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# NONLINEAR FUZZY FRACTIONAL SIGNOMIAL PROGRAMMING PROBLEM: A FUZZY GEOMETRIC PROGRAMMING SOLUTION APPROACH

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**Abstract.** Fuzzy fractional signomial programming problem is a relatively new optimization problem. In real world problems, some variables may vacillate because of various reasons. To tackle these vacillating variables, vagueness is considered in form of fuzzy sets. In this paper, a nonlinear fuzzy fractional signomial programming problem is considered with all its coefficients in objective functions as well as constraints are fuzzy numbers. Two solution approaches are developed based on signomial geometric programming comprising nearest interval approximation with parametric interval valued functions and fuzzy $\alpha$-cut with min–max approach. To demonstrate the proposed methods, two illustrative numerical examples are solved and the results are comparatively discussed showing its feasibility and effectiveness.

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## 1. Introduction

Optimization models for real-life applications such as engineering analysis, risk management, finance, industrial engineering and aircraft design, mostly have signomial terms either in their objective functions or constraints, or in both. Optimization models having signomial terms are known as signomial programming problems (SPPs). Duffin and Peterson [11] introduced the signomial programming problem in 1973. In general, signomial programming problems are considered as generalisations of geometric programming problems in which coefficients as well as the exponents of the variables are unrestricted in sign whereas in geometric programming, the coefficients of variables must be positive real numbers and the exponents of variables are unrestricted in sign. Geometric programming was introduced by Duffin *et al.* [10] in 1967. Signomial terms may also occur in real-life nonlinear fractional programming problems. In an optimization problem, if the objective function exists as the ratio of two linear or nonlinear functions then it is mathematically interpreted as a fractional programming problem. For the first time, balancing fractional programming models were presented by Neumann [33] in 1937 for an expanding economy. As it is quite burdensome to solve fractional programming problems directly, Charnes and Cooper [4] derived an effective variable transformation method to equivalently convert LFP into LPP. By following this masterpiece, Borza *et al.* [2] proposed a method to solve LFPs having interval coefficients. Dorn

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[9], Swarup [31], Wagner and Yuan [34], Sharma *et al.* [29], Pandey and Punnen [26] derived some algorithms based on simplex method to optimise fractional linear programming problems. Practically, the feasible values of parameters involved in an optimization model could not be characterised exactly because of the absence of accessible data. To tackle the fluctuation of parameters, the vagueness of these parameters is considered. The fuzzy set concept is generally suitable to manage such indefinite data. Yang *et al.* [36] developed an algorithm by using the superiority and inferiority measures method (SIMM) to solve a fuzzy multi-objective linear fractional programming problem. Dinkelbach [8] developed an iterative process to solve a non-linear FPP. Nayak and Ojha [24] proposed a solution approach to bi-level nonlinear FPP using the concept of fuzzy and TOPSIS. Zahmatkesh and Cao [37] proposed a method to find the solution to a fuzzy fractional geometric programming problem. Islam and Mandal [13] described some effective methods to find the solution of fuzzy geometric programming problem (FGPP) as well as fuzzy signomial programming problem. They discussed the basics of geometric programming, signomial programming, and fuzzy programming. Yang and Cao [35] discussed the origin of fuzzy GP, some recent research findings and the applications of fuzzy GP in environmental engineering, power systems, economic management, etc. are also presented. Ota and Ojha [25] formulated a multi-objective geometric programming problem and optimised it by using the weighted sum method. In the final step, the obtained results are compared with the results obtained by fuzzy programming. Jafarian *et al.* [14] proposed a method to solve multi-objective nonlinear geometric programming problems by integrating the concept of intuitionistic fuzzy. Mishra and Ota [18] provided a novel way of addressing non-linear fractional programming problems based on the variable transformation method and signomial geometric programming technique. Maiti *et al.* [17] suggested a way to solve multi-objective linear fractional programming problems with fuzzy coefficients and fuzzy variables using Taylor series approximation and a normalisation methodology. The idea of fuzzy $\alpha$-cut was adopted to defuzzify the model. Dey *et al.* [7] developed a method for solving linear fractional bi-level multi-objective decision making problem using a hybrid TOPSIS (order preference by similarity to ideal solution) algorithm and fuzzy goal programming approach. Liu [15] used Zadeh's extension principle to develop a method to transform a fuzzy geometric programming problem into a pair of two-level mathematical programs and considered two geometric programming problems in which each of the parameters of the objective function as well as the constraints are fuzzy numbers. Liu [16] presented a method to find the fuzzy objective value of a fuzzy machining economic problem in which some of the parameters are fuzzy numbers. A rudimentary decision-making framework was provided by Midya *et al.* [20] for the multiple objective fuzzy fractional fixed-charge transportation problem. The multi-objective nonlinear transportation problem is transformed into its linear form using a modified transformation approach and the desirable optimal solution is determined using the fuzzy chance-constrained rough approximation method. Using a green supply chain network system and an intuitionistic fuzzy environment, Midya *et al.* [21] offered a novel research of the multi-stage multi-objective fixed-charge solid transportation problem. Interval programming was used by Midya and Roy [19] to analyze a fixed-charge transportation problem considering interval and rough interval uncertainties. Using fuzzy rough variables as the coefficients of the objective functions and constraints, Roy *et al.* [28] studied the multi-objective multi-item fixed-charge solid transportation problem. Fuzzy-rough expected value operator was used in the process of converting fuzzy-rough model into deterministic form. A multi-objective fixed-charge transportation problem was studied by Roy *et al.* [27]. In this study, the supply and demand parameters are rough variables whereas the parameters in the objective function are random rough variables. The proposed approach used the anticipated value operator to deal with ambiguous parameters. In addition, a method was used for transforming the uncertain problem into deterministic form. Besides, three alternative techniques namely fuzzy programming, global criterion and $\epsilon$-constrained methods were implemented to derive the best compromise solution for the proposed model. Geometric programming problem was used to figure out the upper and lower bounds of the unit production cost at a possibility level of $\alpha$. Banerjee and Roy [1] used intuitionistic fuzzy geometric programming to solve a multi-inventory problem having uniform lead-time demand and some numerical examples are formulated to analyse how fuzzy geometric programming techniques are better than fuzzy nonlinear programming problems. Ghavami *et al.* [12] proposed a new gate sizing method for power minimization of digital circuits. This gate sizing method is based on the fuzzy geometric programming approach. The result of this paper shows that

the power reduction in this proposed technique is greater as compared to a fuzzy linear programming problem. Nasseri and Alizadeh [23] stated the problem of a two-bar truss in which some parameters are fuzzy and they used a fuzzy geometric programming problem to derive a fuzzy objective value. Cao and Wang [3] discussed recent research and development on fuzzy geometric programming problems and also focused on captivating new researchers to the FGPP field.

In this paper, a non-linear fuzzy fractional signomial programming problem (NLFFSPP) is studied and two solution approaches are proposed to generate a set of solutions. To the best of author's knowledge, no other solution methodology is developed in literature to solve such problems which shows the novelty of the proposed work. There are different solution methodologies of nonlinear fractional programming problems (NLFPP) with fraction of posynomials or signomials in literature but it is believed that NLFPP with signomials along with fuzzy parameters has no solution methodology. Besides, Geometric programming approach has been implemented to solve NLFPP in literature but not used so far to solve NLFPP with fraction of signomials and fuzzy parameters. These research gaps motivate the authors to work on such solution methodologies.

The organization of this paper is as follows: accompanying the introduction, preliminary discussions are incorporated in about fuzzy concepts, variable transformation method and min–max approach in Section 2. Mathematical formulation of NLFFSPP is included in Section 3. Proposed solution approaches with detailed algorithmic description and flowchart representation are included in Section 4. Section 5 comprises two illustrative numerical examples solved using the proposed methods and their result analysis. Finally, Section 6 contains the conclusions drawn from this work.

## 2. Preliminaries

In this section, some basic concepts of fuzzy have been discussed which are required in developing the proposed solution approaches.

**Definition 1** ([38])**.** If $X$ is a nonempty set, then a fuzzy set $\bar{A}$ in $X$ is a set of ordered pairs, $\bar{A} = \{(x, \mu_{\bar{A}}(x)) : x \in X\}$, where $\mu_{\bar{A}}(x)$ is called membership function of $x \in \bar{A}$ such that $\mu_{\bar{A}}(x) : X \to [0, 1]$.

**Definition 2** ([38])**.** The support of a fuzzy set $\bar{A}$ is the collection of all points $x$ in $X$ such that $\mu_{\bar{A}}(x) > 0$.

**Definition 3** ([38])**.** The height $h\bar{A}$ of $\bar{A}$ is nothing but the maximum value of the membership degree.

**Definition 4** ([38])**.** If height $h(\bar{A}) = 1$, then the fuzzy set is a normal fuzzy set.

**Definition 5** ([38])**.** The $\alpha$-cut of a fuzzy set $\bar{A}$ is denoted by $\bar{A}_\alpha$ and is described as $\{x : \mu_{\bar{A}}(x) \geq \alpha\}$.

**Definition 6** ([38])**.** A fuzzy set $\bar{A}$ of a set $X$ is called as a convex fuzzy set iff $\lambda x_1 + (1-\lambda)x_2 \in \bar{A},\ \forall\, x_1, x_2 \in \bar{A}$ and $\lambda \in [0, 1]$ which can also be defined in form of membership functions as, $\mu_{\bar{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq min\{\mu_{\bar{A}}(x_1), \mu_{\bar{A}}(x_2)\}$.

**Definition 7** ([38])**.** Fuzzy number is an extension of a real number. It is a special case of a fuzzy set. Consider a fuzzy set $\bar{A}$ which can be defined as a fuzzy number in X if it satisfies the following properties

(1) $\bar{A}$ is a convex fuzzy set.
(2) $\bar{A}$ is a normal fuzzy set *i.e.*, $\mu_{\bar{A}}(x) = 1$.
(3) $\alpha\bar{A}$ must be a closed interval $\forall\, \alpha \in [0, 1]$.
(4) Support of $\bar{A}$ must be bounded.
(5) $\mu_{\bar{A}}$ must be continuous.

**Definition 8** ([38])**.** A triangular fuzzy number $\bar{A}$ is a triplet defined in the form $\bar{A} = (a, b, c)$ where $a \leq b \leq c$. Its associated membership function $\mu_{\bar{A}}$ can be defined as follows (Fig. 1).

$$\mu_{\bar{A}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

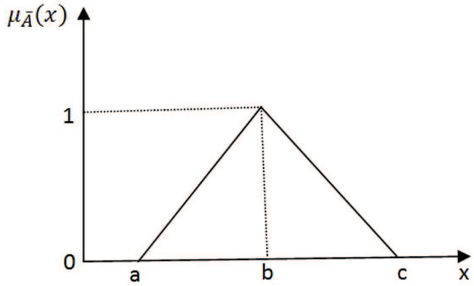


FIGURE 1. Triangular fuzzy membership function.

**Definition 9** ([38])**.** The $\alpha$-cut of the triangular fuzzy number is defined as, $\bar{A} = (a, b, c)$ is, $\bar{A}_\alpha = [a + \alpha(b - a), c - \alpha(c - b)] = [A^L, A^U] \, \forall \, \alpha \in [0, 1]$.

**Definition 10** (Interval analysis [22])**.** Consider the intervals $\bar{A} = [A^L, A^U]$ and $\bar{B} = [B^L, B^U]$ on the set of real numbers $R$. The arithmetic operations on $\bar{A}$ and $\bar{B}$ can be defined as follows.

(1) $\bar{A} + \bar{B} = [A^L + B^L, A^U + B^U]$.

(2) $\bar{A} - \bar{B} = [A^L - B^U, A^U - B^L]$.

(3) $k\bar{A} = \begin{cases} [kA^L, kA^U], k \geq 0 \\ [kA^U, kA^L], k < 0 \end{cases}$.

(4) $\bar{A}\bar{B} = \left[\min\left\{A^L B^L, A^L B^U, A^U B^L, A^U B^U\right\}, \;\; \max\left\{A^L B^L, A^L B^U, A^U B^L, A^U B^U\right\}\right]$.

(5) $\frac{\bar{A}}{\bar{B}} = \left[\min\left\{\frac{A^L}{B^L}, \frac{A^L}{B^U}, \frac{A^U}{B^L}, \frac{A^U}{B^U}\right\}, \;\; \max\left\{\frac{A^L}{B^L}, \frac{A^L}{B^U}, \frac{A^U}{B^L}, \frac{A^U}{B^U}\right\}\right]$.

(6) $\bar{A} = \bar{B}$ iff $A^L = B^L$ and $A^U = B^U$.

(7) Let $x \in [A^L, A^U]$ and $y \in [B^L, B^U]$.

$[A^L, A^U] \leq [B^L, B^U]$ iff *i.e.*, $A^L \leq x \leq y \leq B^U$ for some $x \in \bar{A}$ and $y \in \bar{B}$.

**Definition 11** (Parametric interval valued function [13])**.** As per analytical geometry perspective, an interval can be expressed as a function. The parametric interval-valued function representing an interval $[p, q]$ where $p, q \in R^+$ can be defined as $h(s) = p^{1-s}q^s$, $s \in [0, 1]$ which is a strictly monotone, continuous and invertible function.

## 2.1. Variable transformation method

In 1962 Charnes and Cooper [4] introduced this method to transform linear fractional programming problem into an equivalent linear programming problem by using a proper substitution. The general form of this method can be defined as,

$$\min : \frac{\sum_{j=1}^{n} p_j y_j + \phi}{\sum_{j=1}^{n} q_j y_j + \varphi}$$

subject to

$$\sum_{j=1}^{n} a_{ij} y_j (\leq \text{ or } = \text{ or } \geq) b_i, \qquad i = 1, 2, \ldots, m$$

$$\text{where, } y = (y_j) \geq 0 \text{ and } y \in R^n,$$

$$p_j \text{ and } q_j \in R, \qquad j = 1, 2, \dots, n. \tag{2.2}$$

Substituting $y_j = \frac{z_j}{t}$, assuming $\sum q_j z_j + \varphi t = 1$ and considering it as a constraint, the above model can be formulated as follows,

$$\begin{aligned} \min : & \sum_{j=1}^{n} p_j z_j + \phi t \\ \text{subject to} & \\ & \sum_{j=1}^{n} q_j z_j + \varphi t = 1 \\ & \sum_{j=1}^{n} a_{ij} z_j (\le \text{ or } = \text{ or } \ge) b_i t, \qquad i = 1, 2, \dots, m \\ & \text{where } z_j, t > 0. \end{aligned} \tag{2.3}$$

**Theorem 1** ([30]). *If $(z^*, t^*)$ is an optimal solution of (2.3) then $y^* = \frac{z^*}{t^*}$ is the optimal solution of (2.2).*

**Corollary 1** ([30]). *The transformation $y_j = \frac{z_j}{t}$ establishes an one-to-one correspondence in between the feasible regions of the optimization models (2.2) and (2.3).*

**Definition 12** ([5,6]). Two mathematical programming problems $M_1 : \min f_1(x)$ subject to $x \in \Omega_1$ and $M_2 : \min f_2(x)$ subject to $x \in \Omega_2$ are said to be equivalent iff $\exists$ an one-to-one map $f_3 : \Omega_1 \to \Omega_2$ such that $f_1(x) = f_2(f_3(x))\ \forall\ x \in \Omega_1$.

### 2.2. Min–max approach

Consider the following multi-objective optimization problem(MOOP) of minimization type.

$$\begin{aligned} & \min\ f_i(x), \qquad i = 1, 2, \dots, k \\ & \text{subject to} \\ & \qquad x \in \Omega. \end{aligned} \tag{2.4}$$

Using min–max approach ($\min\{\max_{1 \le i \le k}\{f_i(x)\}\}$), the MOOP (2.4) can be equivalently formulated as follows.

$$\begin{aligned} & \min\ \lambda \\ & \text{subject to} \\ & \qquad f_i(x) \le \lambda,\ x \in \Omega, \qquad i = 1, 2, \dots, k. \end{aligned} \tag{2.5}$$

## 3. Problem formulation: Nonlinear fuzzy fractional signomial programming

Consider the following nonlinear fractional programming problem with signomial functions, comprising triangular fuzzy numbers as the coefficients and the right hand side constants in the constraints.

$$\min : \bar{f}(y) = \frac{\bar{f}_N(y)}{\bar{f}_D(y)} = \frac{\sum_{i=1}^{r_1} \sigma_i^N \bar{c}_i \prod_{j=1}^{n} y_j^{\alpha_{ij}}}{\sum_{i=1}^{r_2} \sigma_i^D \bar{d}_i \prod_{j=1}^{n} y_j^{\beta_{ij}}}$$

subject to

$$\bar{\Omega}(y) = \begin{cases} \bar{f}_r(y) = \sum_{i=1}^{m_r} \sigma_{ri} \bar{a}_{ri} \prod_{j=1}^{n} y_j^{\vartheta_{rij}} \le \bar{\xi}_r, & r = 1, 2, \dots, l \\ y_j \ge 0, \qquad \bar{c}_i, \bar{d}_i, \bar{a}_{ri}, \bar{\xi}_r \in \text{TFN}(R) \end{cases}$$

$$\begin{aligned}
&\text{Assume that, } \bar{f}_D(y) \neq 0 \ \forall \ y \in \bar{\Omega}(y)\\
&\text{where, } y = (y_j) \in R^n, \ \alpha_{ij}, \beta_{ij}, \vartheta_{rij} \in R\\
&\sigma_i^N, \sigma_i^D, \sigma_{ri} \in \{-1, 1\}\\
&\bar{c}_i = \left(c_i^1, c_i^2, c_i^3\right), \ \bar{d}_i = \left(d_i^1, d_i^2, d_i^3\right)\\
&\bar{a}_{ri} = \left(a_{ri}^1, a_{ri}^2, a_{ri}^3\right), \ \bar{\xi}_r = \left(\xi_r^1, \xi_r^2, \xi_r^3\right).
\end{aligned} \tag{3.1}$$

## 4. Proposed solution methodology

In order to solve the NLFFSPP, two different solution approaches are proposed here based on parametric interval valued functions and $\alpha$-cut interval valued functions. Using the substitution $y_j = z_j/t$ and method of variable transformation [4], (3.1) is equivalently transformed into the following model.

$$\begin{aligned}
&\min : \bar{f}(z_j, t) = \sum_{i=1}^{r_1} \sigma_i^N \bar{c}_i \prod_{j=1}^{n} z_j^{\alpha_{ij}} t^{-\alpha_{ij}}\\
&\text{subject to}\\
&\qquad \sum_{i=1}^{r_2} \sigma_i^D \bar{d}_i \prod_{j=1}^{n} z_j^{\beta_{ij}} t^{-\beta_{ij}} = \bar{1}\\
&\qquad \sum_{i=1}^{m_r} \sigma_{ri} \bar{a}_{ri} \prod_{j=1}^{n} z_j^{\vartheta_{rij}} t^{-\vartheta_{rij}} \leq \bar{\xi}_r, \qquad r = 1, 2, \ldots, l\\
&\qquad z_j, t \geq 0.
\end{aligned} \tag{4.1}$$

Fuzzy coefficients are equivalently expressed in form of intervals either using fuzzy $\alpha$-cut or nearest interval approximation method [13] and the model (4.1) is equivalently formulated as,

$$\begin{aligned}
&\min : \bar{f}(z_j, t) = \sum_{i=1}^{r_1} \sigma_i^N \left[c_i^L, c_i^U\right] \prod_{j=1}^{n} z_j^{\alpha_{ij}} t^{-\alpha_{ij}}\\
&\text{subject to}\\
&\qquad \sum_{i=1}^{r_2} \sigma_i^D \left[d_i^L, d_i^U\right] \prod_{j=1}^{n} z_j^{\beta_{ij}} t^{-\beta_{ij}} = \left[1^L, 1^U\right]\\
&\qquad \sum_{i=1}^{m_r} \sigma_{ri} \left[a_{ri}^L, a_{ri}^U\right] \prod_{j=1}^{n} z_j^{\vartheta_{rij}} t^{-\vartheta_{rij}} \leq \left[\xi_r^L, \xi_r^U\right],\\
&\qquad r = 1, 2, \ldots, l \quad z_j \geq 0, \quad t \geq 0
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\text{where, } \bar{c}_i = \left(c_i^1, c_i^2, c_i^3\right) = \left[c_i^L, c_i^U\right]\\
&\bar{d}_i = \left(d_i^1, d_i^2, d_i^3\right) = \left[d_i^L, d_i^U\right]\\
&\bar{a}_{ri} = \left(a_{ri}^1, a_{ri}^2, a_{ri}^3\right) = \left[a_{ri}^L, a_{ri}^U\right]\\
&\bar{\xi}_r = \left(\xi_r^1, \xi_r^2, \xi_r^3\right) = \left[\xi_r^L, \xi_r^U\right]\\
&\bar{1} = \left(1^1, 1^2, 1^3\right) = \left[1^L, 1^U\right].
\end{aligned} \tag{4.3}$$

On solving (4.2), a set of fuzzy efficient solutions of (3.1) can be generated from which one compromise solution is determined by the decision maker based on the objective values. The primal and dual concept of signomial geometric programming is used in the following two solution approaches. Either one of these solution approaches can be used to solve the NLFFSPP.

### 4.1. Signomial geometric programming approach with parametric interval valued functions

Using nearest interval approximation method [13] in model (4.1), all triangular fuzzy numbers are transformed into interval forms. The above interval forms in (4.3) can be written in parametric interval-valued functions *i.e.*,

$$\begin{aligned}
&\left[c_i^L, c_i^U\right] = \left(c_i^L\right)^{1-s}\left(c_i^U\right)^s,\ \left[d_i^L, d_i^U\right] = \left(d_i^L\right)^{1-s}\left(d_i^U\right)^s,\\
&\left[a_{ri}^L, a_{ri}^U\right] = \left(a_{ri}^L\right)^{1-s}\left(a_{ri}^U\right)^s,\ \left[\xi_r^L, \xi_r^U\right] = \left(\xi_r^L\right)^{1-s}\left(\xi_r^U\right)^s,\\
&\left[1^L, 1^U\right] = \left(1^L\right)^{1-s}\left(1^U\right)^s.
\end{aligned} \tag{4.4}$$

The model (4.2) is now formulated as the following signomial geometric programming problem with parametric interval valued functions of $s \in [0,1]$.

$$\begin{aligned}
\min :\ & \bar{f}(z_j, t, s) = \sum_{i=1}^{r_1} \sigma_i^N \left(c_i^L\right)^{1-s}\left(c_i^U\right)^s \prod_{j=1}^{n} z_j^{\alpha_{ij}} t^{-\alpha_{ij}}\\
\text{subject to}\ & \\
& \sum_{i=1}^{r_2} \sigma_i^D \left(d_i^L\right)^{1-s}\left(d_i^U\right)^s \prod_{j=1}^{n} z_j^{\beta_{ij}} t^{-\beta_{ij}} = \left(1^L\right)^{1-s}\left(1^U\right)^s\\
& \sum_{i=1}^{m_r} \sigma_{ri} \left(a_{ri}^L\right)^{1-s}\left(a_{ri}^U\right)^s \prod_{j=1}^{n} (z_j)^{\vartheta_{rij}} (t)^{-\vartheta_{rij}} \le \left(\xi_r^L\right)^{1-s}\left(\xi_r^U\right)^s\\
& r = 1, 2, \ldots, l, z_j \ge 0, \quad t \ge 0, \quad s \in [0,1].
\end{aligned} \tag{4.5}$$

**Dual of model** (4.5)

Dual of above signomial geometric programming problem (4.5) is formulated as follows,

$$\begin{aligned}
\max :\ & \theta_0 \left[\prod_{i=1}^{p} \left(\frac{\left(c_l^L\right)^{1-s}\left(c_l^U\right)^s}{\delta_i}\right)^{\delta_i \sigma_i} \prod_{r=1}^{l} \lambda_r^{\xi_r \lambda_r}\right]^{\theta_0},\\
& p = r_1 + r_2 + m_r\\
\text{subject to}\ & \\
& \sum_{i=1}^{r} \delta_i \sigma_i = \xi_r \lambda_r, \qquad r = 1, 2, \ldots, l\\
& \sum_{i=1}^{l} \sigma_i \vartheta_{rij} \delta_j = 0, \qquad r = 1, 2, \ldots, l, \quad j = 1, 2, \ldots, n\\
& \delta_i > 0,\ \lambda_0 = 1\\
& \text{where, } \xi_r = \pm 1,\ \theta_0 = \pm 1\\
& \sigma_i = \text{sign of coefficients } i.e.,\ 1 \text{ or } -1.
\end{aligned} \tag{4.6}$$

### 4.2. Signomial geometric programming approach with fuzzy parametric valued functions

Using fuzzy $\alpha$-cut, the triangular fuzzy numbers of the model (4.1) can be equivalently expressed in form of intervals as follows,

$$\begin{aligned}
\bar{c}_i &= \left(c_i^1, c_i^2, c_i^3\right)\\
&= \left[c_i^1 + \alpha\left(c_i^2 - c_i^1\right), c_i^3 - \alpha\left(c_i^3 - c_i^2\right)\right]\\
&= \left[c_{iL}(\alpha), c_{iR}(\alpha)\right]\\
\bar{d}_i &= \left(d_i^1, d_i^2, d_i^3\right)\\
&= \left[d_i^1 + \alpha\left(d_i^2 - d_i^1\right), d_i^3 - \alpha\left(d_i^3 - d_i^2\right)\right]\\
&= \left[d_{iL}(\alpha), d_{iR}(\alpha)\right]\\
\bar{a}_{ri} &= \left(a_{ri}^1, a_{ri}^2, a_{ri}^3\right)\\
&= \left[a_{ri}^1 + \alpha\left(a_{ri}^2 - a_{ri}^1\right), a_{ri}^3 - \alpha\left(a_{ri}^3 - a_{ri}^2\right)\right]\\
&= \left[a_{riL}(\alpha), a_{riR}(\alpha)\right]\\
\bar{\xi}_r &= \left(\xi_r^1, \xi_r^2, \xi_r^3\right)\\
&= \left[\xi_r^1 + \alpha\left(\xi_r^2 - \xi_r^1\right), \xi_r^3 - \alpha\left(\xi_r^3 - \xi_r^2\right)\right]\\
&= \left[\xi_{rL}(\alpha), \xi_{rR}(\alpha)\right]\\
\bar{1} &= \left(1^1, 1^2, 1^3\right)\\
&= \left[1^1 + \alpha\left(1^2 - 1^1\right), 1^3 - \alpha\left(1^3 - 1^2\right)\right]\\
&= \left[1_L(\alpha), 1_R(\alpha)\right].
\end{aligned} \tag{4.7}$$

$c_{iL}, d_{iL}, a_{riL}, \xi_{rL}, 1_L$ are used as the lower limits whereas $c_{iR}, d_{iR}, a_{riR}, \xi_{rR}, 1_R$ are used as the upper limits of the $\alpha$-cut interval representation of the fuzzy numbers. Now the above model (4.1) can be expressed in the form (4.2) using fuzzy $\alpha$-cut.

**Definition 13** ([32])**.** $M_1 : \min[f^L(x), f^U(x)]$ subject to $x \in \Omega$ is equivalent to the bi-objective optimization problem $M_2 : \min\{f^L(x), f^U(x)\}$ subject to $x \in \Omega$.

Clearly, $\max\{f^L(x), f^U(x)\} = f^U(x)$ for the interval $[f^L(x), f^U(x)]$. As the problem (4.1) is of minimization type, minimizing the upper limit objective function with a wider feasible region can generate the best compromise solution. Based on the concept of min–max approach (Sect. 2.2) and interval analysis (Def. 10), the model (4.2) can be formulated as the following signomial geometric programming problem with fuzzy parametric valued functions of $\alpha$.

$$\min : \bar{f}(z_j, t, \alpha) = \sum_{i=1}^{r_1} \sigma_i^N \left(c_i^3 - \alpha\left(c_i^3 - c_i^2\right)\right) \prod_{j=1}^{n} z_j^{\alpha_{ij}} t^{-\alpha_{ij}}$$

subject to

$$\begin{aligned}
&\sum_{i=1}^{r_2} \sigma_i^D \left(d_i^1 + \alpha\left(d_i^2 - d_i^1\right)\right) \prod_{j=1}^{n} z_j^{\beta_{ij}} t^{-\beta_{ij}} = 1^1 + \alpha\left(1^2 - 1^1\right)\\
&\sum_{i=1}^{r_2} \sigma_i^D \left(d_i^3 - \alpha\left(d_i^3 - d_i^2\right)\right) \prod_{j=1}^{n} z_j^{\beta_{ij}} t^{-\beta_{ij}} = 1^3 - \alpha\left(1^3 - 1^2\right)\\
&\sum_{i=1}^{m_r} \sigma_{ri} \left(a_{ri}^1 + \alpha\left(a_{ri}^2 - a_{ri}^1\right)\right) \prod_{j=1}^{n} (z_j)^{\vartheta_{rij}} (t)^{-\vartheta_{rij}} \le \xi_r^3 - \alpha\left(\xi_r^3 - \xi_r^2\right),\\
&r = 1, 2, \ldots, l,\ z_j \ge 0,\ t \ge 0,\ \alpha \in [0, 1].
\end{aligned} \tag{4.8}$$

**Dual of model** (4.8)

Dual of the above signomial geometric programming problem (4.8) is formulated as follows,

$$\begin{aligned}
&\max : \theta_0 \left[ \prod_{i=1}^{p} \left( \frac{c_i^3 - \alpha\left(c_i^3 - c_i^2\right)}{\delta_i} \right)^{\delta_i \sigma_i} \prod_{r=1}^{l} \lambda_r^{\xi_r \lambda_r} \right]^{\theta_0}, \\
&p = r_1 + r_2 + m_r \\
\text{subject to} \quad & \\
&\sum_{i=1}^{r} \delta_i \sigma_i = \xi_r \lambda_r, \qquad\qquad r = 1, 2, \ldots, l \\
&\sum_{i=1}^{n+1} \sigma_i \vartheta_{rij} \delta_j = 0, \qquad\qquad r = 1, 2, \ldots, l, \quad j = 1, 2, \ldots, n \\
&\delta_i > 0, \ \lambda_0 = 1 \\
&\text{where } \xi_r = \pm 1, \theta_0 = \pm 1 \\
&\sigma_i = \text{sign of coefficients } i.e., \ 1 \text{ or } -1.
\end{aligned} \tag{4.9}$$

## 4.3. Algorithm and flowchart of the proposed approaches

On solving the above formulated primal and dual models of both the proposed solution approaches, a set of fuzzy efficient solutions can be generated from which one can be choosen as the compromise solution of the NLFFSPP. To explain the steps of the proposed approaches in simple manner, an algorithm and a flowchart are incorporated.

### *4.3.1. Algorithm*

This algorithm comprises the following sequencial steps to solve NLFFSPP using the proposed solution approaches.

**Step 1.** Mathematically formulate the NLFFSPP with triangular fuzzy numbers as defined in (3.1).
**Step 2.** Formulate (4.1) from (3.1) using the proposed substitution of variable transformation method.
**Step 3.** To solve (4.1) using SGP aaproach with parametric interval valued functions, use the following steps.
- **Step 3(1).** Convert the fuzzy numbers into interval forms using nearest interval approximation method and express it as functions of $s \in [0, 1]$ using parametric interval valued functions as defined in (4.4).
- **Step 3(2).** NLFFSPP is now equivalently formulated into the signomial GP as defined in (4.5).
- **Step 3(3).** Find the dual (4.6) of the primal SGP (4.5). Substitute different values of $s \in [0, 1]$ in (4.5), (4.6) and solve the resulting optimization models to generate a set of solutions.

**Step 4.** To solve (4.1) using SGP aaproach with fuzzy parametric valued functions, use the following steps.
- **Step 4(1).** Use fuzzy $\alpha$-cuts in order to convert the fuzzy numbers into intervals of NLFFSPP as defined in (4.7).
- **Step 4(2).** Formulate the SGP (4.8) considering upper limit functions of the parametric interval valued fuzzy numbers obtained in the above step.
- **Step 4(3).** Find the dual (4.9) of the primal SGP (4.8). Substitute different values of $\alpha \in [0, 1]$ in (4.8), (4.9) and solve the resulting optimization models to generate a set of solutions.

**Step 5.** As the problem considered is a fuzzy optimization problem, a set of fuzzy efficient solutions are generated from which one is choosen as the compromise solution based on the objective values.
**Step 6.** Use any one of the above proposed solution approaches to solve the NLFFSPP.

### *4.3.2. Flowchart*

See Figure 2.

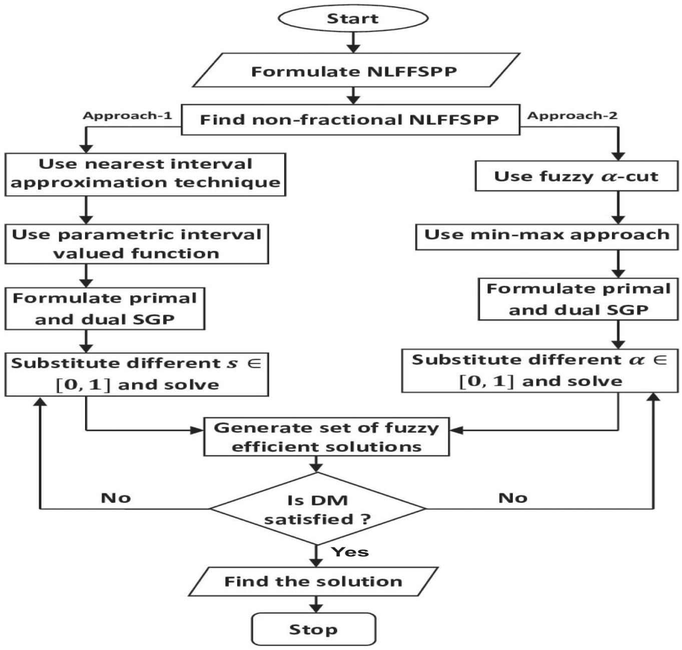


Figure 2. Flowchart of the proposed solution approaches.

## 5. Numerical examples

Consider the following numerical examples of non-linear fuzzy fractional SPP in order to illustrate the proposed solution approaches.

### 5.1. Example 1

The crisp form of the following NLFFSPP was initially solved by Mishra and Ota [18] which is considered here in triangular fuzzy environment.

$$
\begin{aligned}
&\min : f(y) = \frac{\bar{1}y_1^4 - \bar{1}y_2^2}{\bar{1}y_1^2 + \bar{1}y_2^{-1}}\\
&\text{subject to}\\
&\qquad \bar{1}y_1^{-3}y_2^5 \leq \bar{1}\\
&\qquad y_1,\ y_2 \geq 0,\ \ \bar{1} = (0.8, 1, 1.2).
\end{aligned} \tag{5.1}
$$

Using variable transformation method, this nonlinear fractional programming problem is converted into a signomial geometric programming problem by replacing $y_1 = \frac{z_1}{t}$ and $y_2 = \frac{z_2}{t}$ which can be formulated as follows.

$$
\begin{aligned}
&\min : f(z,t) = \bar{1}z_1^4t^{-2} - \bar{1}z_2^2\\
&\text{subject to}\\
&\qquad \bar{1}z_1^2 + \bar{1}t^3z_2^{-1} = \bar{1}\\
&\qquad \bar{1}z_1^{-3}z_2^5t^{-2} \leq \bar{1}\\
&\qquad z_1, z_2, t > 0.
\end{aligned} \tag{5.2}
$$

TABLE 1. Solution of primal (5.3).

| $s$ | $z_1$ | $z_2$ | $t$ | $f(z,t,s)$ |
|---|---|---|---|---|
| 0.1 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2234227 |
| 0.2 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2279514 |
| 0.3 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2325719 |
| 0.4 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2372861 |
| 0.5 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2420958 |
| 0.6 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2470031 |
| 0.7 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2520098 |
| 0.8 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2571179 |
| 0.9 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2623297 |
| 1.0 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2676470 |

On solving this fuzzy model (5.2) by considering it as a crisp model, the optimal solution is obtained as $f(z,t) = -0.2433155$, $z_1 = 0.4930679$, $z_2 = 0.5872001$, $t = 0.7631416$. So the optimal solution of the FPP (5.1) as a crisp model is found to be $y_1 = \frac{z_1}{t}$=0.6461028, $y_2 = \frac{z_2}{t}$=0.7694510 and $f(y) = -0.2433155$.

*5.1.1. SGP approach with parametric interval valued functions (Method-1)*

Using nearest interval approximation method, the triangular fuzzy number $\bar{1}$ is approximated as $\bar{1} = (0.8, 1, 1.2) \simeq [0.9, 1.1] \simeq (0.9)^{1-s}(1.1)^s$. The model (5.2) is formulated as the following primal.

$$\begin{aligned}
&\min : f(z,t,s) = (0.9)^{1-s}(1.1)^s z_1^4 t^{-2} - (0.9)^{1-s}(1.1)^s z_2^2 \\
&\text{subject to} \\
&\qquad (0.9)^{1-s}(1.1)^s z_1^2 + (0.9)^{1-s}(1.1)^s t^3 z_2^{-1} = (0.9)^{1-s}(1.1)^s \\
&\qquad (0.9)^{1-s}(1.1)^s z_1^{-3} z_2^5 t^{-2} \le (0.9)^{1-s}(1.1)^s \\
&\qquad z_1, z_2, t > 0,\ \ s \in [0,1].
\end{aligned} \tag{5.3}$$

Dual of the above primal is,

$$\begin{aligned}
\max : V(\delta) = \sigma \Bigg( &\left(\frac{(0.9)^{1-s}(1.1)^s}{\delta_1}\right)^{\delta_1} \left(\frac{(0.9)^{1-s}(1.1)^s}{\delta_2}\right)^{-\delta_2} \left(\frac{(0.9)^{1-s}(1.1)^s}{(0.9)^{1-s}(1.1)^s\delta_3}\right)^{\delta_3} \left(\frac{(0.9)^{1-s}(1.1)^s}{(0.9)^{1-s}(1.1)^s\delta_4}\right)^{\delta_4} \\
&\times (\delta_3+\delta_4)^{(\delta_3+\delta_4)} \left(\frac{(0.9)^{1-s}(1.1)^s}{(0.9)^{1-s}(1.1)^s\delta_5}\right)^{\delta_5} (\delta_5)^{\delta_5} \Bigg)^{\sigma}
\end{aligned}$$

$$\begin{aligned}
&\text{subject to} \\
&\qquad \delta_1 - \delta_2 = \sigma \\
&\qquad 4\delta_1 + 2\delta_3 - 3\delta_5 = 0 \\
&\qquad -2\delta_2 - \delta_4 + 5\delta_5 = 0 \\
&\qquad -2\delta_1 + 3\delta_4 - 2\delta_5 = 0 \\
&\qquad \text{where } \sigma = \pm 1. \text{ Here } \sigma = -1.
\end{aligned} \tag{5.4}$$

The above primal (5.3) and dual (5.4) of the problem generate the same set of solutions in Tables 1 and 2, which are considered as the fuzzy efficient solutions of (5.1) from which one solution can be determined as the most preferred solution by the DM based on the objective values.

TABLE 2. Solution of dual (5.4).

| $s$ | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\delta_4$ | $\delta_5$ | $V(\delta)$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.4171066 | 1.417107 | 0.2431160 | 0.7568840 | 0.7182194 | −0.2234227 |
| 0.2 | 0.4171066 | 1.417107 | 0.2431160 | 0.7568840 | 0.7182194 | −0.2279514 |
| 0.3 | 0.4171066 | 1.417107 | 0.2431160 | 0.7568840 | 0.7182194 | −0.2325719 |
| 0.4 | 0.4171066 | 1.417107 | 0.2431160 | 0.7568840 | 0.7182194 | −0.2372861 |
| 0.5 | 0.4171066 | 1.417107 | 0.2431160 | 0.7568840 | 0.7182194 | −0.2420958 |
| 0.6 | 0.4171066 | 1.417107 | 0.2431160 | 0.7568840 | 0.7182194 | −0.2470031 |
| 0.7 | 0.4171066 | 1.417107 | 0.2431160 | 0.7568840 | 0.7182194 | −0.2520098 |
| 0.8 | 0.4171066 | 1.417107 | 0.2431160 | 0.7568840 | 0.7182194 | −0.2571179 |
| 0.9 | 0.4171066 | 1.417107 | 0.2431160 | 0.7568840 | 0.7182194 | −0.2623297 |
| 1.0 | 0.4171066 | 1.417107 | 0.2431160 | 0.7568840 | 0.7182194 | −0.2676470 |

### *5.1.2. SGP approach with fuzzy parametric valued functions (Method-2)*

Using fuzzy $\alpha$-cut method, the triangular fuzzy number $\bar{1}$ is approximated as $\bar{1} = (0.8, 1, 1.2) \simeq [0.8 + 0.2\alpha, 1.2 - 0.2\alpha]$. The model (5.2) is formulated as the following primal.

$$
\begin{aligned}
\min:\ & f(z,t,\alpha) = (1.2-0.2\alpha)z_1^4t^{-2} - (1.2-0.2\alpha)z_2^2 \\
\text{subject to}\ & \\
& (0.8+0.2\alpha)z_1^2 + (0.8+0.2\alpha)t^3z_2^{-1} = 0.8+0.2\alpha \\
& (1.2-0.2\alpha)z_1^2 + (1.2-0.2\alpha)t^3z_2^{-1} = 1.2-0.2\alpha \\
& (0.8+0.2\alpha)z_1^{-3}z_2^5t^{-2} \le 1.2-0.2\alpha \\
& z_1, z_2, t > 0,\ \ \alpha \in [0,1].
\end{aligned} \tag{5.5}
$$

Dual of this model is formulated as follows,

$$
\begin{aligned}
\max:\ V(\delta) = \sigma\Bigg(&\left(\frac{1.2-0.2\alpha}{\delta_1}\right)^{\delta_1}\left(\frac{1.2-0.2\alpha}{\delta_2}\right)^{-\delta_2}\left(\frac{1.2-0.2\alpha}{(1.2-0.2\alpha)\delta_3}\right)^{\delta_3}\left(\frac{1.2-0.2\alpha}{(1.2-0.2\alpha)\delta_4}\right)^{\delta_4} \\
&\times(\delta_3+\delta_4)^{(\delta_3+\delta_4)}\left(\frac{0.8+0.2\alpha}{(0.8+0.2\alpha)\delta_5}\right)^{\delta_5}\left(\frac{0.8+0.2\alpha}{(0.8+0.2\alpha)\delta_6}\right)^{\delta_6} \\
&\times(\delta_5+\delta_6)^{(\delta_5+\delta_6)}\left(\frac{0.8+0.2\alpha}{(1.2-0.2\alpha)\delta_7}\right)^{\delta_7}(\delta_7)^{\delta_7}\Bigg)^{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
\text{subject to}\ & \\
& \delta_1 - \delta_2 = \sigma \\
& 4\delta_1 + 2\delta_3 + 2\delta_5 - 3\delta_7 = 0 \\
& -2\delta_2 - \delta_4 - \delta_6 + 5\delta_7 = 0 \\
& -2\delta_1 + 3\delta_4 + 3\delta_6 - 2\delta_7 = 0 \\
& \text{where } \sigma = \pm 1. \text{ Here } \sigma = -1.
\end{aligned} \tag{5.6}
$$

The above primal (5.5) and dual (5.6) of the problem generate the same set of solutions in Tables 3 and 4 which are considered as the fuzzy efficient solutions of (5.1) from which one solution can be determined as the most preferred solution by the DM based on the objective values.

TABLE 3. Solution of primal (5.5).

| $\alpha$ | $z_1$ | $z_2$ | $t$ | $f(z,t,\alpha)$ |
|---|---|---|---|---|
| 0.1 | 0.5233882 | 0.6613456 | 0.7830713 | −0.3717035 |
| 0.2 | 0.5199897 | 0.6526300 | 0.7808832 | −0.3549940 |
| 0.3 | 0.5166039 | 0.6440540 | 0.7786922 | −0.3389710 |
| 0.4 | 0.5132286 | 0.6356081 | 0.7764968 | −0.3235985 |
| 0.5 | 0.5098613 | 0.6272836 | 0.7742954 | −0.3088429 |
| 0.6 | 0.5064998 | 0.6190719 | 0.7720863 | −0.2946735 |
| 0.7 | 0.5031420 | 0.6109651 | 0.7698680 | −0.2810617 |
| 0.8 | 0.4997855 | 0.6029557 | 0.7676388 | −0.2679808 |
| 0.9 | 0.4964282 | 0.5950363 | 0.7653973 | −0.2554064 |
| 1.0 | 0.4930679 | 0.5872001 | 0.7631416 | −0.2433155 |

TABLE 4. Solution of dual (5.6).

| $\alpha$ | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\delta_4$ | $\delta_5$ | $\delta_6$ | $\delta_7$ | $V(\delta)$ |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.3884887 | 1.388489 | 0.0251882 | 0.06676121 | 0.2487470 | 0.6593036 | 0.7006085 | −0.3717035 |
| 0.2 | 0.3917814 | 1.391781 | 0.0110094 | 0.0297074 | 0.2593799 | 0.6999033 | 0.7026347 | −0.3549940 |
| 0.3 | 0.3950403 | 1.395040 | 0.1510698 | 0.4149900 | 0.1158098 | 0.3181303 | 0.7046402 | −0.3389710 |
| 0.4 | 0.3982681 | 1.398268 | 0.2217756 | 0.6201857 | 0.0416279 | 0.1164108 | 0.7066265 | −0.3235985 |
| 0.5 | 0.4014671 | 1.401467 | 0.2181262 | 0.6209545 | 0.0418324 | 0.1190870 | 0.7085951 | −0.3088429 |
| 0.6 | 0.4046395 | 1.404639 | 0.2144852 | 0.6215772 | 0.0420569 | 0.1218807 | 0.7105474 | −0.2946735 |
| 0.7 | 0.4077876 | 1.407788 | 0.0628264 | 0.1853504 | 0.1903254 | 0.5614978 | 0.7124847 | −0.2810617 |
| 0.8 | 0.4109135 | 1.410913 | 0.0091948 | 0.0276159 | 0.2405908 | 0.7225985 | 0.7144083 | −0.2679808 |
| 0.9 | 0.4140191 | 1.414019 | 0.0228566 | 0.0698903 | 0.2235843 | 0.6836687 | 0.7163194 | −0.2554064 |
| 1 | 0.4171066 | 1.417107 | 0.0078729 | 0.0245103 | 0.2352431 | 0.7323737 | 0.7182194 | −0.2433155 |

### 5.2. Example 2

Consider another numerical example of NLFFSPP in the environment of triangular fuzzy numbers.

$$\min : f(y) = \frac{\bar{5}y_1^2y_2^3 - \bar{4}y_1y_3}{\bar{1}y_1^2y_2^{-3}}$$

subject to

$$\begin{aligned}
&\bar{3}y_1^{-7}y_2^2y_3^7 \leq \bar{1}\\
&y_1,\ y_2,\ y_3 \geq 0\\
&\bar{1} = (0.8, 1, 1.2),\ \bar{3} = (1, 3, 5),\ \bar{4} = (2, 4, 6),\ \bar{5} = (3, 5, 7).
\end{aligned} \tag{5.7}$$

Using variable transformation method, this non-linear fractional programming problem is converted into a signomial geometric programming problem by replacing $y_1 = \frac{z_1}{t}$, $y_2 = \frac{z_2}{t}$ and $y_3 = \frac{z_3}{t}$ which can be formulated as follows.

$$\min : f(z,t) = \bar{5}z_1^2z_2^3t^{-5} - \bar{4}z_1z_3t^{-2}$$

subject to

$$\begin{aligned}
&\bar{1}z_1^2z_2^{-3}t = \bar{1}\\
&\bar{3}z_1^{-7}z_2^2z_3^7t^{-2} \leq \bar{1}\\
&z_1, z_2, z_3, t > 0.
\end{aligned} \tag{5.8}$$

On solving this fuzzy model (5.8) by considering it as a crisp model, the optimal solution is obtained as $f(z,t) = -0.7102850$, $z_1 = 64642.17$, $z_2 = 77278.48$, $z_3 = 61187.94$ and $t = 110444.4$. So the optimal solution of the FPP as a crisp model is found to be $y_1 = \frac{z_1}{t}$=0.5852915, $y_2 = \frac{z_2}{t}$=0.6997048, $y_3 = \frac{z_3}{t}$=0.5540157 and $f(y) = -0.7102850$.

### 5.2.1. *SGP approach with parametric interval valued functions (Method-1)*

Using nearest interval approximation method, the triangular fuzzy numbers are approximated as follows.

$$\begin{aligned}
&\bar{1} = (0.8, 1, 1.2) \simeq [0.9, 1.1] \simeq (0.9)^{1-s}(1.1)^s, \quad \bar{3} = (1,3,5) \simeq [2,4] \simeq (2)^{1-s}(4)^s,\\
&\bar{4} = (2,4,6) \simeq [3,5] \simeq (3)^{1-s}(5)^s, \quad \bar{5} = (3,5,7) \simeq [4,6] \simeq (4)^{1-s}(6)^s.
\end{aligned}$$

The model (5.8) is formulated as the following primal using the above approximations of the fuzzy numbers.

$$\begin{aligned}
\min:\ & f(z,t,s) = (4)^{1-s}(6)^s z_1^2 z_2^3 t^{-5} - (3)^{1-s}(5)^s z_1 z_3 t^{-2}\\
\text{subject to}\quad & \\
& (0.9)^{1-s}(1.1)^s z_1^2 z_2^{-3} t = (0.9)^{1-s}(1.1)^s\\
& (2)^{1-s}(4)^s z_1^{-7} z_2^2 z_3^7 t^{-2} \le (0.9)^{1-s}(1.1)^s\\
& z_1, z_2, z_3 t > 0,\ s \in [0,1]. \qquad (5.9)
\end{aligned}$$

Dual of the above primal (5.9) is formulated as follows,

$$\begin{aligned}
\max: V(\delta) = \sigma\Bigg(&\left(\frac{(4)^{1-s}(6)^s}{\delta_1}\right)^{\delta_1}\left(\frac{(3)^{1-s}(5)^s}{\delta_2}\right)^{-\delta_2}\left(\frac{(0.9)^{1-s}(1.1)^s}{(0.9)^{1-s}(1.1)^s\delta_3}\right)^{\delta_3}\\
&\times\ (\delta_3)^{\delta_3}\left(\frac{(2)^{1-s}(4)^s}{(0.9)^{1-s}(1.1)^s\delta_4}\right)^{\delta_4}(\delta_4)^{\delta_4}\Bigg)^{\sigma}
\end{aligned}$$

subject to

$$\begin{aligned}
& \delta_1 - \delta_2 = \sigma\\
& 2\delta_1 - \delta_2 + 2\delta_3 - 7\delta_4 = 0\\
& 3\delta_1 - 3\delta_3 + 2\delta_4 = 0\\
& -5\delta_1 + 2\delta_2 + \delta_3 - 2\delta_4 = 0\\
& -\delta_2 + 7\delta_4 = 0\\
& \text{where } \sigma = \pm 1.\ \text{Here } \sigma = -1. \qquad (5.10)
\end{aligned}$$

The above primal (5.9) and dual (5.10) of the problem generate the same set of solutions in Tables 5 and 6, which are considered as the fuzzy efficient solutions of (5.7) from which one solution can be determined as the most preferred solution by the DM based on the objective values.

### 5.2.2. *SGP approach with fuzzy parametric valued functions (Method-2)*

Using fuzzy $\alpha$-cuts, the triangular fuzzy numbers are approximated as,

$$\begin{aligned}
&\bar{1} = (0.8, 1, 1.2) \simeq [0.8 + 0.2\alpha, 1.2 - 0.2\alpha], \quad \bar{3} = (1,3,5) \simeq [1 + 2\alpha, 5 - 2\alpha]\\
&\bar{4} = (2,4,6) \simeq [2 + 2\alpha, 6 - 2\alpha], \quad \bar{5} = (3,5,7) \simeq [3 + 2\alpha, 7 - 2\alpha].
\end{aligned}$$

The model (5.8) is formulated as the following primal using the above interval forms of the fuzzy numbers.

TABLE 5. Solution of primal (5.9).

| $s$ | $z_1$ | $z_2$ | $z_3$ | $t$ | $f(z,t,s)$ |
|---|---|---|---|---|---|
| 0.1 | 0.8763741 | 1.050589 | 0.8611704 | 1.509799 | −0.5724338 |
| 0.2 | 436.4505 | 522.9339 | 425.7424 | 750.7070 | −0.5999438 |
| 0.3 | 2.875635 | 3.443613 | 2.784570 | 4.938275 | −0.6287760 |
| 0.4 | 877.3821 | 1050.118 | 843.3843 | 1504.306 | −0.6589938 |
| 0.5 | 299.6730 | 358.4803 | 285.9543 | 512.9805 | −0.6906638 |
| 0.6 | 300.1826 | 358.8988 | 284.3459 | 513.0324 | −0.7238558 |
| 0.7 | 443.8839 | 530.4255 | 417.3911 | 757.4161 | −0.7586429 |
| 0.8 | 5.008173 | 5.981402 | 4.674827 | 8.531992 | −0.7951019 |
| 0.9 | 1.970965 | 2.352725 | 1.826322 | 3.352401 | −0.8333130 |
| 1.0 | 6845.172 | 8166.676 | 6296.443 | 11624.31 | −0.8733605 |

TABLE 6. Solution of dual (5.10).

| $s$ | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\delta_4$ | $V(\delta)$ |
|---|---|---|---|---|---|
| 0.1 | 0.8260870 | 1.826087 | 1 | 0.2608696 | −0.5724337 |
| 0.2 | 0.8260870 | 1.826087 | 1 | 0.2608696 | −0.5999438 |
| 0.3 | 0.8260870 | 1.826087 | 1 | 0.2608696 | −0.6287760 |
| 0.4 | 0.8260870 | 1.826087 | 1 | 0.2608696 | −0.6589938 |
| 0.5 | 0.8260870 | 1.826087 | 1 | 0.2608696 | −0.6906638 |
| 0.6 | 0.8260870 | 1.826087 | 1 | 0.2608696 | −0.7238558 |
| 0.7 | 0.8260870 | 1.826087 | 1 | 0.2608696 | −0.7586429 |
| 0.8 | 0.8260870 | 1.826087 | 1 | 0.2608696 | −0.7951019 |
| 0.9 | 0.8260870 | 1.826087 | 1 | 0.2608696 | −0.8333130 |
| 1.0 | 0.8260870 | 1.826087 | 1 | 0.2608696 | −0.8733605 |

$$\begin{aligned}
&\min : f(z,t,\alpha) = (7-2\alpha)z_1^2 z_2^3 t^{-5} - (6-2\alpha) z_1 z_3 t^{-2}\\
&\text{subject to}\\
&\qquad (0.8+0.2\alpha) z_1^2 z_2^{-3} t = 0.8 + 0.2\alpha\\
&\qquad (1.2-0.2\alpha) z_1^2 z_2^{-3} t = 1.2 - 0.2\alpha\\
&\qquad (1+2\alpha) z_1^{-7} z_2^2 z_3^7 t^{-2} \le 1.2 - 0.2\alpha\\
&\qquad z_1, z_2, z_3, t > 0,\ \ \alpha \in [0,1].
\end{aligned} \tag{5.11}$$

Dual of the above primal (5.11) is formulated as follows,

$$\begin{aligned}
\max : V(\delta) = \sigma \Bigg( &\left(\frac{7-2\alpha}{\delta_1}\right)^{\delta_1} \left(\frac{6-2\alpha}{\delta_2}\right)^{-\delta_2} \left(\frac{0.8+0.2\alpha}{(0.8+0.2\alpha)\delta_3}\right)^{\delta_3} (\delta_3)^{\delta_3} \left(\frac{1.2-0.2\alpha}{(1.2-0.2\alpha)\delta_4}\right)^{\delta_4} (\delta_4)^{\delta_4}\\
&\times \left(\frac{1+2\alpha}{(1.2-0.2\alpha)\delta_5}\right)^{\delta_5} (\delta_5)^{\delta_5} \Bigg)^{\sigma}
\end{aligned}$$

subject to

$$\begin{aligned}
&\delta_1 - \delta_2 = \sigma\\
&2\delta_1 - \delta_2 + 2\delta_3 + 2\delta_4 - 7\delta_5 = 0\\
&3\delta_1 - 3\delta_3 - 3\delta_4 + 2\delta_5 = 0
\end{aligned}$$

TABLE 7. Solution of primal (5.11).

| $\alpha$ | $z_1$ | $z_2$ | $z_3$ | $t$ | $f(z,t,\alpha)$ |
|---|---|---|---|---|---|
| 0.1 | 7.598171 | 8.786245 | 8.236073 | 11.74875 | −1.439966 |
| 0.2 | 7.673588 | 8.913652 | 8.137797 | 12.02735 | −1.323829 |
| 0.3 | 49746.85 | 58025.31 | 51753.05 | 78944.40 | −1.221606 |
| 0.4 | 21.68151 | 25.38730 | 22.17234 | 34.80732 | −1.129904 |
| 0.5 | 980.4566 | 1152.246 | 987.1976 | 1591.399 | −1.046461 |
| 0.6 | 126.2162 | 148.8546 | 125.2892 | 207.0413 | −0.9696937 |
| 0.7 | 81.15897 | 96.04514 | 79.51140 | 134.5095 | −0.8984547 |
| 0.8 | 0.9590582 | 1.138816 | 0.9281750 | 1.605724 | −0.8318870 |
| 0.9 | 727.9859 | 867.3509 | 696.5327 | 1231.228 | −0.7693344 |
| 1.0 | 64642.17 | 77278.48 | 61187.94 | 110444 | −0.7102850 |

TABLE 8. Solution of dual (5.12).

| $\alpha$ | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\delta_4$ | $\delta_5$ | $V(\delta)$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.8260870 | 1.826087 | 0.9964709 | $0.3529113E^{-02}$ | 0.2608696 | −1.439966 |
| 0.2 | 0.8260870 | 1.826087 | 0.9964709 | $0.3529113E^{-02}$ | 0.2608696 | −1.323829 |
| 0.3 | 0.8260870 | 1.826087 | 0.9964709 | $0.3529113E^{-02}$ | 0.2608696 | −1.221606 |
| 0.4 | 0.8260870 | 1.826087 | 0.9964709 | $0.3529113E^{-02}$ | 0.2608696 | −1.129904 |
| 0.5 | 0.8260870 | 1.826087 | 0.9964709 | $0.3529113E^{-02}$ | 0.2608696 | −1.046461 |
| 0.6 | 0.8260870 | 1.826087 | 0.9964709 | $0.3529113E^{-02}$ | 0.2608696 | −0.9696937 |
| 0.7 | 0.8260870 | 1.826087 | 0.9964709 | $0.3529113E^{-02}$ | 0.2608696 | −0.8984547 |
| 0.8 | 0.8260870 | 1.826087 | 0.9964709 | $0.3529113E^{-02}$ | 0.2608696 | −0.8318870 |
| 0.9 | 0.8260870 | 1.826087 | 0.9964709 | $0.3529113E^{-02}$ | 0.2608696 | −0.7693344 |
| 1 | 0.8260870 | 1.826087 | 0.9964709 | $0.3529113E^{-02}$ | 0.2608696 | −0.7102850 |

$$
\begin{aligned}
&-5\delta_1 + 2\delta_2 + \delta_3 + \delta_4 - 2\delta_5 = 0 \\
&-\delta_2 + 7\delta_5 = 0 \\
&\text{where } \sigma = \pm 1. \text{ Here } \sigma = -1.
\end{aligned} \tag{5.12}
$$

The above primal (5.11) and dual (5.12) of the problem generate the same set of solutions in Tables 7 and 8 which are considered as the fuzzy efficient solutions of (5.7) from which one solution can be determined as the most preferred solution by the DM based on the objective values.

### 5.3. Comparative result analysis

In example 1, considering the crisp model of NLFFSPP in the numerical example, Mishra and Ota [18] obtained the optimal solution as $y_1$=0.6461028, $y_2$=0.7694510 and the corresponding optimal objective value is $f(y) = -0.2433155$. As this problem is a fuzzy optimization, using the proposed two solution approaches (Method-1 and Method-2) a set of solutions are obtained in Tables 1–4 for $s \in [0,1]$ and $\alpha \in [0,1]$ respectively. The primal and dual optimal objective values obtained are same in Tables 1–4. On varying $s \in [0,1]$ due to proposed Method-1, the optimal range(lower and upper bounds) of the objective value is obtained as $[-0.2676470, -0.2234227]$ in Tables 1 and 2. On varying $\alpha \in [0,1]$ due to proposed Method-2, the optimal range(lower and upper bounds) of the objective value is obtained as $[-0.3717035, -0.2433155]$ in Tables 3 and 4. The following Figure 3 represents that the optimal objective value of the crisp NLFSPP obtained by Mishra and Ota [18] lies in the range of optimal objective values of fuzzy NLFFSPP obtained due to the proposed methods.

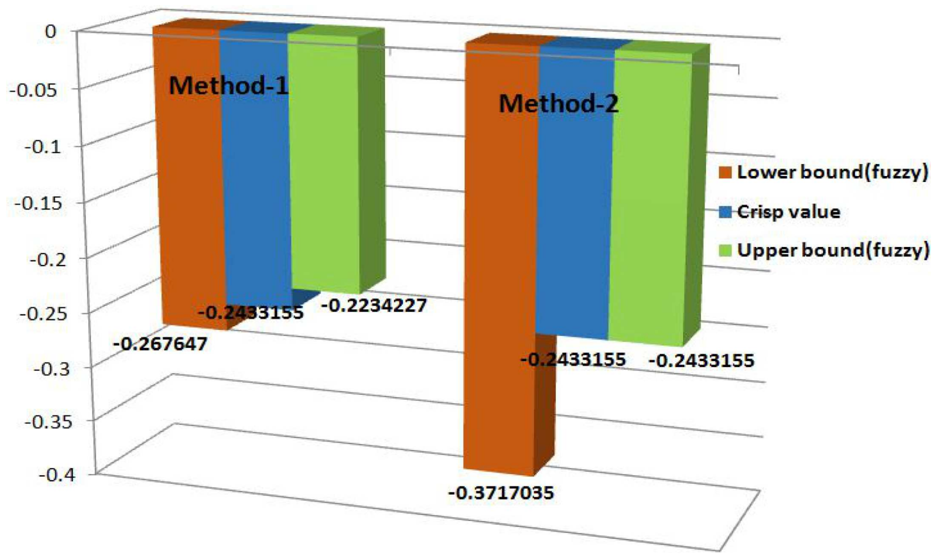


FIGURE 3. Optimal objective values of the crisp and fuzzy NLFFSPP in example 1.

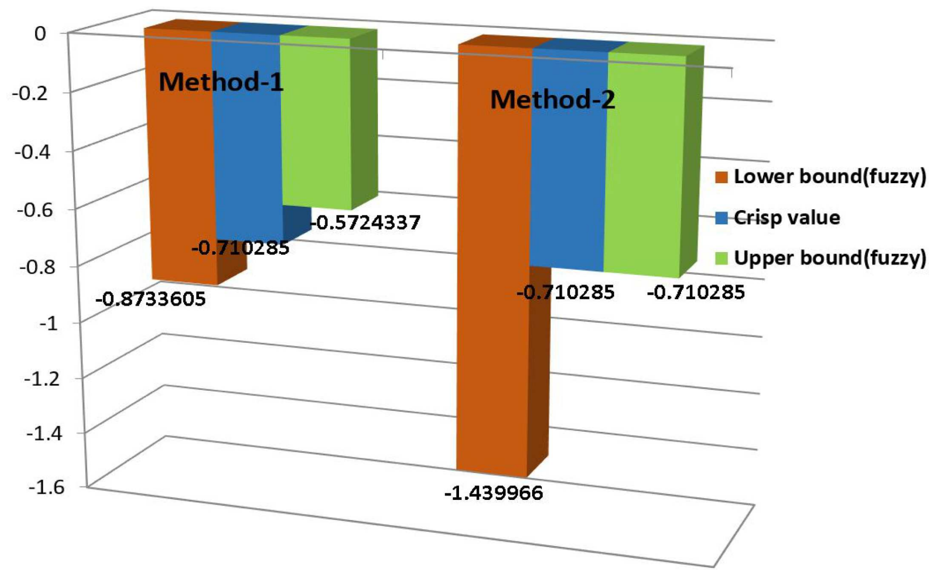


FIGURE 4. Optimal objective values of the crisp and fuzzy NLFFSPP in example 2.

In example 2, considering NLFFSPP as a crisp model the optimal solution is obtained as $y_1$=0.5852915, $y_2$=0.6997048, $y_3$=0.5540157 and the corresponding optimal objective value is $f(y) = -0.7102850$. As this problem is a fuzzy optimization, using the proposed two solution approaches (Method-1 and Method-2) a set of solutions are obtained in Tables 5–8 for $s \in [0,1]$ and $\alpha \in [0,1]$ respectively. The primal and dual optimal objective values obtained are same in Tables 5–8. On varying $s \in [0,1]$ due to proposed Method-1, the optimal range(lower and upper bounds) of the objective value is obtained as $[-0.8733605, -0.5724337]$ in Tables 5 and 6. On varying $\alpha \in [0,1]$ due to proposed Method-2, the optimal range(lower and upper bounds) of the objective value is obtained as $[-1.439966, -0.7102850]$ in Tables 7 and 8. The following Figure 4 represents that the optimal objective value of the crisp NLFFSPP lies in the range of optimal objective values of the fuzzy NLFSPP obtained due to the proposed methods.

It is clearly observed that the objective values obtained in the crisp models of both the examples 1 and 2, closely approach to one of the objective values obtained in Tables 1–4 for example 1, Tables 5–8 for example 2 respectively which justifies the feasibility and effectiveness of the proposed methods.

## 6. Conclusions

This paper develops the methodologies to solve and determine a set of solutions of NLFSPP with fuzzy papameters as no such algorithms can be found in literature to solve this problem in fuzzy environment. The fractional form of NLFFSPP is reduced to non-fractional form implementing an appropriate substitution of variable transformation method so that signomial geometric programming approach is suitably applied to solve it. Fuzzy coefficients are equivalently transformed into intervals either using nearest interval approximation or fuzzy $\alpha$-cuts. Consequently, a nonlinear nonfractional interval valued optimization problem is formulated. In this context, two solution approaches are proposed to generate a set of fuzzy efficient solutions from which one solution can be choosen by the decision maker as the most preferred or compromise solution based on the objective values. As the considered problem is a fuzzy optimization, substitution of different values of the parameter $s \in [0,1]$ of parametric interval-valued functions and $\alpha \in [0,1]$ of fuzzy $\alpha$-cuts generate a set of solutions. The proposed solution approaches are also appropriate to solve NLFFSPP with linear or trapezoidal or pentagonal fuzzy numbers. The limitations of the proposed study can be considered same as the limitations of the SGP as it is used to solve NLFFSPP in the proposed solution methodologies. As the considered problem is defined in fuzzy environment so the main adavantages of the proposed solution approaches are, a set of fuzzy efficient solutions are produced whereas many other methods in literature generate only one solution using ranking function approach. The solutions of the numerical examples and comparative result analysis justify the feasibility and show the effectiveness of the proposed methods. LINGO software is used to obtain the computational results. Multiobjective NLFSPP with fuzzy parameters, NLFSPP in fully fuzzy environment can be studied in future research.

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