) = h(\overline{x})\}$.

Proof. The proof consists of two steps.

– If $f_1(\overline{x}) < f_2(\overline{x})$, then $\mathcal{F}(\overline{x}) = \{1\}$ and $h(x) = f_1(x)$ for each x in a neighborhood of $\overline{x}$. Hence

$$h_d^+(\overline{x}, v) = (f_1)_d^+(\overline{x}, v) \leq \sup_{x^* \in \partial^* f_1(\overline{x})} \langle x^*, v\rangle, \quad \forall v \in \mathbb{R}^n.$$

Consequently, $\partial^* h(\overline{x}) := \partial^* f_1(\overline{x})$ is an upper semi-regular convexificator of h at $\overline{x}$. Similarly, if $f_2(\overline{x}) < f_1(\overline{x})$, we get $\partial^* h(\overline{x}) = \partial^* f_2(\overline{x})$ is an upper semi-regular convexificator of h at $\overline{x}$.

– If $f_1(\overline{x}) = f_2(\overline{x})$, then $\mathcal{F}(\overline{x}) = \{1, 2\}$. Let $v \in \mathbb{R}^n$.

- Since $h(\overline{x}) = f_1(\overline{x}) = f_2(\overline{x})$ and since

$$h_d^+(\overline{x}, v) = \limsup_{t \searrow 0} \frac{h(\overline{x} + tv) - h(\overline{x})}{t}$$

we have

$$h_d^+(\overline{x}, v) = \limsup_{t \searrow 0} \frac{\min\{f_1(\overline{x} + tv) - f_1(\overline{x}), f_2(\overline{x} + tv) - f_1(\overline{x})\}}{t}.$$

Then,

$$h_d^+(\overline{x}, v) \leq \limsup_{t \searrow 0} \frac{\max\{f_1(\overline{x} + tv) - f_1(\overline{x}), f_2(\overline{x} + tv) - f_2(\overline{x})\}}{t}.$$

Consequently,

$$h_d^+(\overline{x}, v) \leq \max\left\{\limsup_{t \searrow 0} \frac{f_1(\overline{x} + tv) - f_1(\overline{x})}{t},\ \limsup_{t \searrow 0} \frac{f_2(\overline{x} + tv) - f_2(\overline{x})}{t}\right\}.$$

Thus,

$$h_d^+(\overline{x}, v) \leq \max\Big\{(f_1)_d^+(\overline{x}, v),\ (f_2)_d^+(\overline{x}, v)\Big\}.$$

- Since $\partial^* f_1(\overline{x})$ and $\partial^* f_2(\overline{x})$ are USRCFs of f_1 and f_2 at $\overline{x}$, we deduce

$$h_d^+(\overline{x}, v) \leq \max\left\{\sup_{x^* \in \partial^* f_1(\overline{x})} \langle x^*, v\rangle,\ \sup_{x^* \in \partial^* f_2(\overline{x})} \langle x^*, v\rangle\right\}.$$

Finally, we obtain

$$h_d^+(\overline{x}, v) \leq \sup_{x^* \in \partial^* f_1(\overline{x}) \cup \partial^* f_2(\overline{x})} \langle x^*, v\rangle, \quad \forall v \in \mathbb{R}^n;$$

which means that

$$\partial^* h(\overline{x}) = \partial^* f_1(\overline{x}) \cup \partial^* f_2(\overline{x})$$

is an upper semi-regular convexificator of h at $\overline{x}$.

□

The following Lemma is needed in the sequel.

Lemma 7. [18] *Let B be a nonempty, convex and compact set and A be a convex cone. If*

$$\sup_{v \in B} \langle v, d\rangle \geq 0, \quad \text{for all } d \in A^-$$

then

$$0 \in B + cl\ A.$$

To progress, we need the following definitions.

Definition 8. [21] A set valued mapping $\psi : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ is upper semicontinuous (u.s.c.) at x, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $x' \in x + \delta\mathbb{B}_{\mathbb{R}^p}$ we have

$$\psi(x') \subseteq \psi(x) + \varepsilon\mathbb{B}_{\mathbb{R}^q},$$

where $\mathbb{B}_{\mathbb{R}^p}$ and $\mathbb{B}_{\mathbb{R}^q}$ are the unit balls in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.

Definition 9. [12] The set-valued mapping $\psi : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ is said to be inner semicompact at $\overline{x}$, $\psi(\overline{x}) \neq \emptyset$, if and only if, for every sequence $x_k \to \overline{x}$ with $\psi(x_k) \neq \emptyset$ there is a sequence of $y_k \in \psi(x_k)$ that contains a convergent subsequence as $k \to \infty$.

Remark 10. [12] The inner semicompactness of the set-valued mapping $\psi : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ holds whenever ψ is uniformly bounded and has nonempty values around $\overline{x}$, *i.e.* there exist a neighbourhood U of $\overline{x}$ and a bounded set $A \subset \mathbb{R}^q$ such that $\psi(x) \subset A$, for all $x \in U$.

3. Necessary optimality conditions

Let $(\overline{x}, \overline{y}) \in \Omega$. For each $(x, y) \in \Omega$, we set

$$\phi(x, y) := \min\{F_L(x, y) - F_L(\overline{x}, \overline{y}),\ F_U(x, y) - F_U(\overline{x}, \overline{y})\}.$$

Next, let us look at how the interval-valued (IVP) is connected to the scalar optimization problem

$$(P^*) : \begin{cases} \min\limits_{x,y} \phi(x, y) \\ \text{s.t.} \ \ (x, y) \in \Omega. \end{cases},$$

Proposition 11. *$(\overline{x}, \overline{y}) \in \Omega$ is a local LU-optimal solution of* (IVP) *if and only if $(\overline{x}, \overline{y})$ is a local optimal solution of* (P^*).

Proof. The proof consists of two parts.

- Suppose that $(\overline{x}, \overline{y})$ be a local LU-optimal solution of (IVP). Then, there exists a neighborhood U_0 of $(\overline{x}, \overline{y})$ such that
$$F_L(x, y) - F_L(\overline{x}, \overline{y}) \geq 0 \text{ and } F_U(x, y) - F_U(\overline{x}, \overline{y}) \geq 0,\ \forall (x, y) \in U_0 \cap \Omega.$$
Since $\phi(\overline{x}, \overline{y}) = 0$, we obtain
$$\phi(x, y) - \phi(\overline{x}, \overline{y}) \geq 0,\ \forall (x, y) \in U_0 \cap \Omega.$$
- Suppose that $(\overline{x}, \overline{y})$ is a local optimal solution of (P^*). By absurdness, suppose that there exists $(x_n, y_n) \in \Omega$ such that $(x_n, y_n) \to (\overline{x}, \overline{y})$ and
$$F_L(\overline{x}, \overline{y}) > F_L(x_n, y_n) \text{ or } F_U(\overline{x}, \overline{y}) > F_U(x_n, y_n).$$
Consequently,
$$0 > F_L(x_n, y_n) - F_L(\overline{x}, \overline{y}) \text{ or } 0 > F_U(x_n, y_n) - F_U(\overline{x}, \overline{y}).$$
Then, $0 > \phi(x_n, y_n)$. Since $\phi(\overline{x}, \overline{y}) = 0$, we obtain
$$\phi(\overline{x}, \overline{y}) > \phi(x_n, y_n);$$
which contradicts the fact that $(\overline{x}, \overline{y})$ is a local optimal solution of (P^*).

□

In order to get necessary optimality conditions of (IVP), we consider the optimal value functions $V_L : \mathbb{R}^{n_1} \to \mathbb{R}$ and $V_U : \mathbb{R}^{n_1} \to \mathbb{R}$ defined by

$$V_L(x) := \min_y \{f_L(x,y) : g_i(x,y) \leq 0, \quad \forall i \in I\},$$

and

$$V_U(x) := \min_y \{f_U(x,y) : g_i(x,y) \leq 0, \quad \forall i \in I\}.$$

Observe that with the help of the optimal value functions V_L and V_U, for each $x \in \mathbb{R}^{n_1}$, the solution sets $\psi_L(x)$ and $\psi_L(x)$ may be formulated as

$$\psi_L(x) = \{y \in \Xi(x) \text{ such that } f_L(x,y) = V_L(x)\}$$

and

$$\psi_L(x) = \{y \in \Xi(x) \text{ such that } f_U(x,y) = V_U(x)\}.$$

Using the optimal value reformulation, (P^*) can be replaced by the optimization problem

$$\left(\widetilde{P}\right) : \begin{cases} \min\limits_{x,y} & \phi(x,y) \\ \text{s.t.} & \begin{cases} (x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \\ G_j(x,y) \leq 0, & \forall j \in J, \\ g_i(x,y) \leq 0, & \forall i \in I, \\ f_L(x,y) - V_L(x) \leq 0, \\ f_U(x,y) - V_U(x) \leq 0. \end{cases} \end{cases}$$

It is worth noting that

$$\Omega = \begin{Bmatrix} (x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \\ G_j(x,y) \leq 0, & \forall j \in J, \\ g_i(x,y) \leq 0, & \forall i \in I, \\ f_L(x,y) - V_L(x) \leq 0, \\ f_U(x,y) - V_U(x) \leq 0 \end{Bmatrix}.$$

Nevertheless, due to the nondifferentiability of the value function, $\left(\widetilde{P}\right)$ is in general a nonconvex and nondifferentiable optimization problems even if all its defining functions are convex and smooth, see [6]. Moreover, due to the bilevel structure, the Mangasarian Fromovitz constraint qualification for $\left(\widetilde{P}\right)$ is never satisfied. To obtain necessary optimality conditions, we suggest the following constraint qualification.

Definition 12. We say that nonsmooth Abadie constraint qualification (ACQ) holds at $(\overline{x},\overline{y}) \in \Omega$ if

$$\Gamma(\overline{x},\overline{y})^- \subseteq T_\Omega(\overline{x},\overline{y}),$$

where

$$\Gamma(\overline{x},\overline{y}) := \left(\bigcup_{j \in J(\overline{x},\overline{y})} \partial^* G_j(\overline{x},\overline{y})\right) \cup \left(\bigcup_{i \in I(\overline{x},\overline{y})} \partial^* g_i(\overline{x},\overline{y})\right)$$
$$\cup \left(\partial^* f_L(\overline{x},\overline{y}) - \partial^* V_L(\overline{x}) \times \{0\}\right) \cup \left(\partial^* f_U(\overline{x},\overline{y}) - \partial^* V_U(\overline{x}) \times \{0\}\right).$$

Remark 13. Under the following hypotheses (H_1), (H_2), (H_3) and (H_4), the optimization problem $\left(\widetilde{P}\right)$ has at least one optimal solution.
$(H_1) : F_L(\cdot,\cdot)$ and $F_U(\cdot,\cdot)$ are l.s.c. on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.
$(H_2) : f_L(\cdot,\cdot)$ and $f_U(\cdot,\cdot)$ are l.s.c and $V_L(\cdot)$ and $V_U(\cdot)$ are upper semicontinuous (u.s.c.) on $\mathbb{R}^{n_1}$.
$(H_3) : G_j(\cdot,\cdot)$ and $g_i(\cdot,\cdot)$ are l.s.c. on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$
(H_4) : The feasible set Ω is nonempty and bounded.

For the rest of the paper, we will make use of the following assumptions.

Assumption 1.

– F_U and F_L, are locally Lipschitz and admit bounded USRCFs $\partial^* F_U(\overline{x},\overline{y})$ and $\partial^* F_L(\overline{x},\overline{y})$ at $(\overline{x},\overline{y})$.
– $G_j, \forall j \in J(\overline{x},\overline{y})$, $g_i, \forall i \in I(\overline{x},\overline{y})$, f_U and f_L admit UCFs $\partial^* G_j(\overline{x},\overline{y}), \forall j \in J(\overline{x},\overline{y})$, and $\partial^* g_i(\overline{x},\overline{y}), \forall i \in I(\overline{x},\overline{y})$, $\partial^* f_U(\overline{x},\overline{y})$ and $\partial^* f_L(\overline{x},\overline{y})$ at $(\overline{x},\overline{y})$. Here, $I(\overline{x},\overline{y})$ and $J(\overline{x},\overline{y})$ denote the sets of the active constraints at $(\overline{x},\overline{y})$.
– V_U and V_L admit UCFs $\partial^* V_U(\overline{x})$ and $\partial^* V_L(\overline{x})$ at $\overline{x}$.

Assumption 2.

– F_U, F_L, f_U and f_L are locally Lipschitz and admit bounded USRCFs $\partial^* F_U(\overline{x},\overline{y})$, $\partial^* F_L(\overline{x},\overline{y})$, $\partial^* f_U(\overline{x},\overline{y})$ and $\partial^* f_L(\overline{x},\overline{y})$ at $(\overline{x},\overline{y})$.
– $G_j, \forall j \in J(\overline{x},\overline{y})$, and $g_i, \forall i \in I(\overline{x},\overline{y})$, are continuous and admit UCFs $\partial^* G_j(\overline{x},\overline{y}), \forall j \in J(\overline{x},\overline{y})$, and $\partial^* g_i(\overline{x},\overline{y}), \forall i \in I(\overline{x},\overline{y})$, at $(\overline{x},\overline{y})$.

Assumption 3.

– There is a neighborhood U of $(\overline{x},\overline{y})$ such that:
  - $\partial^* f_L$, $\partial^* f_U$, $\partial^* g_i, \forall i \in I(\overline{x},\overline{y})$, are bounded upper semi-continuous set valued mappings over $U \cap gr\Xi$.
  - For all $(x,y) \in U \cap gr\Xi$, the sets $\partial^* f_U(x,y)$ and $\partial^* f_L(x,y)$ are USRCFs of f_U and f_L at (x,y).
– The argminimum maps ψ_L and ψ_L are inner semicompact at $\overline{x}$.

Theorem 14. *Let* $(\overline{x},\overline{y}) \in \Omega$ *be a local LU-optimal solution of* (IVP) *where* (ACQ) *holds. Suppose that* $pos\ \Gamma(\overline{x},\overline{y})$ *is closed and that Assumption 1 is satisfied. Then there exist scalars* $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, $\alpha_j \geq 0$, $j \in J(\overline{x},\overline{y})$, $\beta_j \geq 0$, $j \in J(\overline{x},\overline{y})$, $\gamma \geq 0$, $\rho \geq 0$ *such that*

$$(0,0) \in \begin{bmatrix} \lambda_1\ conv\ \partial^* F_L(\overline{x},\overline{y}) + \lambda_2\ conv\ \partial^* F_U(\overline{x},\overline{y}) + \sum_{j \in J(\overline{x},\overline{y})} \alpha_j\ conv\ \partial^* G_j(\overline{x},\overline{y}) \\ + \sum_{i \in I(\overline{x},\overline{y})} \beta_i\ conv\ \partial^* g_i(\overline{x},\overline{y}) + \gamma\ conv\ \partial^* f_L(\overline{x},\overline{y}) + \rho\ conv\ \partial^* f_U(\overline{x},\overline{y}) \\ -\gamma\ conv\ \partial^* V_L(\overline{x}) \times \{0\} - \rho\ conv\ \partial^* V_U(\overline{x}) \times \{0\} \end{bmatrix}. \tag{1}$$

Proof. Since $(\overline{x},\overline{y})$ is a local LU-optimal solution of (IVP), by Proposition 11, it is a local optimal solution of (P^*). Consequently, there exists a neighbourhood U of $(\overline{x},\overline{y})$ such that for all $(x,y) \in U \cap \Omega$

$$\phi(x,y) \geq \phi(\overline{x},\overline{y}). \tag{2}$$

– We claim that

$$\phi^+((\overline{x},\overline{y}),d) \geq 0, \text{ for all } d \in [pos\ \Gamma(\overline{x},\overline{y})]^-. \tag{3}$$

Indeed, let $d = (d_1,d_2) \in T_\Omega(\overline{x},\overline{y})$. Then, there exist $t_n \searrow 0$ and $d_n = (d_{n_1}, d_{n_2})$ such that

$$(\overline{x},\overline{y}) + t_n d_n \in \Omega.$$

For n large enough, we have $(\overline{x},\overline{y}) + t_n d_n \in U \cap \Omega$; and by (2) we get

$$\frac{\phi((\overline{x},\overline{y}) + t_n d_n) - \phi(\overline{x},\overline{y})}{t_n} \geq 0.$$

Since

$$\begin{aligned}\frac{\phi((\overline{x},\overline{y}) + t_n d_n) - \phi(\overline{x},\overline{y})}{t_n} &= \frac{\phi((\overline{x},\overline{y}) + t_n (d_{n_1}, d_{n_2})) - \phi(\overline{x},\overline{y})}{t_n} \\ &= \frac{\phi((\overline{x},\overline{y}) + t_n (d_{n_1}, d_{n_2})) - \phi((\overline{x},\overline{y}) + t_n(d_1,d_2))}{t_n} + \frac{\phi((\overline{x},\overline{y}) + t_n(d_1,d_2)) - \phi(\overline{x},\overline{y})}{t_n}.\end{aligned}$$

And since ϕ is locally Lipschitz, we have

$$\frac{\phi((\overline{x},\overline{y})+t_n(d_{n_1},d_{n_2}))-\phi((\overline{x},\overline{y})+t_n(d_1,d_2))}{t_n}\to 0.$$

Letting $n\to+\infty$, we obtain

$$\phi^+((\overline{x},\overline{y}),d)\geq 0,\ \text{ for all } d\in T_\Omega(\overline{x},\overline{y}).$$

Since (ACQ) holds at $(\overline{x},\overline{y})$, we get

$$\phi^+((\overline{x},\overline{y}),d)\geq 0,\ \text{ for all } d\in \Gamma(\overline{x},\overline{y})^-;$$

which implies inequality (3).

– From inequality (3), we have

$$\sup_{\eta\in\partial^*\phi(\overline{x},\overline{y})}\langle\eta,d\rangle\geq 0,\ \text{ for all } d\in[pos\ \Gamma(\overline{x},\overline{y})]^-,$$

where

$$\partial^*\phi(\overline{x},\overline{y}):=conv\ (\partial^*F_L(\overline{x},\overline{y})\cup\partial^*F_U(\overline{x},\overline{y})) \tag{4}$$

is an upper semi-regular convexificator of ϕ at $(\overline{x},\overline{y})$.

- Using that $\partial^*\phi(\overline{x},\overline{y})$ is convex and compact and Lemma 7, we get

$$(0,0)\in\partial^*\phi(\overline{x},\overline{y})+cl\ pos\ \Gamma(\overline{x},\overline{y}).$$

- By the closedness of $pos\ \Gamma(\overline{x},\overline{y})$, we deduce

$$(0,0)\in\partial^*\phi(\overline{x},\overline{y})+pos\ \Gamma(\overline{x},\overline{y});$$

  which combined with (4), implies the existence of scalars $\lambda_1,\ \lambda_2\in[0,1]$ satisfying $\lambda_1+\lambda_2=1$ and

$$(0,0)\in\lambda_1\ conv\ \partial^*F_L(\overline{x},\overline{y})+\lambda_2\ conv\ \partial^*F_U(\overline{x},\overline{y})+pos\ \Gamma(\overline{x},\overline{y}).$$

  Thus,

$$0\in\left[\begin{array}{c}\lambda_1\ conv\ \partial^*F_L(\overline{x},\overline{y})+\lambda_2\ conv\ \partial^*F_U(\overline{x},\overline{y})+\sum\limits_{j\in J(\overline{x},\overline{y})}pos\ \partial^*G_j(\overline{x},\overline{y})+\sum\limits_{i\in I(\overline{x},\overline{y})}pos\ \partial^*g_i(\overline{x},\overline{y})\\ +pos\ (\partial^*f_L(\overline{x},\overline{y})-\ \partial^*V_L(\overline{x})\times\{0\})+pos\ (\partial^*f_U(\overline{x},\overline{y})-\partial^*V_U(\overline{x})\times\{0\})\end{array}\right].$$

  Consequently, we can find scalars $\alpha_j\geq 0,\ j\in J(\overline{x},\overline{y}),\ \beta_i\geq 0,\ i\in I(\overline{x},\overline{y}),\ \gamma\geq 0,\ \rho\geq 0$ such that

$$(0,0)\in\left[\begin{array}{c}\lambda_1\ conv\ \partial^*F_L(\overline{x},\overline{y})+\lambda_2\ conv\ \partial^*F_U(\overline{x},\overline{y})+\sum\limits_{j\in J(\overline{x},\overline{y})}\alpha_j\ conv\ \partial^*G_j(\overline{x},\overline{y})\\ +\sum\limits_{i\in I(\overline{x},\overline{y})}\beta_i\ conv\ \partial^*g_i(\overline{x},\overline{y})\\ +\gamma\ (conv\ \partial^*f_L(\overline{x},\overline{y})-conv\ \partial^*V_L(\overline{x})\times\{0\})\\ +\rho\ (conv\ \partial^*f_U(\overline{x},\overline{y})-conv\ \partial^*V_U(\overline{x})\times\{0\})\end{array}\right].$$

□

The following example illustrates how to apply Theorem 14.

Example 15. Consider the following bilevel interval-valued optimization problem

$$(\text{IVP}) : \begin{cases} \min\limits_{x,y} & F(x,y) = \left[x - y^2,\ x + xy\right] \\ & G_1(x,y) = x \leq 0, \\ & G_2(x,y) = -x - 1 \leq 0, \\ & y \in \psi(x), \end{cases}$$

and for each $x \in \mathbb{R}$, $\psi(x)$ is the LU-solution set of the following parametric interval-valued optimization problem

$$(\text{IVP}_x) : \begin{cases} \min\limits_{y} & f(x,y) = \left[3y + yx^2 + x,\ -2xy + y\right] \\ & g_1(x,y) = x - y \leq 0, \\ & g_2(x,y) = y \leq 0. \end{cases}$$

– $(\overline{x}, \overline{y}) = (0,0)$ is a feasible solution of (IVP) with $I(\overline{x}, \overline{y}) = \{1\}$, $J(\overline{x}, \overline{y}) = \{1,2\}$, $\psi_L(x) = \{x\}$, $V_L(x) = 4x + x^3$,

$$\psi_U(x) = \begin{cases} \{x\} & \text{if } x \leq 0, \\ \emptyset & \text{if } x > 0, \end{cases} \quad \text{and} \quad V_U(x) = \begin{cases} -2x^2 + x & \text{if } x \leq 0, \\ +\infty & \text{if } x > 0, \end{cases}$$

which implies

$$\Omega = \{(x,y) :\ y = x,\ -1 \leq x \leq 0\} \text{ and } T_\Omega(\overline{x}, \overline{y}) = \{(x,x) : x \leq 0\},$$

with

$$\psi(x) = \begin{cases} \{x\} & \text{if } x \leq 0, \\ \emptyset & \text{if } x > 0. \end{cases}$$

– Assumption 1 is satisfied. Indeed, $\partial^* F_L(\overline{x}, \overline{y}) = \{(1,0)\}$ and $\partial^* F_U(\overline{x}, \overline{y}) = \{(1,0)\}$ are bounded USRCFs of F_L and F_U at $(\overline{x}, \overline{y})$.
  - $\partial^* G_1(\overline{x}, \overline{y}) = \{(1,0)\}$, $\partial^* g_1(\overline{x}, \overline{y}) = \{(1,-1)\}$, $\partial^* g_2(\overline{x}, \overline{y}) = \{(0,1)\}$, $\partial^* f_L(\overline{x}, \overline{y}) = \{(1,3)\}$ and $\partial^* f_U(\overline{x}, \overline{y}) = \{(0,1)\}$ are UCFs of G_1, g_1, g_2, f_L and f_U at $(\overline{x}, \overline{y})$.
  - $\partial^* V_L(\overline{x}) = \{4\}$ and $\partial^* V_U(\overline{x}) = \{1\}$ are UCFs of V_L and V_U at $\overline{x}$.

– (ACQ) holds at $(\overline{x}, \overline{y})$. Indeed, since

$$\Gamma(\overline{x}, \overline{y}) = \{(1,0),\ (1,-1),\ (0,1),\ (-3,3),\ (-1,1)\},$$

we get

$$\Gamma(\overline{x}, \overline{y})^- = \{(x,x) : x \leq 0\} \text{ and } pos\ \Gamma(\overline{x}, \overline{y}) = \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x + y\right\}.$$

Observe that *pos* $\Gamma(\overline{x}, \overline{y})$ is closed and that

$$\Gamma(\overline{x}, \overline{y})^- \subseteq T_\Omega(\overline{x}, \overline{y}).$$

– Notice that for all $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, $\alpha_j \geq 0$, $j \in J(\overline{x}, \overline{y})$, $\beta_i \geq 0$, $i \in I(\overline{x}, \overline{y})$, $\gamma \geq 0$ and $\rho \geq 0$, Inclusion (1) is never satisfied. Taking into account Theorem 14, one sees that $(\overline{x}, \overline{y})$ is not a local LU-optimal solution of (IVP).

Remark 16. The closedness of *pos* $\Gamma(\overline{x}, \overline{y})$ is ensured for bilevel linear optimization problems (where the optimal value functions are piecewise affine) [7]. Notice that if *conv* $\Gamma(\overline{x}, \overline{y})$ is a polyhedral set containing the origin, *pos* $\Gamma(\overline{x}, \overline{y})$ is also closed. Indeed, in this case *pos* $\Gamma(\overline{x}, \overline{y})$ is a polyhedral convex cone [26, Corollary 19.7.1] and, thus, closed. Remark that *pos* $\Gamma(\overline{x}, \overline{y})$ = *pos conv* $\Gamma(\overline{x}, \overline{y})$.

In order to get upper estimates for upper convexificators of the optimal value functions V_L and V_U of the parametric nonsmooth optimization problems $\left(\text{IVP}_x^L\right)$ and $\left(\text{IVP}_x^U\right)$, we need the following constraint qualification [15]: (GCQ) is said to hold at $(\overline{x}, \overline{y}) \in gr\Xi$, for the system of inequalities $g_i(x,y) \leq 0$, $i \in I$, iff

$$[T_{gr\Xi}(\overline{x}, \overline{y})]^- \subseteq cl\ pos\ \Upsilon(\overline{x}, \overline{y})$$

where

$$\Upsilon(\overline{x},\overline{y}) := \bigcup_{i\in I(\overline{x},\overline{y})} \partial^* g_i(\overline{x},\overline{y}).$$

It should be underlined that since f and $g_i, i \in I$, are not necessarily convex, the upper estimates in terms of convex analysis subdifferentials cannot be utilized, as Kohli did in [22]. In addition, in contrast to [22], which utilized a unique optimal value function, we are forced to employ two potentially distinct optimal value functions because the parametric lower-level problem (IVP$_x$) is interval-valued.

Theorem 17. *Let* $(\overline{x},\overline{y}) \in \Omega$ *be a local LU-optimal solution of* (IVP) *where* (*ACQ*) *holds. Suppose that pos* $\Gamma(\overline{x},\overline{y})$ *is closed, that Assumptions 2 and 3 are satisfied and that* (*GCQ*) *holds at all* $(\widetilde{x},\widetilde{y}) \in gr\psi_L \cap gr\psi_L$ *near* $(\overline{x},\overline{y})$. *Then there exist scalars* $\lambda_1,\ \lambda_2 \geq 0,\ \alpha_j \geq 0,\ j \in J(\overline{x},\overline{y}),\ \beta_i \geq 0,\ i \in I(\overline{x},\overline{y}),\ \gamma \geq 0,\ \rho \geq 0,\ \zeta_r \geq 0,\ \chi_r \geq 0,\ y_r \in \psi_L(\overline{x}),\ z_r \in \psi_L(\overline{x}),\ \zeta_r^i \geq 0,\ i \in I(\overline{x},y_r),\ \chi_r^i \geq 0,\ i \in I(\overline{x},z_r),\ r = 1,\dots,\ n_1+1,$ *such that*

$$\lambda_1 + \lambda_2 = 1,\ \sum_{r=1}^{n_1+1} \zeta_r = 1,\ \sum_{r=1}^{n_1+1} \chi_r = 1 \tag{5}$$

and

$$(0,0) \in \begin{bmatrix} \lambda_1\ conv\ \partial^* F_L(\overline{x},\overline{y}) + \lambda_2\ conv\ \partial^* F_U(\overline{x},\overline{y}) + \displaystyle\sum_{j\in J(\overline{x},\overline{y})} \alpha_j\ conv\ \partial^* G_j(\overline{x},\overline{y}) \\ + \displaystyle\sum_{i\in I(\overline{x},\overline{y})} \beta_i\ conv\ \partial^* g_i(\overline{x},\overline{y}) + \gamma\ conv\ \partial^* f_L(\overline{x},\overline{y}) + \rho\ conv\ \partial^* f_U(\overline{x},\overline{y}) \\ -\gamma \displaystyle\sum_{r=1}^{n_1+1} \zeta_r \left(conv\ \partial^* f_L(\overline{x},y_r) + \sum_{i\in I(\overline{x},y_r)} \zeta_r^i\ conv\ \partial^* g_i(\overline{x},y_r) \right) \\ -\rho \displaystyle\sum_{r=1}^{n_1+1} \chi_r \left(conv\ \partial^* f_U(\overline{x},z_r) + \sum_{i\in I(\overline{x},z_r)} \chi_r^i\ conv\ \partial^* g_i(\overline{x},z_r) \right) \end{bmatrix}. \tag{6}$$

Proof. Under the assumptions made in this theorem, the value functions V_L and V_U are locally Lipschitzian around $\overline{x}$. Consequently, we can find convexificators $\partial^* V_L(\overline{x})$ and $\partial^* V_U(\overline{x})$ of V_L and V_U at $\overline{x}$ such that $\partial^* V_L(\overline{x}) \subseteq \partial^c V_L(\overline{x})$ and $\partial^* V_U(\overline{x}) \subseteq \partial^c V_U(\overline{x})$. Notice that $\partial^c V_L(\overline{x})$ and $\partial^c V_U(\overline{x})$ are themselves convexificators of V_L and V_U at $\overline{x}$, respectively. Applying [15, Thm. 12] (its inner semicompact counterpart), we get the following upper estimate of convexificators of $\partial^* V_L(\overline{x})$ and $\partial^* V_U(\overline{x})$:

$$\partial^* V_L(\overline{x}) \subseteq \bigcup_{y\in\psi_L(\overline{x})} \left\{ \alpha^* : (\alpha^*, 0) \in conv\ \partial^* f_L(\overline{x},y) + \sum_{i\in I(\overline{x},y)} pos\ \partial^* g_i(\overline{x},y) \right\} \tag{7}$$

and

$$\partial^* V_U(\overline{x}) \subseteq \bigcup_{z\in\psi_L(\overline{x})} \left\{ \beta^* : (\beta^*, 0) \in conv\ \partial^* f_U(\overline{x},z) + \sum_{i\in I(\overline{x},z)} pos\ \partial^* g_i(\overline{x},z) \right\}. \tag{8}$$

Let $\alpha^* \in conv\ \partial^* V_L(\overline{x})$ and $\beta^* \in conv\ \partial^* V_U(\overline{x})$. By the classical Carathéodory theorem, there exist $\zeta_r \geq 0,\ \chi_r \geq 0,\ \sum_{r=1}^{n_1+1} \zeta_r = 1,\ \sum_{r=1}^{n_1+1} \chi_r = 1$, $\alpha_r^* \in \partial^* V_L(\overline{x})$ and $\beta_r^* \in \partial^* V_U(\overline{x})$ such that

$$\alpha^* = \sum_{r=1}^{n_1+1} \zeta_r\ \alpha_r^* \text{ and } \beta^* = \sum_{r=1}^{n_1+1} \chi_r\ \beta_r^*. \tag{9}$$

Since $\alpha_r^* \in \partial^* V_L(\overline{x})$ and $\beta_r^* \in \partial^* V_U(\overline{x})$, using (7) and (8), we obtain $y_r \in \psi_L(\overline{x})$ and $z_r \in \psi_L(\overline{x})$, $r = 1, \ldots,\ n_1 + 1$, such that

$$(\alpha_r^*, 0) \in conv\ \partial^* f_L(\overline{x}, y_r) + \sum_{i \in I(\overline{x}, y_r)} pos\ \partial^* g_i(\overline{x}, y_r) \tag{10}$$

and

$$(\beta_r^*, 0) \in\ conv\ \partial^* f_L(\overline{x}, z_r) + \sum_{i \in I(\overline{x}, z_r)} pos\ \partial^* g_i(\overline{x}, z_r). \tag{11}$$

Combining (9) with (10) and (11), we get

$$(\alpha^*, 0) \in \sum_{r=1}^{n_1+1} \zeta_r \left(conv\ \partial^* f_L(\overline{x}, y_r) + \sum_{i \in I(\overline{x}, y_r)} pos\ \partial^* g_i(\overline{x}, y_r) \right)$$

and

$$(\beta^*, 0) \in \sum_{r=1}^{n_1+1} \chi_r \left(conv\ \partial^* f_U(\overline{x}, z_r) + \sum_{i \in I(\overline{x}, z_r)} pos\ \partial^* g_i(\overline{x}, z_r) \right).$$

Then, we can find $\zeta_r^i \geq 0,\ i \in I(\overline{x}, y_r),\ \chi_r^i\ \geq 0,\ i \in I(\overline{x}, z_r),\ r = 1, \ldots,\ n_1 + 1$, such that

$$(\alpha^*, 0) \in \sum_{r=1}^{n_1+1} \zeta_r \left(conv\ \partial^* f_L(\overline{x}, y_r) + \sum_{i \in I(\overline{x}, y_r)} \zeta_r^i\ conv\ \partial^* g_i(\overline{x}, y_r) \right) \tag{12}$$

and

$$(\beta^*, 0) \in \sum_{r=1}^{n_1+1} \chi_r \left(conv\ \partial^* f_U(\overline{x}, z_r) + \sum_{i \in I(\overline{x}, z_r)} \chi_r^i\ conv\ \partial^* g_i(\overline{x}, z_r) \right). \tag{13}$$

Using Theorem 14 together with (12) and (13), we arrive at the necessary conditions (5) and (6). □

Example 18 illustrates the necessary optimality conditions given in Theorem 17.

Example 18. Consider the following bilevel interval-valued optimization problem

$$(\text{IVP}) : \begin{cases} \min\limits_{x,y} & F(x, y) = \left[(y+1)^2 - 2x - \dfrac{1}{2},\ 3|x| - |y| + 1 \right] \\ & G_1(x, y) = x \leq 0, \\ & G_2(x, y) = |x| - \frac{1}{2} \leq 0, \\ & y \in \psi(x), \end{cases}$$

and for each $x \in \mathbb{R}$, $\psi(x)$ is the LU-solution set of the following parametric interval-valued optimization problem

$$(\text{IVP}_x) : \begin{cases} \min\limits_{y} & f(x, y) = \left[y + yx^2,\ -xy \right] \\ & g_1(x, y) = x - y \leq 0, \\ & g_2(x, y) = y \leq 0. \end{cases}$$

– $(\overline{x}, \overline{y}) = (0, 0)$ is a LU-optimal solution of (IVP) with $I(\overline{x}, \overline{y}) = \{1\},\ J(\overline{x}, \overline{y}) = \{1, 2\}$,

$$\psi_L(x) = \{x\},\quad V_L(x) = x + x^3,\ \psi_U(x) = \begin{cases} \{x\} & \text{if } x \leq 0, \\ \emptyset & \text{if } x > 0, \end{cases} \text{ and } V_U(x) = \begin{cases} -x^2 & \text{if } x \leq 0, \\ +\infty & \text{if } x > 0, \end{cases}$$

which implies

$$\Omega = \left\{(x,y) :\ y = x,\ -\frac{1}{2} \leq x \leq 0\right\} \text{ and } T_\Omega(\overline{x},\overline{y}) = \{(x,x) : x \leq 0\}$$

with

$$\psi(x) = \begin{cases} \{x\} & \text{if } x \leq 0, \\ \emptyset & \text{if } x > 0. \end{cases}$$

– Assumptions 2 and 3 are satisfied. Indeed, $\partial^* F_L(\overline{x},\overline{y}) = \{(-2,2)\}$ and $\partial^* F_U(\overline{x},\overline{y}) = \{(-3,1),\ (3,-1)\}$ are bounded USRCFs of F_L and F_U at $(\overline{x},\overline{y})$.

- $\partial^* G_1(\overline{x},\overline{y}) = \{(1,0)\}$, $\partial^* g_1(\overline{x},\overline{y}) = \{(1,-1)\}$, $\partial^* g_2(\overline{x},\overline{y}) = \{(0,1)\}$, $\partial^* f_L(\overline{x},\overline{y}) = \{(0,1)\}$ and $\partial^* f_U(\overline{x},\overline{y}) = \{(0,0)\}$ are UCFs of $G_1,\ g_1,\ g_2,\ f_L$ and f_U at $(\overline{x},\overline{y})$.
- $\partial^* V_L(\overline{x}) = \{1\}$ and $\partial^* V_U(\overline{x}) = \{0\}$ are UCFs of V_L and V_U at $\overline{x}$.
- Let $U := \left]-\frac{1}{4}, \frac{1}{4}\right[\times \left]-\frac{1}{4}, \frac{1}{4}\right[$. The sets $\partial^* f_L(\widetilde{x},\widetilde{y}) = \left\{\left(2\widetilde{x}\widetilde{y}, 1+\widetilde{x}^2\right)\right\}$ and $\partial^* f_U(\widetilde{x},\widetilde{y}) = \{(-\widetilde{y},-\widetilde{x})\}$ are USRCFs of f_U and f_L at every $(\widetilde{x},\widetilde{y}) \in U \cap gr\Xi$, where

$$gr\Xi = \left\{(x,y) \in \mathbb{R}^2 : x - y \leq 0 \text{ and } y \leq 0\right\}.$$

  Notice that $\partial^* g_1(\widetilde{x},\widetilde{y}) = \{(1,-1)\}$ and $\partial^* g_2(\widetilde{x},\widetilde{y}) = \{(0,1)\}$ are UCFs of g_1 and g_2 at $(\widetilde{x},\widetilde{y})$. In addition, $\partial^* f_L,\ \partial^* f_U,\ \partial^* g_i, \forall i \in I(\overline{x},\overline{y})$, are bounded upper semi-continuous set-valued mappings over $U \cap gr\Xi$.

  - For each $\varepsilon > 0$, there exists $\delta := \min\left(\frac{\varepsilon}{6},\ \frac{1}{4}\right) > 0$ such that, for each $(x',y') \in (\widetilde{x},\widetilde{y}) + \delta\mathbb{B}_{\mathbb{R}^2}$ we have

$$\partial^* f_L(x',y') \subseteq \partial^* f_L(\widetilde{x},\widetilde{y}) + \varepsilon\mathbb{B}_{\mathbb{R}^2}.$$

  - For each $\varepsilon > 0$, there exists $\delta := \varepsilon > 0$ such that, for each $(x',y') \in (\widetilde{x},\widetilde{y}) + \delta\mathbb{B}_{\mathbb{R}^2}$ we have

$$\partial^* f_U(x',y') \subseteq \partial^* f_U(\widetilde{x},\widetilde{y}) + \varepsilon\mathbb{B}_{\mathbb{R}^2}.$$

  - For each $\varepsilon > 0$, there exists $\delta := \varepsilon > 0$ such that, for each $(x',y') \in (\widetilde{x},\widetilde{y}) + \delta\mathbb{B}_{\mathbb{R}^2}$ we have

$$\partial^* g_1(x',y') \subseteq \partial^* g_1(\widetilde{x},\widetilde{y}) + \varepsilon\mathbb{B}_{\mathbb{R}^2} \text{ and } \partial^* g_2(x',y') \subseteq \partial^* g_2(\widetilde{x},\widetilde{y}) + \varepsilon\mathbb{B}_{\mathbb{R}^2}.$$

– (ACQ) holds at $(\overline{x},\overline{y})$. Indeed, since

$$\Gamma(\overline{x},\overline{y}) = \{(1,0),\ (1,-1),\ (0,1),\ (-1,1)\},$$

we get

$$\Gamma(\overline{x},\overline{y})^- = \{(x,x) : x \leq 0\} \text{ and } pos\ \Gamma(\overline{x},\overline{y}) = \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x + y\right\}.$$

Notice that $pos\ \Gamma(\overline{x},\overline{y})$ is closed and that

$$\Gamma(\overline{x},\overline{y})^- \subseteq T_\Omega(\overline{x},\overline{y}).$$

– (GCQ) holds at all $(\widetilde{x},\widetilde{y}) \in gr\psi_L \cap gr\psi_L$ near $(\overline{x},\overline{y})$. Indeed, for all $(\widetilde{x},\widetilde{y}) \in gr\psi_L \cap gr\psi_L$, we have $\widetilde{y} = \widetilde{x}$ near $\overline{x} = 0$. Notice that either $\widetilde{y} = 0$ or $\widetilde{y} < 0$.

- Suppose that $\widetilde{y} = 0$. Noticing that $(\widetilde{x},\widetilde{y}) = (\overline{x},\overline{y})$, we have

$$T_{gr\Xi}(\widetilde{x},\widetilde{y}) = \left\{(x,y) \in \mathbb{R}^2 : x - y \leq 0 \text{ and } y \leq 0\right\}.$$

  Consequently,

$$[T_{gr\Xi}(\overline{x},\overline{y})]^- = \left\{(x,y) \in \mathbb{R}^2 : x + y \geq 0 \text{ and } x \geq 0\right\}. \tag{14}$$

  Since

$$\Upsilon(\overline{x},\overline{y}) = \partial^* g_1(\widetilde{x},\widetilde{y}) \cup \partial^* g_2(\widetilde{x},\widetilde{y})$$

we have

$$cl\ pos\ \Upsilon(\overline{x},\overline{y}) = cl\ pos\ ((1,-1),\ (0,1)).$$

Consequently,

$$cl\ pos\ \Upsilon(\overline{x},\overline{y}) = \left\{(x,y)\in\mathbb{R}^2 : x+y\geq 0 \text{ and } x\geq 0\right\};$$

which, by (14), implies that

$$[T_{gr\Xi}(\widetilde{x},\widetilde{y})]^- \subseteq cl\ pos\ \Upsilon(\overline{x},\overline{y}).$$

- Suppose that $\widetilde{y}<0$.
  - Since

    $$T_{gr\Xi}(\widetilde{x},\widetilde{y}) = \left\{(x,y)\in\mathbb{R}^2 : x-y\leq 0\right\}$$

    we have

    $$[T_{gr\Xi}(\widetilde{x},\widetilde{y})]^- = \{(x,-x) : x\geq 0\}. \quad (15)$$

  - Since

    $$\Upsilon(\overline{x},\overline{y}) = \partial^* g_1(\widetilde{x},\widetilde{y}) = \{(1,-1)\}$$

    we get

    $$cl\ pos\ \Upsilon(\overline{x},\overline{y}) = cl\ pos\ ((1,-1)).$$

    Consequently,

    $$cl\ pos\ \Upsilon(\overline{x},\overline{y}) = \{(x,-x) :\ x\geq 0\};$$

    which, by (15), implies that

    $$[T_{gr\Xi}(\widetilde{x},\widetilde{y})]^- \subseteq cl\ pos\ \Upsilon(\overline{x},\overline{y}).$$

– Taking $y_1=y_2,\ z_1=z_2=\overline{x},\ \lambda_1=\lambda_2=\zeta_1=\zeta_2=\chi_1=\chi_2=\frac{1}{2},\ \alpha_1=\frac{4}{3},\ \beta_1=2,\ \beta_2=0,\ \gamma=\rho=1,$ $\xi_1^1=\xi_2^1=\chi_1^1=\chi_2^1=\frac{2}{3}$ and $\xi_1^2=\xi_2^2=\chi_1^2=\chi_2^2=\frac{1}{3}$, since $\left(-2,\frac{2}{3}\right)\in conv\ \partial^* F_U(\overline{x},\overline{y})$, we obtain (5) and (6).

If $F_L\equiv F_U$ and $f_L\equiv f_U$, the problem (IVP) collapes with the usual scalar bilevel programming problem

$$(P):\ \min_{x,y} F(x,y)\ \text{ s.t. }\ G_j(x,y)\leq 0,\ j\in J,\ y\in\psi(x),$$

where, for each $x\in\mathbb{R}^{n_1},\ \psi(x)$ is the set of optimal solutions of the following parametric optimization problem

$$\min_{y} f(x,y)\ \text{ s.t. } g_i(x,y)\leq 0,\ i\in I.$$

It is worth mentioning that $\psi\equiv S_L\equiv S_U$ and that $V_L\equiv V_U$. To obtain the necessary optimality conditions for the bilevel optimization problem (P), we need the following assumptions, which are adaptations of Assumptions 2 and 3 to the current situation.

Assumption 4.

– F and f are locally Lipschitz and admit bounded USRCFs $\partial^* F(\overline{x},\overline{y})$ and $\partial^* f(\overline{x},\overline{y})$ at $(\overline{x},\overline{y})$.
– $G_j,\forall j\in J(\overline{x},\overline{y})$, and $g_i,\forall i\in I(\overline{x},\overline{y})$, are continuous and admit UCFs $\partial^* G_j(\overline{x},\overline{y}),\forall j\in J(\overline{x},\overline{y})$, and $\partial^* g_i(\overline{x},\overline{y}),\forall i\in I(\overline{x},\overline{y})$, at $(\overline{x},\overline{y})$.

Assumption 5.

– There is a neighborhood U of $(\overline{x},\overline{y})$ such that:
  - $\partial^* f$ and $\partial^* g_i,\forall i\in I(\overline{x},\overline{y})$, are bounded upper semi-continuous set valued mappings over $U\cap gr\Xi$.
  - For all $(x,y)\in U\cap gr\Xi$, the set $\partial^* f(x,y)$ is an USRCF of f at (x,y).

– The argminimum map ψ is inner semicompact at $\overline{x}$.

The following result follows from Theorem 17. It is worth noting that Corollary 19 provides necessary optimality conditions for the bilevel optimization problem (P) in terms of UCFs and USRCs, while omitting the convexity of both f and $g_i,\ i\in I$, which is required in [22, Thm. 5.1]. Another advantage of this result over Kohli's [22, Thm. 5.1] is that the conditions obtained are not only sequential.

Corollary 19. *Let $(\overline{x},\overline{y})\in\Omega$ be a local optimal solution of (P) where (ACQ) holds. Suppose that pos $\Gamma(\overline{x},\overline{y})$ is closed, that Assumptions 3 and 4 are satisfied and that (GCQ) holds at all $(\widetilde{x},\widetilde{y})\in gr\psi$ near $(\overline{x},\overline{y})$. Then there exist scalars $\alpha_j\geq 0,\ j\in J(\overline{x},\overline{y}),\ \beta_i\geq 0,\ i\in I(\overline{x},\overline{y}),\ \gamma\geq 0,\ \zeta_r\geq 0,\ y_r\in\psi(\overline{x}),\ \zeta_r^i\geq 0,\ i\in I(\overline{x},y_r),\ r=1,\ldots,\ n_1+1$, such that*

$$\sum_{r=1}^{n_1+1}\zeta_r=1, \tag{16}$$

and

$$(0,0)\in\left[\begin{array}{c} conv\ \partial^*F(\overline{x},\overline{y})+\displaystyle\sum_{j\in J(\overline{x},\overline{y})}\alpha_j\ conv\ \partial^*G_j(\overline{x},\overline{y})+\sum_{i\in I(\overline{x},\overline{y})}\beta_i\ conv\ \partial^*g_i(\overline{x},\overline{y})\\ +\gamma\ conv\ \partial^*f(\overline{x},\overline{y})-\gamma\displaystyle\sum_{r=1}^{n_1+1}\zeta_r\left(conv\ \partial^*f(\overline{x},y_r)+\sum_{i\in I(\overline{x},y_r)}\zeta_r^i\ conv\ \partial^*g_i(\overline{x},y_r)\right)\end{array}\right]. \tag{17}$$

The following example provides a case where Corollary 19 is applicable while [22, Thm. 5.1] is not. Observe that in Example 20, the lower-level objective function f is not convex and consequently [22, Thm. 5.1] cannot be used with this last property imposed. Furthermore, the upper estimate of the subdifferential of convex analysis of the optimal value function given by Kohli in [22, Equality 8] is out of use because V is not convex.

Example 20. Consider the following bilevel optimization problem

$$(P):\begin{cases}\min\limits_{x,y} & F(x,y)=(y+1)^2-2x\\ & G_1(x,y)=x\leq 0,\\ & G_2(x,y)=-x-1\leq 0,\\ & y\in\psi(x),\end{cases}$$

and for each $x\in\mathbb{R},\ \psi$ is the solution set of the following parametric optimization problem

$$\begin{cases}\min\limits_{y} & f(x,y)=y+yx^2\\ & g_1(x,y)=x-y\leq 0,\\ & g_2(x,y)=y\leq 0.\end{cases}$$

– On the one hand, $(\overline{x},\overline{y})=(0,0)$ is an optimal solution of (P) with

$$I(\overline{x},\overline{y})=\{1\},\ J(\overline{x},\overline{y})=\{1,2\},\ \psi(x)=\{x\}\text{ and }V(x)=x+x^3.$$

On the other hand,

$$\Omega=\{(x,y):\ y=x,\ -1\leq x\leq 0\}\text{ and }T_\Omega(\overline{x},\overline{y})=\{(x,x):x\leq 0\}.$$

Notice that neither f nor V are convex functions.

– Assumptions 3 and 4 are satisfied. Indeed, $\partial^*F(\overline{x},\overline{y})=\{(-2,2)\}$ is a bounded USRCF of F at $(\overline{x},\overline{y})$.
  - $\partial^*G_1(\overline{x},\overline{y})=\{(1,0)\}$, $\partial^*g_1(\overline{x},\overline{y})=\{(1,-1)\}$, $\partial^*g_2(\overline{x},\overline{y})=\{(0,1)\}$ and $\partial^*f(\overline{x},\overline{y})=\{(0,1)\}$ are UCFs of $G_1,\ g_1,\ g_2$ and f at $(\overline{x},\overline{y})$.
  - $\partial^*V(\overline{x})=\{1\}$ is an UCF of V at $\overline{x}$.

- $\partial^* f_L(\widetilde{x},\widetilde{y}) = \left\{\left(2\widetilde{x}\widetilde{y}, 1+\widetilde{x}^2\right)\right\}$ is an USRCF of f at every $(\widetilde{x},\widetilde{y}) \in U \cap gr\Xi$, where

$$gr\Xi = \left\{(x,y) \in \mathbb{R}^2 : x - y \leq 0 \text{ and } y \leq 0\right\} \text{ and } U := \left]-\frac{1}{4}, \frac{1}{4}\right[\times \left]-\frac{1}{4}, \frac{1}{4}\right[.$$

Furthermore, $\partial^* g_1(\widetilde{x},\widetilde{y}) = \{(1,-1)\}$ and $\partial^* g_2(\widetilde{x},\widetilde{y}) = \{(0,1)\}$ are UCFs of g_1 and g_2 at $(\widetilde{x},\widetilde{y}) \in U \cap gr\Xi$. Notice that $\partial^* f,\ \partial^* g_i,\ i \in I(\overline{x},\overline{y})$, are bounded upper semi-continuous set-valued mappings over $U \cap gr\Xi$.

– (ACQ) holds at $(\overline{x},\overline{y})$ and (GCQ) holds at all $(\widetilde{x},\widetilde{y}) \in gr\psi$ near $(\overline{x},\overline{y})$.
– Taking $y_1 = y_2,\ \zeta_1 = \zeta_2 = \frac{1}{2},\ \alpha_1 = 0,\ \beta_1 = 4,\ \beta_2 = 1,\ \gamma = 3,\ \xi_1^1 = \xi_2^1 = \frac{2}{3}$ and $\xi_1^2 = \xi_2^2 = \frac{1}{3}$, we obtain (16) and (17).

Remark 21. Even if the functions f and $g_i,\ i \in I$, are locally Lipschitz, our results remain more relevant than Kohli's [22] (see Rmks. 3 and 5).

4. Conclusion

Optimization problems are significant because they are used in a wide range of research fields. We get interval-valued optimization problems if we take the coefficients of the objective and constraint functions as closed intervals. This work was about a nonsmooth interval-valued bilevel optimization problem (IVP) in which the functions aren't always convex or locally Lipschitz. Under an appropriate Abadie constraint qualification, given in terms of upper convexificators, using a scalarization technique, we established necessary optimality conditions for (IVP). Afterwards, by using an upper estimate for upper convexificators of the optimal value function, we gave a more detailed result in terms of upper (semi-regular) convexificators of the initial data. Unlike Kohli [*J. Optimiz. Theory Appl.* **152** (2012) 632–651], we achieve our goal without relying on the convexity or local Lipschitzity of the functions that appear in the lower-level problem. Examples illustrating both our findings and the limitations of some earlier research are provided. Our research has left various avenues for future study. For example, it would be interesting to study the sufficient optimality conditions and duality results for the interval-valued bilevel optimization problem (IVP).

Acknowledgements. Our sincere acknowledgements go to the anonymous referees and the editor for their insightful remarks and valuable suggestions, which have helped improve the paper. The work of the second author has been supported by the Alexander-von-Humboldt foundation.

Conflict of interest. The authors declare that they have no conflict of interest.

References

[1] H. Babahadda and N. Gadhi, Necessary optimality conditions for bilevel optimization problems using convexificators. *J. Global Optimiz.* **34** (2006) 535–549.
[2] J.F. Bard, Practical Bilevel Optimization: Algorithms and Applications, Kluwer, Dordrecht (1998).
[3] A.K. Bhurjee and G. Panda, Sufficient optimality conditions and duality theory for interval optimization problem. *Ann .Oper. Res.* **243** (2016) 335–348.
[4] Y. Chalco-Cano, W.A. Lodwick and A. Rufian-Lizana, Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative. *Fuzzy Optim. Decis. Mak.* **12** (2013) 305–322.
[5] F.H. Clarke, Optimization and Nonsmooth Analysis. Wiley-Interscience, New York (1983).
[6] P.A. Clarke and A.W. Westerberg, Optimization for design problems having more than one objective. *Comput.Chem. Eng.* **7** (1983) 259–278.
[7] S. Dempe, Foundations of bilevel programming, in Part of the book series: Nonconvex Optimization and Its Applications, Vol. 61. Springer, New York, NY (2002).
[8] S. Dempe, Foundations of Bilevel Programming, Kluwer, Dordrecht (2002).
[9] S. Dempe, Annotated bibliography on bilevel programming and mathematical programs with equilibrium constraints. *Optimization* **52** (2003) 333–359.
[10] S. Dempe, Bilevel optimization: theory, algorithms, applications and a bibliography, in Bilevel Optimization. Springer Optimization and Its Applications, Vol. 161. Edited by S. Dempe and A. Zemkoho, Springer, Cham (2020).

[11] S. Dempe and S. Franke, Solution of bilevel optimization problems using the KKT approach. *Optimization* **68** (2019) 1471–1489.

[12] S. Dempe and A. Zemkoho, The generalized Mangasarian-Fromowitz constraint qualification and optimality conditions for bilevel programs. *J. Optimiz. Theory Appl.* **148** (2011) 46–68.

[13] S. Dempe, V. Kalashnikov, G.A. Pérez-Valdés and N. Kalashnykova, Bilevel Programming Problems: Theory, Algorithms and Application to Energy Networks, Springer, Berlin (2015).

[14] S. Dempe, O. Khamisov and Yu. Kochetov, A special three-level optimization problem. *J. Global Optimiz.* **76** (2020) 519–531.

[15] S. Dempe, N. Gadhi and L. Lafhim, Optimality conditions for pessimistic bilevel problems using convexificator. *Positivity* **24** (2020) 1399–1417.

[16] S. Dempe, N. Gadhi, M. El idrissi and K. Hamdaoui, Necessary optimality conditions for a semivectorial bilevel problem under a partial calmness condition. *Optimization* **70** (2021) 1937–1957.

[17] J. Dutta and S. Chandra, Convexifactors, generalized convexity, and optimality conditions. *J. Optimiz. Theory Appl.* **113** (2002) 41–64.

[18] N.G. Gadhi, Comments on "A note on the paper "Optimality conditions for optimistic bilevel programming problem using convexifactors"". *J. Optimiz. Theory Appl.* **189** (2021) 938–943.

[19] N. Gadhi and M. Ohda, On multiobjective bilevel optimization using tangential subdifferentials. *J. Ind. Manag. Optim.* **19** (2023) 4949–4958.

[20] A. Jayswal, I. Stancu-Minasian and I. Ahmad, On sufficiency and duality for a class of interval-valued programming problems. *Appl. Math. Comput.* **218** (2011) 4119–4127.

[21] V. Jeyakumar and D.T. Luc, Nonsmooth calculus, minimality, and monotonicity of convexificators. *J. Optimiz. Theory Appl.* **101** (1999) 599–621.

[22] B. Kohli, Optimality conditions for optimistic bilevel programming problem using convexificators. *J. Optimiz. Theory Appl.* **152** (2012) 632–651.

[23] L. Lafhim, N. Gadhi, K. Hamdaoui and F. Rahou, Necessary optimality conditions for a bilevel multiobjective programming problem via a Ψ-reformulation. *Optimization* **67** (2018) 2179–2189.

[24] P. Mehlitz, L.I. Minchenko and A.B. Zemkoho, A note on partial calmness for bilevel optimization problems with linearly structured lower level. *Optimiz. Lett.* **15** (2021) 1277–1291.

[25] B.S. Mordukhovich and Y. Shao, On nonconvex subdifferential calculus in Banach spaces. *J. Convex Anal.* **2** (1995) 211–227.

[26] R.T. Rockafellar, Convex Analysis, Princeton, New Jersey (1970).

[27] H.C. Wu, On interval-valued nonlinear programming problems. *J. Math. Anal. Appl.* **338** (2008) 299–316.

[28] J.J. Ye and D.L. Zhu, Optimality conditions for bilevel programming problems. *Optimization* **33** (1995) 9–27.

[29] G. Zhang, J. Lu and Y. Gao, Multi-level Decision Making: Models, Methods and Applications, Vol. 82. Springer, Berlin (2015).

Please help to maintain this journal in open access!

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting subscribers@edpsciences.org.

More information, including a list of supporters and financial transparency reports, is available at https://edpsciences.org/en/subscribe-to-open-s2o.