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OPTIMALITY, DUALITY AND SADDLE POINT CRITERIA FOR A ROBUST FRACTIONAL INTERVAL-VALUED OPTIMIZATION PROBLEM WITH UNCERTAIN INEQUALITY CONSTRAINTS *VIA* CONVEXIFICATORS

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Abstract. This article focuses on optimality conditions for a robust fractional interval-valued optimization problem with uncertain inequality constraints (RNFIVP) based on convexificators. Using the tools of convexity, an example of sufficient optimality conditions is demonstrated. Robust parametric duality for (RNFIVP) is formulated and utilizing the concept of convexity, usual duality results between the primal and dual problems are investigated. Further, the equivalence between the saddle point criteria of a Lagrangian type function and a robust $\mathcal{LU}$-optimal solution for (RNFIVP) with convexity is also examined.

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1. Introduction

Robust optimization has developed as a powerful conceptual structure for analyzing interval-valued optimization problems involving data uncertainty. This is an emerging area of study that motivates researchers to solve a spectrum of optimization problems concerning real-life situations such as industrial settings, where the data input for an interval-valued program is usually uncertain or noisy due to inaccuracies which occur in measurement or prediction. In the function space, the objective and constraint functions belong to "uncertainty sets". A few examples of real-world applications of robust optimization include topology design problem [25], dynamic power generation [29] and shelter location-allocation problem [14]. For a comprehensive analysis of robust optimization, the readers may refer to [5–7, 17, 18].

Demyanov [11] first proposed the concept of convexificators in 1994 as a generalization of upper convex and lower concave approximations. Convexificators are considered as weaker forms of the concept of subdifferentials as they are often closed sets, unlike the well-known subdifferentials which are compact and convex sets. In the case of extended real-valued functions, the concept of noncompact convexificator and characterization of

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quasiconvexity was proposed by Jeyakumar and Luc [20]. Furthermore, they presented numerous calculus laws which included extremality and mean value properties. Subsequently, Dutta and Chandra [12] introduced a new concept of nonsmooth pseudoconvex function, investigated its properties and also analyzed optimality criteria for vector minimization using convexificators. Moreover, as an application of chain rule for a mathematical programming problem involving inequality constraints, Dutta and Chandra [13] derived necessary optimality conditions. By employing the tools of convexificator, Li and Zhang [22] derived Kuhn–Tucker type necessary optimality criteria for nonsmooth optimization problems using locally Lipschitz functions. Recently, Ahmad *et al.* [3] examined the optimality and duality conditions for nonsmooth minimax programming problems with locally Lipschitz functions using the approach of convexificator. On the other hand, Jayswal *et al.* [19] analyzed optimality criteria for nonsmooth multi-objective programming problems and also discussed the duality results of two types of dual models based on the notion of convexificator.

The objective function in fractional programming problems is the ratio of two functions. When many rates need to be optimized at the same time, such as health care and hospital planning, production planning, and financial and corporate planning, these models naturally occur. There are many applications of fractional programming, namely, engineering designs [27], sustainable management of electric power systems [30] and production planning [4] etc. For more information on fractional programming, one can refer to Stancu-Minasian [26].

In the last few years, there has been a lot of research on fractional interval-valued programming problems. An interval-valued linear fractional programming problem was introduced by Effati and Pakdaman [15], which is reduced to an interval-valued objective function with the bounds being fractional functions. In addition, Ahmad *et al.* [1] studied the Karush–Kuhn–Tucker optimality conditions for a multi-objective programming problem with interval-valued objective functions utilizing generalized convexity and generalized differentiability. Furthermore, Ahmad *et al.* [2] examined the Fritz-John and Kuhn–Tucker type optimality conditions for a non-differentiable interval-valued multi-objective model using the concept of $\mathcal{LU}$-convexity. The Karush–Kuhn–Tucker optimality conditions for multiple objective fractional interval-valued optimization problems were examined by Debnath and Gupta [10], presuming that the functions involved are gH-differentiable. Very recently, Dar *et al.* [8] derived optimality conditions of the Fritz-John and Kuhn-Tucker type for an interval multi-objective fractional model (IVMFP) using the concept of $\mathcal{LU}$-convexity and $\mathcal{LU}$-concavity. Also, they examined parametric duality results. On the other hand, Rani and Kummari [23] discussed optimality and saddle point criteria for a fractional interval-valued optimization problem using convexificator.

We perceive that, in the literature, there is no work addressing robust fractional interval-valued optimization problems with uncertain inequality constraints. Therefore, the focus of this paper is to analyze optimality, duality and saddle point criteria for (RNFIVP). This paper is organized in the following way: Some preliminary and fundamental notions are recalled in Section . Robust optimality criteria for (RNFIVP) are analyzed based on the concept of convexificators in Section 3, while Section 4 deals with the formulation of robust parametric duality for (RNFIVP) and the usual duality results between the primal and dual problems are also investigated. Using the notion of convexificators, Section 5 illustrates how the saddle point conditions of a Lagrange type function and a robust $\mathcal{LU}$-optimal solution for (RNFIVP) are equivalent. Section 6 examines special cases. Finally, Section 7 provides a conclusion to this article.

2. Preliminaries

All spaces in this paper, unless otherwise mentioned, are real Banach spaces with norm denoted by $\|.\|$. Let $\mathcal{X}^*$ be the topological dual of a given real Banach space $\mathcal{X}$ with the canonical dual pairing $\langle ., . \rangle$; which stands for the norm on $\mathcal{X}$ and $\mathcal{X}^*$. Let $\mathbb{R}^n_+$ be the non-negative orthant of an n-dimensional Euclidean space $\mathbb{R}^n$. Let $\zeta : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. Then the lower and upper Dini directional derivatives of ζ at $\alpha \in \mathcal{X}$ in the direction of β is defined as follows as:

$$\zeta^-(\alpha, \beta) = \lim_{t \to 0+} \inf \frac{\zeta(\alpha + t\beta) - \zeta(\alpha)}{t},$$

$$\zeta^+(\alpha,\beta) = \lim_{t\to 0+}\sup \frac{\zeta(\alpha+t\beta)-\zeta(\alpha)}{t}\cdot$$

Definition 2.1 (Jeyakumar and Luc's [20]). A function $\zeta : \mathcal{X} \to \mathbb{R}\cup\{+\infty\}$ is said to have a convexificator $\partial^*\zeta(\alpha)$ at α provided $\partial^*\zeta(\alpha) \subset \mathcal{X}^*$ is weak* closed and

$$\zeta^+(\alpha,\beta) \geq \inf_{\alpha^*\in\partial^*\zeta(\alpha)}\langle\alpha^*,\beta\rangle \quad \text{and} \quad \zeta^-(\alpha,\beta) \leq \sup_{\alpha^*\in\partial^*\zeta(\alpha)}\langle\alpha^*,\beta\rangle, \quad \forall\,\beta\in\mathcal{X}.$$

On the lines of Ahmad *et al.* [3], the following Definitions 2.2–2.6 are considered.

Definition 2.2. A function $\zeta : \mathcal{X}\to\mathbb{R}$ is said to be $\partial^*\zeta$-convex at $\hat\alpha\in\mathcal{X}$, if

$$\zeta(\alpha)-\zeta(\hat\alpha) \geq \langle\rho,\alpha-\hat\alpha\rangle, \quad \text{for all } \rho\in\partial^*\zeta(\hat\alpha) \text{ and } \alpha\in\mathcal{X}.$$

Definition 2.3. A function $\zeta : \mathcal{X}\to\mathbb{R}$ is said to be strict $\partial^*\zeta$-convex at $\hat\alpha\in\mathcal{X}$, if

$$\zeta(\alpha)-\zeta(\hat\alpha) > \langle\rho,\alpha-\hat\alpha\rangle, \quad \text{for all } \rho\in\partial^*\zeta(\hat\alpha),\ \alpha\in\mathcal{X} \text{ and } \alpha\neq\hat\alpha.$$

Definition 2.4. A function $\zeta : \mathcal{X}\to\mathbb{R}$ is said to be $\partial^*\zeta$-pseudoconvex at $\hat\alpha\in\mathcal{X}$, if

$$\zeta(\alpha)<\zeta(\hat\alpha) \Rightarrow \langle\rho,\alpha-\hat\alpha\rangle<0, \quad \text{for all } \rho\in\partial^*\zeta(\hat\alpha) \text{ and } \alpha\in\mathcal{X},$$

identically

$$\langle\rho,\alpha-\hat\alpha\rangle\geq 0 \Rightarrow \zeta(\alpha)\geq\zeta(\hat\alpha), \quad \text{for all } \rho\in\partial^*\zeta(\hat\alpha) \text{ and } \alpha\in\mathcal{X}.$$

Definition 2.5. A function $\zeta : \mathcal{X}\to\mathbb{R}$ is said to be strict $\partial^*\zeta$-pseudoconvex at $\hat\alpha\in\mathcal{X}$, if

$$\zeta(\alpha)\leq\zeta(\hat\alpha) \Rightarrow \langle\rho,\alpha-\hat\alpha\rangle<0, \quad \text{for all } \rho\in\partial^*\zeta(\hat\alpha) \text{ and } \alpha\in\mathcal{X},$$

identically

$$\langle\rho,\alpha-\hat\alpha\rangle\geq 0 \Rightarrow \zeta(\alpha)>\zeta(\hat\alpha), \quad \text{for all } \rho\in\partial^*\zeta(\hat\alpha) \text{ and } \alpha\in\mathcal{X}.$$

Definition 2.6. A function $\zeta : \mathcal{X}\to\mathbb{R}$ is said to be $\partial^*\zeta$-quasiconvex at $\hat\alpha\in\mathcal{X}$, if

$$\zeta(\alpha)\leq\zeta(\hat\alpha) \Rightarrow \langle\rho,\alpha-\hat\alpha\rangle\leq 0, \quad \text{for all } \rho\in\partial^*\zeta(\hat\alpha) \text{ and } \alpha\in\mathcal{X},$$

identically

$$\langle\rho,\alpha-\hat\alpha\rangle>0 \Rightarrow \zeta(\alpha)>\zeta(\hat\alpha), \quad \text{for all } \rho\in\partial^*\zeta(\hat\alpha) \text{ and } \alpha\in\mathcal{X}.$$

Let $\mathbb{I} = \left\{a=\left[a^{\mathcal{L}},a^{\mathcal{U}}\right] : a^{\mathcal{L}},\ a^{\mathcal{U}}\in\mathbb{R},\ a^{\mathcal{L}}\leq a^{\mathcal{U}}\right\}$. The fundamental concepts of interval mathematics are as follows:

Let $\dfrac{\mathbb{E}}{\mathbb{H}} = \dfrac{[\theta_1^{\mathcal{L}},\theta_1^{\mathcal{U}}]}{[\delta_1^{\mathcal{L}},\delta_1^{\mathcal{U}}]} = \left[\dfrac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}},\dfrac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}}\right]$ and $\dfrac{\mathbb{J}}{\mathbb{G}} = \dfrac{[\theta_2^{\mathcal{L}},\theta_2^{\mathcal{U}}]}{[\delta_2^{\mathcal{L}},\delta_2^{\mathcal{U}}]} = \left[\dfrac{\theta_2^{\mathcal{L}}}{\delta_2^{\mathcal{U}}},\dfrac{\theta_2^{\mathcal{U}}}{\delta_2^{\mathcal{L}}}\right]$ be two fractional closed intervals with $\dfrac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}}\leq\dfrac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}}$ and $\dfrac{\theta_2^{\mathcal{L}}}{\delta_2^{\mathcal{U}}}\leq\dfrac{\theta_2^{\mathcal{U}}}{\delta_2^{\mathcal{L}}}$, $\quad\delta_1^{\mathcal{L}},\ \delta_1^{\mathcal{U}},\ \delta_2^{\mathcal{L}},\ \delta_2^{\mathcal{U}}\neq 0$.

(i) $\dfrac{\mathbb{E}}{\mathbb{H}}+\dfrac{\mathbb{J}}{\mathbb{G}} = \dfrac{[\theta_1^{\mathcal{L}},\theta_1^{\mathcal{U}}]}{[\delta_1^{\mathcal{L}},\delta_1^{\mathcal{U}}]}+\dfrac{[\theta_2^{\mathcal{L}},\theta_2^{\mathcal{U}}]}{[\delta_2^{\mathcal{L}},\delta_2^{\mathcal{U}}]} = \left[\dfrac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}},\dfrac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}}\right]+\left[\dfrac{\theta_2^{\mathcal{L}}}{\delta_2^{\mathcal{U}}},\dfrac{\theta_2^{\mathcal{U}}}{\delta_2^{\mathcal{L}}}\right] = \left[\dfrac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}}+\dfrac{\theta_2^{\mathcal{L}}}{\delta_2^{\mathcal{U}}},\dfrac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}}+\dfrac{\theta_2^{\mathcal{U}}}{\delta_2^{\mathcal{L}}}\right]$,

(ii) $\dfrac{-\,\mathbb{E}}{\mathbb{H}} = -\left[\theta_1^{\mathcal{L}},\theta_1^{\mathcal{U}}\right]\times\left[\dfrac{1}{\delta_1^{\mathcal{L}}},\dfrac{1}{\delta_1^{\mathcal{U}}}\right] = [-\theta_1^{\mathcal{U}},-\theta_1^{\mathcal{L}}]\times\left[\dfrac{1}{\delta_1^{\mathcal{L}}},\dfrac{1}{\delta_1^{\mathcal{U}}}\right] = \left[\dfrac{-\,\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}},\dfrac{-\,\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}}\right]$, with $\delta_1^{\mathcal{L}}>0,\delta_1^{\mathcal{U}}>0$,

(iii) $\dfrac{\mathbb{E}}{\mathbb{H}}-\dfrac{\mathbb{J}}{\mathbb{G}} = \dfrac{\mathbb{E}}{\mathbb{H}}+\left(\dfrac{-\,\mathbb{J}}{\mathbb{G}}\right) = \left[\dfrac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}},\dfrac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}}\right]+\left[\dfrac{-\,\theta_2^{\mathcal{U}}}{\delta_2^{\mathcal{L}}},\dfrac{-\,\theta_2^{\mathcal{L}}}{\delta_2^{\mathcal{U}}}\right] = \left[\dfrac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}}-\dfrac{\theta_2^{\mathcal{U}}}{\delta_2^{\mathcal{L}}},\dfrac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}}-\dfrac{\theta_2^{\mathcal{L}}}{\delta_2^{\mathcal{U}}}\right]$,

(iv) $\gamma\left(\frac{\mathbb{E}}{\mathbb{H}}\right) = \begin{cases} \left[\frac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}}, \frac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}}\right], & \text{if } \gamma \geq 0, \\ \left[\frac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}}, \frac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}}\right], & \text{if } \gamma < 0. \end{cases}$

An order relation $\preceq_{\mathcal{LU}}$ between two intervals $\frac{\mathbb{E}}{\mathbb{H}}$ and $\frac{\mathbb{J}}{\mathbb{G}}$ is stated as given below:

(i) $\frac{\mathbb{E}}{\mathbb{H}} \preceq_{\mathcal{LU}} \frac{\mathbb{J}}{\mathbb{G}}$ iff $\frac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}} \leq \frac{\theta_2^{\mathcal{L}}}{\delta_2^{\mathcal{U}}}$ and $\frac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}} \leq \frac{\theta_2^{\mathcal{U}}}{\delta_2^{\mathcal{L}}}$.

(ii) $\frac{\mathbb{E}}{\mathbb{H}} \prec_{\mathcal{LU}} \frac{\mathbb{J}}{\mathbb{G}}$ iff $\frac{\mathbb{E}}{\mathbb{H}} \preceq_{\mathcal{LU}} \frac{\mathbb{J}}{\mathbb{G}}$ and $\frac{\mathbb{E}}{\mathbb{H}} \neq \frac{\mathbb{J}}{\mathbb{G}}$, identically

$$\begin{cases} \frac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}} < \frac{\theta_2^{\mathcal{L}}}{\delta_2^{\mathcal{U}}} \\ \frac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}} \leq \frac{\theta_2^{\mathcal{U}}}{\delta_2^{\mathcal{L}}}, \end{cases} \text{ or } \begin{cases} \frac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}} \leq \frac{\theta_2^{\mathcal{L}}}{\delta_2^{\mathcal{U}}} \\ \frac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}} < \frac{\theta_2^{\mathcal{U}}}{\delta_2^{\mathcal{L}}}, \end{cases} \text{ or } \begin{cases} \frac{\theta_1^{\mathcal{L}}}{\delta_1^{\mathcal{U}}} < \frac{\theta_2^{\mathcal{L}}}{\delta_2^{\mathcal{U}}} \\ \frac{\theta_1^{\mathcal{U}}}{\delta_1^{\mathcal{L}}} < \frac{\theta_2^{\mathcal{U}}}{\delta_2^{\mathcal{L}}}. \end{cases}$$

Consider the subsequent robust non-differentiable fractional interval-valued optimization problem with data uncertainty in constraints:

$$\text{(UNFIVP-1)} \qquad \min \frac{[\zeta^{\mathcal{L}}(\alpha), \zeta^{\mathcal{U}}(\alpha)]}{[\nu^{\mathcal{L}}(\alpha), \nu^{\mathcal{U}}(\alpha)]}$$
$$\text{subject to}$$
$$\omega_s(\alpha, \kappa_s) \leq 0, \ \forall\, s \in \mathcal{S}, \ \forall\, \kappa_s \in K_s,$$
$$\alpha \in \mathcal{X} \subseteq \mathbb{R}^n,$$

which reduces to the problem

$$\text{(UNFIVP-2)} \qquad \min \left[\frac{\zeta^{\mathcal{L}}(\alpha)}{\nu^{\mathcal{U}}(\alpha)}, \frac{\zeta^{\mathcal{U}}(\alpha)}{\nu^{\mathcal{L}}(\alpha)}\right]$$
$$\text{subject to}$$
$$\omega_s(\alpha, \kappa_s) \leq 0, \ \forall\, s \in \mathcal{S}, \ \forall\, \kappa_s \in K_s,$$
$$\alpha \in \mathcal{X} \subseteq \mathbb{R}^n,$$

where $\mathcal{S}$ is an arbitrary index set (possibly infinite) and $\zeta^{\mathcal{L}}(\alpha), \zeta^{\mathcal{U}}(\alpha) \geq 0$, $\nu^{\mathcal{L}}(\alpha), \nu^{\mathcal{U}}(\alpha) > 0$, and $\omega_s : \mathcal{X} \times \mathbb{R}^p \to \mathbb{R}$ are continuous functions on $\mathcal{X}$ and $\kappa_s \in \mathbb{R}^p$ is an uncertain parameter which pertains to the convex compact set $K_s \subset \mathbb{R}^p$, $s \in \mathcal{S}$. The uncertainty set-valued function $K : \mathcal{S} \rightrightarrows \mathbb{R}^p$, is given by $K(s) := K_s$, $\forall\, s \in \mathcal{S}$, so,

$$\text{graph } (K) = \{(s, \kappa_s) : s \in \mathcal{S}, \ \kappa_s \in K_s\},$$

and $\kappa \in K$ implies that κ is a choice of K, that is, $\kappa : \mathcal{S} \rightrightarrows \mathbb{R}^p$ and $\kappa_s \in K_s$, $\forall\, s \in \mathcal{S}$.
By setting $\zeta^{\mathcal{L}} = {h_0}^{\mathcal{L}}$, $\nu^{\mathcal{U}} = {t_0}^{\mathcal{L}}$, $\zeta^{\mathcal{U}} = {h_0}^{\mathcal{U}}$, $\nu^{\mathcal{L}} = {t_0}^{\mathcal{U}}$, then the above problem transforms to

$$\text{(RNFIVP)} \qquad \min \left[\frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha), \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)\right]$$
$$\text{subject to}$$
$$\omega_s(\alpha, \kappa_s) \leq 0, \ \forall\, s \in \mathcal{S}, \ \forall\, \kappa_s \in K_s,$$
$$\alpha \in \mathcal{X} \subseteq \mathbb{R}^n.$$

Let $\mathbb{M}$ denote the robust feasible set for the problem (RNFIVP). That is,

$$\mathbb{M} = \{\alpha \in \mathbb{R}^n : \omega_s(\alpha, \kappa_s) \leq 0, \ \forall\, s \in \mathcal{S}, \ \forall\, \kappa_s \in K_s\}.$$

Definition 2.7 (Wu [28])**.** A robust feasible solution $\hat{\alpha}$ is termed as a robust $\mathcal{LU}$-optimal solution for (RNFIVP) if and only if there exists no robust feasible solution α such that

$$\left[\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha), \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha)\right] \prec_{\mathcal{LU}} \left[\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}), \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha})\right].$$

3. Robust optimality conditions

Consider the subsequent two fractional problems for the given robust feasible solution $\hat{\alpha}$:

$$\text{(RFP1)} \quad \min\ \Psi^{\mathcal{L}}(\alpha) = \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha)$$
subject to
$$\omega_s(\alpha, \kappa_s) \le 0,\ \forall\, s \in \mathcal{S},\ \forall\, \kappa_s \in K_s,$$
$$\frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) \le \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha}),$$
$$\alpha \in \mathcal{X} \subseteq \mathbb{R}^n.$$

$$\text{(RFP2)} \quad \min\ \Psi^{\mathcal{U}}(\alpha) = \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha)$$
subject to
$$\omega_s(\alpha, \kappa_s) \le 0,\ \forall\, s \in \mathcal{S},\ \forall\, \kappa_s \in K_s,$$
$$\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) \le \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}),$$
$$\alpha \in \mathcal{X} \subseteq \mathbb{R}^n.$$

On the lines of Debnath and Gupta [9], we give the below mentioned Lemmas 3.1 and 3.2.

Lemma 3.1. *Let $\hat{\alpha}$ be a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP) if and only if $\hat{\alpha}$ is a robust optimal solution for the problems (RFP1) and (RFP2).*

Proof. Let $\hat{\alpha}$ be a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP). Then there is no robust feasible solution α such that

$$\left[\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha), \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha)\right] \prec_{\mathcal{LU}} \left[\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}), \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha})\right].$$

This implies that there is no robust feasible solution α such that

$$\begin{cases} \dfrac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) < \dfrac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}) \\ \dfrac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) \le \dfrac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha}) \end{cases}, \text{ or } \begin{cases} \dfrac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) \le \dfrac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}) \\ \dfrac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) < \dfrac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha}) \end{cases}, \text{ or } \begin{cases} \dfrac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) < \dfrac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}) \\ \dfrac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) < \dfrac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha}). \end{cases}$$

Suppose that $\hat{\alpha}$ is not a robust optimal solution for the problem (RFP1), then there exists an α satisfying the constraints of the problem (RFP1), that is, $\omega_s(\alpha, \kappa_s) \le 0,\ \forall\, s \in \mathcal{S},\ \forall\, \kappa_s \in K_s$ and $\frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) \le \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha})$ such that $\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) < \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha})$. By $\omega_s(\alpha, \kappa_s) \le 0,\ \forall\, s \in \mathcal{S},\ \forall\, \kappa_s \in K_s$, it follows that α is a robust feasible solution of the problem (RNFIVP). Hence, the inequalities $\frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) \le \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha})$ and $\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) < \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha})$ contradict the presumption that $\hat{\alpha}$ is a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP).

Conversely, let $\hat{\alpha}$ be a robust optimal solution for the problems (RFP1) and (RFP2). Suppose that $\hat{\alpha}$ is not a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP), then there exists a robust feasible solution α, that is, $\omega_s(\alpha, \kappa_s) \le 0,\ \forall\, s \in \mathcal{S},\ \forall\, \kappa_s \in K_s$ such that

$$\left[\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha), \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha)\right] \prec_{\mathcal{LU}} \left[\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}), \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha})\right].$$

Thus, by the definition of the relation $\prec_{\mathcal{LU}}$, we get

$$\left(\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) < \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}) \ \text{ and } \ \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) \leq \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha})\right), \tag{3.1}$$

$$\text{or} \quad \left(\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) \leq \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}) \ \text{ and } \ \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) < \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha})\right), \tag{3.2}$$

$$\text{or} \quad \left(\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) < \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}) \ \text{ and } \ \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) < \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha})\right). \tag{3.3}$$

By $\omega_s(\alpha, \kappa_s) \leq 0, \ \forall\, s \in \mathcal{S}, \ \forall\, \kappa_s \in K_s$ and second inequality of (3.1), which implies that α is a robust feasible solution of the problem (RFP1). Thus, the first inequality of (3.1) contradicts the presumption that $\hat{\alpha}$ is a robust optimal solution of the problem (RFP1). Further, by $\omega_s(\alpha, \kappa_s) \leq 0, \ \forall\, s \in \mathcal{S}, \ \forall\, \kappa_s \in K_s$ and first inequality of (3.2), which implies that α is a robust feasible solution of the problem (RFP2). Thus, the second inequality of (3.2) contradicts the presumption that $\hat{\alpha}$ is a robust optimal solution of the problem (RFP2). □

Lemma 3.2. *Let $\hat{\alpha}$ be a robust $\mathcal{LU}$-optimal of the problem (RNFIVP) if and only if $\hat{\alpha}$ minimizes $\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha)$ on the subsequent uncertainty constraint set*

$$\mathcal{F}_{\mathcal{L}} = \left\{\alpha \in \mathcal{X} \,|\, \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) \leq \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha}), \ \omega_s(\alpha, \kappa_s) \leq 0, \ \forall\, s \in \mathcal{S}, \ \forall\, \kappa_s \in K_s\right\}.$$

Proof. Let $\hat{\alpha}$ minimizes $\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha)$ on the subsequent uncertainty constraint set

$$\mathcal{F}_{\mathcal{L}} = \left\{\alpha \in \mathcal{X} \,|\, \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) \leq \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha}), \ \omega_s(\alpha, \kappa_s) \leq 0, \ \forall\, s \in \mathcal{S}, \ \forall\, \kappa_s \in K_s\right\}.$$

Suppose that $\hat{\alpha}$ is not a robust $\mathcal{LU}$-optimal of the problem (RNFIVP), then there exists a robust feasible solution α such that

$$\left[\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha), \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha)\right] \prec_{\mathcal{LU}} \left[\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}), \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha})\right].$$

Thus, by the definition of the relation $\prec_{\mathcal{LU}}$, we get

$$\begin{cases} \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) < \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}) \\ \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) \leq \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha}) \end{cases}, \ \text{or} \ \begin{cases} \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) \leq \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}) \\ \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) < \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha}) \end{cases}, \ \text{or} \ \begin{cases} \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha) < \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}) \\ \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\alpha) < \frac{h_0{}^{\mathcal{U}}}{t_0{}^{\mathcal{U}}}(\hat{\alpha}) \end{cases},$$

which contradicts that $\hat{\alpha}$ minimizes $\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha)$ on $\mathcal{F}_{\mathcal{L}}$.

Conversely, let $\hat{\alpha}$ be a robust $\mathcal{LU}$-optimal of the problem (RNFIVP).

Suppose that $\hat{\alpha}$ does not minimizes $\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha)$ on $\mathcal{F}_{\mathcal{L}}$, then there exists a robust feasible solution α_0 such that

$$\frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\alpha_0) < \frac{h_0{}^{\mathcal{L}}}{t_0{}^{\mathcal{L}}}(\hat{\alpha}),$$

which contradicts that $\hat{\alpha}$ is a robust $\mathcal{LU}$-optimal of the problem (RNFIVP). □

Let $\hat{\alpha}$ be a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP).

Slater's constraint qualification for the problem (RFP1):

We say that the Slater type constraint qualification is satisfied at $\hat{\alpha}$, if there exists $\alpha \in \mathcal{X}$ such that $\omega_s(\alpha, \kappa_s) < 0$, $s \in \mathcal{S}$, $\kappa_s \in K_s$ and $h_0^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0^{\mathcal{U}}(\alpha) < 0$.

Slater's constraint qualification for the problem (RFP2):

We say that the Slater type constraint qualification is satisfied at $\hat{\alpha}$, if there exists $\alpha \in \mathcal{X}$ such that $\omega_s(\alpha, \kappa_s) < 0$, $s \in \mathcal{S}$, $\kappa_s \in K_s$ and $h_0^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0^{\mathcal{L}}(\alpha) < 0$.

Consider the subsequent single-objective fractional uncertainty constraint problem:

$$
\begin{aligned}
(G) \qquad & \min\ \Psi(\alpha) = \frac{h_1}{t_1}(\alpha) \\
& \text{subject to} \\
& \omega_s(\alpha, \kappa_s) \le 0,\ \forall\, s \in \mathcal{S},\ \forall\, \kappa_s \in K_s, \\
& \alpha \in \mathcal{X} \subseteq \mathbb{R}^n,
\end{aligned}
$$

where h_1, t_1 and $\omega_s : \mathcal{X} \times \mathbb{R}^p \to \mathbb{R}$ are continuous functions on $\mathcal{X}$ such that $h_1(\alpha) \ge 0$ and $t_1(\alpha) > 0$, for all $\alpha \in \mathcal{X}$ and $\kappa_s \in \mathbb{R}^p$ is an uncertain parameter which pertains to the convex compact set $K_s \subset \mathbb{R}^p$, $s \in \mathcal{S}$.

The following theorem for the problem (G) is stated, based on Theorem 6 of Gadhi [16] and Lee and Lee [21].

Theorem 3.3. *Suppose that $\hat{\alpha}$ is a robust optimal solution of the problem (G) and the Slater's constraint qualification holds at $\hat{\alpha}$. Assume that h_1, t_1 and ω_s, $s \in \mathcal{S}$ are continuous and admit bounded convexificators $\partial^* h_1(\hat{\alpha})$, $\partial^* t_1(\hat{\alpha})$ and $\partial^* \omega_s(\hat{\alpha}, \kappa_s)$, $s \in \mathcal{S}$, $\kappa_s \in K_s$ at $\hat{\alpha}$ respectively and also $\partial^* h_1(\hat{\alpha})$, $\partial^* t_1(\hat{\alpha})$ and $\partial^* \omega_s(\hat{\alpha}, \kappa_s)$, $s \in \mathcal{S}$, $\kappa_s \in K_s$ are upper semicontinuous at $\hat{\alpha}$, then there exist $\tau > 0$ and $(\varrho_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$ such that*

$$0 \in \tau(\partial^* h_1(\hat{\alpha}) - \Psi(\hat{\alpha})\partial^* t_1(\hat{\alpha})) + \sum_{s\in\mathcal{S}} \varrho_s \partial^* \omega_s(\hat{\alpha}, \kappa_s), \tag{3.4}$$

$$\varrho_s \omega_s(\hat{\alpha}, \kappa_s) = 0, \qquad s \in \mathcal{S}, \tag{3.5}$$

$$\varrho_s \ge 0 \text{ and } \omega_s(\hat{\alpha}, \kappa_s) \le 0, \qquad s \in \mathcal{S}. \tag{3.6}$$

Theorem 3.4 (Karush–Kuhn–Tucker robust necessary $\mathcal{LU}$-optimality conditions)**.** *Suppose that $\hat{\alpha}$ is a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP) and both the Slater's constraint qualification for the problems (RFP1) and (RFP2) are satisfied at $\hat{\alpha}$. Assume that ${h_0}^{\mathcal{L}}$, ${t_0}^{\mathcal{L}}$, ${h_0}^{\mathcal{U}}$, ${t_0}^{\mathcal{U}}$ and ω_s, $s \in \mathcal{S}$ are continuous and admit bounded convexificators $\partial^* {h_0}^{\mathcal{L}}(\hat{\alpha})$, $\partial^* {t_0}^{\mathcal{L}}(\hat{\alpha})$, $\partial^* {h_0}^{\mathcal{U}}(\hat{\alpha})$, $\partial^* {t_0}^{\mathcal{U}}(\hat{\alpha})$ and $\partial^* \omega_s(\hat{\alpha}, \kappa_s)$, $s \in \mathcal{S}$, $\kappa_s \in K_s$ at $\hat{\alpha}$ respectively and that $\partial^* {h_0}^{\mathcal{L}}(\hat{\alpha})$, $\partial^* {t_0}^{\mathcal{L}}(\hat{\alpha})$, $\partial^* {h_0}^{\mathcal{U}}(\hat{\alpha})$, $\partial^* {t_0}^{\mathcal{U}}(\hat{\alpha})$ and $\partial^* \omega_s(\hat{\alpha}, \kappa_s)$, $s \in \mathcal{S}$, $\kappa_s \in K_s$ are upper semicontinuous at $\hat{\alpha}$, then there exist $\tau^{\mathcal{L}} > 0$, $\tau^{\mathcal{U}} > 0$ and $(\varrho_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$ such that*

$$
\begin{aligned}
0 \in \tau^{\mathcal{L}}\Big(\partial^* {h_0}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{L}}(\hat{\alpha})\Big) &+ \tau^{\mathcal{U}}\Big(\partial^* {h_0}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{U}}(\hat{\alpha})\Big) \\
&+ \sum_{s\in\mathcal{S}} \varrho_s \partial^* \omega_s(\hat{\alpha}, \kappa_s),
\end{aligned} \tag{3.7}
$$

$$\varrho_s \omega_s(\hat{\alpha}, \kappa_s) = 0, \qquad s \in \mathcal{S}, \tag{3.8}$$

$$\varrho_s \ge 0 \text{ and } \omega_s(\hat{\alpha}, \kappa_s) \le 0, \qquad s \in \mathcal{S}. \tag{3.9}$$

Proof. Since $\hat{\alpha}$ is a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP), by Lemma 3.1, $\hat{\alpha}$ is also a robust optimal solution for the problems (RFP1) and (RFP2). Hence, by utilizing Lemma 3.2 at $\hat{\alpha}$, the minimum value of $\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha)$ is attained on the uncertainty constraint set

$$\mathcal{F}_{\mathcal{L}} = \left\{ \alpha \in \mathcal{X} \,\middle|\, \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha) \le \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha}),\ \omega_s(\alpha, \kappa_s) \le 0,\ s \in \mathcal{S},\ \kappa_s \in K_s \right\}$$

and the minimum value of $\frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)$ is attained at $\hat{\alpha}$ on the uncertainty constraint set

$$\mathcal{F}_{\mathcal{U}} = \left\{ \alpha \in \mathcal{X} \middle| \frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha) \leq \frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha}),\ \omega_s(\alpha, \kappa_s) \leq 0,\ s \in \mathcal{S},\ \kappa_s \in K_s \right\}.$$

According to Theorem 3.3, there exist $\tau^{\mathcal{LL}} > 0$, $\tau^{\mathcal{LU}} > 0$, $(\varrho_s^{\mathcal{L}})_{s\in\mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$ and $\tau^{\mathcal{UL}} > 0$, $\tau^{\mathcal{UU}} > 0$, $(\varrho_s^{\mathcal{U}})_{s\in\mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$ such that

$$\begin{aligned} 0 \in\ & \tau^{\mathcal{LL}}\Big(\partial^* {h_0}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{L}}(\hat{\alpha})\Big) + \tau^{\mathcal{LU}}\Big(\partial^* {h_0}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{U}}(\hat{\alpha})\Big) \\ & + \sum_{s\in S} \varrho_s^{\mathcal{L}} \partial^* \omega_s(\hat{\alpha}, \kappa_s), \end{aligned} \tag{3.10}$$

$$\varrho_s^{\mathcal{L}} \omega_s(\hat{\alpha}, \kappa_s) = 0, \qquad s \in \mathcal{S}, \tag{3.11}$$

$$\varrho_s^{\mathcal{L}} \geq 0 \text{ and } \omega_s(\hat{\alpha}, \kappa_s) \leq 0, \qquad s \in \mathcal{S} \tag{3.12}$$

and

$$\begin{aligned} 0 \in\ & \tau^{\mathcal{UL}}\Big(\partial^* {h_0}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{L}}(\hat{\alpha})\Big) + \tau^{\mathcal{UU}}\Big(\partial^* {h_0}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{U}}(\hat{\alpha})\Big) \\ & + \sum_{s\in \mathcal{S}} \varrho_s^{\mathcal{U}} \partial^* \omega_s(\hat{\alpha}, \kappa_s), \end{aligned} \tag{3.13}$$

$$\varrho_s^{\mathcal{U}} \omega_s(\hat{\alpha}, \kappa_s) = 0, \qquad s \in \mathcal{S}, \tag{3.14}$$

$$\varrho_s^{\mathcal{U}} \geq 0 \text{ and } \omega_s(\hat{\alpha}, \kappa_s) \leq 0, \qquad s \in \mathcal{S}. \tag{3.15}$$

From (3.10) to (3.15), we obtain

$$\begin{aligned} 0 \in\ & \big[\tau^{\mathcal{LL}} + \tau^{\mathcal{UL}}\big]\Big(\partial^* {h_0}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{L}}(\hat{\alpha})\Big) + \big[\tau^{\mathcal{LU}} + \tau^{\mathcal{UU}}\big] \\ & \Big(\partial^* {h_0}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{U}}(\hat{\alpha})\Big) + \sum_{s\in \mathcal{S}} \big[\varrho_s^{\mathcal{L}} + \varrho_s^{\mathcal{U}}\big] \partial^* \omega_s(\hat{\alpha}, \kappa_s), \end{aligned} \tag{3.16}$$

$$\big[\varrho_s^{\mathcal{L}} + \varrho_s^{\mathcal{U}}\big] \omega_s(\hat{\alpha}, \kappa_s) = 0, \qquad s \in \mathcal{S}, \tag{3.17}$$

$$\big[\varrho_s^{\mathcal{L}} + \varrho_s^{\mathcal{U}}\big] \geq 0 \text{ and } \omega_s(\hat{\alpha}, \kappa_s) \leq 0, \qquad s \in \mathcal{S}. \tag{3.18}$$

The equations from (3.16) to (3.18) along with $\tau^{\mathcal{LL}} + \tau^{\mathcal{UL}} = \tau^{\mathcal{L}}$, $\tau^{\mathcal{LU}} + \tau^{\mathcal{UU}} = \tau^{\mathcal{U}}$ and $\varrho_s^{\mathcal{L}} + \varrho_s^{\mathcal{U}} = \varrho_s$ yield

$$\begin{aligned} 0 \in\ & \tau^{\mathcal{L}}\Big(\partial^* {h_0}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})\partial^* {t_0}^{L}(\hat{\alpha})\Big) + \tau^{\mathcal{U}}\Big(\partial^* {h_0}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})\partial^* {t_0}^{U}(\hat{\alpha})\Big) \\ & + \sum_{s\in S} \varrho_s \partial^* \omega_s(\hat{\alpha}, \kappa_s), \\ & \varrho_s \omega_s(\hat{\alpha}, \kappa_s) = 0, \qquad s \in \mathcal{S}, \\ & \varrho_s \geq 0 \text{ and } \omega_s(\hat{\alpha}, \kappa_s) \leq 0, \qquad s \in \mathcal{S}. \end{aligned}$$

Thus, this concludes the proof of the theorem. □

Theorem 3.5 (Robust sufficient $\mathcal{LU}$-optimality conditions)**.** *Suppose that $\hat{\alpha}$ is a robust feasible solution of the problem (RNFIVP), then there exists $(\varrho_s)_{s\in S} \in \mathbb{R}_+^{(S)}$ such that the conditions (3.7)–(3.9) hold at $\hat{\alpha}$. Also, presume that*

(i) $\tau^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(.) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(.)\Big] + \tau^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(.) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(.)\Big]$ *is* $\tau^{\mathcal{L}}\Big[\partial^* {h_0}^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{L}}\Big] + \tau^{\mathcal{U}}\Big[\partial^* {h_0}^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{U}}\Big]$*-convex at* $\hat{\alpha}$,

(ii) $\varrho_s\omega_s(\hat{\alpha},\kappa_s)$, *for* $s\in\mathcal{S},\kappa_s\in K_s$ *is* $\partial^*\omega_s(.,\kappa_s)$*-convex at* $\hat{\alpha}$,

then $\hat{\alpha}$ *is a robust* $\mathcal{LU}$*-optimal solution of the problem (RNFIVP).*

Proof. The conditions (3.7)–(3.9) hold at $\hat{\alpha}$ with Lagrange multipliers $\tau^{\mathcal{L}}>0,\tau^{\mathcal{U}}>0,(\varrho_s)_{s\in\mathcal{S}}\in\mathbb{R}_+^{(\mathcal{S})}$. Therefore, it follows from (3.7), that there exist $\iota^{\mathcal{L}}\in\partial^*{h_0}^{\mathcal{L}}(\hat{\alpha})$, $\upsilon^{\mathcal{L}}\in\partial^*{t_0}^{\mathcal{L}}(\hat{\alpha})$, $\iota^{\mathcal{U}}\in\partial^*{h_0}^{\mathcal{U}}(\hat{\alpha})$, $\upsilon^{\mathcal{U}}\in\partial^*{t_0}^{\mathcal{U}}(\hat{\alpha})$ and $\varsigma_s\in\partial^*\omega_s(\hat{\alpha},\kappa_s)$, $s\in\mathcal{S}$, $\kappa_s\in K_s$ such that

$$\tau^{\mathcal{L}}\left[\iota^{\mathcal{L}}-\Psi^{\mathcal{L}}(\hat{\alpha})\upsilon^{\mathcal{L}}\right]+\tau^{\mathcal{U}}\left[\iota^{\mathcal{U}}-\Psi^{\mathcal{U}}(\hat{\alpha})\upsilon^{\mathcal{U}}\right]+\sum_{s\in\mathcal{S}}\varrho_s\varsigma_s=0. \tag{3.19}$$

Suppose that $\hat{\alpha}$ is not a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP), then there exists a robust feasible solution α such that

$$\left[\frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha),\frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)\right]\prec_{\mathcal{LU}}\left[\frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha}),\frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha})\right].$$

That is,

$$\begin{cases}\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha)<\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha})\\ \dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)\leq\dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha})\end{cases},\ \text{or}\ \begin{cases}\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha)\leq\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha})\\ \dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)<\dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha})\end{cases},\ \text{or}\ \begin{cases}\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha)<\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha})\\ \dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)<\dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha})\end{cases},$$

which gives

$$\begin{aligned}{h_0}^{\mathcal{L}}(\alpha)-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha)&<{h_0}^{\mathcal{L}}(\hat{\alpha})-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\hat{\alpha}),\\ {h_0}^{\mathcal{U}}(\alpha)-\Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\alpha)&\leq{h_0}^{\mathcal{U}}(\hat{\alpha})-\Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\hat{\alpha}),\end{aligned}$$

or

$$\begin{aligned}{h_0}^{\mathcal{L}}(\alpha)-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha)&\leq{h_0}^{\mathcal{L}}(\hat{\alpha})-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\hat{\alpha}),\\ {h_0}^{\mathcal{U}}(\alpha)-\Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\alpha)&<{h_0}^{\mathcal{U}}(\hat{\alpha})-\Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\hat{\alpha}),\end{aligned}$$

or

$$\begin{aligned}{h_0}^{\mathcal{L}}(\alpha)-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha)&<{h_0}^{\mathcal{L}}(\hat{\alpha})-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\hat{\alpha}),\\ {h_0}^{\mathcal{U}}(\alpha)-\Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\alpha)&<{h_0}^{\mathcal{U}}(\hat{\alpha})-\Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\hat{\alpha}).\end{aligned}$$

By using $\tau^{\mathcal{L}}>0$, $\tau^{\mathcal{U}}>0$, the above inequalities yield

$$\begin{aligned}&\tau^{\mathcal{L}}\left[{h_0}^{\mathcal{L}}(\alpha)-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha)\right]+\tau^{\mathcal{U}}\left[{h_0}^{\mathcal{U}}(\alpha)-\Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\alpha)\right]\\ <&\tau^{\mathcal{L}}\left[{h_0}^{\mathcal{L}}(\hat{\alpha})-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\hat{\alpha})\right]+\tau^{\mathcal{U}}\left[{h_0}^{\mathcal{U}}(\hat{\alpha})-\Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\hat{\alpha})\right].\end{aligned}$$

From assertion (i), that is, $\tau^{\mathcal{L}}\left[{h_0}^{\mathcal{L}}(.)-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(.)\right]+\tau^{\mathcal{U}}\left[{h_0}^{\mathcal{U}}(.)-\Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(.)\right]$ is $\tau^{\mathcal{L}}\left[\partial^*{h_0}^{\mathcal{L}}-\Psi^{\mathcal{L}}(\hat{\alpha})\partial^*{t_0}^{\mathcal{L}}\right]+\tau^{\mathcal{U}}\left[\partial^*{h_0}^{\mathcal{U}}-\Psi^{\mathcal{U}}(\hat{\alpha})\partial^*{t_0}^{\mathcal{U}}\right]$-convex at $\hat{\alpha}$, we obtain

$$\left\langle\tau^{\mathcal{L}}\left[\iota^{\mathcal{L}}-\Psi^{\mathcal{L}}(\hat{\alpha})\upsilon^{\mathcal{L}}\right]+\tau^{\mathcal{U}}\left[\iota^{\mathcal{U}}-\Psi^{\mathcal{U}}(\hat{\alpha})\upsilon^{\mathcal{U}}\right],\alpha-\hat{\alpha}\right\rangle<0, \tag{3.20}$$

for all $\iota^{\mathcal{L}}\in\partial^*{h_0}^{\mathcal{L}}(\hat{\alpha}),\upsilon^{\mathcal{L}}\in\partial^*{t_0}^{\mathcal{L}}(\hat{\alpha}),\iota^{\mathcal{U}}\in\partial^*{h_0}^{\mathcal{U}}(\hat{\alpha})$ and $\upsilon^{\mathcal{U}}\in\partial^*{t_0}^{\mathcal{U}}(\hat{\alpha})$.

By utilizing the robust feasibility of α, $\varrho_s \geq 0$, $s \in \mathcal{S}$ and (3.8), we get

$$\varrho_s \omega_s(\alpha, \kappa_s) \leq \varrho_s \omega_s(\hat{\alpha}, \kappa_s), \quad s \in \mathcal{S}.$$

The above inequality along with assertion (ii), gives

$$\langle \varrho_s \varsigma_s, \alpha - \hat{\alpha} \rangle \leq 0, \quad \text{for all } \varsigma_s \in \partial^* \omega_s(\hat{\alpha}, \kappa_s),\ s \in \mathcal{S}. \tag{3.21}$$

Combining (3.20) and (3.21), we have

$$\left\langle \tau^{\mathcal{L}}\left[\iota^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\upsilon^{\mathcal{L}}\right] + \tau^{\mathcal{U}}\left[\iota^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\upsilon^{\mathcal{U}}\right] + \sum_{s \in S} \varrho_s \varsigma_s, \alpha - \hat{\alpha} \right\rangle < 0,$$

which contradicts (3.19). Hence, $\hat{\alpha}$ is a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP). □

The sufficient $\mathcal{LU}$-optimality conditions specified in Theorem 3.5 is demonstrated by the subsequent example.

Example 3.6.

$$\text{(RFP3)} \qquad \min \frac{[\zeta_1^{\mathcal{L}}(\alpha), \zeta_1^{\mathcal{U}}(\alpha)]}{[\nu_1^{\mathcal{L}}(\alpha), \nu_1^{\mathcal{U}}(\alpha)]} = \min \frac{[2\alpha^2, 2\alpha^2 + 1]}{[\alpha^2 + 1, \alpha^2 + 2]}$$
$$\text{subject to}$$
$$\omega_s(\alpha, \kappa_s) = s\alpha^2 - \alpha\kappa_s - 2s \leq 0,\ s \in \mathcal{S} = [0, 1],\ \kappa_s \in [-s, s],$$
$$\alpha \in \mathcal{X} = \mathbb{R}.$$

The optimization problem under consideration is now rewritten as follows:

$$\min \left[\frac{2\alpha^2}{\alpha^2 + 2}, \frac{2\alpha^2 + 1}{\alpha^2 + 1}\right]$$
$$\text{subject to}$$
$$\omega_s(\alpha, \kappa_s) = s\alpha^2 - \alpha\kappa_s - 2s \leq 0,\ s \in \mathcal{S} = [0, 1],\ \kappa_s \in [-s, s],$$
$$\alpha \in \mathcal{X} = \mathbb{R}.$$

where $\Psi^{\mathcal{L}}(\alpha) = \frac{h_1^{\mathcal{L}}}{t_1^{\mathcal{L}}}(\alpha) = \frac{2\alpha^2}{\alpha^2+2}$, $\Psi^{\mathcal{U}}(\alpha) = \frac{h_1^{\mathcal{U}}}{t_1^{\mathcal{U}}}(\alpha) = \frac{2\alpha^2+1}{\alpha^2+1}$. Clearly, the robust feasible set is $[0, 1]$.

For the robust $\mathcal{LU}$-optimal solution $\hat{\alpha} = 0$, $\partial^* h_1^{\mathcal{L}}(\hat{\alpha}) = \{0\}$, $\partial^* t_1^{\mathcal{L}}(\hat{\alpha}) = \{0\}$, $\partial^* h_1^{\mathcal{U}}(\hat{\alpha}) = \{0\}$, $\partial^* t_1^{U}(\hat{\alpha}) = \{0\}$ and $\partial^* \omega_s(\hat{\alpha}, \kappa_s) = \{0\}$, $\kappa_s = o$, for $s = 0$. Also, for the robust $\mathcal{LU}$-optimal solution $\hat{\alpha} = \{0\}$, there exist $\tau^{\mathcal{L}} > 0, \tau^{\mathcal{U}} > 0, (\varrho_s)_{s \in S} \in \mathbb{R}_+^{(S)}, \kappa_s = o,$ for $s = 0$ such that the conditions (3.7)–(3.9) are satisfied at $\hat{\alpha}$ and it is easy to see that

(i) $h_1^{\mathcal{L}}(.) - \Psi_1^{\mathcal{L}}(\hat{\alpha}) t_1^{\mathcal{L}}(.)$ and $h_1^{\mathcal{U}}(.) - \Psi_1^{\mathcal{U}}(\hat{\alpha}) t_1^{\mathcal{U}}(.)$ are respectively $\partial^* h_1^{\mathcal{L}} - \Psi_1^{\mathcal{L}}(\hat{\alpha})$ $\partial^* t_1^{\mathcal{L}}$-convex and $\partial^* h_1^{\mathcal{U}} - \Psi_1^{\mathcal{U}}(\hat{\alpha}) \partial^* t_1^{U}$-convex at $\hat{\alpha}$,
(ii) $\varrho_s \omega_s(\hat{\alpha}, \kappa_s)$, for $s \in \mathcal{S}$, $\kappa_s \in K_s$ is $\partial^* \omega_s(., \kappa_s)$-convex at $\hat{\alpha}$.

Therefore, by Theorem 3.5, $\hat{\alpha} = \{0\}$ is a robust $\mathcal{LU}$-optimal solution of (RFP3).

Theorem 3.7 (Robust sufficient $\mathcal{LU}$-optimality conditions)**.** *Suppose that $\hat{\alpha}$ is a robust feasible solution of the problem (RNFIVP), then there exists $(\varrho_s)_{s \in \mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$ such that the conditions (3.7)–(3.9) hold at $\hat{\alpha}$. Also, presume that*

(i) $\tau^{\mathcal{L}}\left[{h_0}^{\mathcal{L}}(.) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(.)\right] + \tau^{\mathcal{U}}\left[{h_0}^{\mathcal{U}}(.) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(.)\right]$ *is* $\tau^{\mathcal{L}}\left[\partial^* {h_0}^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{L}}\right] + \tau^{\mathcal{U}}\left[\partial^* {h_0}^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{U}}\right]$*-pseudoconvex at $\hat{\alpha}$,*

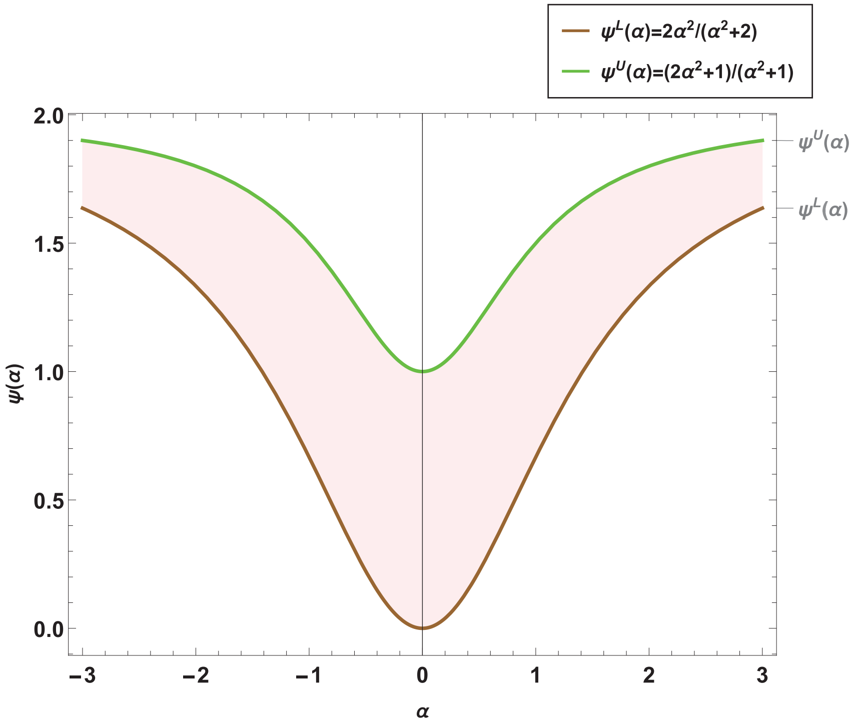

FIGURE 1. Graphical view of the objective function of the problem (RFP3).

(ii) *$\varrho_s\omega_s(\hat{\alpha},\kappa_s)$, for $s\in\mathcal{S}$, $\kappa_s\in K_s$ is $\partial^*\omega_s(.,\kappa_s)$-quasiconvex at $\hat{\alpha}$,*

then $\hat{\alpha}$ is a robust $\mathcal{LU}$-optimal solution of the (RNFIVP).

Proof. The conditions (3.7)–(3.9) hold at $\hat{\alpha}$ with Lagrange multipliers $\tau^{\mathcal{L}}>0$, $\tau^{\mathcal{U}}>0$, $(\varrho_s)_{s\in\mathcal{S}}\in\mathbb{R}_+^{(\mathcal{S})}$. Therefore, it follows from (3.7), that there exist $\iota^{\mathcal{L}}\in\partial^*{h_0}^{\mathcal{L}}(\hat{\alpha})$, $\upsilon^{\mathcal{L}}\in\partial^*{t_0}^{\mathcal{L}}(\hat{\alpha})$, $\iota^{\mathcal{U}}\in\partial^*{h_0}^{\mathcal{U}}(\hat{\alpha})$, $\upsilon^{\mathcal{U}}\in\partial^*{t_0}^{\mathcal{U}}(\hat{\alpha})$, and $\varsigma_s\in\partial^*\omega_s(\hat{\alpha},\kappa_s)$, $s\in\mathcal{S}$, $\kappa_s\in K_s$ such that

$$\tau^{\mathcal{L}}\left[\iota^{\mathcal{L}}-\Psi^{\mathcal{L}}(\hat{\alpha})\upsilon^{\mathcal{L}}\right]+\tau^{\mathcal{U}}\left[\iota^{\mathcal{U}}-\Psi^{\mathcal{U}}(\hat{\alpha})\upsilon^{\mathcal{U}}\right]+\sum_{s\in\mathcal{S}}\varrho_s\varsigma_s=0. \tag{3.22}$$

Suppose that $\hat{\alpha}$ is not a robust $\mathcal{LU}$-optimal solution of (RNFIVP), then there exists a robust feasible solution α such that

$$\left[\frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha),\frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)\right]\prec_{\mathcal{LU}}\left[\frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha}),\frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha})\right].$$

That is,

$$\begin{cases}\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha)<\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha})\\ \dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)\leq\dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha})\end{cases},\quad\text{or}\quad\begin{cases}\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha)\leq\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha})\\ \dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)<\dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha})\end{cases},\quad\text{or}\quad\begin{cases}\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha)<\dfrac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha})\\ \dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)<\dfrac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha})\end{cases},$$

which gives

$${h_0}^{\mathcal{L}}(\alpha)-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha)<{h_0}^{\mathcal{L}}(\hat{\alpha})-\Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\hat{\alpha}),$$

$$h_0{}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\alpha) \leq h_0{}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\hat{\alpha}),$$

or

$$h_0{}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\alpha) \leq h_0{}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\hat{\alpha}),$$
$$h_0{}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\alpha) < h_0{}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\hat{\alpha}),$$

or

$$h_0{}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\alpha) < h_0{}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\hat{\alpha}),$$
$$h_0{}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\alpha) < h_0{}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\hat{\alpha}).$$

By using $\tau^{\mathcal{L}} > 0$, $\tau^{\mathcal{U}} > 0$, the above inequalities yield

$$\begin{aligned}&\tau^{\mathcal{L}}\Big[h_0{}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\alpha)\Big] + \tau^{\mathcal{U}}\Big[h_0{}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\alpha)\Big]\\ <&\tau^{\mathcal{L}}\Big[h_0{}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\hat{\alpha})\Big] + \tau^{\mathcal{U}}\Big[h_0{}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\hat{\alpha})\Big].\end{aligned}$$

From assertion (i), that is, $\tau^{\mathcal{L}}\Big[h_0{}^{\mathcal{L}}(.) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(.)\Big] + \tau^{\mathcal{U}}\Big[h_0{}^{\mathcal{U}}(.) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(.)\Big]$ is $\tau^{\mathcal{L}}\Big[\partial^* h_0{}^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\partial^* t_0{}^{\mathcal{L}}\Big] + \tau^{\mathcal{U}}\Big[\partial^* h_0{}^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\partial^* t_0{}^{\mathcal{U}}\Big]$-pseudoconvex at $\hat{\alpha}$, we obtain

$$\big\langle \tau^{\mathcal{L}}\big[\iota^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\upsilon^{\mathcal{L}}\big] + \tau^{\mathcal{U}}\big[\iota^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\upsilon^{\mathcal{U}}\big], \alpha - \hat{\alpha}\big\rangle < 0, \tag{3.23}$$

for all $\iota^{\mathcal{L}} \in \partial^* h_0{}^{\mathcal{L}}(\hat{\alpha}), \upsilon^{\mathcal{L}} \in \partial^* t_0{}^{\mathcal{L}}(\hat{\alpha}), \iota^{\mathcal{U}} \in \partial^* h_0{}^{\mathcal{U}}(\hat{\alpha})$ and $\upsilon^{\mathcal{U}} \in \partial^* t_0{}^{\mathcal{U}}(\hat{\alpha})$.

By utilizing the robust feasibility of α, $\varrho_s \geq 0$, $s \in \mathcal{S}$ and (3.8), we get

$$\varrho_s \omega_s(\alpha, \kappa_s) \leq \varrho_s \omega_s(\hat{\alpha}, \kappa_s), \quad s \in \mathcal{S}.$$

The above inequality along with assertion (ii), gives

$$\langle \varrho_s \varsigma_s, \alpha - \hat{\alpha}\rangle \leq 0, \quad \text{for all } \varsigma_s \in \partial^* \omega_s(\hat{\alpha}, \kappa_s), \quad s \in \mathcal{S}. \tag{3.24}$$

Combining (3.23) and (3.24), we have

$$\left\langle \tau^{\mathcal{L}}\big[\iota^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\upsilon^{\mathcal{L}}\big] + \tau^{\mathcal{U}}\big[\iota^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\upsilon^{\mathcal{U}}\big] + \sum_{s\in\mathcal{S}} \varrho_s \varsigma_s, \alpha - \hat{\alpha}\right\rangle < 0,$$

which contradicts (3.22). Hence, $\hat{\alpha}$ is a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP). □

4. Robust parametric duality

The aforementioned robust parametric duality of the primal problem (RNFIVP) is discussed in the present section.

(RNFIVD)

$$\max\ \vartheta = \big[\vartheta^{\mathcal{L}}, \vartheta^{\mathcal{U}}\big] \tag{4.1}$$

subject to (4.2)

$$\begin{aligned}0 \in\ &\tau^{\mathcal{L}}\Big(\partial^* h_0{}^{\mathcal{L}}(\beta) - \vartheta^{\mathcal{L}}\partial^* t_0{}^{\mathcal{L}}(\beta)\Big) + \tau^{\mathcal{U}}\Big(\partial^* h_0{}^{\mathcal{U}}(\beta) - \vartheta^{\mathcal{U}}\partial^* t_0{}^{\mathcal{U}}(\beta)\Big)\\ &+ \sum_{s\in\mathcal{S}} \varrho_s \partial^* \omega_s(\beta, \kappa_s),\end{aligned} \tag{4.3}$$

$$h_0{}^{\mathcal{L}}(\beta) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(\beta) \geq 0, \tag{4.4}$$
$$h_0{}^{\mathcal{U}}(\beta) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{U}}(\beta) \geq 0, \tag{4.5}$$
$$\varrho_s \omega_s(\beta, \kappa_s) = 0, \qquad s \in \mathcal{S}, \tag{4.6}$$
$$\varrho_s \geq 0 \text{ and } \omega_s(\beta, \kappa_s) \leq 0, \quad s \in \mathcal{S}, \tag{4.7}$$

where $\beta \in \mathcal{X} \subseteq \mathbb{R}^n$, $\tau^{\mathcal{L}} > 0$, $\tau^{\mathcal{U}} > 0$, $\kappa_s \in K_s$, $\vartheta \in \mathbb{I}$, $\vartheta^{\mathcal{L}} \geq 0$, $\vartheta^{\mathcal{U}} \geq 0$. The robust feasible set of (RNFIVD) is represented by $\mathbb{M}^D$, consisting of all points which takes the form $\big(\beta, \kappa_s, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}, \vartheta\big) \in \mathcal{X} \times K_s \times (\mathbb{R}_+ \setminus \{0\}) \times (\mathbb{R}_+ \setminus \{0\}) \times \mathbb{I}$ that satisfies the constraints of (RNFIVD).

Theorem 4.1 (Weak duality)**.** *Suppose that $\alpha \in \mathbb{M}$ and $\big(\beta, \kappa_s, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}, \vartheta\big) \in \mathbb{M}^D$. Also, presume that*

(i) $\tau^{\mathcal{L}}\Big[h_0{}^{\mathcal{L}}(.) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(.)\Big] + \tau^{\mathcal{U}}\Big[h_0{}^{\mathcal{U}}(.) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{U}}(.)\Big]$ *is* $\tau^{\mathcal{L}}\Big[\partial^* h_0{}^{\mathcal{L}} - \vartheta^{\mathcal{L}} \partial^* t_0{}^{\mathcal{L}}\Big] + \tau^{\mathcal{U}}\Big[\partial^* h_0{}^{\mathcal{U}} - \vartheta^{\mathcal{U}} \partial^* t_0{}^{\mathcal{U}}\Big]$*-convex at* β*,*

(ii) $\varrho_s \omega_s(\beta, \kappa_s)$*, for* $s \in \mathcal{S}, \kappa_s \in K_s$ *is* $\partial^* \omega_s(., \kappa_s)$*-convex at* β*.*

Then

$$\Psi(\alpha) \nprec_{\mathcal{L}\mathcal{U}} \vartheta.$$

Proof. From the condition (4.3), there exist $\iota^{\mathcal{L}} \in \partial^* h_0{}^{\mathcal{L}}(\beta)$, $\upsilon^{\mathcal{L}} \in \partial^* t_0{}^{\mathcal{L}}(\beta)$, $\iota^{\mathcal{U}} \in \partial^* h_0{}^{\mathcal{U}}(\beta)$, $\upsilon^{\mathcal{U}} \in \partial^* t_0{}^{\mathcal{U}}(\beta)$ and $\varsigma_s \in \partial^* \omega_s(\beta, \kappa_s)$, $s \in \mathcal{S}, \kappa_s \in K_s$ such that

$$\tau^{\mathcal{L}}\big[\iota^{\mathcal{L}} - \vartheta^{\mathcal{L}} \upsilon^{\mathcal{L}}\big] + \tau^{\mathcal{U}}\big[\iota^{\mathcal{U}} - \vartheta^{\mathcal{U}} \upsilon^{\mathcal{U}}\big] + \sum_{s \in \mathcal{S}} \varrho_s \varsigma_s = 0. \tag{4.8}$$

Let us assume that $\Psi(\alpha) \prec_{\mathcal{L}\mathcal{U}} \vartheta$, then by the application of (4.4) and (4.5), we get

$$h_0{}^{\mathcal{L}}(\alpha) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(\alpha) < h_0{}^{\mathcal{L}}(\beta) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(\beta),$$
$$h_0{}^{\mathcal{U}}(\alpha) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{U}}(\alpha) \leq h_0{}^{\mathcal{U}}(\beta) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{U}}(\beta),$$

or

$$h_0{}^{\mathcal{L}}(\alpha) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(\alpha) \leq h_0{}^{\mathcal{L}}(\beta) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(\beta),$$
$$h_0{}^{\mathcal{U}}(\alpha) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{U}}(\alpha) < h_0{}^{\mathcal{U}}(\beta) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{U}}(\beta),$$

or

$$h_0{}^{\mathcal{L}}(\alpha) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(\alpha) < h_0{}^{\mathcal{L}}(\beta) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(\beta),$$
$$h_0{}^{\mathcal{U}}(\alpha) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{U}}(\alpha) < h_0{}^{\mathcal{U}}(\beta) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{L}}(\beta).$$

By using $\tau^{\mathcal{L}} > 0$, $\tau^{\mathcal{U}} > 0$, the above inequalities yield

$$\tau^{\mathcal{L}}\Big[h_0{}^{\mathcal{L}}(\alpha) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(\alpha)\Big] + \tau^{\mathcal{U}}\Big[h_0{}^{\mathcal{U}}(\alpha) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{U}}(\alpha)\Big] < \tau^{\mathcal{L}}\Big[h_0{}^{\mathcal{L}}(\beta) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(\beta)\Big] + \tau^{\mathcal{U}}\Big[h_0{}^{\mathcal{U}}(\beta) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{U}}(\beta)\Big].$$

From assertion (i), that is, $\tau^{\mathcal{L}}\Big[h_0{}^{\mathcal{L}}(.) - \vartheta^{\mathcal{L}} t_0{}^{\mathcal{L}}(.)\Big] + \tau^{\mathcal{U}}\Big[h_0{}^{\mathcal{U}}(.) - \vartheta^{\mathcal{U}} t_0{}^{\mathcal{U}}(.)\Big]$ is $\tau^{\mathcal{L}}\Big[\partial^* h_0{}^{\mathcal{L}} - \vartheta^{\mathcal{L}} \partial^* t_0{}^{\mathcal{L}}\Big] + \tau^{\mathcal{U}}\Big[\partial^* h_0{}^{\mathcal{U}} - \vartheta^{\mathcal{U}} \partial^* t_0{}^{\mathcal{U}}\Big]$-convex at β, we obtain

$$\big\langle \tau^{\mathcal{L}}\big[\iota^{\mathcal{L}} - \vartheta^{\mathcal{L}} \upsilon^{\mathcal{L}}\big] + \tau^{\mathcal{U}}\big[\iota^{\mathcal{U}} - \vartheta^{\mathcal{U}} \upsilon^{\mathcal{U}}\big], \alpha - \beta \big\rangle < 0, \tag{4.9}$$

for all $\iota^{\mathcal{L}} \in \partial^* {h_0}^{\mathcal{L}}(\beta)$, $\upsilon^{\mathcal{L}} \in \partial^* {t_0}^{\mathcal{L}}(\beta)$, $\iota^{\mathcal{U}} \in \partial^* {h_0}^{\mathcal{U}}(\beta)$, $\upsilon^{\mathcal{U}} \in \partial^* {t_0}^{\mathcal{U}}(\beta)$.

By utilizing the robust feasibility of β, $\varrho_s \geq 0$, $s \in \mathcal{S}$ and (4.6), we get

$$\varrho_s \omega_s(\alpha, \kappa_s) \leq \varrho_s \omega_s(\beta, \kappa_s), \ s \in \mathcal{S}.$$

The above inequality along with assertion (ii), gives

$$\langle \varrho_s \varsigma_s, \alpha - \beta \rangle \leq 0, \quad \text{for all } \varsigma_s \in \partial^* \omega_s(\beta, \kappa_s), \ s \in \mathcal{S}. \tag{4.10}$$

Combining (4.9) and (4.10), we have

$$\left\langle \tau^{\mathcal{L}}\left[\iota^{\mathcal{L}} - \vartheta^{\mathcal{L}} \upsilon^{\mathcal{L}}\right] + \tau^{\mathcal{U}}\left[\iota^{\mathcal{U}} - \vartheta^{\mathcal{U}} \upsilon^{\mathcal{U}}\right] + \sum_{s \in S} \varrho_s \varsigma_s, \alpha - \beta \right\rangle < 0,$$

which contradicts (4.8). Thus, this concludes the proof of the theorem. □

Corollary 4.2. *If in addition to the presumptions of Theorem 4.1, we consider* $\hat{\alpha} \in \mathbb{M}$ *and* $\left(\hat{\beta}, \hat{\kappa}_s, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\vartheta}\right) \in \mathbb{M}^D$ *such that* $\hat{\vartheta}^{\mathcal{L}} = \Psi^{\mathcal{L}}(\hat{\beta})$, $\hat{\vartheta}^{\mathcal{U}} = \Psi^{\mathcal{U}}(\hat{\beta})$, *that is,* $\hat{\vartheta}^{\mathcal{L}} = \frac{{h_0}^{\mathcal{L}}(\hat{\beta})}{{t_0}^{\mathcal{L}}(\hat{\beta})}$, $\hat{\vartheta}^{\mathcal{U}} = \frac{{h_0}^{\mathcal{U}}(\hat{\beta})}{{t_0}^{\mathcal{U}}(\hat{\beta})}\cdot$ *Then* $\hat{\alpha}$ *and* $\left(\hat{\beta}, \hat{\kappa}_s, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\vartheta}\right)$ *are robust* $\mathcal{LU}$*-optimal solution of the problems (RNFIVP) and (RNFIVD), respectively.*

Theorem 4.3 (Strong duality)**.** *. Suppose that* $\hat{\alpha}$ *is a robust* $\mathcal{LU}$*-optimal solution of the problem (RNFIVP) and both the Slater's constraint qualification for the problems (RFP1) and (RFP2) are satisfied at* $\hat{\alpha}$*. Then there exist* $\hat{\tau}^{\mathcal{L}} > 0$, $\hat{\tau}^{\mathcal{U}} > 0$, $\hat{\kappa}_s \in K_s, \ s \in S$ *and* $\hat{\vartheta} \in \mathbb{I}$ *such that* $\left(\hat{\alpha}, \hat{\kappa}_s, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\vartheta}\right) \in \mathbb{M}^D$*. Also, if the presumptions of Theorem 4.1 hold for all robust feasible solution of the problems (RNFIVP) and (RNFIVD), respectively then* $\left(\hat{\alpha}, \hat{\kappa}_s, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\vartheta}\right)$ *is a robust* $\mathcal{LU}$*-optimal solution of the problem (RNFIVD).*

Proof. Since $\hat{\alpha}$ is a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP) and both the Slater's constraint qualification for the problems (RFP1) and (RFP2) are satisfied at $\hat{\alpha}$, therefore, by the Theorem 3.4, there exist $\hat{\tau}^{\mathcal{L}} > 0$, $\hat{\tau}^{\mathcal{U}} > 0$ $\hat{\kappa}_s \in K_s, \ s \in S$, $\hat{\vartheta} \in \mathbb{I}$ and $(\hat{\varrho}_s)_{s \in \mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$ such that $\left(\hat{\alpha}, \hat{\kappa}_s, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\vartheta}\right)$ satisfies the conditions (3.7)–(3.9). We set $\hat{\vartheta}^{\mathcal{L}} = \Psi^{\mathcal{L}}(\hat{\alpha})$, $\hat{\vartheta}^{\mathcal{U}} = \Psi^{\mathcal{U}}(\hat{\alpha})$. This implies $\left(\hat{\alpha}, \hat{\kappa}_s, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\vartheta}\right) \in \mathbb{M}^D$. Therefore, by application of Corollary 4.2, we conclude that $\left(\hat{\alpha}, \hat{\kappa}_s, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\vartheta}\right)$ is a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVD). Thus, this concludes the proof of the theorem. □

Theorem 4.4 (Strict converse duality)**.** *Suppose that* $\hat{\alpha}$ *and* $\left(\hat{\beta}, \hat{\kappa}_s, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\vartheta}\right)$ *are robust* $\mathcal{LU}$*-optimal solution of the problems (RNFIVP) and (RNFIVD), respectively with* $\hat{\vartheta}^{\mathcal{L}} = \Psi^{\mathcal{L}}(\hat{\alpha})$, $\hat{\vartheta}^{\mathcal{U}} = \Psi^{\mathcal{U}}(\hat{\alpha})$*. Also, presume that*

(i) $\hat{\tau}^{\mathcal{L}}\left[{h_0}^{\mathcal{L}}(.) - \hat{\vartheta}^{\mathcal{L}}{t_0}^{\mathcal{L}}(.)\right] + \hat{\tau}^{\mathcal{U}}\left[{h_0}^{\mathcal{U}}(.) - \hat{\vartheta}^{\mathcal{U}}{t_0}^{\mathcal{U}}(.)\right]$ *is* $\hat{\tau}^{\mathcal{L}}\left[\partial^* {h_0}^{\mathcal{L}} - \hat{\vartheta}^{\mathcal{L}}\partial^* {t_0}^{\mathcal{L}}\right] + \hat{\tau}^{\mathcal{U}}\left[\partial^* {h_0}^{\mathcal{U}} - \hat{\vartheta}^{\mathcal{U}}\partial^* {t_0}^{\mathcal{U}}\right]$*-strictly convex at* $\hat{\beta}$*,*

(ii) $\hat{\varrho}_s \omega_s(\hat{\beta}, \hat{\kappa}_s)$*, for* $s \in \mathcal{S}, \hat{\kappa}_s \in K_s$ *is* $\partial^* \omega_s(., \hat{\kappa}_s)$*-convex at* $\hat{\beta}$*,*

then $\hat{\alpha} = \hat{\beta}$*.*

Proof. Presume that $\hat{\alpha} \neq \hat{\beta}$. Since $\left(\hat{\beta}, \hat{\kappa}_s, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\vartheta}\right) \in \mathbb{M}^D$, from (4.3) to (4.6), we have the following observations

$$\hat{\tau}^{\mathcal{L}}\left[\iota^{\mathcal{L}} - \hat{\vartheta}^{\mathcal{L}} \upsilon^{\mathcal{L}}\right] + \hat{\tau}^{\mathcal{U}}\left[\iota^{\mathcal{U}} - \hat{\vartheta}^{\mathcal{U}} \upsilon^{\mathcal{U}}\right] + \sum_{s \in \mathcal{S}} \hat{\varrho}_s \varsigma_s = 0. \tag{4.11}$$

$$\hat{\tau}^{\mathcal{L}}[{h_0}^{\mathcal{L}}(\hat{\beta}) - \hat{\vartheta}^{\mathcal{L}}{t_0}^{\mathcal{L}}(\hat{\beta})] + \hat{\tau}^{\mathcal{U}}[{h_0}^{\mathcal{U}}(\hat{\beta}) - \hat{\vartheta}^{\mathcal{U}}{t_0}^{\mathcal{U}}(\hat{\beta})] + \hat{\varrho}_s\omega_s(\hat{\beta}, \hat{\kappa}_s) \geq 0, \tag{4.12}$$

where $\iota^{\mathcal{L}} \in \partial^* {h_0}^{\mathcal{L}}(\hat{\beta})$, $\upsilon^{\mathcal{L}} \in \partial^* {t_0}^{\mathcal{L}}(\hat{\beta})$, $\iota^{\mathcal{U}} \in \partial^* {h_0}^{\mathcal{U}}(\hat{\beta})$, $\upsilon^{\mathcal{U}} \in \partial^* {t_0}^{\mathcal{U}}(\hat{\beta})$, $\varsigma_s \in \partial^*\omega_s(\hat{\beta}, \hat{\kappa}_s)$.

Using the presumptions, that is, $\hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(.) - \hat{\vartheta}^{\mathcal{L}}{t_0}^{\mathcal{L}}(.)\Big] + \hat{\tau}^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(.) - \hat{\vartheta}^{\mathcal{U}}{t_0}^{\mathcal{U}}(.)\Big]$ is $\hat{\tau}^{\mathcal{L}}\Big[\partial^* {h_0}^{\mathcal{L}} - \hat{\vartheta}^{\mathcal{L}}\partial^* {t_0}^{\mathcal{L}}\Big] + \hat{\tau}^{\mathcal{U}}\Big[\partial^* {h_0}^{\mathcal{U}} - \hat{\vartheta}^{\mathcal{U}}\partial^* {t_0}^{\mathcal{U}}\Big]$-strictly convex at $\hat{\beta}$ and $\hat{\varrho}_s\omega_s(\hat{\beta}, \hat{\kappa}_s)$, for $s \in \mathcal{S}, \hat{\kappa}_s \in K_s$ is $\partial^*\omega_s(., \hat{\kappa}_s)$-convex at $\hat{\beta}$ and applying (4.11), we obtain

$$\begin{aligned} &\hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(\hat{\alpha}) - \hat{\vartheta}^{\mathcal{L}}{t_0}^{\mathcal{L}}(\hat{\alpha})\Big] + \hat{\tau}^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(\hat{\alpha}) - \hat{\vartheta}^{\mathcal{U}}{t_0}^{\mathcal{U}}(\hat{\alpha})\Big] + \hat{\varrho}_s\omega_s(\hat{\alpha}, \hat{\kappa}_s) - \\ &\Big(\hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(\hat{\beta}) - \hat{\vartheta}^{\mathcal{L}}{t_0}^{\mathcal{L}}(\hat{\beta})\Big] + \hat{\tau}^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(\hat{\beta}) - \hat{\vartheta}^{\mathcal{U}}{t_0}^{\mathcal{U}}(\hat{\beta})\Big] + \hat{\varrho}_s\omega_s(\hat{\beta}, \hat{\kappa}_s)\Big) > 0. \end{aligned} \tag{4.13}$$

Further, by using the presumptions $\hat{\vartheta}^{\mathcal{L}} = \Psi^{\mathcal{L}}(\hat{\alpha})$, $\hat{\vartheta}^{\mathcal{U}} = \Psi^{\mathcal{U}}(\hat{\alpha})$, that is, $\hat{\vartheta}^{\mathcal{L}} = \frac{{h_0}^{\mathcal{L}}(\hat{\alpha})}{{t_0}^{\mathcal{L}}(\hat{\alpha})}$, $\hat{\vartheta}^{\mathcal{U}} = \frac{{h_0}^{\mathcal{U}}(\hat{\alpha})}{{t_0}^{\mathcal{U}}(\hat{\alpha})}$ and $\hat{\varrho}_s\omega_s(\hat{\alpha}, \hat{\kappa}_s) = 0$ in (4.13), we get a contradiction to (4.12). Thus, this concludes the proof of the theorem. □

5. Lagrangian type function and saddle-point analysis

In the present section, for the robust feasible solution $\hat{\alpha} \in \mathbb{M}$, we define the Lagrangian type function for the primal problem (RNFIVP) as stated below:

$$\begin{aligned} \mathfrak{L}\big(\alpha, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}, \varrho_s\big) = \tau^{\mathcal{L}}\Big({h_0}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha)\Big) + \tau^{\mathcal{U}}\Big({h_0}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\alpha)\Big) \\ + \sum_{s \in \mathcal{S}} \varrho_s\omega_s(\alpha, \kappa_s), \end{aligned}$$

where $\alpha \in \mathcal{X}$, $\tau^{\mathcal{L}} \geq 0$, $\tau^{\mathcal{U}} \geq 0$ and $(\varrho_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$. A saddle-point of $\mathfrak{L}\big(\alpha, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}, \varrho_s\big)$ and its relation to the problem (RNFIVP) is defined as follows.

Definition 5.1. A point $\big(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\big) \in \mathcal{X} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^{(\mathcal{S})}$ is termed as a saddle point for $\mathfrak{L}\big(\alpha, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}, \varrho_s\big)$, if

(i) $\mathfrak{L}\big(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \varrho_s\big) \leq \mathfrak{L}\big(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\big)$, for all $(\varrho_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$,
(ii) $\mathfrak{L}\big(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\big) \leq \mathfrak{L}\big(\alpha, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\big)$, for all $\alpha \in \mathcal{X}$.

Theorem 5.2. *Let $\hat{\tau}^{\mathcal{L}} > 0$, $\hat{\tau}^{\mathcal{U}} > 0$ and $\big(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\big)$ be a saddle point for $\mathfrak{L}\big(\alpha, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}, \varrho_s\big)$. Then $\hat{\alpha}$ is a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP).*

Proof. Suppose that $\hat{\alpha}$ is not a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP), then there exists a robust feasible solution α such that

$$\left[\frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha), \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha)\right] \prec_{\mathcal{LU}} \left[\frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha}), \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha})\right].$$

That is,

$$\begin{cases} \frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha) < \frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha}) \\ \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha) \leq \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha}) \end{cases}, \text{ or } \begin{cases} \frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha) \leq \frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha}) \\ \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha) < \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha}) \end{cases}, \text{ or } \begin{cases} \frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\alpha) < \frac{{h_0}^{\mathcal{L}}}{{t_0}^{\mathcal{L}}}(\hat{\alpha}) \\ \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\alpha) < \frac{{h_0}^{\mathcal{U}}}{{t_0}^{\mathcal{U}}}(\hat{\alpha}) \end{cases},$$

which implies

$$\begin{aligned} {h_0}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha) < {h_0}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\hat{\alpha}), \\ {h_0}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\alpha) \leq {h_0}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\hat{\alpha}), \end{aligned}$$

or

$$h_0{}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\alpha) \leq h_0{}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\hat{\alpha}),$$
$$h_0{}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\alpha) < h_0{}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\hat{\alpha}),$$

or

$$h_0{}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\alpha) < h_0{}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\hat{\alpha}),$$
$$h_0{}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\alpha) < h_0{}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\hat{\alpha}).$$

The above inequalities along with $\hat{\tau}^{\mathcal{L}} > 0$, $\hat{\tau}^{\mathcal{U}} > 0$ yield

$$\begin{aligned}&\hat{\tau}^{\mathcal{L}}\left[h_0{}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\alpha)\right] + \hat{\tau}^{\mathcal{U}}\left[h_0{}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\alpha)\right]\\ <&\hat{\tau}^{\mathcal{L}}\left[h_0{}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\hat{\alpha})\right] + \hat{\tau}^{\mathcal{U}}\left[h_0{}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\hat{\alpha})\right].\end{aligned} \tag{5.1}$$

Since $\left(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\right)$ is a saddle point for $\mathfrak{L}\left(\alpha, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}, \varrho_s\right)$, then by Definition 5.1(i), we get

$$\mathfrak{L}\left(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \varrho_s\right) \leq \mathfrak{L}\left(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\right),$$

that is,

$$\sum_{s\in\mathcal{S}} \varrho_s \omega_s(\hat{\alpha}, \hat{\kappa}_s) \leq \sum_{s\in\mathcal{S}} \hat{\varrho}_s \omega_s(\hat{\alpha}, \hat{\kappa}_s). \tag{5.2}$$

Taking $(\varrho_s)_{s\in\mathcal{S}} = (\hat{\varrho}_s + 1)_{s\in\mathcal{S}}$ in the above inequality (5.2), we obtain

$$\omega_s(\hat{\alpha}, \hat{\kappa}_s) \leq 0, \quad s \in \mathcal{S}.$$

This implies, $\hat{\alpha}$ is a robust feasible solution of the problem (RNFIVP).

Using $\hat{\varrho}_s \in \mathbb{R}_+^{(\mathcal{S})}$, the above inequality gives

$$\hat{\varrho}_s \omega_s(\hat{\alpha}, \hat{\kappa}_s) \leq 0, \quad s \in \mathcal{S}. \tag{5.3}$$

Using $(\varrho_s)_{s\in\mathcal{S}} = 0$, $s \in \mathcal{S}$ in the inequality (5.2), we obtain

$$\hat{\varrho}_s \omega_s(\hat{\alpha}, \hat{\kappa}_s) \geq 0, \quad s \in \mathcal{S}. \tag{5.4}$$

The inequalities (5.3) and (5.4) yield

$$\hat{\varrho}_s \omega_s(\hat{\alpha}, \hat{\kappa}_s) = 0, \quad s \in \mathcal{S}. \tag{5.5}$$

Since $\left(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\right)$ is a saddle point for $\mathfrak{L}(\alpha, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}, \varrho_s)$, then by Definition 5.1 (ii), we get

$$\mathfrak{L}\left(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\right) \leq \mathfrak{L}\left(\alpha, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\right).$$

That is,

$$\begin{aligned}&\hat{\tau}^{\mathcal{L}}\left(h_0{}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\hat{\alpha})\right) + \hat{\tau}^{\mathcal{U}}\left(h_0{}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\hat{\alpha})\right) + \sum_{s\in\mathcal{S}}\hat{\varrho}_s \omega_s(\hat{\alpha}, \hat{\kappa}_s)\\ \leq&\hat{\tau}^{\mathcal{L}}\left(h_0{}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha})t_0{}^{\mathcal{L}}(\alpha)\right) + \hat{\tau}^{\mathcal{U}}\left(h_0{}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha})t_0{}^{\mathcal{U}}(\alpha)\right) + \sum_{s\in\mathcal{S}}\hat{\varrho}_s \omega_s(\alpha, \hat{\kappa}_s).\end{aligned}$$

By utilizing the robust feasibility of α of the problem (RNFIVP) together with $(\hat{\varrho}_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$ and (5.5), the above inequality yield

$$\hat{\tau}^{\mathcal{L}}\Big({h_0}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\hat{\alpha})\Big) + \hat{\tau}^{\mathcal{U}}\Big({h_0}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\hat{\alpha})\Big)$$
$$\leq \hat{\tau}^{\mathcal{L}}\Big({h_0}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha)\Big) + \hat{\tau}^{\mathcal{U}}\Big({h_0}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\alpha)\Big),$$

which contradicts (5.1). Thus, this concludes the proof of the theorem. □

Theorem 5.3. *Suppose that $\hat{\alpha}$ is a robust $\mathcal{LU}$-optimal solution of the problem (RNFIVP), there exists $(\hat{\varrho}_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$ such that the conditions (3.7)–(3.9) hold at $\hat{\alpha}$. Also, presume that*

(i) $\hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(.) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(.)\Big] + \hat{\tau}^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(.) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(.)\Big]$ *is* $\hat{\tau}^{\mathcal{L}}\Big[\partial^* {h_0}^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{L}}\Big] + \hat{\tau}^{\mathcal{U}}\Big[\partial^* {h_0}^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{U}}\Big]$*-convex at* $\hat{\alpha}$.

(ii) $\hat{\varrho}_s\omega_s(\hat{\alpha}, \hat{\kappa})$, *for* $s \in \mathcal{S}, \hat{\kappa}_s \in K_s$ *is* $\partial^*\omega_s(., \hat{\kappa}_s)$*-convex at* $\hat{\alpha}$

then $\big(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\big)$ *is a saddle point for* $\mathfrak{L}(\alpha, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}, \varrho_s)$.

Proof. The conditions (3.7)–(3.9) hold at $\hat{\alpha}$ with Lagrange multipliers $\hat{\tau}^{\mathcal{L}} > 0$, $\hat{\tau}^{\mathcal{U}} > 0$, $(\hat{\varrho}_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$. Therefore, it follows from (3.7), that there exist $\iota^{\mathcal{L}} \in \partial^* {h_0}^{\mathcal{L}}(\hat{\alpha})$, $\upsilon^{\mathcal{L}} \in \partial^* {t_0}^{\mathcal{L}}(\hat{\alpha})$, $\iota^{\mathcal{U}} \in \partial^* {h_0}^{\mathcal{U}}(\hat{\alpha})$, $\upsilon^{\mathcal{U}} \in \partial^* {t_0}^{\mathcal{U}}(\hat{\alpha})$ and $\varsigma_s \in \partial^*\omega_s(\hat{\alpha}, \kappa_s)$, $s \in \mathcal{S}$ such that

$$\hat{\tau}^{\mathcal{L}}\big[\iota^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\upsilon^{\mathcal{L}}\big] + \hat{\tau}^{\mathcal{U}}\big[\iota^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\upsilon^{\mathcal{U}}\big] + \sum_{s\in\mathcal{S}} \hat{\varrho}_s\varsigma_s = 0. \tag{5.6}$$

From assertion (i), that is, $\hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(.) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(.)\Big] + \hat{\tau}^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(.) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(.)\Big]$ is $\hat{\tau}^{\mathcal{L}}\Big[\partial^* {h_0}^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{L}}\Big] + \hat{\tau}^{\mathcal{U}}\Big[\partial^* {h_0}^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\partial^* {t_0}^{\mathcal{U}}\Big]$-convex at $\hat{\alpha}$, we obtain

$$\begin{aligned}&\hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha)\Big] - \hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\hat{\alpha})\Big] + \hat{\tau}^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\alpha)\Big]\\ &-\hat{\tau}^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\hat{\alpha})\Big] \geq \big\langle \hat{\tau}^{\mathcal{L}}\big[\iota^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\upsilon^{\mathcal{L}}\big] + \hat{\tau}^{\mathcal{U}}\big[\iota^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\upsilon^{\mathcal{U}}\big], \alpha - \hat{\alpha}\big\rangle,\end{aligned} \tag{5.7}$$

for all $\iota^{\mathcal{L}} \in \partial^* {h_0}^{\mathcal{L}}(\hat{\alpha}), \upsilon^{\mathcal{L}} \in \partial^* {t_0}^{\mathcal{L}}(\hat{\alpha}), \iota^{\mathcal{U}} \in \partial^* {h_0}^{\mathcal{U}}(\hat{\alpha})$ and $\upsilon^{\mathcal{U}} \in \partial^* {t_0}^{\mathcal{U}}(\hat{\alpha})$

From assertion (ii), we get

$$\sum_{s\in\mathcal{S}} \hat{\varrho}_s\omega_s(\alpha, \hat{\kappa}_s) - \sum_{s\in\mathcal{S}} \hat{\varrho}_s\omega_s(\hat{\alpha}, \hat{\kappa}_s) \geq \Big\langle \sum_{s\in\mathcal{S}} \hat{\varrho}_s\varsigma_s, \alpha - \hat{\alpha}\Big\rangle, \tag{5.8}$$

for all $\varsigma_s \in \partial^*\omega_s(\hat{\alpha}, \hat{\kappa}_s)$, $s \in \mathcal{S}$.

Combining (5.7) and (5.8), we have

$$\begin{aligned}&\hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha)\Big] + \hat{\tau}^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\alpha)\Big] + \sum_{s\in\mathcal{S}} \hat{\varrho}_s\omega_s(\alpha, \hat{\kappa}_s)\\ &-\Bigg[\hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\hat{\alpha})\Big] + \hat{\tau}^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\hat{\alpha})\Big] + \sum_{s\in\mathcal{S}} \hat{\varrho}_s\omega_s(\hat{\alpha}, \hat{\kappa}_s)\Bigg]\\ &\geq \Big\langle \hat{\tau}^{\mathcal{L}}\big[\iota^{\mathcal{L}} - \Psi^{\mathcal{L}}(\hat{\alpha})\upsilon^{\mathcal{L}}\big] + \hat{\tau}^{\mathcal{U}}\big[\iota^{\mathcal{U}} - \Psi^{\mathcal{U}}(\hat{\alpha})\upsilon^{\mathcal{U}}\big] + \sum_{s\in\mathcal{S}} \hat{\varrho}_s\varsigma_s,\ \alpha - \hat{\alpha}\Big\rangle.\end{aligned}$$

Using (5.6) in the above inequality, we get

$$\hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(\alpha) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\alpha)] + \hat{\tau}^{\mathcal{U}}[{h_0}^{\mathcal{U}}(\alpha) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\alpha)\Big] + \sum_{s\in\mathcal{S}} \hat{\varrho}_s \omega_s(\alpha, \hat{\kappa}_s)$$

$$\geq \left[\hat{\tau}^{\mathcal{L}}\Big[{h_0}^{\mathcal{L}}(\hat{\alpha}) - \Psi^{\mathcal{L}}(\hat{\alpha}){t_0}^{\mathcal{L}}(\hat{\alpha})\Big] + \hat{\tau}^{\mathcal{U}}\Big[{h_0}^{\mathcal{U}}(\hat{\alpha}) - \Psi^{\mathcal{U}}(\hat{\alpha}){t_0}^{\mathcal{U}}(\hat{\alpha})\Big] + \sum_{s\in\mathcal{S}} \hat{\varrho}_s \omega_s(\hat{\alpha}, \hat{\kappa}_s)\right],$$

that is,

$$\mathfrak{L}\big(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\big) \leq \mathfrak{L}\big(\alpha, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\big). \tag{5.9}$$

By utilizing the robust feasibility of $\hat{\alpha}$ of the problem (RNFIVP) and $(\varrho_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{(\mathcal{S})}$, we have

$$\varrho_s \omega_s(\hat{\alpha}, \hat{\kappa}_s) \leq 0, \quad s \in \mathcal{S}. \tag{5.10}$$

By using (5.10) and the optimality conditions (3.8), we get

$$\mathfrak{L}\big(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \varrho_s\big) \leq \mathfrak{L}\big(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\big). \tag{5.11}$$

The inequalities (5.9) and (5.11) implies $\big(\hat{\alpha}, \hat{\tau}^{\mathcal{L}}, \hat{\tau}^{\mathcal{U}}, \hat{\varrho}_s\big)$ is a saddle point for $\mathfrak{L}\big(\alpha, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}, \varrho_s\big)$. Thus, this concludes the proof of the theorem. □

6. Special Cases

(i) If $\nu^{\mathcal{L}}(\alpha) = \nu^{\mathcal{U}}(\alpha) = 1$, then (RNFIVP) model reduces to (P1) model of Jaichander *et al.* [18].

$$\text{(P1)} \qquad \min\ \big[\zeta^{\mathcal{L}}(\alpha), \zeta^{\mathcal{U}}(\alpha)\big]$$
$$\text{subject to}$$
$$\omega_s(\alpha, \kappa_s) \leq 0,\ \forall\ s \in \mathcal{S},\ \kappa_s \in K_s,$$
$$\alpha \in \mathcal{X} \subseteq \mathbb{R}^n.$$

(ii) When the uncertain parameter κ is absent in the constraints, then (UNFIVP-1) model takes the form of (P2) model of Debnath and Gupta [9] and Rani and Kummari [23, 24].

$$\text{(P2)} \qquad \min\ \left[\frac{\zeta^{\mathcal{L}}(\alpha)}{\nu^{\mathcal{U}}(\alpha)}, \frac{\zeta^{\mathcal{U}}(\alpha)}{\nu^{\mathcal{L}}(\alpha)}\right]$$
$$\text{subject to}$$
$$\omega_s(\alpha) \leq 0, \quad \forall\ s = 1, 2, \ldots, m,$$
$$\alpha \in \mathcal{X} \subseteq \mathbb{R}^n.$$

(iii) From Dutta and Chandra [12, 13] and Jeykumar and Luc [20], it is evident that the convexificators are not necessarily compact or convex. These exemptions permit applications to an extensive category of nonsmooth continuous functions. Thus, the results of this paper are sharper when compared to the other papers like robust interval-valued optimization problems with uncertain inequality constraints [17, 18].

7. Conclusion

In this article, the notion of convexificator is employed to discuss the optimality conditions for a robust fractional interval-valued optimization problem with uncertainty constraints (RNFIVP) and an example is provided to illustrate sufficient optimality criteria. Furthermore, the robust parametric duality of (RNFIVP) is described and the duality results between the primal and dual problems are investigated by utilizing the concept of convexity. Finally, the equivalence between the saddle point criteria of a Lagrangian type function and a robust $\mathcal{LU}$-optimal solution of (RNFIVP) with convexity is also examined. The approaches used in this study, from our viewpoint, can be used to demonstrate equivalent outcomes for other types of fractional programming problems involving convexificators. This could be the focus of research in the future.

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