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# TOUGHNESS AND BINDING NUMBER BOUNDS OF STAR-LIKE AND PATH FACTOR

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**Abstract.** Let $\mathcal{L}$ be a set which consists of some connected graphs. Let $E$ be a spanning subgraph of graph $G$. It is called a $\mathcal{L}$-factor if every component of it is isomorphic to the element in $\mathcal{L}$. In this contribution, we give the lower bounds of four parameters $(t(G), I(G), I'(G), \text{bind }(G))$ of $G$, which force the graph $G$ admits a $(\{K_{1,i} : q \leq i \leq 2q-1\} \cup \{K_{2q+1}\})$-factor for $q \geq 2$ and a $\{P_2, P_{2q+1}\}$-factor for $q \geq 3$ respectively. The tightness of the bounds are given.

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## 1. Introduction

In the network design stage, one of the problems that must be considered is the ruggedness and vulnerability of computer network. This issue is so critical that it has attracted the attention of scholars. One can refer to [2,4]. We can view a network as a graph of graph theory. For example, in a data transmission network, each site is regarded as the vertex of the graph, and the connections between sites are regarded as the edges of the graph. Thus, we can study ruggedness and vulnerability from the perspective of graph theory. Graph factor theory, an important branch of graph theory, is often applied to solve problems related to computer networks and network transmission. Because the more star structures in the graph, the more fragile the network corresponds to the graph; whether the sites in the network can communicate depends on whether there is a way between the sites. In other words, the theories of the star factor and path factor can help for solving the problem of vulnerability and transmission rate in network design. In the contribution, we give sufficient conditions for the existence of a star-like factor, $(\{K_{1,i} : q \leq i \leq 2q-1\} \cup \{K_{2q+1}\})$-factor for $q \geq 2$, and path factor of $G$, $\{P_2, P_{2q+1}\}$-factor for $q \geq 3$, by using the above four parameters which characterized the vulnerability of graphs.

Let $\mathcal{L}$ be a set which consists of some connected graphs. When $E$ is a spanning subgraph of some graph $G$ and every component of $E$ is isomorphic to the element in $\mathcal{L}$, then, it is called a $\mathcal{L}$-factor of $G$. For instance, if $G$ has a spanning subgraph $E$ with every component being a $K_2$, then $E$ is a 1-factor of $G$.

In the contribution, we only consider finite graph with no loops and multiple edges (say simple graph). Let $G$ be a finite simple graph whose vertex and edge set denoted by $V(G)$ and $E(G)$ respectively. $|V(G)|$ is the order of $G$. For every $x \in V(G)$, let $N_G(x)$ denote the set of vertices adjacent to $x$ in $G$. For $U \subseteq V(G)$, the set $N_G(U) = \{u | u \text{ is adjacent to some elements of } U\}$, that is $N_G(U) = \bigcup_{x \in U} N_G(x)$. Let $c(G-U)$ and

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iso$(G-U)$ denote the number of components and the number of isolated vertices of $G-U$, respectively. Let $i$ be a positive integer and $c_i(G-U)$ denote the number of connected components with $i$ vertices in $G-U$. If the context is clear, we just rewrite $c_i(G-U)$ as $c_i$. Let $\mathscr{C}_i(G-U)$ be the set of connected components with order $i$ in $G-U$. Notice that $c_1(G-U)$ is equivalent to iso$(G-U)$.

As usual, $K_n$ denotes the complete graph with order $n$ and $P_n$ denotes the path with order $n$. $K_{m,n}$ is a complete bipartite graph with two color classes of $m, n$ vertices. Particularly, we call a kind of complete bipartite graphs star, when one of its color classes has only one vertex, *i.e.*, $K_{1,n}$ for $n \geq 1$.

A subset $S \subseteq V(G)$ is called a vertex cut of $G$ if the graph $G \setminus S$ became disconnected. The vertex connectivity number $\kappa(G)$ of $G$ is defined as the minimum size of $S$ such that $G \setminus S$ is disconnected. Notice that $\kappa(G) = 0$ if and only if $G$ is disconnected or a $K_1$. $\kappa(K_n) = n-1, n \geq 1$. Other notation and terminology not defined refer to [15].

The following four parameters characterized the vulnerability of graphs which are important parameter used in computer networks. Then, the relationships can be established between graph factors and these parameters.

If $|U| \geq t \cdot c(G-U)$ for any $U \subset V(G)$ and $c(G-U) \geq 2$, the graph $G$ is called $t$-tough. If $G$ is a complete graph, $G$ is $t$-tough for any positive real number $t$. When $G \neq K_n$, there exists a maximum number $t$ (denoted by $t(G)$) such that $G$ is $t$-tough. The number $t(G)$, which was introduced by Chvátal [5], is called the toughness of $G$. If $G = K_n$, then define $t(G) = +\infty$, otherwise

$$t(G) = \min\left\{\frac{|U|}{c(G-U)} \,\middle|\, U \subseteq V(G), c(G-U) \geq 2\right\}.$$

In [20], Zhou *et al.* proved that $G$ is a $(P_{\geq 3}, m)$-factor-critical covered graph when $G$ is $(m+1)$-connected and $t(G) > \frac{2+m}{3}$ with its order at least $m+3$. Generally, a graph $G$ is named a $(P_{\geq n}, m)$-factor-critical covered graph if for any subset $U \subseteq V(G)$ such that $|U| = m$ and any edge $e \in E(G \setminus U)$, $G-U$ has a $P_{\geq n}$-factor covering the edge $e$. And $P_{\geq n}$ means $\{P_i : i \geq n\}$. More results on $t(G)$ were studied in [7, 10, 12].

The definition of isolated toughness $I(G)$, introduced by Yang *et al.* [14], is as follows: When $G = K_n$, $I(G) = +\infty$, else

$$I(G) = \min\left\{\frac{|U|}{\mathrm{iso}(G-U)} \,\middle|\, U \subseteq V(G), \mathrm{iso}(G-U) \geq 2\right\}.$$

Kano *et al.* [11] proved that $G$ has a $P_{\geq 3}$-factor when $G$ satisfies $I(G) \geq \frac{3}{2}\cdot$ If for any edge $e \in E(G)$, the resulting subgraph $G \setminus e$ admits a $P_{\geq 3}$-factor, then $G$ is called a $P_{\geq 3}$-factor deleted graph. Gao *et el.* [9] proved that if $G$ admits $\kappa(G) \geq 2$ and $I(G) > \frac{3}{2}$, then $G$ is a $P_{\geq 3}$-factor deleted graph. A graph $G$ is named $P_{\geq k}$-factor uniform if for every pair of edges $e_1, e_2 \in E(G)$, $G$ admits a $P_{\geq k}$-factor that contains $e_1$ and avoids $e_2$. Recently, Zhou *et al.* [18] proved that a graph $G$ is $P_{\geq 2}$-factor uniform when $G$ is 3-edge-connected graph and $I(G) > 1$. Moreover, they also obtained that $G$ is $P_{\geq 3}$-factor uniform for 3-edge-connected graph $G$ and $I(G) > 2$.

As a generalization of $I(G)$, Zhang and Liu [16] defined $I'(G)$: if $G = K_n$, then $I'(G) = +\infty$ and else

$$I'(G) = \min\left\{\frac{|U|}{\mathrm{iso}(G-U)-1} \,\middle|\, U \subseteq V(G),\ \mathrm{iso}(G-U) \geq 2\right\}.$$

Gao *et al.* [9] also showed that $G$ is a $P_{\geq 3}$-factor deleted graph if $G$ satisfies $\kappa(G) \geq 2$ and $I'(G) > 2$. Guan *et al.* [10] showed that $G$ has a $\{K_2, C_{2i+1} | i \geq 2\}$-factor if $I'(G) > 5$.

The last parameter is called the binding number bind $(G)$, which is defined by Woodall [13] as follows:

$$\mathrm{bind}\ (G) = \min\left\{\frac{|N_G(U)|}{|U|} \,\middle|\, \emptyset \neq U \subseteq V(G),\ N_G(U) \neq V(G)\right\}.$$

Anderson [1] showed that $G$ has a $K_2$-factor if $G$ is a graph with even vertices and bind $(G) \geq \frac{4}{3}\cdot$ Guan *et al.* [10] showed $G$ has a $\{K_2, C_{2i+1} | i \geq 2\}$-factor if bind $(G) > \frac{4}{3}\cdot$ Recently, Zhou *et al.* showed some new results with respect to this parameter one can find in [21].

Other toughness parameters can be found in [3, 19].

We proceed as follows. In Section 2, we give the lower bounds of the above four parameters such that the graph $G$ admits a $(\{K_{1,i} : q \le i \le 2q-1\} \cup \{K_{2q+1}\})$-factor for $q \ge 2$. This factor is called star-like factor as it including large stars and a complete component $K_{2q+1}$. We also give the lower bounds of the above four parameters for graph $G$ admitting a path factor that is $\{P_2, P_{2q+1}\}$-factor for $q \ge 3$ in Section 3.

## 2. Conditions for the presence of star-like factor

Let $q \ge 2$ be a positive integer, the spanning subgraph $E$ of a graph $G$ is a $(\{K_{1,i} : q \le i \le 2q-1\} \cup \{K_{2q+1}\})$-factor of $G$ if all the components of $E$ are isomorphic to $K_{1,i}$ for $q \le i \le 2q-1$ or $K_{2q+1}$. Theorem 2.1 will be leveraged to prove four lemmas in this part.

**Theorem 2.1** ([17])**.** *For $2 \le q (\in \mathbb{Z}^+)$, a graph $G$ admits a $(\{K_{1,i} : q \le i \le 2q-1\} \cup \{K_{2q+1}\})$-factor if $G$ satisfies* $\mathrm{iso}(G-P) \le \frac{1}{q}|P|$ *for all $P \subset V(G)$.*

The following theorem provides four sufficient conditions for the existence of $(\{K_{1,i} : q \le i \le 2q-1\} \cup \{K_{2q+1}\})$-factor of $G$ with the help of the four parameters mentioned earlier.

**Theorem 2.2.** *Let $2 \le q\ (\in \mathbb{Z}^+)$. When $G$ is a graph with order at least $q+1$ and $\kappa(G) \ge q$. If it satisfies one of the following conditions:*

(i) $t(G) \ge q$;
(ii) $I(G) \ge q$; (Gao and Wang [8])
(iii) $I'(G) \ge 2q$;
(iv) bind $(G) \ge q$,

*then $G$ admits a $(\{K_{1,i} : q \le i \le 2q-1\} \cup \{K_{2q+1}\})$-factor.*

We proved the theorem by four lemmas.

**Lemma 2.1.** *Let $2 \le q\ (\in \mathbb{Z}^+)$. When $G$ has at least $q+1$ vertices and $t(G) \ge q$, $G$ admits a $(\{K_{1,i} : q \le i \le 2q-1\} \cup \{K_{2q+1}\})$-factor.*

*Proof.* Assume $G$ has no $(\{K_{1,i} : q \le i \le 2q-1\} \cup \{K_{2q+1}\})$-factor, then we will show that $t(G) < q$. There exists a subset $U \subset V(G)$ such that $\mathrm{iso}(G-U) \ge \frac{1}{q}|U| + \frac{1}{q}$ by Theorem 2.1. We can easily get that

$$c(G-U) \ge \mathrm{iso}(G-U) \ge \frac{1}{q}|U| + \frac{1}{q}\cdot$$

Notice that $c(G-U) \ge 2$. Otherwise, $c(G-U) = 1$ implies $\mathrm{iso}(G-U) = 1$ and $|U| \le q-1$. Then we can obtain $|V(G)| = 1 + |U| \le q$, which is contradict with $G$ having at least $q+1$ vertices. Thus, by the definition of $t(G)$ we have $c(G-U) \ge 2$, and

$$t(G) \le \frac{|U|}{c(G-U)} \le \frac{q \cdot c(G-U) - 1}{c(G-U)} = q - \frac{1}{c(G-U)} < q.$$

Therefore, $t(G) < q$. □

**Remark 2.1.** We construct a kind of graphs $G_k$ for $k \ge 1$ to show that $t(G) < q$ is tight. Let $K$ be a complete graph with order $kq$ and $G_k = K \vee (k+1)K_1$ with "$\vee$" denoting "joining". Notice that $|V(G_k)| = kq+k+1 > q+1$ and $G_k$ has no $(\{K_{1,i} : q \le i \le 2q-1\} \cup \{K_{2q+1}\})$-factor since $k+1 = \mathrm{iso}(G_k - V(K)) > \frac{1}{q}|V(K)| = k$. Let

$U$ be a subset of $V(G_k)$ such that $c(G-U) \geq 2$, then $V(K) \subseteq U$. Thus, we have $|U| \geq kq$ and $c(G_k - U) = kq + k + 1 - |U|$. Since

$$\begin{aligned}\frac{|U|}{c(G_k-U)} &= \frac{|U|}{kq+k+1-|U|}\\ &= -1 + \frac{kq+k+1}{kq+k+1-|U|}\\ &\geq -1 + \frac{kq+k+1}{k+1}\\ &= \frac{kq}{k+1},\end{aligned}$$

we have $t(G_k) = \frac{kq}{k+1} < q$. That implies $\lim\limits_{k\to+\infty} t(G_k) = q$.

**Lemma 2.2.** *[8] Let $2 \leq q$ $(\in \mathbb{Z}^+)$. For any connected graph $G$ with order at least 2, if $I(G) \geq q$, $G$ has a $(\{K_{1,i} : q \leq i \leq 2q-1\} \cup \{K_{2q+1}\})$-factor.*

This theorem had been proved by Gao and Wang [8]. Here we give a kind of graphs to prove that $I(G) < q$ is tight when $G$ has no $(\{K_{1,i} : q \leq i \leq 2q-1\} \cup \{K_{2q+1}\})$-factor.

**Remark 2.2.** To show $I(G) < q$ is tight, we use the graphs $G_k$ which are mentioned in Remark 2.1. If $U \subset V(G)$ and $\mathrm{iso}(G_k - U) \geq 2$, then $V(K) \subseteq U$, which means $|U| \geq kq$ and $\mathrm{iso}(G_k - U) \leq k+1$. Therefore, we have $\frac{|U|}{\mathrm{iso}(G_k-U)} \geq \frac{kq}{k+1}$. In particular, take $V(K)$ as $U'$, then $I(G_k) = \frac{|U'|}{\mathrm{iso}(G_k-U')} = \frac{kq}{k+1}$ and $\lim\limits_{k\to+\infty} I(G_k) = q$.

**Lemma 2.3.** *Let $q \geq 2$ and $G$ be a connected graph with $|V(G)| \geq q+1$ and $\kappa(G) \geq q$. If $I'(G) \geq 2q$, then $G$ admits a $(\{K_{1,i} : q \leq i \leq 2q-1\} \cup \{K_{2q+1}\})$-factor.*

*Proof.* Assume that $G$ has no $(\{K_{1,i} : q \leq i \leq 2q-1\} \cup \{K_{2q+1}\})$-factor, then we will show $I'(G) < 2q$. According to Theorem 2.1, there exists a subset $U \subset V(G)$ such that $\mathrm{iso}(G-U) > \frac{1}{q}|U|$, which is equivalent to $\mathrm{iso}(G-U) \geq \frac{1}{q}|U| + \frac{1}{q}$. We claim that $\mathrm{iso}(G-U) \geq 2$. Otherwise, $\mathrm{iso}(G-U) = 1$ implies $|U| \leq q-1$. If graph $G$ has only one vertex when removing subset of vertex $U$, then $|V(G)| = 1 + |U| \leq q$, a contradiction. If $G$ has other components excepting $U$ and the isolated vertex, then $\kappa(G) \leq |U| \leq q-1$, also a contradiction. Thus, we know that $\mathrm{iso}(G-U) \geq 2$, and by the definition of $I'(G)$ we have

$$\begin{aligned}I'(G) &\leq \frac{|U|}{\mathrm{iso}(G-U)-1}\\ &< \frac{q \cdot iso(G-U)}{\mathrm{iso}(G-U)-1}\\ &= \frac{q\cdot(\mathrm{iso}(G-U)-1)+q}{\mathrm{iso}(G-U)-1}\\ &= q + \frac{q}{\mathrm{iso}(G-U)-1}\\ &\leq 2q.\end{aligned}$$

Therefore, $I'(G) < 2q$. The proof is completed. $\square$

**Remark 2.3.** Now, we will state that $I'(G) \geq 2q$ in Lemma 2.3 is all the best possible. We construct a graph $M_1 = K_{2q^2} \vee (2q+1)K_1$. Firstly, this graph admits the condition that $|V(M_1)| = 2q^2 + 2q + 1 > q+1$ and $\kappa(M_1) = 2q^2 > q$. Let $U = V(K_{2q^2})$. Thus, $|U| = 2q^2$ and $\mathrm{iso}(M_1 - U) = 2q+1$. By the definition of $I'(M_1)$, we have $I'(M_1) = \frac{|U|}{\mathrm{iso}(M_1-U)-1} = \frac{2q^2}{(2q+1)-1} = q < 2q$. Moreover, it is obvious that $M_1$ has no $(\{K_{1,i} : q \leq i \leq 2q-1\} \cup \{K_{2q+1}\})$-factor since $2q+1 = \mathrm{iso}(M_1 - U) > \frac{|U|}{q} = 2q$.

**Lemma 2.4.** *Let* $2 \leq q$ $(\in \mathbb{Z}^+)$ *and* $G$ *be a graph. If* bind $(G) \geq q$, *then* $G$ *has a* $(\{K_{1,i} : q \leq i \leq 2q-1\} \cup \{K_{2q+1}\})$*-factor.*

*Proof.* Assume that $G$ has no $(\{K_{1,i} : q \leq i \leq 2q-1\} \cup \{K_{2q+1}\})$-factor, then we will show that bind $(G) < q$. Because Theorem 2.1, we have a subset $U \subset V(G)$ such that $\mathrm{iso}(G-U) \geq \frac{1}{q}|U| + \frac{1}{q}\cdot$ Let $\mathscr{C}_1$ be the set of isolated vertices in $G-U$ and $|\mathscr{C}_1| = c_1$. We can get $\mathrm{iso}(G-U) = c_1 \geq \frac{1}{q}|U| + \frac{1}{q}$. Notice that $c_1 \geq 1$ and $|V(G)| \geq c_1 + |U|$. Let $Y = V(G) \setminus U$. Then $N_G(Y) = V(G) \setminus V(\mathscr{C}_1) \neq V(G)$.

$$\begin{aligned}
\text{bind } (G) &\leq \frac{|N_G(Y)|}{|Y|} = \frac{|V(G)| - c_1}{|V(G)| - |U|}\\
&= 1 + \frac{|U| - c_1}{|V(G)| - |U|}\\
&\leq 1 + \frac{c_1 \cdot (q-1) - 1}{c_1}\\
&= q - \frac{1}{c_1} < q.
\end{aligned}$$

Therefore, bind $(G) < q$. The proof is completed. $\square$

**Remark 2.4.** For showing the tightness of bind $(G) < q$, we use the graph $G_k$ appeared in Remark 2.1. Let $Y \subset V(G_k)$ such that $N_{G_k}(Y) \neq V(G_k)$, then $|Y \cap V(K)| \leq 1$. If $|Y \cap V(K)| = 1$, then $|Y \cap (V(G_k) \setminus V(K))| = 0$ since $N_{G_k}(Y) \neq V(G_k)$. Hence, we have $\frac{|N_{G_k}(Y)|}{|Y|} = \frac{|V(G_k)|-1}{1} > k$. If $|Y \cap V(K)| = 0$, then $Y \subseteq V(G_k) \setminus V(K)$. Let $k_1$ be the number of independent vertices in $Y$, *i.e.*, $|Y| = k_1$. Notice that $1 \leq k_1 \leq k+1$ and $|N_{G_k}(Y)| = qk$. Therefore,

$$\frac{|N_{G_k}(Y)|}{|Y|} = \frac{qk}{k_1} \geq \frac{qk}{k+1}\cdot$$

When $|Y| = k+1$, we have bind $(G_k) = \frac{qk}{k+1}$ and $\lim\limits_{k \to +\infty}$ bind $(G_k) = q$.

## 3. Conditions for the presence of path factor

Let $3 \leq q$ $(\in \mathbb{Z}^+)$. A spanning subgraph $E$ of $G$ is a $\{P_2, P_{2q+1}\}$-factor if all the components of $E$ are isomorphic to $P_2$ or $P_{2q+1}$. We also use four parameters to give the criterions of graph $G$ to have a $\{P_2, P_{2q+1}\}$-factor, see Theorem 3.1.

**Theorem 3.1.** *Let* $3 \leq q$ $(\in \mathbb{Z}^+)$. *$G$ is a graph with* $|V(G)| \geq \frac{6}{5}q^2 + 2q - \frac{1}{5}$ *and* $\kappa(G) \geq \frac{6}{5}q^2 + \frac{4}{5}\cdot$ *If $G$ admits one of the following conditions:*

(i) $t(G) \geq \frac{6}{5}q^2$;
(ii) $I(G) \geq \frac{6}{5}q^2 + 2q - 2$;
(iii) $I'(G) \geq \frac{12}{5}q^2 + 4q - 4$;
(iv) bind $(G) \geq \frac{6}{5}q^2 + 2q - 2$,

*then $G$ has a* $\{P_2, P_{2q+1}\}$*-factor.*

The rest part we will use four lemmas to prove Theorems 3.1 and 3.2 is very useful to our proof:

**Theorem 3.2** (Egawa, Furuya and Ozeki [6])**.** *Let* $3 \leq q$ $(\in \mathbb{Z}^+)$ *and $G$ be a graph. If* $\sum_{0 \leq i \leq q-1} c_{2i+1}(G-U) \leq \frac{5}{6q^2}|U|$ *for all* $U \subseteq V(G)$, *then $G$ has a* $\{P_2, P_{2q+1}\}$*-factor.*

**Lemma 3.1.** *Let* $3 \leq q$ $(\in \mathbb{Z}^+)$ *and $G$ be a graph with* $|V(G)| \geq \frac{6}{5}q^2 + 2q - \frac{1}{5}\cdot$ *If $G$ satisfies* $t(G) \geq \frac{6}{5}q^2$, *then $G$ admits a* $\{P_2, P_{2q+1}\}$*-factor.*

*Proof.* Assume that $G$ has no $\{P_2, P_{2q+1}\}$-factor, then by Theorem 3.2, there exists a subset $U \subseteq V(G)$ such that $\sum_{0\le i\le q-1} c_{2i+1}(G-U) \ge \frac{5}{6q^2}|U| + \frac{1}{6q^2}$. Our aim is to show that $t(G) < \frac{6}{5}q^2$. Recall the definition of $c(G-U)$ and $c_{2i+1}(G-U)$, we can obtain that

$$c(G-U) \ge \sum_{0\le i\le q-1} c_{2i+1}(G-U) \ge \frac{5}{6q^2}|U| + \frac{1}{6q^2}. \tag{1}$$

**Case 1.** $c(G-U) \ge 2$.

If $c(G-U) \ge 2$, then

$$t(G) \le \frac{|U|}{c(G-U)} \le \frac{6q^2 \cdot c(G-U) - 1}{5 \cdot c(G-U)} = \frac{6}{5}q^2 - \frac{1}{5 \cdot c(G-U)} < \frac{6}{5}q^2.$$

**Case 2.** $c(G-U) = 1$.

By inequation (1), we have $\sum_{0\le i\le q-1} c_{2i+1}(G-U) = 1$ and $1 \ge \frac{5}{6q^2}|U| + \frac{1}{6q^2}$, that is, $|U| \le \frac{6}{5}q^2 - \frac{1}{5}$. Let $c_{2s+1}(G-U) = 1$ for $0 \le s \le q-1$, then $|V(G)| = |U| + 2s + 1 \le \frac{6}{5}q^2 + 2q - \frac{6}{5}$ which is a contradiction.

Therefore, we proved $t(G) < \frac{6}{5}q^2$. □

**Remark 3.1.** We will give a kind of graphs to show that $t(G) < \frac{6}{5}q^2$ is tight. Let $H_k = K' \vee (5k+1)K_1$, *i.e.* $H_k$ is a complete graph with order $6q^2k$ by joining $5k+1$ isolated vertices to each vertex of $K'$ for $k \ge 1$. Clearly, $H_k$ has no $\{P_2, P_{2q+1}\}$-factor as $5k+1 = \sum_{0\le i\le q-1} c_{2i+1}(H_k - V(K')) = c_1(H_k - V(K')) > \frac{5}{6q^2}|V(K')| = 5k$. If $U$ is a vertex set satisfies $c(H_k - U) \ge 2$, then there must be $V(K') \subseteq U$. Thus, we have $|U| \ge 6q^2k$ and $c(H_k - U) = 6q^2k + 5k + 1 - |U|$. Hence,

$$\frac{|U|}{c(H_k - U)} = \frac{|U|}{6q^2k + 5k + 1 - |U|} = -1 + \frac{6q^2k + 5k + 1}{6q^2k + 5k + 1 - |U|} \ge \frac{6q^2k}{5k+1}.$$

Therefore, we have $t(H_k) = \frac{6q^2k}{5k+1}$ and $\lim\limits_{k\to+\infty} t(H_k) = \frac{6}{5}q^2$.

**Lemma 3.2.** *Let* $3 \le q$ $(\in \mathbb{Z}^+)$ *and* $G$ *be a graph with* $|V(G)| \ge \frac{6}{5}q^2 + 2q - \frac{1}{5}$ *and* $\kappa(G) \ge \frac{6}{5}q^2 + \frac{4}{5}$. *If* $G$ *satisfies* $I(G) \ge \frac{6}{5}q^2 + 2q - 2$, *then* $G$ *admits a* $\{P_2, P_{2q+1}\}$*-factor.*

*Proof.* Assume that $G$ has no $\{P_2, P_{2q+1}\}$-factor. Then, there exists a subset $U \subseteq V(G)$ such that $\sum_{0\le i\le q-1} c_{2i+1}(G-U) \ge \frac{5}{6q^2}|U| + \frac{1}{6q^2}$. It is sufficient to show that $I(G) < \frac{6}{5}q^2 + 2q - 2$.

**Case 1.** $\sum_{0\le i\le q-1} c_{2i+1}(G-U) \ge 2$.

Let $b_{2i+1}(j)$ denote the number of elements with the minimum degree $j$ for $1 \le j \le 2i$ in $\mathscr{C}_{2i+1}(G-U)$. Clearly, we have $\sum_{j=1}^{2i} b_{2i+1}(j) = c_{2i+1}(G-U)$. Let $X_i$ be the set of vertices which are adjacent to the vertex of minimum degree in every elements of $\mathscr{C}_{2i+1}(G-U)$. Let $X = \cup_{i=1}^{q-1} X_i$ and $Y = U \cup X$. Therefore, we have $|Y| = |U| + \sum_{i=1}^{q-1}\sum_{j=1}^{2i} j \cdot b_{2i+1}(j)$ and $\mathrm{iso}(G-Y) \ge \sum_{0\le i\le q-1} c_{2i+1}(G-U) \ge 2$. Notice that

$$\sum_{i=1}^{q-1}\sum_{j=1}^{2i} j \cdot b_{2i+1}(j) \le \sum_{1\le i\le q-1} 2i \cdot c_{2i+1}(G-U) \le \sum_{1\le i\le q-1} (2q-2) \cdot c_{2i+1}(G-U).$$

Hence,

$$\begin{aligned} I(G) &\leq \frac{|Y|}{\mathrm{iso}(G-Y)} \\ &\leq \frac{|U| + \sum_{i=1}^{q-1}\sum_{j=1}^{2i} j\cdot b_{2i+1}(j)}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)} \\ &< \frac{\sum_{0\leq i\leq q-1}\frac{6}{5}q^2 c_{2i+1}(G-U) + \sum_{i=1}^{q-1}\sum_{j=1}^{2i} j\cdot b_{2i+1}(j)}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)} \\ &= \frac{6}{5}q^2 + \frac{\sum_{i=1}^{q-1}\sum_{j=1}^{2i} j\cdot b_{2i+1}(j)}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)} \\ &\leq \frac{6}{5}q^2 + \frac{\sum_{0\leq i\leq q-1}(2q-2)\cdot c_{2i+1}(G-U) - (2q-2)\cdot c_1}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)} \\ &= \frac{6}{5}q^2 + 2q - 2 - \frac{(2q-2)\cdot c_1}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)} \\ &\leq \frac{6}{5}q^2 + 2q - 2. \end{aligned}$$

**Case 2.** $\sum_{0\leq i\leq q-1} c_{2i+1}(G-U) = 1$.

Notice that $0 \leq |U| \leq \frac{6}{5}q^2 - \frac{1}{5}$ since $1 \geq \frac{5}{6q^2}|U| + \frac{1}{6q^2}\cdot$ Without loss of generality, let $c_{2q-1} = 1$. If $|V(G)| = |U| + 2q - 1$, then contradict with $|V(G)| \geq \frac{6}{5}q^2 + 2q - \frac{1}{5}\cdot$ Therefore, there are other components excepting $S$ and the above component with $2q-1$ vertices, that is to say $\kappa(G) \leq |U| \leq \frac{6}{5}q^2 - \frac{1}{5}$, also a contradiction.

Thus, we proved $I(G) < \frac{6}{5}q^2 + 2q - 2$. □

**Remark 3.2.** To show that $I(G) < \frac{6}{5}q^2+2q-2$ is tight. We construct a kind of graphs $F_k = K' \vee (5k+1)K_{2q-1}$ for $k \geq 1$. $K'$ is a complete graph with order $6q^2k$. This graph satisfies $|V(F_k)| = 6q^2k + (2q-1)(5k+1) > \frac{6}{5}q^2+2q-\frac{1}{5}$ and $\kappa(F_k) = 6q^2k > \frac{6}{5}q^2+\frac{4}{5}\cdot$ Clearly, $F_k$ has no $\{P_2, P_{2q+1}\}$-factor as $\sum_{0\leq i\leq q-1} c_{2i+1}(F_k - V(K')) = c_{2q-1}(F_k - V(K')) > \frac{5}{6q^2}|V(K')|$. Let $U$ be a vertex subset such that $\mathrm{iso}(F_k - U) \geq 2$, then all vertices in complete graph $K'$ must be in $U$. Then, we have $|U| \geq 6q^2k + (2q-2)\cdot \mathrm{iso}(F_k - U)$ and $\mathrm{iso}(F_k - U) \leq 5k+1$. Hence,

$$\begin{aligned} \frac{|U|}{\mathrm{iso}(F_k-U)} &\geq \frac{6q^2k + (2q-2)\cdot \mathrm{iso}(F_k-U)}{\mathrm{iso}(F_k-U)} \\ &= 2q - 2 + \frac{6q^2k}{\mathrm{iso}(F_k-U)} \\ &\geq 2q - 2 + \frac{6q^2k}{5k+1}\cdot \end{aligned}$$

Take $U'$ as $V(K')$, we have $I(F_k) = 2q - 2 + \frac{6q^2k}{5k+1}$ and $\lim\limits_{k\to+\infty} I(F_k) = 2q - 2 + \frac{6}{5}q^2\cdot$

**Lemma 3.3.** *Let $3 \leq q$ $(\in \mathbb{Z}^+)$. Let $G$ be a graph with $|V(G)| \geq \frac{6}{5}q^2+2q-\frac{1}{5}$ and $\kappa(G) \geq \frac{6}{5}q^2+\frac{4}{5}\cdot$ If $G$ satisfies $I'(G) \geq \frac{12}{5}q^2 + 4q - 4$, then $G$ admits a $\{P_2, P_{2q+1}\}$-factor.*

*Proof.* Assume that $G$ has no $\{P_2, P_{2q+1} : q \geq 3\}$-factor, we will show that $I'(G) < \frac{6}{5}q^2 + 4q - 4$. Because Theorem 3.2, we obtained a subset $U \subseteq V(G)$ such that $\sum_{0\leq i\leq q-1} c_{2i+1}(G-U) \geq \frac{5}{6q^2}|U| + \frac{1}{6q^2}\cdot$ The method is the same as Lemma 3.2 that we should discuss two cases.

**Case 1.** $\sum_{0\leq i\leq q-1} c_{2i+1}(G-U) \geq 2$.

According to the proof of Lemma 3.2, we have

$$\begin{aligned}
I'(G) &\leq \frac{|Y|}{\mathrm{iso}(G-Y)-1} \\
&\leq \frac{|U| + \sum_{i=1}^{q-1}\sum_{j=1}^{2i} j\cdot b_{2i+1}(j)}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)-1} \\
&< \frac{\sum_{0\leq i\leq q-1}\frac{6}{5}q^2 c_{2i+1}(G-U) + \sum_{i=1}^{q-1}\sum_{j=1}^{2i} j\cdot b_{2i+1}(j)}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)-1} \\
&= \frac{6}{5}q^2 + \frac{\frac{6}{5}q^2 + \sum_{i=1}^{q-1}\sum_{j=1}^{2i} j\cdot b_{2i+1}(j)}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)-1} \\
&\leq \frac{6}{5}q^2 + \frac{\frac{6}{5}q^2 + (2q-2)\cdot\sum_{0\leq i\leq q-1} c_{2i+1}(G-U) - (2q-2)\cdot c_1}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)-1} \\
&= \frac{6}{5}q^2 + \frac{\frac{6}{5}q^2 + (2q-2)\cdot\left(\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)-1\right) + (2q-2)\cdot(1-c_1)}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)-1} \\
&= \frac{6}{5}q^2 + 2q - 2 + \frac{\frac{6}{5}q^2 + (2q-2)\cdot(1-c_1)}{\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)-1} \\
&\leq \frac{12}{5}q^2 + 2q - 2 + (2q-2)\cdot(1-c_1) \\
&\leq \frac{12}{5}q^2 + 4q - 4.
\end{aligned}$$

**Case 2.** $\sum_{0\leq i\leq q-1} c_{2i+1}(G-U) = 1$.

In this case, we can obtain $0 \leq |U| \leq \frac{6}{5}q^2 - \frac{1}{5}$. Then $c_{2q-1} = 1$ is one possible way such that $\sum_{0\leq i\leq q-1} c_{2i+1}(G-U) = 1$. If $|V(G)| = |U| + 2q - 1$, then contradict with $|V(G)| \geq \frac{6}{5}q^2 + 2q - \frac{1}{5}$. If there are other components excepting $U$ and the above component with $2q-1$ vertices, then $\kappa(G) \leq |U| \leq \frac{6}{5}q^2 - \frac{1}{5}$, a contradiction.

From the above discuss, we finally get that $I'(G) < \frac{12}{5}q^2 + 4q - 4$. $\square$

**Remark 3.3.** Let us show that $I'(G) \geq \frac{12}{5}q^2 + 4q - 4$ in Lemma 3.3 is all the best possible. We construct a graph $M_2 = K_{6q^2} \vee 6K_1$. Clearly, $M_2$ has no $\{P_2, P_{2q+1}\}$-factor as $\sum_{0\leq i\leq q-1} c_{2i+1}(M_2 - V(K_{6q^2})) = c_1(M_2 - V(K_{6q^2})) > \frac{5}{6q^2}\left|V(K_{6q^2})\right|$. Also, this graph $G$ satisfies $|V(G)| = 6q^2 + 6 > \frac{6}{5}q^2 - \frac{1}{5} + 2q$ and $\kappa(G) = 6q^2 > \frac{6}{5}q^2 - \frac{1}{5} + 1$. Let $U$ be a vertex subset satisfies $\mathrm{iso}(M_2 - U) \geq 2$, we have $V(K_{6q^2}) \subseteq U$. Take $U$ as $V(K_{6q^2})$, then $I'(M_2) = \frac{|U|}{\mathrm{iso}(G-U)-1} = \frac{6q^2}{5} < \frac{12}{5}q^2 + 4q - 4$.

**Lemma 3.4.** *Let $G$ be a graph and $3 \leq q$ ($\in \mathbb{Z}^+$). If $G$ satisfies* bind $(G) \geq \frac{6}{5}q^2 + 2q - 2$, *then $G$ has a $\{P_2, P_{2q+1}\}$-factor.*

*Proof.* Assume that $G$ has no $\{P_2, P_{2q+1}\}$-factor, we have a subset $U \subseteq V(G)$ such that $\sum_{0\leq i\leq q-1} c_{2i+1}(G-U) > \frac{5}{6q^2}|U|$. It is sufficient to prove that bind $(G) < \frac{6}{5}q^2 + 2q - 2$.

If $c_1 \geq 1$, then we take $Y = V(G) \setminus U$. Notice that $N_G(Y) = V(G) \setminus V(\mathcal{C}_1(G-U)) \neq V(G)$ and $|V(G)| \geq \sum_{0\leq i\leq q-1}(2i+1)\cdot c_{2i+1}(G-U)+|U|$. Hence,

$$\begin{aligned}
\text{bind }(G) &\leq \frac{|N_G(Y)|}{|Y|} \\
&= \frac{|V(G)|-c_1}{|V(G)|-|U|} \\
&= 1+\frac{|U|-c_1}{|V(G)|-|U|} \\
&< 1+\frac{\sum_{0\leq i\leq q-1}\frac{6}{5}q^2\cdot c_{2i+1}(G-U)-c_1}{\sum_{0\leq i\leq q-1}(2i+1)\cdot c_{2i+1}(G-U)} \\
&= 1+\frac{\sum_{0\leq i\leq q-1}\frac{6}{5}q^2(2i+1)\cdot c_{2i+1}(G-U)-\sum_{0\leq i\leq q-1}\frac{12}{5}q^2 i\cdot c_{2i+1}(G-U)-c_1}{\sum_{0\leq i\leq q-1}(2i+1)\cdot c_{2i+1}(G-U)} \\
&= 1+\frac{6}{5}q^2-\frac{\sum_{0\leq i\leq q-1}\frac{12}{5}q^2 i\cdot c_{2i+1}(G-U)+c_1}{\sum_{0\leq i\leq q-1}(2i+1)\cdot c_{2i+1}(G-U)} \\
&= 1+\frac{6}{5}q^2-\frac{\sum_{0\leq i\leq q-1}(2i+1)\cdot c_{2i+1}(G-U)+\sum_{0\leq i\leq q-1}\left(\frac{12}{5}q^2 i-2i-1\right)\cdot c_{2i+1}(G-U)+c_1}{\sum_{0\leq i\leq q-1}(2i+1)\cdot c_{2i+1}(G-U)} \\
&= \frac{6}{5}q^2-\frac{\sum_{0\leq i\leq q-1}\left(\frac{12}{5}q^2 i-2i-1\right)\cdot c_{2i+1}(G-U)+c_1}{\sum_{0\leq i\leq q-1}(2i+1)\cdot c_{2i+1}(G-U)}.
\end{aligned}$$

Since $q \geq 3$, it follows that $\sum_{0\leq i\leq q-1}\left(\frac{12}{5}q^2 i-2i-1\right)\cdot c_{2i+1}(G-U)+c_1 > 0$. Thus, we can obtain bind $(G) < \frac{6}{5}q^2$ in this case.

If $c_1 = 0$, the following two cases are discussed further.

**Case 1.** $\sum_{2\leq i\leq q-1} c_{2i+1}(G-U) = 0$.

Notice that $c_3 \geq 1$ since $\sum_{0\leq i\leq q-1} c_{2i+1}(G-U) \geq \frac{5}{6q^2}|U| + \frac{1}{6q^2}$ and $\sum_{0\leq i\leq q-1} c_{2i+1}(G-U)$ must be an integer. We firstly consider $c_3 = 1$. Let $T$ be the component with 3 vertices and $v$ be one of the vertices in $T$. Because $N_G(v) \subseteq (T\setminus\{v\})\cup U \neq V(G)$, we have $|N_G(v)| \leq |T\setminus\{v\}|+|U| \leq \frac{6}{5}q^2+\frac{9}{5}$. Thus,

$$\text{bind }(G) \leq \frac{|N_G(v)|}{|\{v\}|} \leq \frac{6}{5}q^2+\frac{9}{5}.$$

The rest is to consider $c_3 \geq 2$. Let $Y$ be a set of all vertices from $c_3-1$ elements in $\mathscr{C}_3(G-U)$. Obviously, $|Y| = 3(c_3-1)$. Because $N_G(Y) \subseteq Y\cup U \neq V(G)$, then we have

$$\begin{aligned}
\text{bind }(G) &\leq \frac{|N_G(Y)|}{|Y|} \leq \frac{|Y|+|U|}{|Y|} \\
&= 1+\frac{|U|}{|Y|} \leq 1+\frac{6q^2c_3-1}{15(c_3-1)} \\
&= 1+\frac{6q^2(c_3-1)+6q^2-1}{15(c_3-1)} \\
&= 1+\frac{2q^2}{5}+\frac{6q^2-1}{15(c_3-1)} \\
&\leq \frac{4}{5}q^2+\frac{14}{15}.
\end{aligned}$$

**Case 2.** $\sum_{2\le i\le q-1} c_{2i+1}(G-U)\ge 1$.

Notice that there must exist an integer $s$ for $2\le s\le q-1$ such that $c_{2s+1}\ne 0$ since $\sum_{2\le i\le q-1} c_{2i+1}(G-U)\ge 1$. We can calculate the minimum degree of all elements in $\mathscr{C}_{2s+1}(G-U)$ and let $T_s$ is the element with the least minimum degree in $\mathscr{C}_{2s+1}(G-U)$. Let $v$ be the minimum degree vertex in $T_s$ and $Y=V(G)\setminus(S\cup N_{T_s}(v))$, then $d_G(v)\le 2s$ and $N_G(Y)\subseteq V(G)\setminus\{v\}\ne V(G)$. We can easily get $|Y|=|V(G)|-|U|-|N_{T_s}(v)|$ and $|N_G(Y)|\le|V(G)|-1$. Hence,

$$
\begin{aligned}
\text{bind }(G) &\le \frac{|N_G(Y)|}{|Y|}\\
&\le \frac{|V(G)|-1}{|V(G)|-|U|-|N_{T_s}(v)|}\\
&= 1+\frac{|U|+|N_{T_s}(v)|-1}{|V(G)|-|U|-|N_{T_s}(v)|}\\
&< 1+\frac{\sum_{0\le i\le q-1}\frac{6}{5}q^2\cdot c_{2i+1}(G-U)+|N_{T_s}(v)|-1}{\sum_{0\le i\le q-1}(2i+1)\cdot c_{2i+1}(G-U)-|N_{T_s}(v)|}\\
&= 1+\frac{6}{5}q^2-\frac{\sum_{0\le i\le q-1}\frac{12}{5}q^2 i\cdot c_{2i+1}(G-U)-\left(\frac{6}{5}q^2+1\right)|N_{T_s}(v)|+1}{\sum_{0\le i\le q-1}(2i+1)\cdot c_{2i+1}(G-U)-|N_{T_s}(v)|}\\
&= 1+\frac{6}{5}q^2+\frac{\frac{6}{5}q^2\left(|N_{T_s}(v)|-\sum_{0\le i\le q-1}2i\cdot c_{2i+1}(G-U)\right)+|N_{T_s}(v)|-1}{\sum_{0\le i\le q-1}(2i+1)\cdot c_{2i+1}(G-U)-|N_{T_s}(v)|}\\
&\le \frac{6}{5}q^2+|N_{T_s}(v)|\\
&\le \frac{6}{5}q^2+2q-2.
\end{aligned}
$$

In the penultimate equality, we use the fact that $|N_{T_s}(v)|-\sum_{0\le i\le q-1}2i\cdot c_{2i+1}(G-U)\le 0$ and $\sum_{0\le i\le q-1}(2i+1)\cdot c_{2i+1}(G-U)-|N_{T_s}(v)|\ge 1$.

Therefore, we have bind $(G)<\frac{6}{5}q^2+2q-2$. □

**Remark 3.4.** We will show that bind $(G)\le\frac{6}{5}q^2+2q-2$ in Lemma 3.4 is all the best possible. We construct a graph $M_3=K_{6q^3}\vee(5q+1)K_1$. By the definition of bind $(M_3)$, let $Y$ be a vertex set that containing all the independent vertices in $M_3$, then $|Y|=5q+1$. It is obvious that bind $(M_3)=\frac{|N_{M_3}(Y)|}{|Y|}=\frac{6q^3}{5q+1}<\frac{6q^2}{5}+2q-2$. Moreover, graph $M_3$ has no $\{P_2,P_{2q+1}\}$-factor since $5q+1=c_1(M_3-V(K_{6q^3}))>\frac{5}{6q^2}|K_{6q^3}|=5q$.

## 4. Conclusion

In this contribution, we have shown some sufficient conditions of $G$ for having $(\{K_{1,i}:q\le i\le 2q-1\}\cup\{K_{2q+1}\})$-factor and $\{P_2,P_{2q+1}\}$-factor, respectively. But note that each remark came after each lemma can only tell us that the value we obtained is the best upper bound when graph $G$ has no $(\{K_{1,i}:q\le i\le 2q-1\}\cup\{K_{2q+1}\})$-factor and $\{P_2,P_{2q+1}\}$-factor, respectively. Because the Theorems 2.1 and 3.2 are only the sufficient conditions.

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## References


[1] I. Anderson, Perfect matchings of a graph. *J. Graph Theory* **10** (1971) 183–186.

[2] S. Abdullah and M. Aslam, New multicriteria group decision support systems for small hydropower plant locations selection based on intuitionistic cubic fuzzy aggregation information. *Int. J. Intell. Syst.* **35** (2020) 983–1020.

[3] C. Chen, K.M. Koh and Y. Peng, On the higher-order edge toughness of a graph. *Discrete Math.* **111** (1993) 113–123.

[4] F. Chiclana, F. Mata, L. Pérez and E. Herrera-Viedma, Type-1 OWA unbalanced fuzzy linguistic aggregation methodology: application to eurobonds credit risk evaluation. *Int. J. Intell. Syst.* **33** (2018) 1071–1088.

[5] V. Chvátal, Tough graphs and Hamiltonian circuits. *Discrete Math.* **5** (1973) 215–228.

[6] Y. Egawa, M. Furuya and K. Ozeki, Sufficient conditions for the existence of a path-factor which are related to odd components. *J. Graph Theory.* **89** (2018) 327–340.

[7] W. Gao and Y. Gao, Toughness condition for a graph to be a fractional $(g, f, n)$-critical deleted graph. *Sci. World J.* (2014) 369–798.

[8] W. Gao and W.F. Wang, Remarks on component factors. *J. Oper. Res. Soc. China* (2021) 1–11.

[9] W. Gao, W.F. Wang and Y.J. Chen, Tight bounds for the existence of path factors in network vulnerability parameter settings. *Int. J. Intell. Syst.* **36** (2021) 1133–1158.

[10] X. Guan, T. Ma and C. Shi, Tight toughness, isolated toughness and binding number bounds for the $\{K_2, C_n\}$-factors. Preprint arXiv:2204.04373 (2022).

[11] M. Kano, H. Lu and Q. Yu, Component factors with large components in graphs. *Appl. Math. Lett.* **23** (2010) 385–389.

[12] C. Liu and L. Zhang, Toughness and the existence of fractional $k$-factors of graphs. *Discrete Math.* **308** (2008) 1741–1748.

[13] D. Woodall, The binding number of a graph and its Anderson number. *J. Graph Theory B* **15** (1973) 225–255.

[14] J. Yang, Y. Ma and G. Liu, Fractional $(g, f)$-factors in graphs. *Appl. Math. J. Chinese Univ. Ser. A* **16** (2001) 385–390.

[15] Q. Yu and G. Liu, Graph Factors and Matching Extensions, Springer-Verlag, Berlin (2009) xii+353.

[16] L. Zhang and G. Liu, Fractional $k$-factor of graphs. *J. Syst. Sci. Math. Sci.* **21** (2001) 88–92.

[17] Y. Zhang, G. Yan and M. Kano, Star-like factors with large components. *J. Oper. Res. Soc. China* **3** (2015) 81–88.

[18] S. Zhou, Z. Sun and H. Liu, Isolated toughness and path-factor uniform graphs. *RAIRO: OR* **55** (2021) 1279–1290.

[19] S. Zhou, J. Wu and Q. Bian, On path-factor critical deleted (or covered) graphs. *Aequ. Math.* **96** (2022) 795–802.

[20] S. Zhou, J. Wu and Y. Xu, Toughness, isolated toughness and path factors in graphs. *Bull. Aust. Math. Soc.* **106** (2022) 195–202.

[21] S. Zhou, Q. Bian and Q. Pan, Path factors in subgraphs. *Discrete Appl. Math.* **319** (2022) 183–191.