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MODIFICATION OF SOME SCALARIZATION APPROACHES FOR MULTIOBJECTIVE OPTIMIZATION

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Abstract. In this paper, we propose revisions of two existing scalarization approaches, namely the feasible-value constraint and the weighted constraint. These methods do not easily provide results on proper efficient solutions of a general multiobjective optimization problem. By proposing some novel modifications for these methods, we derive some interesting results concerning proper efficient solutions. These scalarization approaches need no convexity assumption of the objective functions. We also demonstrate the efficiency of the proposed method using numerical experiments. In particular, a rocket injector design problem involving four objective functions illustrates the performance of the proposed method.

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1. Introduction

Multiobjective optimization refers to the mathematical problems in which we deal with minimization or maximization of competing objective functions over a feasible set of decisions. One challenging issue is that it is usually not possible to optimize multiple objective functions at the same time. This type of problem has been playing an important role in many applied fields, including economics, engineering, medicine, management, and etc. (see [6, 20, 23, 26–29, 33]). One of the most important methods for solving multiobjective optimization problems is to use scalarization approaches. In these methods, a parametric single objective optimization problem corresponding to the multiobjective optimization problem is solved, and the relationship between optimal solutions of the single objective problem and efficient solutions of the multiobjective optimization problem is investigated. A broad range of scalarization approaches exists in the literature, including the weighted-sum scalarization approach [14, 30], the epsilon-constraint approach [7], the Pascoletti Serafini approach [24], the modified Pascoletti Serafini methods [1, 8, 11, 12, 16], the weighted-constraint method [4] and the feasible-value-constraint approach [5]. It is important to note that references [4, 5] provided results that were necessary and/or sufficient for (weakly) efficient solutions. However, these scalarization techniques have no result on proper efficiency. In this respect, our aim is to propose a modification of these methods by adding slack and surplus variables in the constraints and penalizing the violations in the objective function, the inflexibility of the con-

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straints is resolved. Here, we will investigate how adding slack and surplus variables in the constraints can help to identify conditions for (weak, proper) efficiency. Furthermore, we find several necessary and sufficient conditions for different types of efficient solutions of a multiobjective optimization problem. The proposed scalarization methods require no convexity assumption on objective functions. We demonstrate the efficiency of the proposed method using numerical experiments. Our first and second numerical examples are three-objective problems with nonconvex constraints. For these problems approximating the nondominated set (Pareto front) is a difficult task. As a practical problem, the rocket injector design problem will be investigated in the last numerical example. This problem was first presented in [18] with four objective functions. In [18], this problem is reduced to a three-objective problem. However, our obtained results show that such a reduction may lead to loosing some parts of the nondominated set.

The rest of the paper is organized as follows. We review preliminaries and basic definitions in Section 2. In Sections 3 and 4, we propose two modifications of the feasible-value constraint and weighted constraint approaches. Some necessary and sufficient conditions relating to weak efficient, efficient, and proper efficient solutions are proved by using these modifications. Section 5 provides some numerical examples that demonstrate the efficiency of our method in practice.

2. Basic notations and preliminaries

This section contains some notations and standard definitions from multiobjective optimization which are used throughout this paper. The general form of a multiobjective optimization problem is given by

$$\text{(MOP)} \qquad \begin{cases} \min\ f(x) := (f_1(x), f_2(x), \dots, f_\ell(x)) \\ \text{s.t.}\ \ x \in X := \{x \in \mathbb{R}^n \mid g_j(x) \le 0, \quad j = 1, \dots, m\}, \end{cases}$$

with vector-valued continuous functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell,\ g : \mathbb{R}^n \longrightarrow \mathbb{R}^m (\ell, n, m \in \mathbb{N},\ \ell \ge 2)$ and a nonempty set of feasible points $X \subseteq \mathbb{R}^n$. We assume that the functions f_i are bounded from below on the set X, with a known lower bound. A vector $x \in \mathbb{R}^\ell$ with entries $x_i \in \mathbb{R}$, $i = 1, \dots, \ell$, is written as $(x_1, \dots, x_\ell)$. All relations in this paper should be read as component-wise, *i.e.*, for $x, x' \in \mathbb{R}^\ell$ it is

$$\begin{aligned} x < x' &\iff x_i < x'_i, \quad \text{for all } i = 1, \dots, \ell, \\ x \le x' &\iff x_i \le x'_i, \quad \text{for all } i = 1, \dots, \ell, \text{ and } x \ne x', \\ x \leqq x' &\iff x_i \le x'_i, \quad \text{for all } i = 1, \dots, \ell. \end{aligned}$$

The following definitions are standard in multiobjective optimization [9, 15].

Definition 2.1. Let $\check{x} \in X$.

(a) $\check{x}$ is called an efficient solution for the MOP, if there exists no $x \in X$ such that $f(x) \le f(\check{x})$.
(b) $\check{x}$ is a weakly efficient solution for the MOP, if there is no other $x \in X$ such that $f(x) < f(\check{x})$.
(c) $\check{x}$ is a strictly efficient solution for the MOP, if there exists no $x \in X, x \ne \check{x}$ such that $f(x) \leqq f(\check{x})$.
(d) $\check{x}$ is said to be a proper efficient solution in the Geoffrion's sense for the MOP, if it is efficient and there exists a real positive number $M > 0$ such that for each $x \in X$ and $i \in \{1, 2, \dots, \ell\}$ satisfying $f_i(x) < f_i(\check{x})$ there exists at least one $j \in \{1, 2, \dots, \ell\}$ such that $f_j(\check{x}) < f_j(x)$ and

$$\frac{f_i(\check{x}) - f_i(x)}{f_j(x) - f_j(\check{x})} \le M.$$

There are different definitions for proper efficiency in the literature; see [2, 3, 13, 17, 21, 25, 32]. Throughout this paper, all proper efficient solutions of the MOP refers to Geoffrions proper efficient solutions. Hereafter, the set of weakly efficient, efficient, strictly efficient and proper efficient solutions of the MOP are denoted by $X_{\mathrm{WE}}, X_{\mathrm{E}}, X_{\mathrm{SE}}$ and X_{PE}, respectively. The image of the feasible set under the objective functions is called the image space, and it is denoted by Y. The images of efficient solutions are called nondominated solutions. The set of nondominated solution is denoted by Y_{N}.

Definition 2.2. (a) Let $Y \subseteq \mathbb{R}^\ell$. The set Y_{N} is called externally stable if $Y \subseteq Y_{\mathrm{N}} + \mathbb{R}^\ell$.

(b) An ideal point $y^I = (y_1^I, y_2^I, \ldots, y_\ell^I) \in \mathbb{R}^\ell$, associated with the MOP, is defined as $y_i^I = \min_{x \in X} f_i(x),\ i = 1, \ldots, \ell$.

(c) The point $u = (u_1, u_2, \ldots, u_\ell) \in \mathbb{R}^\ell$ in which $u_i = y_i^I - \varepsilon_i$, where $\varepsilon_i > 0$, for every $i = 1, \ldots, \ell$, is called a utopia point of the MOP.

In the following, we give a brief review of the two exiting scalarization approaches, namely the feasible-value constraint and the weighted constraint.

The feasible-value constraint approach. For some $\widehat{x} \in X$, set the weight

$$W(\widehat{x}) := \left\{ w \in \mathbb{R}^\ell \,\middle|\, w_i > 0,\ w_i = \frac{1/f_i(\widehat{x})}{\sum_{j=1}^{\ell} 1/f_j(\widehat{x})}, \quad i = 1, \ldots, \ell \right\}. \tag{2.1}$$

For each fixed $k \in \{1, 2, \ldots, \ell\}$ and for a given $w \in W(\widehat{x})$, the mathematical model of the feasible-value constraint approach is stated as follows (see [5]):

$$\left(P_{\widehat{x}}^k\right) \qquad \begin{cases} \min\ f_k(x) \\ \text{s.t. } w_i f_i(x) \le w_k f_k(\widehat{x}), \quad i = 1, \ldots, \ell,\ i \ne k, \\ \qquad x \in X. \end{cases}$$

Define the feasible and solution set of $(P_{\widehat{x}}^k)$:

$$X_k(\widehat{x}) := \{x \in X :\ w_i f_i(x) \le w_k f_k(\widehat{x}), \quad i \ne k\}, \qquad S_{\widehat{x}}^k := \left\{x \in X \mid x \text{ solves } \left(P_{\widehat{x}}^k\right)\right\}.$$

The main characteristics of the feasible-value-constraint approach are given in the following two main theorems, which can be found in [5].

Theorem 2.3. *Let $\widehat{x} \in X$ and $w \in W(\widehat{x})$.*

- *Suppose that for some k, $\check{x} \in S_{\widehat{x}}^k$. Then $\check{x} \in X_{\mathrm{WE}}$.*
- *If $\check{x}$ is the unique solution of $(P_{\widehat{x}}^k)$, for some k, then $\check{x} \in X_E$.*
- *If $\check{x}$ is an optimal solution of $(P_{\widehat{x}}^k)$ for every $k = 1, \ldots, \ell$, then $\check{x} \in X_{\mathrm{E}}$.*

Theorem 2.4. *Assume $\check{x} \in X_{\mathrm{E}}$ and $w \in W(\widehat{x})$. Then $\check{x} \in S_{\check{x}}^k$, for all $k \in \{1, \ldots, \ell\}$.*

The weighted-constraint approach. For each fixed $k \in \{1, 2, \ldots, \ell\}$, the mathematical model of the weighted-constraint technique is stated as follows (see [4]):

$$\left(P_w^k\right) \qquad \begin{cases} \min\ f_k(x) \\ \text{s.t. } w_i f_i(x) \le w_k f_k(x), \quad i = 1, \ldots, \ell,\ i \ne k, \\ \qquad x \in X. \end{cases}$$

Define the set of positive weights and solution set of (P_w^k) as:

$$W := \left\{ w \in \mathbb{R}^\ell \mid w_i > 0,\ \sum_{i=1}^{\ell} w_i = 1 \right\}, \qquad S_w^k := \left\{x \in X \mid x \text{ solves } \left(P_w^k\right)\right\}.$$

What follows, we state the principal results of the weighted-constraint approach obtained by Burachik *et al.* (see [4]).

Theorem 2.5. *For every $k \in \{1, 2, \ldots, \ell\}$ and some $w \in W$, one has $\check{x} \in S_w^k$ if and only if $\check{x} \in X_{\mathrm{WE}}$.*

We note that the scalarized problems $(P_{\widehat{x}}^k)$ and (P_w^k) do not easily provide results on proper efficiency of optimal solutions. In the following, we present a modification of the scalarized problems $(P_{\widehat{x}}^k)$ and (P_w^k) by adding slack and surplus variables, and derive necessary and sufficient conditions for weak efficient, efficient, and proper efficient solutions of the MOP.

3. The flexible and modified feasible-value constraint problems

By using Ehrgott and Ruzika's idea [10], we propose the following two scalarized problem by adding slack and surplus variables. In [10], the objective functions are bounded from above by a parameter ε. Therefore, improper selection of ε may lead to poor performance of the approach. So, we need an informed choice of ε using the structure of the problem. We allow the added constraints of $(P^k_{\widehat{x}})$ to be violated and then penalize these violations in the objective function of $(P^k_{\widehat{x}})$.

For any fixed $k \in \{1, 2, \ldots, \ell\}$, let $\widehat{x} \in X$ and $w \in W(\widehat{x})$. The modified feasible-value-constraint scalarization problem $(\mathrm{MP}^k_{\widehat{x}})$ is formulated as follows:

$$\left(\mathrm{MP}^k_{\widehat{x}}\right) \qquad \begin{cases} \min\ w_k f_k(x) - \displaystyle\sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i \\ \text{s.t. } w_i(f_i(x) - u_i + s_i) \le w_k(f_k(\widehat{x}) - u_k), \quad i = 1, \ldots, \ell,\ i \neq k, \\ \qquad x \in X,\ s_i \ge 0, \quad i = 1, \ldots, \ell,\ i \neq k, \end{cases}$$

where u is a utopia vector respective to problem $(\mathrm{MP}^k_{\widehat{x}})$, and $\gamma_i,\ i \neq k$, are nonnegative weights. Recently, Hoseinpoor and Ghaznavi [19] proposed a modified objective-constraint scalarization technique and established sufficient conditions for (weakly, properly) efficient solutions of a general multiobjective optimization problem. This scalarizing can be obtained by choosing appropriate values for the scalarized problem $(\mathrm{MP}^k_{\bar{x}})$ parameters $(u = (0, 0, \ldots, 0), w_i s_i = t_i)$. The reference point information of the decision maker is taken into consideration by the $(\mathrm{MP}^k_{\bar{x}})$ method, which is one of the characteristics of the $(\mathrm{MP}^k_{\bar{x}})$ scalarization method.

The flexible feasible-value-constraint scalarization problem $(\mathrm{FP}^k_{\widehat{x}})$ is formulated as follows:

$$\left(\mathrm{FP}^k_{\widehat{x}}\right) \qquad \begin{cases} \min\ w_k f_k(x) + \displaystyle\sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i s_i \\ \text{s.t. } w_i(f_i(x) - u_i - s_i) \le w_k(f_k(\widehat{x}) - u_k), \quad i = 1, \ldots, \ell,\ i \neq k, \\ \qquad x \in X,\ s_i \ge 0, \quad i = 1, \ldots, \ell,\ i \neq k, \end{cases}$$

where u is a utopia vector respective to problem $(\mathrm{FP}^k_{\widehat{x}})$ and $\nu_i,\ i \neq k$, are nonnegative weights.

It is recalled that we denote by $\mathrm{SMP}^k_{\widehat{x}}$ the set of solutions of problem $(\mathrm{MP}^k_{\widehat{x}})$. In the following, we provide results which characterize (proper, weak) efficiency solutions of the MOP utilizing the scalarized problems $(\mathrm{MP}^k_{\widehat{x}})$.

Theorem 3.1. *Fix $\widehat{x} \in X$, and let $w \in W(\widehat{x})$.*

(1) *If $(\check{x}, \check{s}) \in \mathrm{SMP}^k_{\check{x}}$ for some k and $\gamma \geqq 0$, then $\check{x} \in X_{\mathrm{WE}}$.*
(2) *If $(\check{x}, \check{s})$ is a unique optimal solution of $(\mathrm{MP}^k_{\widehat{x}})$ with $\gamma \ge 0$, for some k, then $\check{x} \in X_{\mathrm{SE}}$.*
(3) *If $(\check{x}, \check{s})$ is an optimal solution of $(\mathrm{MP}^k_{\widehat{x}})$ and $\gamma > 0$, then*

$$w_i(f_i(\check{x}) - u_i + \check{s}_i) = w_k(f_k(\widehat{x}) - u_k), \quad i = 1, \ldots, \ell,\ i \neq k.$$

Proof. (1) Suppose that $\check{x} \notin X_{\mathrm{WE}}$. Therefore, there exists $\widetilde{x} \in X$ such that

$$f_j(\widetilde{x}) < f_j(\check{x}), \quad j = 1, \ldots, \ell. \tag{3.1}$$

we break the proof into two cases.

Case I. Let $(\widetilde{x}, \check{s})$ be feasible for $(\mathrm{MP}^k_{\widehat{x}})$. we have

$$w_i(f_i(\widetilde{x}) - u_i + \check{s}_i) \le w_k(f_k(\widehat{x}) - u_k), \quad i = 1, \ldots, \ell,\ i \neq k. \tag{3.2}$$

Since $w_k > 0$, from (3.1) we obtain

$$w_k f_k(\widetilde{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i < w_k f_k(\check{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i. \tag{3.3}$$

Hence, equations (3.2) and (3.3) are contradictory to that $(\check{x}, \check{s}) \in \mathrm{SMP}^k_{\check{x}}$. Therefore, case I cannot happen, and we are left only with Case II.

Case II. Suppose that $(\widetilde{x}, \check{s})$ is not feasible for $(\mathrm{MP}^k_{\widehat{x}})$. Thus, there exists $i \in \{1, 2, \ldots, \ell\}\backslash\{k\}$ such that

$$w_i(f_i(\widetilde{x}) - u_i + \check{s}_i) > w_k(f_k(\widehat{x}) - u_k). \tag{3.4}$$

Since $(\check{x}, \check{s})$ is feasible for $(\mathrm{MP}^k_{\widehat{x}})$, one can see

$$w_i(f_i(\check{x}) - u_i + \check{s}_i) \le w_k(f_k(\widehat{x}) - u_k), \quad i \neq k. \tag{3.5}$$

By (3.4) and (3.5), we conclude

$$w_i(f_i(\widetilde{x}) - u_i + \check{s}_i) > w_i(f_i(\check{x}) - u_i + \check{s}_i), \quad i \neq k.$$

Therefor, for $i \neq k$, we have $f_i(\widetilde{x}) > f_i(\check{x})$. This contradicts (3.1) for $j = i$. Consequently, $\check{x} \in X_{\mathrm{WE}}$.

(2) Let $\check{x}$ and k be as in the hypothesis, and assume x satisfies

$$f_i(x) \le f_i(\check{x}), \quad \text{for all } i \in \{1, 2, \ldots, \ell\}, \tag{3.6}$$

and there exists index r such that

$$f_r(x) < f_r(\check{x}). \tag{3.7}$$

Now, we consider two cases.

Case I ($r = k$)**.** From (3.6) and the fact that $(\check{x}, \check{s})$ is feasible for $(\mathrm{MP}^k_{\widehat{x}})$, we deduce

$$w_i(f_i(x) - u_i + \check{s}_i) \le w_i(f_i(\check{x}) - u_i + \check{s}_i) \le w_k(f_k(\widehat{x}) - u_k), \quad \text{for all } i \neq k \tag{3.8}$$

Therefore, equations (3.7) and (3.8) contradict the optimality of $(\check{x}, \check{s})$ to $(\mathrm{MP}^k_{\widehat{x}})$.

Case II ($r \neq k$)**.** From (3.6) and (3.7), and the fact that $(\check{x}, \check{s})$ is feasible for $(\mathrm{MP}^k_{\widehat{x}})$, one can write

$$w_i(f_i(x) - u_i + \check{s}_i) \le w_i(f_i(\check{x}) - u_i + \check{s}_i) \le w_k(f_k(\widehat{x}) - u_k), \quad i = 1, \ldots, \ell,\ i \neq r, k,$$

and

$$w_r(f_r(x) - u_r + \check{s}_r) < w_r(f_r(\check{x}) - u_r + \check{s}_r) \le w_k(f_k(\widehat{x}) - u_k).$$

Hence, $(x, \check{s})$ is feasible for $(\mathrm{MP}^k_{\widehat{x}})$. Moreover, by (3.6) we obtain

$$w_k f_k(x) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \check{s}_i \le w_k f_k(\check{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \check{s}_i.$$

Since the objective function value of $(x, \check{s})$ is equal to that of $(\check{x}, \check{s})$, uniqueness of the optimal solution leads to $x = \check{x}$. Hence, $\check{x} \in X_{\mathrm{SE}}$.

(3) Suppose to the contrary that $w_{i_0}(f_{i_0}(\check{x}) - u_{i_0} + \check{s}_{i_0}) < w_k(f_k(\widehat{x}) - u_k)$, for some $i_0 \in \{1, 2, \ldots, \ell\}\backslash\{k\}$. Then, there exists a scalar δ_{i_0} such that $\check{s}_{i_0} + \delta_{i_0} > 0$ and $w_{i_0}(f_{i_0}(\check{x}) - u_{i_0} + \check{s}_{i_0} + \delta_{i_0}) \le w_k(f_k(\widehat{x}) - u_k)$. Define

$$\widetilde{s}_i := \begin{cases} \check{s}_i, & i \neq i_0, k, \\ \check{s}_i + \delta_i, & i = i_0. \end{cases}$$

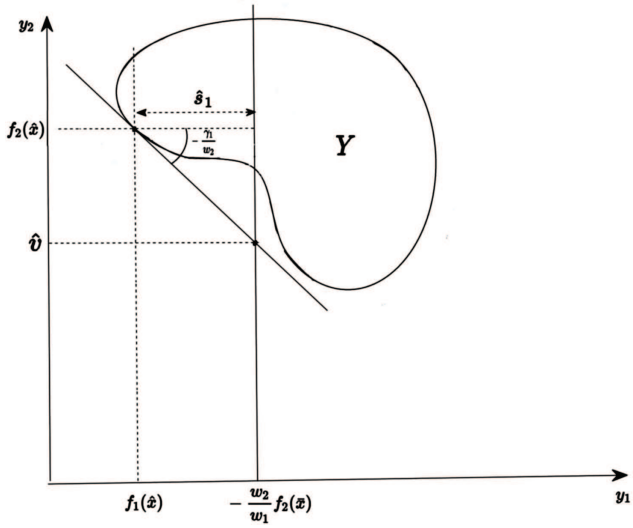

FIGURE 1. Geometrical illustration of the $(\mathrm{MP}^k_{\widehat{x}})$ for a bicriteria example.

Clearly, $(\check{x}, \widetilde{s})$ is feasible for $(\mathrm{MP}^k_{\widehat{x}})$. Since $\gamma > 0$ and $\widetilde{s}_{i_0} > \check{s}_i$, we have

$$w_k f_k(\check{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \widetilde{s}_i < w_k f_k(\check{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \check{s}_i,$$

which contradicts $(\check{x}, \check{s}) \in \mathrm{SMP}^k_{\check{x}}$.

□

Due to part 3 of Theorem 3.1, after introducing the slack variables, the inequalities $(w_i(f_i(x) - u_i + s_i) \le w_k(f_k(\widehat{x}) - u_k), \quad i = 1, \ldots, \ell,\ i \neq k)$, do not change; the inequalities in $(\mathrm{MP}^2_{\widehat{x}})$ are always active at optimality. Before presenting theoretical findings, we first offer a geometric interpretation of the proposed method. part 3 of Theorem 3.1 allows us to visualize the implementations in the objective space by examining the specific case of $l = 2$. The point $(0, 0)$ is chosen as the utopia point, and we consider a given feasible solution (x^*, s^*). Denote the objective function value of $(\mathrm{MP}^2_{\widehat{x}})$ as $d = w_2 f_2(x^*) - \gamma_1 s_1^*$. The level set of $v = \frac{d}{w_2} = f_2(x) - \frac{\gamma_1}{w_2} s_1$ may be represented as a line in the $f_2 - s_1$-space that passes through $(s_1^*, f_2(x^*))$ and has the slope $-\frac{\gamma_1}{w_2}$.

Let $(\hat{x}, \hat{s}_1) \in \mathrm{SMP}^2_{\bar{x}}$ and $\hat{d} = w_2 f_2(\hat{x}) - \gamma_1 \hat{s}_1$ denote the optimal value of $(\mathrm{MP}^2_{\bar{x}})$. Observe that, owing to part 3 of Theorem 3.1, the added constraints hold with equality at optimality, therefore $\hat{s}_1 = \frac{w_2}{w_1} f_2(\bar{x}) - f_1(\hat{x})$. If $\hat{s}_1 = 0$, we have $f_1(\hat{x}) = \frac{w_2}{w_1} f_2(\bar{x})$ and $f_2(\hat{x}) = \hat{v}$. Suppose that $\hat{s}_1 \neq 0$. If we substitute $\hat{s}_1$ into the objective function of $(\mathrm{MP}^2_{\bar{x}})$, we attain

$$-\frac{\gamma_1}{w_2} = \frac{f_2(\hat{x}) - \hat{v}}{f_1(\hat{x}) - \frac{w_2}{w_1} f_2(\bar{x})}.$$

The negative slope of the line through $(f_1(\hat{x}), f_2(\hat{x}))$ and $(\frac{w_2}{w_1} f_2(\bar{x}), \hat{v})$ is equal to the scalar $\frac{\gamma_1}{w_2}$. View Figure 1 to see this observation in visual form. The additional constraint $w_1 f_1(x) \le w_2 f_2(\bar{x})$, reduces the feasible set of the MOP. As a result, a line with the slope $-\frac{\gamma_1}{w_2}$ is transformed parallel to the origin until it supports the restricted nondominated set. Hence, the point of support is the nondominated solution $f(\hat{x})$.

In the following, by utilizing the proof of the part 3 of Theorem 3.1, we establish a sufficient condition for efficient and proper efficient solutions of the MOP. As the proof of the following theorem is similar to [10], the proof of parts 1 and 2 are omitted.

Theorem 3.2. *Fix $\widehat{x} \in X$ and let $w \in W(\widehat{x})$.*

(1) *If $(\check{x}, \check{s}) \in \mathrm{SMP}^k_{\check{x}}$ and $\gamma > 0$, then $\check{x} \in X_{\mathrm{E}}$.*
(2) *If $\check{x}$ is a proper efficient solution of the problem*

$$\begin{cases} \min\ f(x) := (f_1(x), \dots, f_\ell(x)) \\ \text{s.t. } x \in X_k(\widehat{x}), \\ \qquad w_i f_i(\check{x}) < w_k f_k(\widehat{x}), \quad i = 1, \dots, \ell,\ i \neq k, \end{cases}$$

then $\check{x} \in X_{\mathrm{PE}}$.
(3) *If $(\check{x}, \check{s}) \in \mathrm{SMP}^k_{\check{x}}$ and $(\gamma, \check{s}) > 0$, then $\check{x} \in X_{\mathrm{PE}}$.*

Proof. (1) The proof is similar to the Theorem 3.1 in [10].

(2) The proof is similar to that of Lemma 3.2 in [10].
(3) From part 1 we have $\check{x} \in X_{\mathrm{E}}$. By part 3 of Theorem 3.1, we can rewrite the objective function as

$$w_k f_k(\check{x}) - \sum_{\substack{1 \le i \le \ell \\ i \neq k}} \gamma_i \left(\frac{w_k}{w_i}(f_k(\widehat{x}) - u_k) - (f_i(\check{x}) - u_i) \right) = \sum_{i=1}^{\ell} \gamma_i f_i(\check{x}) - \sum_{\substack{1 \le i \le \ell \\ i \neq k}} \gamma_i \left(\frac{w_k}{w_i}(f_k(\widehat{x}) - u_k) + u_i \right),$$

in which $\gamma_k := w_k$. Since the term

$$\sum_{\substack{1 \le i \le \ell \\ i \neq k}} \gamma_i \left(\frac{w_k}{w_i}(f_k(\widehat{x}) - u_k) + u_i \right),$$

is constant, we conclude

$$\check{x} \in \arg\min \left\{ \sum_{i=1}^{\ell} \gamma_i f_i(x) \mid x \in X_k(\widehat{x}) \right\},$$

with $\gamma > 0$. Using Geoffrion's theorem [20], $\check{x}$ is proper efficient for the MOP with feasible set $X_k(\widehat{x})$. In view of part 2, $\check{x}$ is a proper efficient solution of the MOP with feasible set X.

□

The next theorem shows that, if Y_{N} is not externally stable, the result of part 3 of Theorem 3.2 is no longer true.

Theorem 3.3. *If Y_{N} in problem $(\mathrm{MP}^k_{\widehat{x}})$ is not externally stable, then the MOP does not have any proper efficient solution.*

Proof. The proof can be found in [31]. □

In the next theorem, we present a necessary condition for proper efficient solutions of the MOP.

Theorem 3.4. (1) *Let $\check{x} \in X_{\mathrm{E}}$ and $w \in W(\widehat{x})$. Then we conclude $(\check{s}, \gamma) \geqq 0$ such that for every $k \in \{1, 2, \dots, \ell\}$, $(\check{x}, \check{s}) \in \mathrm{SMP}^k_{\check{x}}$.*
(2) *Let $\check{x} \in X_{\mathrm{PE}}$ and $w \in W(\check{x})$. Then there exist $\gamma > 0$ and $\check{s}$ such that for every $k \in \{1, 2, \dots, \ell\}$, $(\check{x}, 0) \in \mathrm{SMP}^k_{\check{x}}$.*

Proof. (1) It is sufficient to set $\check{s} = 0$ and $\gamma = 0$. Therefore, by Theorem 2.4 $(\check{x}, 0) \in \mathrm{SMP}^k_{\check{x}}$.

(2) Let $\check{x} \in X_{\mathrm{PE}}$, $w \in W(\check{x})$ and $u = 0$.
Case I. Suppose that there is not $i \neq k$ and $x \in X$ such that $w_i f_i(x) < w_k f_k(\check{x})$. Let $x \in X_k(\check{x})$. Hence, $w_i f_i(x) = w_k f_k(\check{x})$, for all $i \neq k$ and, since $\check{x} \in X_{\mathrm{E}}$, $w_k f_k(x) \geq w_k f_k(\check{x})$. It follows that (x, s) does not yield a better objective function value for $(\mathrm{MP}^k_{\check{x}})$ for any choice of $\gamma > 0$.

TABLE 1. Summary of the modified feasible-value-constraint approach ($\mathrm{MP}^k_{\widehat{x}}$).

Reference	Parameters & conditions	Implication for $\check{x}$
Part 1 of Theorem 3.1	$\gamma \geqq 0$	$\check{x} \in X_{\mathrm{WE}}$
Part 2 of Theorem 3.1	$\gamma \geqq 0$ & (x, s) is unique	$\check{x} \in X_{\mathrm{SE}}$
Part 1 of Theorem 3.2	$\gamma > 0$	$\check{x} \in X_{\mathrm{E}}$
Part 3 of Theorem 3.2	$(\gamma, \check{s}) > 0$	$\check{x} \in X_{pE}$
Reference	Type of solution	Optimality for ($\mathrm{MP}^k_{\widehat{x}}$)
Part 1 of Theorem 3.4	$\check{x} \in X_{\mathrm{E}}$ & Y_{N} be externally stable	$\exists(\gamma, u)$ s.t. $(\check{x}, 0) \in \mathrm{SMP}^k_{\check{x}}$
Part 2 of Theorem 3.4	$\check{x} \in X_{pE}$ & Y_{N} be externally stable	$\exists(\gamma, \check{s}, u)$ s.t. $(\check{x}, \check{s}) \in \mathrm{SMP}^k_{\check{x}}$

Case II. Assume that there exists $i \neq k$ and $x \in X$ such that $w_i f_i(x) < w_k f_k(\check{x})$. Let $x \in X_k(\check{x})$, then for all $i \in \{1, 2, \ldots, \ell\} \backslash \{k\}$, one can observe $w_i f_i(x) < w_k f_k(\check{x})$. since $x \in X_{\mathrm{PE}}$, there exists a positive scalar M such that for each $x \in X_k(\check{x})$ and i with $w_i f_i(x) < w_k f_k(\check{x})$, we obtain $w_k f_k(x) > w_k f_k(\check{x})$ and

$$\frac{w_k f_k(\check{x}) - w_i f_i(x)}{w_k f_k(x) - w_k f_k(\check{x})} \leq M.$$

Define weights $\gamma_i := \frac{w_i}{M(p-1)}$. Then $\gamma > 0$. If $(\check{x}, 0)$ is not an optimal solution for ($\mathrm{MP}^k_{\check{x}}$), there exist a feasible (x, s) with

$$w_k f_k(x) - \sum_{\substack{1 \leq i \leq \ell \\ i \neq k}} \gamma_i s_i < w_k f_k(\check{x}) - \sum_{\substack{1 \leq i \leq \ell \\ i \neq k}} \gamma_i \check{s}_i = w_k f_k(\check{x}). \tag{3.9}$$

Due to part 3 of Theorem 3.1,

$$w_k(f_k(x) - f_k(\check{x})) < \sum_{\substack{1 \leq i \leq \ell \\ i \neq k}} \gamma_i \left(\frac{w_k}{w_i} f_k(\check{x}) - f_i(x) \right). \tag{3.10}$$

We have

$$\gamma_i = \frac{w_i}{M(p-1)} \leq \frac{w_k}{p-1} \frac{f_k(x) - f_k(\check{x})}{f_k(\check{x}) - f_i(x)}, \qquad i \neq k.$$

Consequently, one can write

$$\gamma_i \left(\frac{w_k}{w_i} f_k(\check{x}) - f_i(x) \right) \leq \frac{w_k}{p-1} (f_k(x) - f_k(\check{x})), \qquad i \neq k. \tag{3.11}$$

Summing (3.11) over all $i \in \{1, 2, \ldots, n\} \backslash \{k\}$ yields

$$\sum_{\substack{1 \leq i \leq \ell \\ i \neq k}} \gamma_i \left(\frac{w_k}{w_i} f_k(\check{x}) - f_i(x) \right) \leq w_k(f_k(x) - f_k(\check{x})). \tag{3.12}$$

Therefore, equation (3.10) contradicts (3.12).

□

In Table 1, we summarize some of the results obtained for the scalarized problem ($\mathrm{MP}^k_{\widehat{x}}$).

3.1. The revised feasible-value constraint problem

Using the idea of Ehrgott and Ruzika [10], we propose the following scalarized problem by a combination of $(\mathrm{MP}^k_{\check{x}})$ and $(\mathrm{FP}^k_{\check{x}})$, which is considered as follows.

For any fixed $k \in \{1, 2, \ldots, \ell\}$, let $\check{x} \in X$ and $w \in W(\check{x})$. The Revised feasible-value constraint problem $(\mathrm{RP}^k_{\check{x}})$ is stated as follows:

$$\left(\mathrm{RP}^k_{\check{x}}\right) \qquad \begin{cases} \min\ w_k f_k(x) - \sum\limits_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i^+ + \sum\limits_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i s_i^- \\ \text{s.t. } w_i\big(f_i(x) - u_i + s_i^+ - s_i^-\big) \le w_k(f_k(\check{x}) - u_k), \quad i \neq k, \\ \qquad x \in X,\ s_i^+, s_i^- \ge 0, \quad i \neq k, \end{cases}$$

in which u is a utopia vector respective to problem $(\mathrm{RP}^k_{\check{x}})$, and $\nu_i, \gamma_i,\ i \neq k$, are nonnegative weights.

Based on the work of Ehrgott and Ruzika [10], it should be noted that if there exist some $w \in W(\check{x})$ for a given feasible solution (x, s^+, s^-) of $(\mathrm{RP}^k_{\check{x}})$, $(x, s^+ + \alpha, s^- + \alpha)$ is also feasible for $(\mathrm{RP}^k_{\check{x}})$, where $\alpha \in \mathbb{R}^p_{\geqq}$. Then,

$$w_k f_k(x) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i\big(s_i^+ + \alpha_i\big) + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i\big(s_i^- + \alpha_i\big) = w_k f_k(x) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i s_i^- + \sum_{\substack{1\le i\le \ell\\ i\neq k}} (\nu_i - \gamma_i)\alpha_i.$$

This shows that the objective function value depends on $\alpha \in \mathbb{R}^\ell_{\geqq}$. Here, α can be chosen arbitrarily in $\mathbb{R}^\ell_{\geqq}$. If $\nu < \gamma$, then the objective function value of $(\mathrm{RP}^k_{\check{x}})$ is unbounded. In this respect, in what follows, we assume $\nu \geqq \gamma$.

We recall that the set of solutions of problem $(\mathrm{RP}^k_{\widehat{x}})$ is denoted by $\mathrm{SRP}^k_{\check{x}}$. Below, we provide results characterizing (proper, weak) efficiency solutions of the MOP utilizing the scalarized problem $(\mathrm{RP}^k_{\check{x}})$.

Theorem 3.5. *Let $\check{x} \in X$ and $w \in W(\check{x})$.*

(1) *Suppose that for some k, $(\widehat{x}, \widehat{s}^+, \widehat{s}^-) \in \mathrm{SRP}^k_{\check{x}}$ with $\gamma \geqq 0$. Then $\widehat{x} \in X_{\mathrm{WE}}$.*
(2) *If $(\widehat{x}, \widehat{s}^+, \widehat{s}^-)$ is the unique solution of $(\mathrm{RP}^k_{\check{x}})$ with $\gamma \geqq 0$ for some k, then $\check{x} \in X_{\mathrm{SE}}$.*
(3) *If $(\widehat{x}, \widehat{s}^+, \widehat{s}^-) \in \mathrm{SRP}^k_{\check{x}}$ and $\gamma > 0$, then*

$$w_i\big(f_i(\widehat{x}) - u_i + \widehat{s}_i^+ - \widehat{s}_i^-\big) = w_k(f_k(\check{x}) - u_k).$$

Proof. (1) The proof follows from the case 1 of Theorem 3.1.

(2) The proof is similar to case 2 of Theorem 3.1.
(3) Suppose to the contrary that $w_{i_0}(f_{i_0}(\widehat{x}) - u_{i_0} + \widehat{s}_{i_0}^+ - \widehat{s}_{i_0}^-) < w_k(f_k(\check{x}) - u_k)$ for some $i_0 \in \{1, 2, \ldots, \ell\}\backslash\{k\}$. Then, there exists a scalar δ_{i_0} such that $\widehat{s}_{i_0}^+ + \delta_{i_0} > 0$ and $w_{i_0}(f_{i_0}(\widehat{x}) - u_{i_0} + \widehat{s}_{i_0}^+ \delta_{i_0} - \widehat{s}_{i_0}^-) \le w_k(f_k(\check{x}) - u_k)$. Define

$$\widetilde{s}_i^+ := \begin{cases} \widehat{s}_i^+, & i \neq i_0, k, \\ \widehat{s}_i^+ + \delta_i, & i = i_0. \end{cases}$$

$(\widehat{x}, \widetilde{s}^+, \widehat{s}^-)$ is feasible for $(\mathrm{RP}^k_{\check{x}})$. Since $\gamma > 0$ and $\widetilde{s}_{i_0}^+ > \widehat{s}_i^+$, we have

$$\begin{aligned} w_k f_k(\widehat{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \widehat{s}_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i \widehat{s}_i^- &= w_k f_k(\widehat{x}) - \sum_{\substack{i\neq k\\ i\neq i_0}} \gamma_i \widetilde{s}_i^+ - \gamma_{i_0} \widetilde{s}_{i_0}^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i \widehat{s}_i^- \\ &= w_k f_k(\widehat{x}) - \sum_{\substack{i\neq k\\ i\neq i_0}} \gamma_i \widehat{s}_i^+ - \gamma_{i_0}\big(\widehat{s}_{i_0}^+ + \delta_{i_0}\big) + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i \widehat{s}_i^- \end{aligned}$$

$$= w_k f_k(\widehat{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \widehat{s_i^+} + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i \widehat{s_i^-} - \gamma_{i_0}\delta_{i_0}$$
$$< w_k f_k(\widehat{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \widehat{s_i^+} + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i \widehat{s_i^-}.$$

which contradicts with $(\widehat{x}, \widehat{s}^+, \widehat{s}^-) \in \mathrm{SRP}^k_{\check{x}}$.

□

In the following, utilizing the proof of Theorem 3.5, we establish a sufficient condition for proper efficient solutions of the MOP.

Theorem 3.6. *Let $\check{x} \in X$ and $w \in W(\widehat{x})$.*

(1) *If $(\widehat{x}, \widehat{s}^+, \widehat{s}^-) \in \mathrm{SRP}^k_{\check{x}}$ and $\gamma > 0$, then $\widehat{x} \in X_{\mathrm{E}}$*
(2) *If $(\widehat{x}, \widehat{s}^+, \widehat{s}^-)$ is an optimal solution of $(\mathrm{RP}^k_{\check{x}})$ and $\gamma > 0$, then $\widehat{x} \in X_{\mathrm{PE}}$*

Proof. (1) The proof follows from Theorem 3.5 and part 2 of Theorem 3.1.
(2) The proof is similar to that of Theorem 5.1 in [10], so we omit it here.

□

Similar to Theorem 3.4, we present a necessary condition for proper efficient solutions of the MOP.

Theorem 3.7. *Let $w \in W(\widehat{x})$.*

(1) *Let $\widehat{x} \in X_{\mathrm{E}}$. Then exist $(\gamma, \nu) \geqq 0$ such that $(\widehat{x}, \widehat{s}^+ = 0, \widehat{s}^- = 0) \in \mathrm{SRP}^k_{\check{x}}$ for every $k \in \{1, 2, \ldots, \ell\}$.*
(2) *Let Y_{N} be externally stable and $\widehat{x} \in X_{\mathrm{PE}}$. Then there exist $(\gamma, \nu) \geqq 0$ such that $(\widehat{x}, \widehat{s}^+ = 0, \widehat{s}^- = 0) \in \mathrm{SRP}^k_{\check{x}}$.*

Proof. The proof follows from of Theorem 3.4. □

3.2. Relationship between the $(\mathrm{RP}^k_{\widehat{x}})$ and $(\mathrm{RP}^k(\varepsilon))$

Ehrgott and Ruzika [10] proposed the following scalar optimization problem, called the Improved ε-Constraint method, for determining efficient solutions of the MOP:

$$\Big(\mathrm{RP}^k(\varepsilon)\Big) \qquad \begin{cases} \min\ f_k(x) - \sum\limits_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i^+ + \sum\limits_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i s_i^- \\ \text{s.t. } f_i(x) + s_i^+ - s_i^- \le \varepsilon_i, \quad i \neq k, \\ \qquad s_i^+, s_i^- \ge 0, \quad i \neq k,\ x \in X, \end{cases}$$

where $\nu_i, \gamma_i \ge 0,\ i \neq k$.

The following theorem establishes the relationship between the $(\mathrm{RP}^k_{\widehat{x}})$ and $(\mathrm{RP}^k(\varepsilon))$. Recall that we denote the set of solutions of Problem $(\mathrm{RP}^k(\varepsilon))$ by $\mathrm{SRP}^k(\varepsilon)$.

Theorem 3.8. *Fix $\check{x} \in X$ and $k \in \{1, 2, \ldots, \ell\}$. Let $\widehat{x} \in X$ and set $\varepsilon := f(\check{x})$. Then $\mathrm{SRP}^k(\varepsilon) = \mathrm{SRP}^k_{\check{x}}$.*

Proof. Assume that w is as in (2.1). We begin by showing that $\mathrm{SRP}^k(\varepsilon) \subseteq \mathrm{SRP}^k_{\check{x}}$. Let $(\widehat{x}, \widehat{s}^+, \widehat{s}^-) \in \mathrm{SRP}^k(\varepsilon)$, then it satisfies

$$f_k(\widehat{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \widehat{s_i^+} + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i \widehat{s_i^-} \le f_k(x) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i s_i^-, \tag{3.13}$$

and, since $\varepsilon := f(\check{x})$, one can write $f_i(\widehat{x}) + \widehat{s_i^+} - \widehat{s_i^-} \le f(\check{x})$, $i \neq k$.

Suppose by contradiction that $(\widehat{x}, \widehat{s}^+, \widehat{s}^-) \notin \mathrm{SRP}^k(\varepsilon)$. Then there exist (x, s^+, s^-) such that

$$w_k f_k(x) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i s_i^- < w_k f_k(\widehat{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \widehat{s}_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i \widehat{s}_i^-,$$

and

$$w_i\big(f_i(x) + s_i^+ - s_i^-\big) \le w_k f_k(\check{x}) = w_i f_i(\check{x}), \quad i \neq k.$$

Since $w_i > 0$, for all $i = 1, \dots, \ell$

$$f_k(x) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \frac{\gamma_i}{w_k} s_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \frac{\nu_i}{w_k} s_i^- \le f_k(\widehat{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \frac{\gamma_i}{w_k} \widehat{s}_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \frac{\nu_i}{w_k} \widehat{s}_i^-, \tag{3.14}$$

and

$$f_i(x) + s_i^+ - s_i^- \le f_i(\check{x}), \quad i \neq k.$$

Therefore, equation (3.13) contradicts (3.14). Hence $(\widehat{x}, \widehat{s}^+, \widehat{s}^-) \in \mathrm{SRP}^k(\varepsilon)$. Let us now show that $\mathrm{SRP}^k_{\check{x}} \subseteq \mathrm{SRP}^k(\varepsilon)$. Let $(\widehat{x}, \widehat{s}^+, \widehat{s}^-) \in \mathrm{SRP}^k(\varepsilon)$. By hypothesis for all $x \in X$ and $s_i^+, s_i^- \ge 0,\ i \neq k$, we obtain

$$w_k f_k(\widehat{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \widehat{s}_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i \widehat{s}_i^- \le w_k f_k(x) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i s_i^-,$$

and by definition of w, we also have that

$$w_i\big(f_i(x) + s_i^+ - s_i^-\big) \le w_k f_k(\check{x}) = w_i f_i(\check{x}), \quad i \neq k.$$

Since $w_i > 0$, for all $i = 1, \dots, \ell$

$$f_k(\widehat{x}) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i \widehat{s}_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i \widehat{s}_i^- \le f_k(x) - \sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i^+ + \sum_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i s_i^-,$$

such that for all x,

$$f_i(x) + s_i^+ - s_i^- \le f_i(\check{x}), \quad i \neq k.$$

So, we conclude $x \in \mathrm{SRP}^k(\varepsilon)$. □

4. The revised weighted-constraint problem

Burachik *et al.* [4] proposed the following scalarization method, named the weighted-constraint approach

$$\big(P_w^k\big) \qquad \begin{cases} \min\ f_k(x) \\ \text{s.t. } w_i f_i(x) \le w_k f_k(x), \quad i = 1, \dots, \ell, i \neq k, \\ \qquad x \in X. \end{cases}$$

The scalarized problem (P_w^k) does not easily provide results on (proper)efficiency of optimal solutions. In the following, motivated by the idea of Ehrgott and Ruzika [10], we allow the added constraints of (P_w^k) to be violated, and then penalize these violations in the objective function of (P_w^k).

The modified weighted-constraint approach is formulated as follows:

$$\big(\mathrm{MP}_w^k\big) \qquad \begin{cases} \min\ w_k f_k(x) - \displaystyle\sum_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i \\ \text{s.t. } w_i(f_i(x) + s_i) \le w_k f_k(x), \quad i = 1, \dots, \ell,\ i \neq k, \\ \qquad x \in X,\ s_i \ge 0, \quad i = 1, \dots, \ell,\ i \neq k, \end{cases}$$

in which $\gamma_i,\ i=1,\dots,\ell,\ i\neq k$, are nonnegative weights.

The flexible weighted-constraint approach is formulated as follows:

$$\left(\mathrm{FP}_w^k\right)\qquad \begin{cases} \min\ w_k f_k(x)+\sum\limits_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i s_i \\ \text{s.t. } w_i(f_i(x)-s_i)\le w_k f_k(x), \quad i=1,\dots,\ell,\ i\neq k,\\ \qquad x\in X,\ s_i\ge 0, \quad i=1,\dots,\ell,\ i\neq k, \end{cases}$$

in which $\nu_i\ge 0,\ i=1,\dots,\ell,\ i\neq k$.

In the present research, we purpose a combination of the two modifications (MP_w^k) and (FP_w^k) to generate proper efficient solutions of the MOP.

$$\left(\mathrm{RP}_w^k\right)\qquad \begin{cases} \min\ w_k f_k(x)-\sum\limits_{\substack{1\le i\le \ell\\ i\neq k}} \gamma_i s_i^+ +\sum\limits_{\substack{1\le i\le \ell\\ i\neq k}} \nu_i s_i^- \\ \text{s.t. } w_i(f_i(x)+s_i^+-s_i^-)\le w_k f_k(x), \quad i=1,\dots,\ell,\ i\neq k,\\ \qquad x\in X,\ s_i^+,s_i^-\ge 0, \quad i=1,\dots,\ell,\ i\neq k, \end{cases}$$

in which $\nu_i,\gamma_i\ge 0,\ i=1,\dots,\ell,\ i\neq k$.

In the following, we provide results characterizing (proper, weak) efficiency solutions of the MOP utilizing the scalarized problem (RP_w^k).

Theorem 4.1. *Assume that there exists some $w\in W$ such that $(\widehat{x},\widehat{s}^+,\widehat{s}^-)$ is a solution of (RP_w^k).*

(1) *If $\gamma>0$, then*
$$w_i(f_i(\widehat{x})+\widehat{s}_i^+-\widehat{s}_i^-)=w_k f_k(\widehat{x}).$$
(2) *Suppose that for some k, $(\widehat{x},\widehat{s}^+,\widehat{s}^-)$ is an optimal solution of (RP_w^k) with $\gamma\geqq 0$. Then $\widehat{x}\in X_{\mathrm{WE}}$.*
(3) *If $(\widehat{x},\widehat{s}^+,\widehat{s}^-)$ is the unique solution of (RP_w^k) with $\gamma\geqq 0$, for some k, then $\widehat{x}\in X_{\mathrm{SE}}$.*
(4) *If $\gamma>0$, then $\widehat{x}\in X_{\mathrm{E}}$.*

Proof. The proof is similar to Theorem 3.1. So we omit it here. □

In the following, utilizing the proof of Theorem 4.1, we establish a sufficient condition for proper efficient solutions of the MOP.

Theorem 4.2. *If there is some $w\in W$ such that $(\widehat{x},\widehat{s}^+,\widehat{s}^-)$ be a solution of (RP_w^k) with $\gamma>0$ and there exists a partition $J\cap\check{J}$ of $\{1,2,\dots,\ell\}\backslash\{k\}$ such that $\widehat{s}_i^+>0$ for $i\in J$, and $\widehat{s}_i^->0$ for $i\in\check{J}$, then $\widehat{x}\in X_{\mathrm{PE}}$.*

Proof. The proof is similar to the Theorem 5.2 in [10] and will be omitted here. □

At the end of this section, similar to Theorems 3.4 and 3.7, we present a necessary condition for proper efficient solutions of the MOP.

Theorem 4.3. *Assume that there exist some $w\in W$.*

(1) *If $\widehat{x}\in X_{\mathrm{E}}$, then there exist $(\gamma,\nu)\geqq 0$, and $\widehat{s}^+,\widehat{s}^-$ such that $(\widehat{x},\widehat{s}^+,\widehat{s}^-)$ is an optimal solution of (RP_w^k), for all $k\in\{1,2,\dots,\ell\}$.*
(2) *If $\widehat{x}\in X_{\mathrm{PE}}$, then there exist $\nu>0$, $0<\gamma<\infty$ and $\widehat{s}^+,\widehat{s}^-$ such that $(\widehat{x},\widehat{s}^+,\widehat{s}^-)$ is an optimal solution of (RP_w^k), for every $k\in\{1,2,\dots,\ell\}$.*

5. Numerical results

In this section, we demonstrate the efficiency of the proposed approaches through the results of some numerical experiments. To this end, we divide this section into three experiments. All experiments have been implemented in MATLAB (R2017b). MATLAB's optimization solver fmincon has been used, with default options, for solving the associated nonlinear problems. All numerical tests have been performed on PC Intel Core i7 2700 k CPU 3.4 GHz and 8 GB of RAM.

For comparison, Algorithms 1 and 2 in [5] are utilized, which implements the $(P_{\bar{x}}^k)$ and (P_w^k), respectively. In addition, for the proposed approaches, the same steps as 1–5 of Algorithms 1 and 2 in [5] are done, except that in step 4 of those, our proposed approaches are solved instead of $(P_{\bar{x}}^k)$ and (P_w^k).

Example 5.1. We consider the following nonconvex three objective problem which has a Pareto front with a boundary that is difficult to construct [5].

$$\min\ (x_1, x_2, x_3)$$
$$\text{s.t.}\quad \begin{cases} (x_1-2)^2+(x_2-2)^2+(x_3-2)^2 \le 4, \\ (1.05)^2(x_1+x_2+x_3)^2 - 4(x_1^2+x_2^2+x_3^2) + 7.18 \le 0, \\ 0 \le x_i \le 4,\ i=1,2,3. \end{cases}$$

For the algorithms that implement $(\mathrm{MP}_{\bar{x}}^k)((\mathrm{MP}_w^k))$ and $(\mathrm{FP}_{\bar{x}}^k)$ $((\mathrm{FP}_w^k))$, the nonnegative weights γ_i for $i = 1,2,3$ are selected as random. The point $(-10,-10,-10)$ is taken as the utopia point. The algorithms are implemented with $N = 21$ $(N = 35)$. Figure 2 shows the Pareto points generated by our proposed approaches applied to the given algorithms are extremely successful compared to Figures 5a and 5c of [5]. All of the outer and inner end points of the Pareto front are generated among these produced points, as shown in the figure, and the provided approaches generate Pareto points that are distributed rather evenly in the approximation of the Pareto front. The distribution of points obtained by our approaches and the proposed method in [5] is depicted in Figure 2.

Example 5.2. The original problem in [22] has earlier been studied in [11]. The Pareto front of the following test problem is non-convex and gaps appear in the boundary of the Pareto front.

$$\min\ \left(-x_1, -x_2, -x_3^2\right)$$
$$\text{s.t.}\quad \begin{cases} -\cos(x_1) - \exp(-x_2) + x_3 \le 0, \\ 0 \le x_1 \le \pi, \\ x_2 \ge 0, \\ x_3 \ge 1.2. \end{cases}$$

The point $(-100,-100,-100)$ is taken as the utopia point and we set $N = 20$ $(N = 30)$. Uniformly distributed weights are provided and the resulting Pareto points are shown in Figure 3. Note that we plot again the negative objective function values.

5.1. Application to a liquid-rocket injector design

The liquid rocket single element injector design problem was previously studied as a multiobjective optimization problem in [5, 18]. This is a computationally expensive engineering design problem, involving design of a hybrid Boeing element injector. This problem has two primary goals: (i) the improvement of the performance (by minimizing combustion length), (ii) material sustainability (by minimizing face temperature). For the rocket injector design problem, four design variables are defined in [18], which are listed below:

α : the hydrogen flow angle,
ΔHA: the hydrogen area,

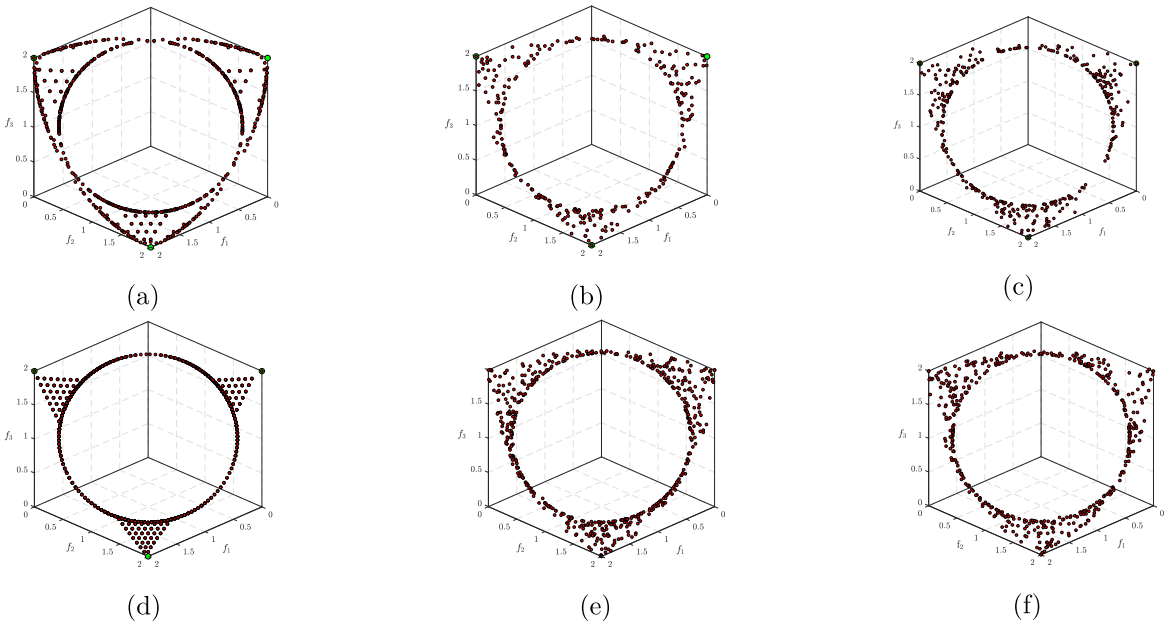

FIGURE 2. Example 5.1: Pareto front approximations by the scalarization techniques, with $N = 21(N = 35)$. (a) points found by $(p_{\tilde{x}}^k)$, (b) points found by $(Fp_{\tilde{x}}^k)$, (c) points found by $(Mp_{\tilde{x}}^k)$, (d) points found by (P_w^k), (e) points found by (FP_w^k), (f) points found by (MP_w^k).

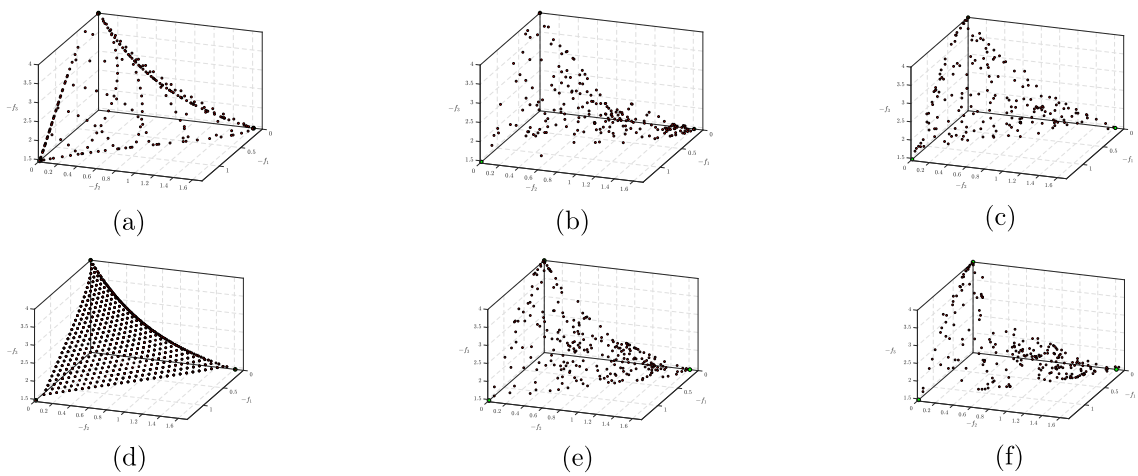

FIGURE 3. Example 5.1: Pareto front approximations by the scalarization techniques, with $N = 21(35)$. (a) points found by $(p_{\tilde{x}}^k)$, (b) points found by $(Fp_{\tilde{x}}^k)$, (c) points found by $(Mp_{\tilde{x}}^k)$, (d) points found by (P_w^k), (e) points found by (FP_w^k), (f) points found by (MP_w^k).

TABLE 2. Range of design variables (α is an acute angle in degrees and x is the thickness of OPTT in inches).

	Minimum		Maximum	
Variable	Actual	Normalized	Actual	Normalized
α	0	0	α	1
ΔHA	Baseline	0	Baseline +25%	1
ΔOA	Baseline −40%	0	Baseline	1
OPTT	x in.	0	$2x$ in.	1

ΔOA: the oxygen area,
OPTT: the oxidizer post tip thickness.

The variables range considered in the mathematical model of the rocket injector design problem are shown in Table 2. Note that the variables of the problem have been normalized. According to Goel *et al.* [18], the four objective functions to be considered for a rocket injector design problem consist of:

$\mathrm{TF}_{\max}$: the face temperature, which is the maximum temperature of the injector face,
$\mathrm{TT}_{\max}$: the tip temperature, which is the maximum temperature on the post tip of the injector,
X_{CC}: the combustion length, which is the distance from the inlet where 99% of the combustion is complete,
TW_4: the wall temperature, which is the wall temperature at three inches (at the fourth probe) from the injector face.

The rocket injector design problem can be written as [18]:

$$\begin{aligned} &\min\ f(x) := (f_1(x), f_2(x), f_3(x), f_4(x)) \\ &\text{s.t.}\ \ x \in X = \left\{x \in \mathbb{R}^4 \mid 0 \le x \le 1\right\}, \end{aligned}$$

where

$$\begin{aligned} x &= (\alpha, \Delta HA, \Delta OA, OPTT) \\ (f_1(x), f_2(x), f_3(x), f_4(x)) &= (\mathrm{TF}_{\max}(x), \mathrm{TT}_{\max}(x), X_{\mathrm{CC}}(x), \mathrm{TW}_4(x)) \end{aligned}$$

$$\begin{aligned} f_1(x) = {} & 0.692 + 0.477x_1 - 0.687x_2 - 0.080x_3 - 0.0650x_4 - 0.167x_1^2 - 0.0129x_1x_2 \\ & + 0.0796x_2^2 - 0.0634x_1x_3 - 0.0257x_2x_3 + 0.0877x_3^2 - 0.0521x_1x_4 \\ & + 0.00156x_2x_4 + 0.00198x_3x_4 + 0.0184x_4^2, \\ f_2(x) = {} & 0.370 - 0.205x_1 + 0.0307x_2 + 0.108x_3 + 1.019x_4 - 0.135x_1^2 + 0.0141x_1x_2 \\ & + 0.0998x_2^2 + 0.208x_1x_3 - 0.0301x_2x_3 - 0.226x_3^2 + 0.353x_1x_4 - 0.0497x_3x_4 \\ & - 0.423x_4^2 + 0.202x_1^2x_2 - 0.281x_1^2x_3 - 0.342x_2^2x_1 - 0.245x_2^2x_3 + 0.281x_3^2x_2 \\ & \quad 0.184x_1x_4^2 - 0.281x_1x_2x_3, \\ f_3(x) = {} & 0.153 - 0.322x_1 + 0.396x_2 + 0.424x_3 + 0.0226x_4 + 0.175x_1^2 + 0.0185x_1x_2 \\ & - 0.0701x_2^2 - 0.251x_1x_3 + 0.179x_2x_3 + 0.0150x_3^2 + 0.0134x_1x_4 + 0.0296x_2x_4 \\ & + 0.0752x_3x_4 + 0.0192x_4^2, \\ f_4(x) = {} & 0.758 + 0.358x_1 - 0.807x_2 + 0.0925x_3 - 0.0468x_4 - 0.172x_1^2 + 0.0106x_1x_2 \\ & + 0.0697x_2^2 - 0.146x_1x_3 - 0.0416x_2x_3 + 0.102x_3^2 - 0.0694x_1x_4 \\ & - 0.00503x_2x_4 + 0.0151x_3x_4 + 0.0173x_4^2. \end{aligned}$$

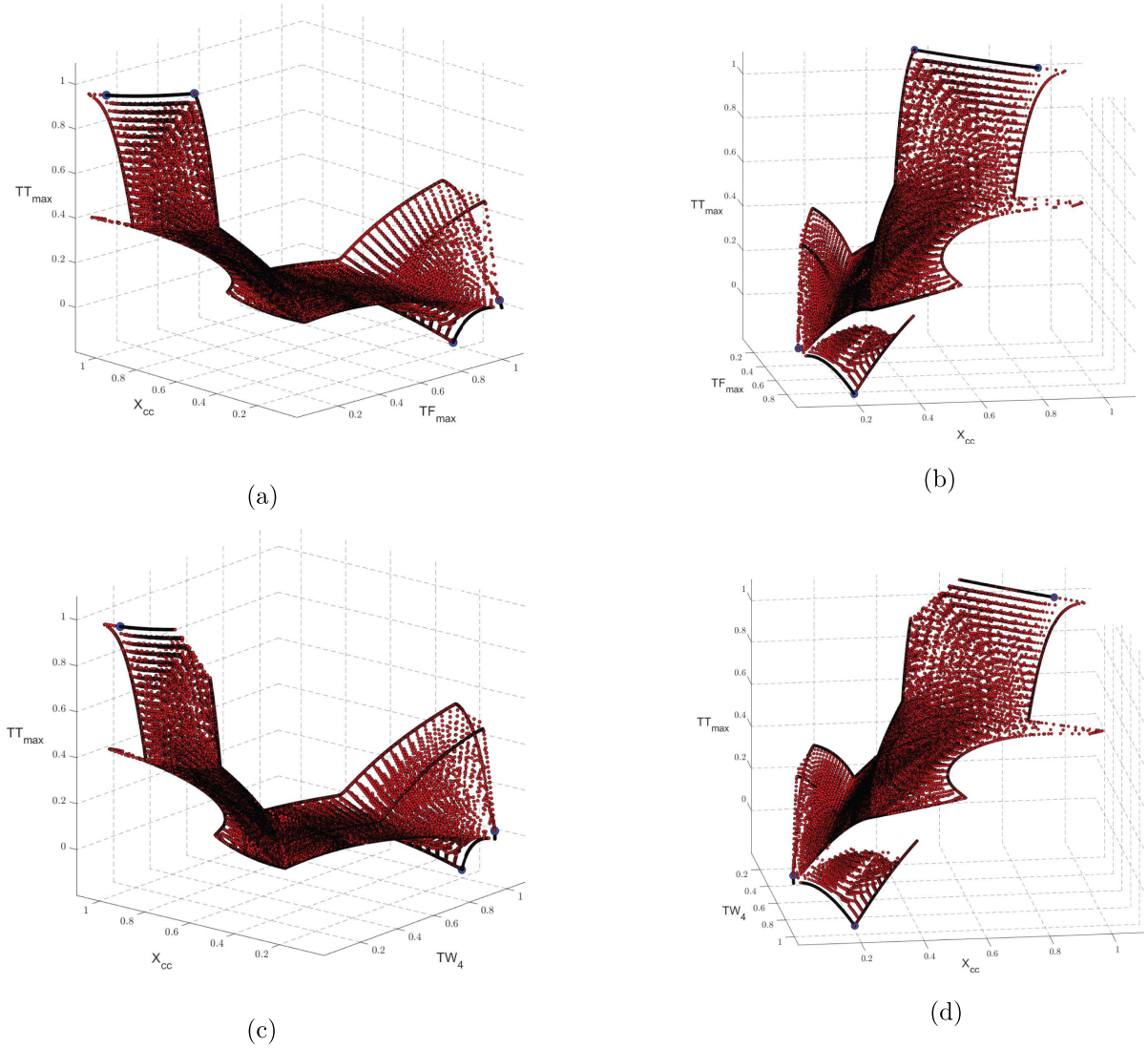

FIGURE 4. Projected Nondominated set for the rocket injector design problem. (a) Projection of the nondominated set to the $f_1f_3f_2$-space. (b) Rotated view of the front in (a). (c) Projection of the nondominated set to the $f_4f_3f_2$-space. (d) Rotated view of the front in (c).

In obtaining an approximation of the nondominated set of the rocket injector design problem, we have used Algorithm 9 in [5], except solve problem $(Fp_{\bar{x}}^k)$ instead of problem $(p_{\bar{x}}^k)$. We choose to display projections $(\mathrm{TF}_{\max}, \mathrm{TT}_{\max}, X_{\mathrm{CC}}, \mathrm{TW}_4) \mapsto (\mathrm{TF}_{\max}, \mathrm{TT}_{\max}, X_{\mathrm{CC}}, 0)$ and $(\mathrm{TF}_{\max}, \mathrm{TT}_{\max}, X_{\mathrm{CC}}, \mathrm{TW}_4) \mapsto (0, \mathrm{TT}_{\max}, X_{\mathrm{CC}}, \mathrm{TW}_4)$ in $f_1f_3f_2$- and $f_4f_3f_2$-spaces in Figures 4a and 4c, respectively.

The rocket injector design problem was solved in [18] with three objectives, namely, with $\mathrm{TF}_{\max}, \mathrm{TT}_{\max}$, and X_{CC}. This reduced problem (with TW_4 eliminated out) is justified by a claim in [18] that $\mathrm{TF}_{\max}$ and TW_4 are correlated after a correlation analysis. However, Figures 4c and 4d show that the projection of the nondominated set to $f_4f_3f_2$-space clearly differs from that in Figures 4a and 4b: the points facing f_3f_2-plane, changes more

rapidly with $\mathrm{TT}_{\max}$ and with X_{CC}. This implies that incorporation of TW_4 in this multiobjective optimization problem is essential, or necessary, as opposed to its exclusion in [18].

6. Conclusions

In the present research, we proposed a modification of the scalarization approach introduced by Burachik *et al.* [4,5] for solving multiobjective programming problems. We proved necessary and sufficient conditions for various types of efficiency, in particular for proper efficiency. The proposed approach was applied to solve problems with convex, nonconvex, connected and disconnected feasible sets. We have demonstrated the efficiency of the proposed method using numerical experiments. Since the modifications resolve the inflexibility of the constraints of the feasible-value constraint and the weighted constraint methods leading to computational advantages, it will also be interesting to study applications of our approach to multiobjective mixed integer optimization problems.

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