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GRAPHS WITH SMALL OR LARGE ROMAN {3}-DOMINATION NUMBER

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Abstract. For an integer $k \geq 1$, a Roman $\{k\}$-dominating function (R$\{k\}$DF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, k\}$ such that for every vertex $v \in V$ with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq k$, where $N(v)$ is the set of vertices adjacent to v. The weight of an R$\{k\}$DF is the sum of its function values over the whole set of vertices, and the Roman $\{k\}$-domination number $\gamma_{\{kR\}}(G)$ is the minimum weight of an R$\{k\}$DF on G. In this paper, we will be focusing on the case $k = 3$, where trivially for every connected graphs of order $n \geq 3$, $3 \leq \gamma_{\{kR\}}(G) \leq n$. We characterize all connected graphs G of order $n \geq 3$ such that $\gamma_{\{3R\}}(G) \in \{3, n-1, n\}$, and we improve the previous lower and upper bounds. Moreover, we show that for every tree T of order $n \geq 3$, $\gamma_{\{3R\}}(T) \geq \gamma(T) + 2$, where $\gamma(T)$ is the domination number of T, and we characterize the trees attaining this bound.

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1. Introduction

Throughout this paper, G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The order $n = n(G)$ of a graph G is the number of its vertices. Two vertices u and v are *neighbors* in G if they are adjacent, that is, if $uv \in E(G)$. For any vertex $u \in V(G)$, let $N_G(u)$ be the set of neighbors of u and let $N_G[u] = N_G(u) \cup \{u\}$. The *degree* of a vertex v of G is $\deg_G(v) = |N_G(v)|$. The *maximum degree* of G is denoted by $\Delta(G)$. Let u and v be two vertices in G. A *uv-path* is a path with end-vertices u and v, and the *distance* between u and v is the length of a shortest uv-path. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum distance between vertices of G. A *diametral path* in G is a path with length $\mathrm{diam}(G)$. A *leaf* in a graph is a vertex of degree one and its unique neighbor is a *support vertex*. A *tree* is a connected acyclic graph, while a *unicyclic graph* is a connected graph that contains exactly one cycle. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with p and q leaves attached to each support vertex, respectively, is denoted by $DS_{p,q}$. The *subgraph* of G induced by a set of vertices $S \subseteq V(G)$ is denoted by $G[S]$.

Keywords. Roman domination, Roman $\{k\}$-domination number.

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A set $S \subseteq V(G)$ is a *dominating set* of a graph G if every vertex not in S is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set.

For an integer $k \geq 1$, let f be a function that assigns to each vertex a color chosen from the set $\{0, 1, 2, \ldots, k\}$ and let $(V_0, V_1, \ldots, V_k)$ be the ordered partition of V induced by f, where $V_i = \{v \in V \mid f(v) = i\}$ for each $i \in \{0, 1, \ldots, k\}$. Since there is a 1–1 correspondence between the functions $f : V \to \{0, 1, \ldots, k\}$ and the ordered partitions $(V_0, V_1, \ldots, V_k)$ of V, we will simply write $f = (V_0, V_1, \ldots, V_k)$. Moreover, the weight of f is $\sum_{v \in V} f(v) = \sum_{i=1}^{k} i|V_i|$.

A function $f = (V_0, V_1, \ldots, V_k)$ defined on a graph G is a *Roman* $\{k\}$*-dominating function*, abbreviated as R$\{k\}$DF, if for every vertex $v \in V_0$ we have $\sum_{u \in N(v)} f(u) \geq k$. The minimum weight of an R$\{k\}$DF on G is called the *Roman* $\{k\}$*-domination number* of G, denoted by $\gamma_{\{kR\}}(G)$. An R$\{k\}$DF on G with weight $\gamma_{\{kR\}}(G)$ is called a $\gamma_{\{kR\}}(G)$*-function*. Roman $\{k\}$-domination was introduced by Wang *et al.* [17] as a generalization of Roman $\{2\}$-domination defined by Chellali *et al.* [5] and known also as the Italian domination in [10]. Italian domination and its variants have been studied by several authors [1, 2, 11–13, 15, 16]. The authors [17] showed in particular that the Roman $\{k\}$-domination problem is NP-complete for some classes of graphs including bipartite planar and chordal bipartite graphs. Moreover, the particular case $k = 2$ has been handled in [9] by Haynes *et al.* who characterized the graphs with large Roman $\{2\}$-domination number. For further details on Roman domination and its variations we refer the reader the book chapters [6, 8] and survey [7].

Trivially, if G is a connected graph on two vertices, then $\gamma_{\{3R\}}(G) = 2$. Hence we will consider connected graphs G of order $n \geq 3$. Since assigning a 1 to every vertex of G provides an R$\{3\}$DF, it follows that $\gamma_{\{3R\}}(G) \leq n$. Moreover, from the definition of an R$\{3\}$DF, we must have $\gamma_{\{3R\}}(G) \geq 3$.

In this paper, we improve the lower and upper bounds, and we provide a characterization of all connected graphs G of order $n \geq 3$ such that $\gamma_{\{3R\}}(G) \in \{3, n-1, n\}$. We also show that for every tree T of order $n \geq 3$, $\gamma_{\{3R\}}(T) \geq \gamma(T) + 2$ and we characterize the extremal trees attaining this bound. Before presenting all these results it is worthwhile to recall some additional definitions and results.

For an integer $k \geq 1$, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$, and the *$k$-rainbow domination number* $\gamma_{rk}(G)$ is the minimum weight of a kRDF of G. The k-rainbow domination was introduced by Shao *et al.* [14] as generalization of 2-rainbow domination introduced by Brešar *et al.* [3]. We note that if f is a kRDF on a graph G, then the function h defined on G by $h(v) = |f(v)|$ for each $v \in V(G)$ is obviously an RkDF of G. Therefore, $\gamma_{\{kR\}}(G) \leq \gamma_{rk}(G)$ holds for every graph G and any integer $k \geq 1$. In [4], Chang *et al.* showed that the equality $\gamma_{\{kR\}}(G) = \gamma_{rk}(G)$ holds for every strongly chordal graphs which includes in particular the class of trees. Consequently, using this result and the exact value of $\gamma_{r3}(G)$ for paths established in [14] we state the following corollary.

Corollary 1. *For $n \geq 1$,*

$$\gamma_{\{3R\}}(P_n) = \begin{cases} \lceil \frac{3n}{4} \rceil + 1 & n \equiv 0 \pmod 4 \\ \lceil \frac{3n}{4} \rceil & \text{otherwise.} \end{cases}$$

Furthermore, using Corollary 1, the following result can be shown for cycles.

Corollary 2. $\gamma_{\{3R\}}(C_3) = 3$, $\gamma_{\{3R\}}(C_4) = 3$ *and for $n \geq 5$,* $\gamma_{\{3R\}}(C_n) = \lceil \frac{3n}{4} \rceil$.

Proof. The result is immediate for $n \in \{3, 4\}$. Hence we assume that $n \geq 5$. Let $C_n = (v_1 v_2 \ldots v_n)$ be a cycle on n vertices. Since the path $P_n = v_1 v_2 \ldots v_n$ is a spanning graph of C_n, and any R$\{3\}$DF of P_n is an R$\{3\}$DF of C_n, we deduce from Corollary 1 that if $n \not\equiv 0 \pmod 4$, then $\gamma_{\{3R\}}(C_n) \leq \gamma_{\{3R\}}(P_n) = \lceil \frac{3n}{4} \rceil$. Now, if $n \equiv 0 \pmod 4$, then the function g defined on $V(C_n)$ by $g(v_{4i+1}) = 1$, $g(v_{4i+3}) = 2$, $g(v_{4i+2}) = g(v_{4i+4}) = 0$ for $0 \leq i \leq \frac{n}{4} - 1$, is an R$\{3\}$DF of C_n of weight $3n/4$ and so $\gamma_{\{3R\}}(C_n) \leq \lceil \frac{3n}{4} \rceil$. In either case, $\gamma_{\{3R\}}(C_n) \leq \lceil \frac{3n}{4} \rceil$.

To prove the inverse inequality, let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{\{3R\}}(C_n)$-function. If $V_0 = \emptyset$, then $\gamma_{\{3R\}}(C_n) = n \geq \lceil \frac{3n}{4} \rceil$. Hence we assume that $V_0 \neq \emptyset$. If there exists an index i such that $f(v_i) = f(v_{i+1}) = 0$ or $\min\{f(v_i), f(v_{i+1})\} \geq 1$, then the function f restricted to $C_n - v_i v_{i+1}$ is an R{3}DF of $C_n - v_i v_{i+1}$ and Corollary 1 implies that $\omega(f) \geq \gamma_{\{3R\}}(C_n - v_i v_{i+1}) \geq \lceil \frac{3n}{4} \rceil$. Hence we assume that such an i does not exist. Let, without loss of generality, that $v_2 \in V_0$ and let $f(v_3) \geq f(v_1)$. By definition and our earlier assumption, we have $f(v_3) \geq 2$, $f(v_4) = 0$ and $f(v_5) \geq 1$. If $n \equiv 1 \pmod 4$, then the function f restricted to $V(C_n) - \{v_2\}$ is an R{3}DF of $C_n - v_2$ and Corollary 1 implies that $\omega(f) \geq \gamma_{\{3R\}}(C_n - v_2) = \lceil \frac{3(n-1)}{4} \rceil + 1 \geq \lceil \frac{3n}{4} \rceil$. If $n \equiv 2 \pmod 4$, then the function g defined on $V(C_n) - \{v_2, v_3\}$ by $g(v_4) = 1$ and $g(x) = f(x)$ for $x \in V(C_n) - \{v_2, v_3, v_4\}$ is an R{3}DF of $C_n - \{v_2, v_3\}$ and by Corollary 1 we obtain $\omega(f) \geq \gamma_{\{3R\}}(C_n - \{v_2, v_3\}) + 1 = \lceil \frac{3(n-2)}{4} \rceil + 1 + 1 \geq \lceil \frac{3n}{4} \rceil$. If $n \equiv 3 \pmod 4$, then the function f restricted to $V(C_n) - \{v_2, v_3, v_4\}$ is an R{3}DF of $C_n - \{v_2, v_3, v_4\}$ and by Corollary 1 we get $\omega(f) \geq \gamma_{\{3R\}}(C_n - \{v_2, v_3, v_4\}) + 2 = \lceil \frac{3(n-3)}{4} \rceil + 1 + 2 \geq \lceil \frac{3n}{4} \rceil$. Finally, assume that $n \equiv 0 \pmod 4$. Then $n \geq 8$ and the function g defined on $V(C_n) - \{v_2, v_3, v_4, v_5\}$ by $g(v_6) = 1$ and $g(x) = f(x)$ otherwise, is an R{3}DF of $C_n - \{v_2, v_3, v_4, v_5\}$ and by Corollary 1 we get $\omega(f) \geq \gamma_{\{3R\}}(C_n - \{v_2, v_3, v_4, v_5\}) + 2 = \lceil \frac{3(n-4)}{4} \rceil + 1 + 2 = \lceil \frac{3n}{4} \rceil$. In either case, $\gamma_{\{3R\}}(C_n) \geq \lceil \frac{3n}{4} \rceil$ and the equality follows. This completes the proof. □

2. Lower bounds

We begin by giving a characterization of connected graphs G of order n such that $\gamma_{\{3R\}}(G) = 3$.

Theorem 3. *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{\{3R\}}(G) = 3$ if and only if one of the following conditions holds.*

(1) $\Delta(G) = n - 1$.
(2) $\Delta(G) = n - 2$ *and there are two non-adjacent vertices u and v, each of degree $n - 2$.*
(3) $\Delta(G) < n - 1$ *and there are three vertices u, v and w such that $V(G) - \{u, v, w\} \subseteq N(u) \cap N(v) \cap N(w)$.*

Proof. Assume that $\gamma_{\{3R\}}(G) = 3$ and let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{\{3R\}}(G)$-function such that $|V_3|$ is maximized. Since $3 = \gamma_{\{3R\}}(G) = |V_1| + 2|V_2| + 3|V_3|$, we deduce that either $|V_3| = 1, |V_1| = |V_2| = 0$ or $|V_3| = 0, |V_1| = |V_2| = 1$ or $|V_3| = |V_2| = 0$ and $|V_1| = 3$. Let us examine each of these three situations. Assume first that $|V_3| = 1$, with $V_3 = \{v\}$, and $|V_1| = |V_2| = 0$. Then $V_0 = V(G) - \{v\} \neq \emptyset$ and since each vertex of V_0 is adjacent to v, it follows that $\deg_G(v) = n - 1 = \Delta(G)$. Hence (1) follows. From now on we can assume that $|V_3| = 0$. Assume that $|V_1| = |V_2| = 1$, and let $V_1 = \{u\}$ and $V_2 = \{v\}$. By the choice of f, we have $\Delta(G) < n - 1$. Clearly $V_0 \neq \emptyset$, since $n \geq 3$. Moreover, since vertex of V_0 must be adjacent to u and v, we deduce that $\deg_G(u) \geq n - 2$ and $\deg_G(v) \geq n - 2$. Since $\Delta(G) < n - 1$, it follows that $uv \notin E(G)$ and $\deg_G(u) = \deg_G(v) = n - 2$. Hence (2) follows. Finally, assume that $|V_3| = |V_2| = 0$, $|V_1| = 3$, and let $V_1 = \{u, v, w\}$. The choice of f and the connectedness of G implies that $V_0 \neq \emptyset$, and hence each vertex is adjacent to all V_1. Therefore $V(G) - \{u, v, w\} \subseteq N(u) \cap N(v) \cap N(w)$, and hence (3) follows.

The converse is easy to show and we omit the details. □

Theorem 4. *If G is a connected graph of order n and maximum degree $\Delta(G)$, then $\gamma_{\{3R\}}(G) \geq \frac{2n+\Delta(G)+1}{\Delta(G)+3}$.*

Proof. If $\Delta(G) = 1$, then $G = K_2$ and $\gamma_{\{3R\}}(G) = 2 \geq \frac{6}{4}$. If $\Delta(G) = 2$, then G is either a path or a cycle of order at least three and by Corollaries 1 and 2, $\gamma_{\{3R\}}(G) \geq \lceil \frac{3n}{4} \rceil \geq \frac{2n+3}{5}$. Hence we assume that $\Delta(G) \geq 3$ and let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{\{3R\}}(G)$-function. Let $V_0' = \{x \in V_0 \mid N(x) \cap V_3 \neq \emptyset\}$ and $V'' = V_0 - V_0'$. Since every vertex of V_3 has at most $\Delta(G)$ neighbors in V_0', we have $|V_0'| \leq \Delta(G)|V_3|$. Also, each vertex of $V_1 \cup V_2$ may have at most $\Delta(G)$ neighbors in V_0''. We consider two cases.

Case 1. $|V_2| = 0$.

Since each vertex of V_0'' has at least three neighbors in V_1, we have $3|V_0''| \leq \Delta(G)|V_1|$, and therefore $|V_0| = |V_0'| + |V_0''| \leq \Delta(G)|V_3| + \frac{\Delta(G)}{3}|V_1|$. It follows that

$$\frac{\Delta(G) + 3}{3}\gamma_{\{3R\}}(G) = \frac{\Delta(G) + 3}{3}(|V_1| + 3|V_3|)$$

$$
\begin{aligned}
&= \Delta(G)|V_3| + \frac{\Delta(G)}{3}|V_1| + |V_1| + 3|V_3| \\
&\geq |V_0| + |V_1| + 3|V_3| \\
&\geq n.
\end{aligned}
$$

Consequently, $\gamma_{\{3R\}}(G) \geq \frac{3n}{\Delta(G)+3} \geq \frac{2n+\Delta(G)+1}{\Delta(G)+3}$.

Case 2. $|V_2| \geq 1$.

Since each vertex of V_0'' has at least two neighbors in $V_1 \cup V_2$, we have $2|V_0''| \leq \Delta(G)(|V_1| + |V_2|)$. Hence $|V_0| = |V_0'| + |V_0''| \leq \Delta(G)|V_3| + \frac{\Delta(G)}{2}(|V_1| + |V_2|)$, and thus

$$
\begin{aligned}
\frac{\Delta(G)+3}{2}\gamma_{\{3R\}}(G) &= \frac{\Delta(G)+3}{2}(|V_1| + 2|V_2| + 3|V_3|) \\
&= \frac{\Delta(G)}{2}|V_1| + \Delta(G)|V_2| + \frac{3\Delta(G)}{2}|V_3| + \frac{3}{2}|V_1| + 3|V_2| + \frac{9}{2}|V_3| \\
&\geq |V_0| + \frac{\Delta(G)}{2}|V_2| + \frac{\Delta(G)}{2}|V_3| + |V_1| + \frac{1}{2}|V_1| + |V_2| + 2|V_2| + |V_3| + \frac{7}{2}|V_3| \\
&\geq n + \frac{\Delta(G)}{2}(|V_2| + |V_3|) + \frac{1}{2}|V_1| \geq n + \frac{\Delta(G)}{2} + \frac{1}{2} \\
&= \frac{2n + \Delta(G) + 1}{2}.
\end{aligned}
$$

Therefore $\gamma_{\{3R\}}(G) \geq \frac{2n+\Delta(G)+1}{\Delta(G)+3}$ and the desired bound follows in either case.

□

We now turn our attention to the class of trees for which we provide a lower bound on the Roman {3}-domination in terms of the domination number. Moreover, we characterize the extremal trees attaining this lower bound. For this purpose, for any integers $k \geq 1$ and $t \geq 0$ with $k \geq t$, let $S_{t,k}$ be the tree obtained from the star $K_{1,k}$ by subdividing t edges of the star. The tree $S_{t,k}$ is called a *wounded spider* if $t \leq k-1$ and a *healthy spider* when $t = k$. We will sometimes say that we have a spider no matter if it is wounded or healthy. Let $\mathcal{T}$ be the family of all trees $S_{t,k}$. Before stating the result, it is worth noting that if T is a trivial tree, then $\gamma_{\{3R\}}(T) = \gamma(T) = 1$ while it has order 2, then $\gamma_{\{3R\}}(T) = 2 = \gamma(T) + 1$.

Theorem 5. *Let T be a tree of order $n \geq 3$. Then $\gamma_{\{3R\}}(T) \geq \gamma(T) + 2$ with equality if and only if $T \in \mathcal{T}$.*

Proof. We proceed by induction on the order n. If $n = 3$, then T is a path $P_3 \in \mathcal{T}$, where $\gamma(P_3) = 1$ and $\gamma_{\{3R\}}(G) = 3$ yielding $\gamma_{\{3R\}}(G) = \gamma(G) + 2$. This establishes the base case. Let $n \geq 4$ and assume that the result holds for any tree T' of order $n' < n$. Let T be a tree of order n. Among all $\gamma_{\{3R\}}$-functions, let $f = (V_0, V_1, V_2, V_3)$ be one such that $\sum_{v \in L(T)} f(v)$ is as small as possible, where $L(T)$ is the set of leaves of T. Note that $V_1 \cup V_2 \cup V_3$ is a dominating set of T. We consider the following two cases.

Case 1. $V_3 \neq \emptyset$.

If $|V_3| \geq 2$ or $V_2 \neq \emptyset$, then $\gamma_{\{3R\}}(G) = |V_1| + 2|V_2| + 3|V_3| \geq |V_1| + |V_2| + |V_3| + 3 \geq \gamma(G) + 3$. Hence we can assume that $|V_3| = 1$ and $V_2 = \emptyset$. As above, one can see that

$$\gamma_{\{3R\}}(G) = |V_1| + 3|V_3| = |V_1| + |V_3| + 2 \geq \gamma(G) + 2. \tag{1}$$

Further if $\gamma_{\{3R\}}(T) = \gamma(G) + 2$, then we have equality throughout the inequality chain (1). In particular, $V_1 \cup V_3$ is a minimum dominating set of T where $|V_3| = 1$, say $V_3 = \{v\}$. Recall that each vertex is adjacent to either the unique vertex of V_3 or to at least three vertices of V_1. Note also that since T is a tree, no two vertices of V_0 can have more than one common neighbor in V_1. Now, if some vertices $u \in V_0$ has at least two neighbors in V_1, then $\{u\} \cup V_3 \cup V_1 - (N(u) \cap V_1)$ is a dominating set of T smaller than $V_1 \cup V_3$, a contradiction. Thus each vertex of V_0 has at most one neighbor in V_1. Moreover, if some vertices $u \in V_1$ has a neighbor in $V_1 \cup V_3$, then $(V_1 \cup V_3) - \{u\}$ is a dominating set of T smaller than $V_1 \cup V_3$, a contradiction too. Therefore $V_1 \cup V_3$ is an independent set, and since each vertex of V_0 is adjacent to v and has at most one neighbor in V_1, we conclude that T is a spider. Now if v is not a support vertex, then T would be a healthy spider but in that case reassigning v the value 2 instead of 3 provides an R{3}DF of weight $\gamma_{\{3R\}}(T) - 1$, a contradiction. Consequently, T is a wounded spider and thus $T \in \mathcal{T}$.

Case 2. $V_3 = \emptyset$ and $V_2 \neq \emptyset$.

Let $v \in V_2$. Clearly $N(v) \cap V_0 \neq \emptyset$, otherwise reassigning a 1 to v provides an R{3}DF of T of weight less than $\gamma_{\{3R\}}(T)$, a contradiction. Moreover, by the choice of f, v is not a leaf (otherwise reassign v and its support vertex the value 1 instead of 2 and 0, respectively). Let $u \in N(v) \cap V_0$ and let $N(u) = \{u_1 = v, u_2, \ldots, u_k\}$. By definition we must have $f(u_i) \geq 1$ for some $i \geq 2$. Without loss of generality, let $f(u_i) \geq 1$ for every $i \in \{1, 2, \ldots, t\}$, where $t \geq 2$. In addition, let T_i be the component of $T - u$ containing u_i. Since v is not a leaf, $|V(T_1)| \geq 2$. Moreover, if $|V(T_1)| = 2$, then the unique neighbor of v in T_1, say w, is a leaf neighbor in T with $f(w) = 1$. But then by reassigning v and w the values 3 and 0 instead of 2 and 1, respectively, we get a $\gamma_{\{3R\}}(T)$-function which contradicts the choice of f. Therefore $|V(T_1)| \geq 3$. In this case, the restriction of f on $V(T_1)$ is an R{3}DF of T_1 and by the induction hypothesis on T_1, we have $\omega(f|_{V(T_1)}) \geq \gamma(T_1) + 2$. On the other hand, if $k \geq t + 1$, then T_i is nontrivial for each $i \geq t + 1$, and in this case the restriction of f to T_i is an R{3}DF of T_i, yielding $f(V(T_i)) \geq \gamma(T_i) + 1$. In the following let S_i be a $\gamma(T_i)$-set for each i and let $S = \cup_{i=1}^{k} S_i$.

Assume first that $t \geq 3$, and let T' be the subtree of T induced by $V(T) - V(T_1)$. We already know that $\omega(f|_{V(T_1)}) \geq \gamma(T_1) + 2$. Observe that T' has order at least three. Now if the restriction of f to $V(T')$ is an R{3}DF, then clearly $\omega(f|_{V(T')}) \geq \gamma(T') + 2$. Hence assume that $f|_{V(T')}$ is not an R{3}DF of T'. Then we must have $t = 3$, and so considering the restriction of f to T' and reassigning vertex u the value 1 instead of 0 provides an R{3}DF f' on T', yielding by the inductive hypothesis that $\omega(f|_{V(T')}) + 1 = \omega(f') \geq \gamma(T') + 2$. Therefore regardless of the situation, we have $\omega(f|_{V(T')}) \geq \gamma(T') + 1$. It follows that

$$\begin{aligned} \omega(f) &= \omega\big(f|_{V(T_1)}\big) + \omega\big(f|_{V(T')}\big) \\ &\geq \gamma(T_1) + 2 + \gamma(T') + 1 \\ &\geq \gamma(T) + 3. \end{aligned}$$

Assume now that $t = 2$. If $|V(T_2)| \geq 3$, then since the restriction of f on T_2 is an R{3}DF of T_2, we deduce from the induction hypothesis on T_2 that $\omega(f_{V(T_2)}) \geq \gamma(T_2) + 2$. Recall that for each $i > t = 2$, the tree T_i is nontrivial and satisfies $f(V(T_i)) \geq \gamma(T_i) + 1$. Now, since $S \cup \{u\}$ is a dominating set of T, $|S| \geq \gamma(T) - 1$ and thus

$$\begin{aligned} \omega(f) &\geq \sum_{i=1}^{k} f(V(T_i)) \geq (|S_1| + 2) + (|S_1| + 2) + \sum_{i=3}^{k} (|S_i| + 1) \\ &= 4 + (k-2) + \sum_{i=1}^{k} |S_i| = 4 + (k-2) + |S| \\ &\geq 3 + (k-2) + \gamma(T) \geq \gamma(T) + 3. \end{aligned}$$

Hence it can assume now that $|V(T_2)| \leq 2$. If $|V(T_2)| = 2$, then we have $f(V(T_2)) \geq \gamma(T_2) + 1$ and, without loss of generality, we may assume that $u_2 \in S_2$. Hence S is a dominating set of T and as above, we have

$$\begin{aligned} \omega(f) &\geq \sum_{i=1}^{k} f(V(T_i)) \geq 3 + (k-2) + \sum_{i=1}^{k} |S_i| = 3 + (k-2) + |S| \\ &\geq 3 + (k-2) + \gamma(T) \geq \gamma(T) + 3. \end{aligned}$$

Finally, assume that $|V(T_2)| = 1$. Then $f(u_2) = 1$ and $S_2 = \{u_2\}$. If $k \geq 3$, then as above S dominates T and thus

$$\omega(f) \geq \sum_{i=1}^{k} f(V(T_i)) \geq 2 + (k-2) + \sum_{i=1}^{k} |S_i| \geq 3 + |S| \geq 3 + \gamma(T).$$

Therefore let $k = 2$ and so $\deg_T(u) = 2$. By symmetry, we can also assume that every neighbor v assigned a 0 plays the same role as u, and thus it has degree two and it is a support vertex. Now, if v has a leaf neighbor z assigned a non-zero value, then reassigning v and z the values 3 and 0 provides a $\gamma_{\{3R\}}$-function which contradicts the choice of f. Consequently, since $f(v) = 2$ we deduce that v is not a support vertex. On the other hand, if v has a non-leaf neighbor w assigned a non-zero value, then let T' and T'' be the components of $T - vw$ containing v and w respectively. Clearly, f restricted to T' is an R{3}DF of T' and likewise f restricted to T'' is an R{3}DF of T''. Using the fact that T'' is nontrivial, we obtain that $\omega(f) = \omega(f|_{V(T')}) + \omega(f|_{V(T'')}) \geq \gamma(T') + \gamma(T'') + 3 \geq \gamma(T) + 3$. Hence we may assume that no neighbor of v is assigned a non-zero value. Therefore T is a healthy spider that belongs to $\mathcal{T}$ and satisfies $\omega(f) = \gamma(T) + 2$.

Case 3. $V_2 \cup V_3 = \emptyset$.

If $V_0 = \emptyset$, then $\gamma_{\{3R\}}(T) = n$ and since $\gamma(T) \leq \lfloor n/2 \rfloor$, we get $\gamma_{\{3R\}}(T) = n = \lfloor n/2 \rfloor + \lceil n/2 \rceil \geq \gamma(T) + \lceil n/2 \rceil \geq \gamma(T) + 2$ with equality if and only if $n \in \{3, 4\}$, and the choice of f implies that $T \in \{P_3, P_4\}$. In either case, $T \in \mathcal{T}$. In the sequel, we can assume that $V_0 \neq \emptyset$. First let $|V_0| = 1$. Then $\gamma_{\{3R\}}(T) = n - 1$ and so $n \geq 4$. If $n = 4$, then we must have $T = K_{1,3}$ and this situation is excluded by the choice of f. Let $n \geq 5$. Then we have $\gamma_{\{3R\}}(T) = n - 1 \geq \lfloor n/2 \rfloor + 2$ with equality if and only if $n \in \{5, 6\}$. If $n \in \{5, 6\}$, the considering all trees of order 5 and 6 and the assumption $\gamma_{\{3R\}}(T) = n - 1$, one can easily see that $\gamma_{\{3R\}}(T) \geq \gamma(T) + 2$ with equality if and only if $T \in \{S_{2,3}, S_{1,3}, S_{2,2}\}$. In either case, $T \in \mathcal{T}$.

Now let $|V_0| \geq 2$ and let $u \in V_0$. Assume that $N(u) = \{u_1, \ldots, u_k\}$. Clearly $k \geq 3$. If $f(u_i) = 0$ for some i, the let T_1 and T_2 be the components of $T - uu_i$ containing u and u_i, respectively. Then the restriction of f on each T_i is an R{3}DF of T_i and by the induction hypothesis we have $f(V(T_i)) \geq \gamma(T_i) + 2$ for each $i \in \{1, 2\}$. It follows that $\omega(f) = f(V(T_1)) + f(V(T_2)) \geq 4 + \gamma(T_1) + \gamma(T_2) \geq \gamma(T) + 4$. Hence we assume that $f(u_i) = 1$ for each i. Root T at u. Let $w \in V_0 - \{u\}$ be a vertex at minimum distance from u. Since u and w cannot be adjacent, let w' be the parent of w in T. Then $f(w') = 1$. Let T' and T'' be the components of $T - ww'$ containing w and w' respectively. Clearly, f restricted to T'' is an R{3}DF of T'' and by the induction hypothesis, $\omega(f|_{V(T'')}) \geq \gamma(T'') + 2$. Since T' has order at least three, by a similar argument to that used in Case 2 (when $t \geq 3$) we can see that $\omega(f|_{V(T')}) \geq \gamma(T') + 1$. Therefore, $\omega(f) = \omega(f|_{V(T')}) + \omega(f|_{V(T'')}) \geq \gamma(T') + \gamma(T'') + 3 \geq \gamma(T) + 3$. This completes the proof. □

3. Upper bounds

In this section, we present two upper bounds on the Roman {3}-domination number of a connected graph that improve the trivial upper bound given in Section 1. The first one is in terms of the order and maximum degree, while the second one is in terms of the order and diameter of the graph. □

Proposition 6. *Let G be a connected graph of order n with maximum degree $\Delta \geq 2$. Then $\gamma_{\{3R\}}(G) \leq n - \Delta + 2$.*

Proof. Let v be a vertex with maximum degree Δ and define the function f on G by $f(v) = 3$, $f(x) = 0$ for $x \in N(v)$ and $f(x) = 1$ for the remaining vertices. Clearly, f is an R{3}DF of G and thus $\gamma_{\{3R\}}(G) \leq n - \Delta + 2$. □

Proposition 7. *Let G be a connected graph of order $n \geq 2$. Then*

$$\gamma_{\{3R\}}(G) \leq \begin{cases} n - 1 + \left\lceil \frac{3 - \mathrm{diam}(G)}{4} \right\rceil & \text{if } \mathrm{diam}(G) \equiv 1, 2, 3 \pmod 4, \\ n + \left\lceil \frac{3 - \mathrm{diam}(G)}{4} \right\rceil & \text{if } \mathrm{diam}(G) \equiv 0 \pmod 4. \end{cases}$$

Proof. Let $P = v_1 v_2 \dots v_{\mathrm{diam}(G)+1}$ be a diametral path in G and let f be a $\gamma_{\{3R\}}(P)$-function. Define g on G by $g(x) = f(x)$ for $x \in V(P)$ and $g(x) = 1$ for the remaining vertices of G. Clearly g is an R{3}DF of G and the result follows from Corollary 1. □

In the remaining part of this section, we provide some sufficient conditions for special connected graphs G of order n to have $\gamma_{\{3R\}}(G) \le n-2$. These results will be useful in the next section to characterize the connected graphs G of order n such that $\gamma_{\{3R\}}(G) = n-1$.

Observation 8. *If G is a connected graph of order n with two non-adjacent vertices x, y each of degree at least three, then $\gamma_{\{3R\}}(G) \le n-2$.*

Proof. By assigning a 0 to x and y, and a 1 to each of the remaining vertices, we get an R{3}DF on G with weight $n-2$ and thus $\gamma_{\{3R\}}(G) \le n-2$. □

Proposition 9. *Let G be a connected graph of order n having a cycle $C = (v_1 v_2 v_3)$.*

(1) *If G has two disjoint paths $v_1 y_1 y_2 y_3$ and $v_2 y_4 y_5 y_6$ such that no y_i belongs to C, then $\gamma_{\{3R\}}(G) \le n-2$.*
(2) *If G has two disjoint paths $v_1 y_1$ and $v_2 y_2 y_3 y_4 y_5$ such that no y_i belongs to C, then $\gamma_{\{3R\}}(G) \le n-2$.*
(3) *If G has a path $v_1 y_1 y_2 y_3 y_4 y_5$ such that no y_i belongs to C, then $\gamma_{\{3R\}}(G) \le n-2$.*
(4) *If G has three disjoint paths $v_1 y_1, v_2 y_2, v_3 y_3 y_4$ such that no y_i belongs to C, then $\gamma_{\{3R\}}(G) \le n-2$.*

Proof. Let f be an R{3}DF defined on $V(G)$ as follows for each case.

(1) $f(y_1) = f(y_4) = 2$, $f(v_1) = f(v_2) = f(y_2) = f(y_5) = 0$ and $f(x) = 1$ for the remaining vertices. Then $\omega(f) = n-2$ and thus $\gamma_{\{3R\}}(G) \le n-2$.
(2) $f(y_3) = 2$, $f(v_1) = f(y_2) = f(y_4) = 0$ and $f(x) = 1$ for the remaining vertices. Then $\omega(f) = n-2$ and thus $\gamma_{\{3R\}}(G) \le n-2$.
(3) $f(y_3) = 2$, $f(v_1) = f(y_2) = f(y_4) = 0$ and $f(x) = 1$ for the remaining vertices. Then $\omega(f) = n-2$ and thus $\gamma_{\{3R\}}(G) \le n-2$.
(4) $f(v_3) = 2$, $f(v_1) = f(v_2) = f(y_3) = 0$ and $f(x) = 1$ for the remaining vertices. Then $\omega(f) = n-2$ and thus $\gamma_{\{3R\}}(G) \le n-2$.

□

Proposition 10. *Let G be a connected graph of order n having a cycle $C = (v_1 v_2 \dots v_t)$ and a path $v_1 y_1 y_2$ where $t \ge 4$ and $y_1, y_2 \notin V(C)$. Then $\gamma_{\{3R\}}(G) \le n-2$.*

Proof. Define the function f on G by $f(v_1) = 2$, $f(v_2) = f(v_t) = f(y_1) = 0$ and $f(x) = 1$ for the remaining vertices. Then f is an R{3}RD of G with weight $n-2$ and thus $\gamma_{\{3R\}}(G) \le n-2$. □

Proposition 11. *Let G be a connected graph of order n having a cycle of order m such that $6 \le m \le n-1$. Then $\gamma_{\{3R\}}(G) \le n-2$.*

Proof. Let $C = (v_1 v_2 \dots v_m)$ be a cycle on m vertices in G with $6 \le m \le n-1$. Since G is connected, we may assume, without loss of generality, that v_1 has at least one neighbor that does not belong to C. Define the function f on G by $f(v_4) = 2$, $f(v_3) = f(v_5) = f(v_1) = 0$ and $f(x) = 1$ for the remaining vertices. Then f is an R{3}RD of G with weight $n-2$ and thus $\gamma_{\{3R\}}(G) \le n-2$. □

4. Graphs G of order n with $\gamma_{\{3R\}}(G) \in \{n-1, n\}$

We begin by characterizing the connected graphs G of order n such that $\gamma_{\{3R\}}(G) = n$.

Theorem 12. *Let G be a connected graph of order n. Then $\gamma_{\{3R\}}(G) = n$ if and only if $G \in \{P_2, P_3, P_4, C_3\}$.*

Proof. Assume that $\gamma_{\{3R\}}(G) = n$. By Proposition 6, we deduce that $\Delta \leq 2$ and thus G is either a path or a cycle. It follows from Corollaries 1 and 2 that $G \in \{P_2, P_3, P_4, C_3\}$.

The converse is trivial. □

Our next aim is to characterize the connected graphs of order n such that $\gamma_{\{3R\}}(G) = n-1$. For this purpose we start with the class of trees. In what follows, let P_n^r denote the tree obtained from a path $P_n = v_1v_2 \dots v_n$ by adding for some $r \in \{1, \dots, n\}$ a new vertex v_r' and the edge $v_r'v_r$. Also let $P_n^{r,m}$ denote the tree obtained from P_n by adding for some $r, m \in \{1, \dots, n\}$ two new vertices v_r' and v_m' and the edges $v_r'v_r$ and $v_m'v_m$.

Theorem 13. *Let T be a tree of order $n \geq 4$ different from a path P_4. Then $\gamma_{\{3R\}}(T) = n - 1$ if and only if $T \in \{P_5, P_6, P_7, P_8\} \cup \{P_3^2, P_4^2, P_5^2, P_5^3, P_6^2, P_6^3, P_7^3, P_7^4, P_8^5\} \cup \{P_4^{2,3}, P_5^{2,3}, P_6^{2,3}, P_6^{3,4}, P_7^{3,4}\}$.*

Proof. Assume that $\gamma_{\{3R\}}(T) = n - 1$. According to Proposition 6, we have $\Delta(T) \leq 3$. If $\Delta(T) = 2$, then T is a path and by Corollary 1 we have $T \in \{P_5, P_6, P_7, P_8\}$. Hence assume that $\Delta(T) = 3$. By Proposition 7, $\mathrm{diam}(T) \leq 7$.

Let $P = v_1v_2 \dots v_t$ be a diametral path in T, where $t = \mathrm{diam}(T) + 1$. If $t = 3$, then T is the star $K_{1,3} = P_3^2$, while if $t = 4$, then T is a double star $DS_{r,s}$ with $2 \geq r \geq s \geq 1$ and thus $T \in \{P_4^2, P_4^{2,3}\}$. Hence assume that $t \geq 5$, and consider the following cases.

Case 1. $t = 8$.

If $\deg_T(v_2) \geq 3$, then the function f defined on T by $f(v_2) = f(v_4) = f(v_6) = 0$, $f(v_5) = 2$ and $f(x) = 1$ for the remaining vertices, is an R{3}DF of T with weight $n-2$ which is a contradiction. Therefore $\deg_T(v_2) = 2$. Likewise, we have $\deg_T(v_7) = 2$. Now, if $\deg_T(v_3) \geq 3$, then the function f defined on T by $f(v_3) = f(v_5) = f(v_7) = 0$, $f(v_6) = 2$ and $f(x) = 1$ for any other vertex x of T is an R{3}DF of T of weight $n - 2$, a contradiction. Hence $\deg_T(v_3) = 2$, and likewise $\deg_T(v_6) = 2$. If $\deg_T(v_4) \geq 3$ and v_4 has a child w with depth at least 1 other than v_3, then the function f defined on T by $f(v_3) = f(v_5) = f(w) = 0$, $f(v_4) = 2$ and $f(x) = 1$ for any other vertex x of T is an R{3}DF of T of weight $n - 2$, a contradiction. Therefore v_4 has degree two or it has degree three with a unique leaf neighbor (since $\Delta(T) = 3$). By symmetry, $\deg_T(v_5) = 2$ or $\deg_T(v_5) = 3$ and has a unique leaf-neighbor. Now, if $\deg_T(v_4) = 3$ and $\deg_T(v_5) = 3$, then the function f defined on T by $f(v_2) = f(v_4) = f(v_5) = f(v_7) = 0$, $f(v_3) = f(v_6) = 2$ and $f(x) = 1$ for the remaining vertices is an R{3}DF of T of weight $n-2$, a contradiction. Consequently we may assume that $\deg_T(v_4) = 2$ and v_5 is a support vertex, and therefore $T = P_8^5$.

Case 2. $t = 7$.

As in the Case 1, we can see that $\deg_T(v_2) = 2$ and $\deg_T(v_6) = 2$. If v_i has a child w with depth at least one for some $i \in \{3, 4, 5\}$, then the function f defined on T by $f(v_{i-1}) = f(v_{i+1}) = f(w) = 0$, $f(v_i) = 2$ and $f(x) = 1$ for the remaining vertices, is an R{3}DF of T of weight $n - 2$, a contradiction. Thus any child (if any) of v_i is a leaf for all $i \in \{3, 4, 5\}$. Moreover, by Observation 8, $\min\{\deg_T(v_3), \deg_T(v_5)\} = 2$. Without loss of generality, let $\deg_T(v_5) = 2$ and therefore $T \in \{P_7^3, P_7^4, P_7^{3,4}\}$.

Case 3. $t = 6$.

As in the Case 2, we can see that any child of v_i (if any) is a leaf for $i \in \{3, 4\}$. Moreover, by Observation 8, $\min\{\deg_T(v_2), \deg_T(v_5)\} = 2$. Without loss of generality, let $\deg_T(v_5) = 2$. Observation 8 also implies that $\min\{\deg_T(v_2), \deg_T(v_4)\} = 2$. Therefore, up to isomorphism, $T \in \{P_6^2, P_6^3, P_6^{3,4} P_6^{2,3}\}$.

Case 4. $t = 5$.

Again as in the Case 2, we can see that any child (if any) of v_3 is a leaf. Since by Observation 8, $\min\{\deg_T(v_2), \deg_T(v_4)\} = 2$, let us assume that $\deg_T(v_4) = 2$. Also, since $\Delta(T) = 3$, it follows that $\{P_5^2, P_5^3, P_5^{2,3}\}$, and this completes the proof.

□

The following corollary is an immediate consequence of Theorems 12 and 13.

Corollary 14. *If G is a connected graph of order $n \geq 10$, then $\gamma_{\{3R\}}(G) \leq n - 2$.*

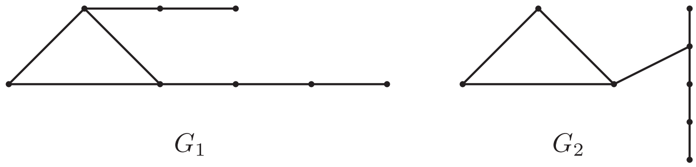

FIGURE 1. Two unicyclic graphs of order 8 with $\gamma_{\{3R\}}(G) = 7$.

Proof. It suffices to consider any spanning tree T of G. Clearly, $\gamma_{\{3R\}}(G) \leq \gamma_{\{3R\}}(T)$. Now by Theorems 12 and 13, since trees T of order n with $\gamma_{\{3R\}}(T) \in \{n-1, n\}$ have order at most 9, we deduce that $\gamma_{\{3R\}}(T) \leq n-2$, and thus $\gamma_{\{3R\}}(G) \leq n-2$. □

According to Corollary 14, to achieve the characterization of connected graphs G of order n such that $\gamma_{\{3R\}}(G) = n-1$, we consider in the remainder of this section only the connected graphs of order $n \in \{4, 5, \ldots, 9\}$.

Lemma 15. *Let G be a connected graph of order 9. Then $\gamma_{\{3R\}}(G) = 8$ if and only if $G \in \{P_8^5, P_7^{3,4}\}$.*

Proof. If $G \in \{P_8^5, P_7^{3,4}\}$, then clearly $\gamma_{\{3R\}}(G) = 8$. To prove the necessity, assume that $\gamma_{\{3R\}}(G) = 8$. By Proposition 6, $\Delta(G) \leq 3$. We deduce from Corollaries 1 and 2 that $\Delta(G) = 3$. Let T be a spanning subtree of G. By Theorem 12, $\gamma_{\{3R\}}(T) \leq 8$, and since $8 = \gamma_{\{3R\}}(G) \leq \gamma_{\{3R\}}(T) \leq 8$, we obtain $\gamma_{\{3R\}}(T) = 8$. According to Theorem 13, we deduce that every spanning tree of G is isomorphic to P_8^5 or $P_7^{3,4}$.

Assume first that $T = P_8^5$, where T is obtained from a path $v_1v_2\ldots v_8$ by adding a new vertex w and the edge v_5w. Observation 8 implies that $\deg_G(v_2) = \deg_G(v_3) = \deg_G(v_7) = 2$. Furthermore, by Propositions 9–11, we have $wv_1, wv_8, wv_4, wv_6 \notin E(G)$. Therefore $\deg_G(w) = 1$. Again by Propositions 10 and 11 we deduce that $\deg_G(v_1) = 1$, and likewise $\deg_G(v_8) = 1$. Therefore $G = P_8^5$.

Assume now that $T = P_7^{3,4}$, where T is obtained from a path $v_1v_2\ldots v_7$ by adding two new vertices w and w' and the edges v_3w and v_4w'. By Observation 8, we have $\deg_G(v_2) = \deg_G(v_5) = \deg_G(v_6) = 2$. Moreover, by Propositions 10 and 11 we must have $\deg_G(w) = \deg_G(w') = \deg_G(v_1) = \deg_G(v_7) = 1$. Therefore $G = P_7^{3,4}$, and the proof is complete. □

Let G_1 be the graph obtained from a cycle $(v_1v_2v_3)$ by adding two disjoint paths $v_1x_1x_2$ and $v_2x_3x_4x_5$ such that no x_i belongs to $\{v_1, v_2, v_3\}$, and let G_2 be the graph obtained from a cycle $(v_1v_2v_3)$ by adding a path $x_1x_2x_3x_4x_5$ and the edge vx_2 (see Fig. 1).

Lemma 16. *Let G be a connected graph of order 8. Then $\gamma_{\{3R\}}(G) = 7$ if and only if $G \in \{P_8, P_7^3, P_7^4, P_6^{2,3}, P_6^{3,4}, G_1, G_2\}$.*

Proof. If $G \in \{P_8, P_7^3, P_7^4, P_6^{2,3}, P_6^{3,4}, G_1, G_2\}$, then one can easily see that $\gamma_{\{3R\}}(G) = 7$. To prove the necessity, let G be a connected graph of order 8 with $\gamma_{\{3R\}}(G) = 7$. By Proposition 6, $\Delta(G) \leq 3$. If $\Delta(G) = 2$, then G is a path or cycle of order 8 and by Corollary 1, $G = P_8$. In the next, we can assume that $\Delta(G) = 3$. As in the proof of Lemma 15 we can see that any spanning tree of G is isomorphic to one of the trees $P_7^3, P_7^4, P_6^{2,3}$ or $P_6^{3,4}$. Let T be a spanning tree of G with a diameter as small as possible. We consider the following situations.

Case 1. Every spanning tree of G is isomorphic to P_7^3.

Assume that T is obtained from a path $v_1v_2\ldots v_7$ by adding a new vertex w and the edge v_3w. If $G = P_7^3$, then we are done. Hence suppose that $G \neq P_7^3$. Observation 8 implies that $\deg_G(v_6) = \deg_G(v_5) = 2$ and Proposition 9-(4) implies that $v_2v_4 \notin E(G)$. Moreover, by Propositions 10 and 9-(2) we deduce that $v_7w, v_1w, v_2w \notin E(G)$. Hence $\deg_G(w) \leq 2$. If $\deg_G(w) = 1$, then Propositions 10 and 11 imply $v_1v_4, v_1v_7, v_2v_7, v_4v_7 \notin E(G)$ and thus $G = P_7^3$, contradicting our assumption that $G \neq P_7^3$. Therefore $\deg_G(w) = 2$, and so $wv_4 \in E(G)$. A similar argument as above shows that $v_1v_7, v_2v_7 \notin E(G)$ and thus $G = G_1$ which is obtained from P_7^3 by adding the edge wv_4.

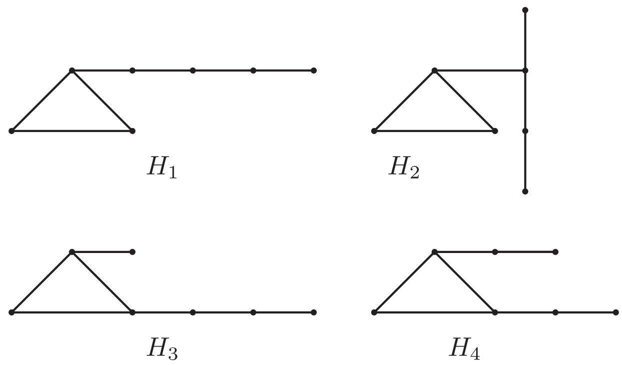

FIGURE 2. Four unicyclic graphs of order 7 with $\gamma_{\{3R\}}(G) = 6$.

Case 2. Every spanning tree of G is isomorphic to P_7^4.

Assume that T is obtained from a path $v_1v_2\dots v_7$ by adding a new vertex w and the edge v_4w. If $G = P_7^4$, then we are done. Hence suppose that $G \neq P_7^4$. Observation 8 implies that $\deg_G(v_6) = \deg_G(v_2) = 2$. If $wv_3 \in E(G)$ or $wv_5 \in E(G)$, then we are in the position of Case 1, and thus $G = G_1$. Hence assume that $wv_5, wv_3 \notin E(G)$. By the choice of T we have $v_1v_3, v_1v_5, v_7v_5, v_7v_3 \notin E(G)$. Now from Propositions 9–11, we deduce that $v_3v_5, v_1w, v_7w, v_1v_7 \notin E(G)$. Therefore $\deg_G(w) = \deg_G(v_1) = \deg_G(v_7) = 1$, and thus $G = P_7^4$, contradicting our assumption that $G \neq P_7^4$.

Case 3. Every spanning tree of G is isomorphic to $P_6^{2,3}$.

Assume that T is obtained from a path $v_1v_2\dots v_6$ by adding two new vertices w' and w and the edges v_2w' and v_3w. Observation 8 implies that $\deg_G(v_4) = \deg_G(v_5) = 2$ and Propositions 10 and 11 imply that $\deg_G(v_6) = \deg_G(w) = 1$ and $\deg_G(v_1), \deg_G(w') \leq 2$. If $v_1w' \in E(G)$, then $G = G_2$ while if $v_1w' \notin E(G)$, then $G = P_6^{2,3}$.

Case 4. Every spanning tree of G is isomorphic to $P_6^{3,4}$.

Assume that T is obtained from a path $v_1v_2\dots v_6$ by adding two new vertices w' and w and the edges v_3w and v_4w'. Observation 8 implies that $\deg_G(v_2) = \deg_G(v_5) = 2$ and Proposition 10 yields $\deg_G(w) = \deg_G(w') = \deg_G(v_1) = \deg_G(v_6) = 1$. Therefore $G = P_6^{3,4}$. This completes the proof.

□

Let $\mathcal{H}$ be the collection of the four unicyclic graphs in Figure 2.

Lemma 17. *Let G be a connected graph of order 7. Then $\gamma_{\{3R\}}(G) = 6$ if and only if $G \in \{P_7, C_7, P_6^3, P_6^2, P_5^{2,3}\} \cup \mathcal{H}$.*

Proof. If $G \in \{P_7, P_6^3, P_6^2, P_5^{2,3}\} \cup \mathcal{H}$, then the result is immediate. To prove the necessity, let G be a connected graph of order 7 with $\gamma_{\{3R\}}(G) = 6$. By Proposition 6, $\Delta(G) \leq 3$. If $\Delta(G) = 2$, then G is a path or cycle of order 7 and by Corollaries 1 and 2, we have $G \in \{P_7, C_7\}$. Hence assume that $\Delta(G) = 3$. As in the proof of Lemma 15, we can see that any spanning tree of G is isomorphic to $P_7, P_6^3, P_6^2, P_5^{2,3}$. Let T be a spanning tree of G with a diameter as small as possible. Clearly, by our choice of T and the fact $\Delta(G) = 3$, we have $T \neq P_7$. Consider the following situations.

Case 1. $T = P_5^{2,3}$.

Assume that T is obtained from a path $v_1v_2v_3v_4v_5$ by adding two new vertices w' and w and the edges v_2w' and v_3w. Observation 8 implies that $\deg_G(v_4) = 2$ while Proposition 10 implies that $v_1v_5, wv_5, wv_1, ww', w'v_5 \notin E(G)$. Hence $\deg_G(v_5) = \deg_G(w) = 1$ and either $\deg_G(w') = 1$ or $v_1w' \in E(G)$. If $\deg_G(w') = 1$, then $G = P_5^{2,3}$ while if $w'v_1 \in E(G)$, then $G = H_2$.

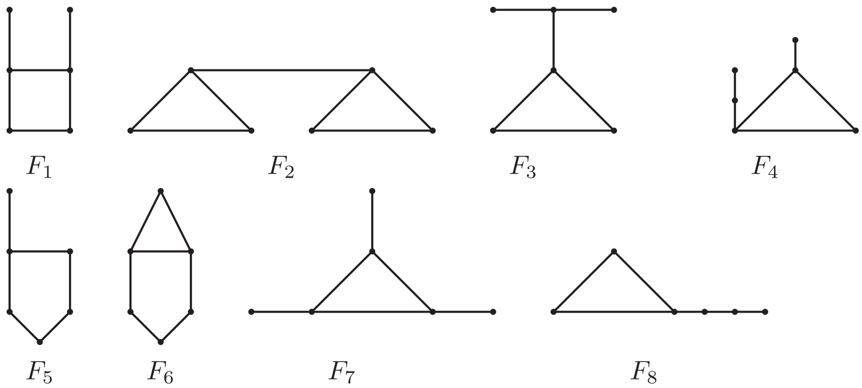

FIGURE 3. Eight graphs of order 6 with $\gamma_{\{3R\}}(G) = 5$.

Case 2. $T = P_6^2$.

Assume that T is obtained from a path $v_1v_2v_3v_4v_5v_6$ by adding a new vertex w and the edge v_2w. As in Case 1, Observation 8 implies that $\deg_G(v_4) = \deg_G(v_5) = 2$ and Proposition 10 leads that $wv_6, v_3v_6, v_1v_6 \notin E(G)$. Hence $\deg_G(v_6) = 1$ and $\deg_G(v_1) \leq 2$. If $wv_1, wv_3 \in E(G)$, then the function g defined on $V(G)$ by $g(v_3) = 2$, $g(w) = g(v_2) = g(v_4) = 0$ and $g(x) = 1$ otherwise, is an R{3}DF of T of weight $n-2$, a contradiction. Thus $wv_1 \notin E(G)$ or $wv_3 \notin E(G)$. First let $v_1v_3 \notin E(G)$. Clearly if $wv_1 \notin E(G)$ and $wv_3 \notin E(G)$, then $G = P_6^2$. If $wv_1 \in E(G)$, then $G = H_1$ while if $wv_3 \in E(G)$, then $G = H_3$. Assume now that $v_1v_3 \in E(G)$. We must have $wv_1 \notin E(G)$ (otherwise assigning a 2 to v_3 and a 1 to w, v_5, v_6 and a 0 to v_1, v_2, v_4 provides an R{3}DF of G of weight $n - 2 = 5$). Therefore $G = H_3$.

Case 3. $T = P_6^3$.

Assume that T is obtained from a path $v_1v_2v_3v_4v_5v_6$ by adding a new vertex w and the edge v_3w. If $wv_2 \in E(G)$, then we are in the position of Case 2. Hence we assume that $wv_2 \notin E(G)$. By Observation 8 and Propositions 10 and 9 we have $\deg_G(v_5) = 2$ and $wv_1, wv_6, v_1v_6, v_2v_6, v_1v_4, v_2v_4 \notin E(G)$. If $wv_4 \in E(G)$, then we must have $\deg_G(v_2) = 2$ and $\deg_G(v_1) = \deg(v_6) = 1$ yielding $G = H_4$. Hence assume that $wv_4 \notin E(G)$ and so $\deg_G(w) = 1$. Then $\deg_G(v_1) = 1$. Now, if $v_4v_6 \notin E(G)$, then $G = P_6^3$ while if $v_4v_6 \in E(G)$, then $G = H_2$. This completes the proof.

□

Let $\mathcal{F}$ be the collection of the eight graphs in Figure 3.

Lemma 18. *Let G be a connected graph of order 6. Then $\gamma_{\{3R\}}(G) = 5$ if and only if $G \in \{P_6, C_6, P_5^3, P_5^2, P_4^{2,3}\} \cup \mathcal{F}$.*

Proof. If $G \in \{P_6, C_6, P_5^3, P_5^2, P_4^{2,3}\} \cup \mathcal{F}$, then the result is immediate. To prove the necessity, let G be a connected graph of order 6 with $\gamma_{\{3R\}}(G) = 5$. By Proposition 6, $\Delta(G) \leq 3$. If $\Delta(G) = 2$, then G is a path or cycle of order 6 and by Corollaries 1 and 2, we have $G \in \{P_6, C_6\}$. Hence assume that $\Delta(G) = 3$. A similar argument to that used in the proof of Lemma 15 shows that any spanning tree of G is isomorphic to $P_6, P_5^3, P_5^2, P_4^{2,3}$. Let T be a spanning tree of G with a diameter as small as possible. Clearly, by our choice of T and the fact $\Delta(G) = 3$, we have $T \neq P_6$. Consider the three possible cases.

Case 1. $T = P_4^{2,3}$.

Assume that T is obtained from a path $v_1v_2v_3v_4$ by adding two new vertices w' and w and the edges v_2w' and v_3w. It follows from Proposition 8 that the degree of v_1 and v_4 is at most two. Now if $wv_1, w'v_4 \in E(G)$, then the function g defined by $g(v_2) = 2, g(v_1) = g(v_3) = g(w') = 0$, $g(x) = 1$ otherwise, is an R{3}DF of

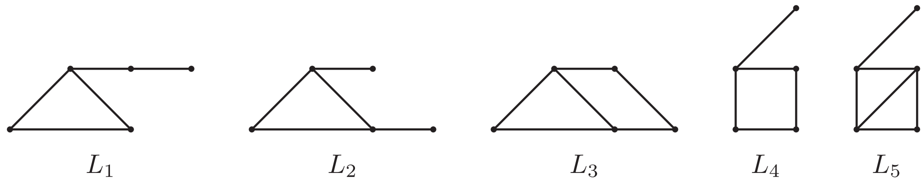

FIGURE 4. Five graphs of order 5 with $\gamma_{\{3R\}}(G) = 4$.

T of weight 4, a contradiction. Hence $|E(G) \cap \{wv_1, w'v_4\}| \le 1$. If $wv_1 \in E(G)$ (the case $w'v_4 \in E(G)$ is similar), then clearly $G = F_1$. Hence we assume that $|E(G) \cap \{wv_1, w'v_4\}| = 0$. If $wv_4, w'v_1 \in E(G)$, then $G = F_2$ while if $wv_4 \in E(G)$ and $w'v_1 \notin E(G)$, then $G = F_3$. Finally, if $wv_4, w'v_1 \notin E(G)$, then $G = P_4^{2,3}$ when $ww' \notin E(G)$ and $G = F_1$ when $ww' \in E(G)$.

Case 2. $T = P_5^3$.

Assume that T is obtained from a path $v_1v_2v_3v_4v_5$ by adding a new vertex w and the edge v_3w. If $v_1v_4 \in E(G)$ (the case $v_2v_5 \in E(G)$ is similar), then deleting the edge v_1v_2 from T and adding the edge v_1v_4 provides a spanning tree of G with diameter smaller than that of T which contradicts our choice. Hence we can assume that $v_1v_4, v_2v_5 \notin E(G)$. If $v_2v_4 \in E(G)$, then $wv_5 \notin E(G)$ (else we can get an R{3}DF of G of weight $n-2=4$) and thus likewise $wv_1 \notin E(G)$. Now if $v_1v_5 \notin E(G)$, then $G = F_7$, while if $v_1v_5 \in E(G)$, then one can easily see that $\gamma_{\{3R\}}(G) < 5$. Hence we can assume for the next that $v_2v_4 \notin E(G)$. By Proposition 10 and Observation 8, we have $wv_1, wv_5 \notin E(G)$ and that $|E(G) \cap \{wv_2, wv_4\}| \le 1$. If $v_1v_5 \in E(G)$, then one can see that $G \in \{F_5, F_6\}$. Hence let $v_1v_5 \notin E(G)$. Then $\deg_G(v_1) = \deg_G(v_5) = 1$. Since $v_2v_4 \notin E(G)$, we get $G = F_4$ (when wv_2 or $wv_4 \in E(G)$) or $G = P_5^3$ otherwise.

Case 3. $T = P_5^2$.

Assume that T is obtained from a path $v_1v_2v_3v_4v_5$ by adding a new vertex w and the edge v_2w. By Observation 8, $\deg_G(v_4) = 2$. If $v_1v_5 \in E(G)$, then considering the spanning subtree $(T + v_1v_5) - v_4v_5$, we will be in the position of Case 2 and the result follows. If $wv_5 \in E(G)$, then again considering the spanning subtree $(T + wv_5) - v_5v_4$, we will be in the position of Case 2 and the result follows. The same situation also occurs when $v_3v_5 \in E(G)$. Hence we can assume that $v_1v_5, wv_5, v_3v_5 \notin E(G)$. Hence $\deg_G(v_5) = 1$. Now since the edges wv_1 and wv_3 cannot both belong to $E(G)$ (for otherwise we can get an R{3}DF of weight 4), one can see that $G \in \{F_4, F_8\}$. This completes the proof.

□

Let $\mathcal{L}$ be the collection of the five graphs in Figure 4.

Lemma 19. *Let G be a connected graph of order 5. Then $\gamma_{\{3R\}}(G) = 4$ if and only if $G \in \{P_5, C_5, P_4^2,\} \cup \mathcal{L}$.*

Proof. If $G \in \{P_5, C_5, P_4^2\} \cup \mathcal{L}$, then the result is immediate. To prove the necessity, let G be a connected graph of order 5 with $\gamma_{\{3R\}}(G) = 4$. By Proposition 6, $\Delta(G) \le 3$. If $\Delta(G) = 2$, then G is a path or cycle of order 5, and by Corollaries 1 and 2, $G \in \{P_5, C_5\}$. Hence assume that $\Delta(G) = 3$, and let v be a vertex of degree 3, with $N(v) = \{v_1, v_2, v_3\}$. Since G is connected of order 5, we may assume, without loss of generality, that $v_1w \in E(G)$, where w is the fifth vertex of G. Now, if $\deg_G(w) = 3$, then by Observation 8, $\gamma_{\{3R\}}(G) \le n - 2 = 3$. Hence $\deg_G(w) \le 2$. Assume first that $\deg_G(w) = 1$. If $\deg_G(v_1) = 2$, then clearly $G \in \{P_4^2, L_1\}$. Thus assume that $\deg_G(v_1) = 3$ and let $v_1v_2 \in E(G)$. Then $G = L_2$ if $v_2v_3 \notin E(G)$ while $G = L_5$ if $v_2v_3 \in E(G)$. Finally, assume that $\deg_G(w) = 2$, and let, without loss of generality, $v_2w \in E(G)$. Clearly, if $v_1v_2 \in E(G)$, then $G = L_5$. Hence we assume that $v_1v_2 \notin E(G)$. If $|\{v_1v_3, v_2v_3\} \cap E(G)| = 2$, then Observation 8 applied to v_1 and v_2 leads that $\gamma_{\{3R\}}(G) \le n - 2 = 3$. Hence $|\{v_1v_3, v_2v_3\} \cap E(G)| \le 1$. Up to isomorphism, let $v_1v_3 \notin E(G)$. It follows that $G \in \{L_3, L_4\}$ and the proof is complete. □

Let G_3 be the graph of order 4 obtained from a star $K_{1,3} = P_3^2$ by adding an edge between two leaves of the star.

Lemma 20. *Let G be a connected graph of order 4. Then $\gamma_{\{3R\}}(G) = 3$ if and only if $G \in \{K_{1,3}, C_4, K_4, K_4 - e, G_3\}$.*

Proof. Assume that G is a connected graph of order 4 with $\gamma_{\{3R\}}(G) = 3$. If $\Delta(G) = 2$, then clearly $G = C_4$ and $G \neq P_4$ (by Cor. 1). If $\Delta(G) = 3$, then G is certainly one of the graphs belonging to $\{P_3^2, K_4, K_4 - e, G_3\}$. The converse is trivial. □

Now we are ready to state the main result of this section whose proof follows from Theorem 13, Corollary 14 and Lemmas 15, 16, . . . ,20.

Theorem 21. *Let G be a connected graph of order $n \geq 4$. Then $\gamma_{\{3R\}}(G) = n - 1$ if and only if G is isomorphic to one of graphs in $\{P_{i+1}, C_i \mid 4 \leq i \leq 7\} \cup \{P_i^2 \mid 3 \leq i \leq 6\} \cup \{P_8^5, P_i^3 \mid 5 \leq i \leq 7\} \cup \{P_7^{3,4}, P_6^{3,4}, P_6^{2,3}, P_5^{2,3}, P_4^{2,3}\} \cup \{G_1, G_2, G_3\} \cup \{K_4, K_4 - e\} \cup \mathcal{F} \cup \mathcal{H} \cup \mathcal{L}$.*

Observe that any graph given in Theorems 12 and 21 is connected having size equal to at most the order plus 2, and consequently the following corollary is obtained.

Corollary 22. *If G is a connected graph of order n and size $m \geq n + 3$, then $\gamma_{\{3R\}}(G) \leq n - 2$.*

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