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AN EFFICIENT MULTI PARAMETRIC KERNEL FUNCTION FOR LARGE AND SMALL-UPDATE METHODS INTERIOR POINT ALGORITHM FOR $P_*(\kappa)$-HORIZONTAL LINEAR COMPLEMENTARITY PROBLEM

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Abstract. In this paper, we propose the first efficient multi parametric kernel function with logarithmic barrier term. A class of polynomial interior-point algorithms for $P_*(\kappa)$-horizontal linear complementarity problem based on this kernel function, with parameters $p_i > 0$ for all $i \in 1, 2, , m$, are presented. Then by using some simple analysis tools, we present a primal-dual interior point method (**IPM**) for $P_*(\kappa)$-horizontal linear complementarity problems based on this kernel function. At the same time, we derive the complexity bounds small and large-update methods, respectively. In particular, if we take many different values of the parameters, we obtain the best known iteration bounds for the algorithms with large- and small-update methods are derived, namely, $\mathbf{O}\big((1+2\kappa)\sqrt{n}(\log n)\log\frac{n}{\epsilon}\big)$ and $\mathbf{O}\big((1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon}\big)$ respectively. We illustrate the performance of the proposed kernel function by some numerical results that are derived by applying our algorithm.

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1. Introduction

The $P_*(\kappa)$-horizontal linear complementarity problem ($P_*(\kappa)$-HLCP) consists of determining a pair of vectors $(x, y) \in \mathbb{R}^{2n}$ that satisfies the following conditions:

$$\begin{aligned} -Mx + Ny = q, &\qquad (x, y) \geq 0, \\ xy = 0. & \end{aligned} \tag{1}$$

where $q \in \mathbb{R}^n; M, N \in \mathbb{R}^{n\times n}$ and the matrix pair (M, N) belongs to P_* the class of matrices was first introduced by Kojima *et al.* [12], *i.e.*, for some nonnegative κ,

$$-Mx + Ny = 0 \Longrightarrow (1+4\kappa)\sum_{i\in I_+(x)} x_i y_i + \sum_{i\in I_-(x)} x_i y_i \geq 0$$

where $\kappa \geq 0, I_+(x) = \{i \in I : x_i y_i \geq 0\}, I_-(x) = \{i \in I : x_i y_i < 0\},$ and $I = \{1, 2, \ldots, n\}$

Keywords. $P_*(\kappa)$-horizontal linear complementarity problem, Multi parametric kernel function, Large- and small-update methods, Interior point methods, Complexity bound.

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In recent years, several interior point methods (**IPMs**) for solving $P_*(\kappa)$-HLCP have been used. Wang *et al.* [21] presented an interior point algorithm based on the parametric kernel function which was first introduced in [7]. The theoretical results show that their algorithm has the best known iteration complexity bound. Asadi *et al.* [1] proposed the new kernel function for $P_*(\kappa)$ and computed the worst case iteration complexity bounds for large and small-update methods. They showed that their algorithm has $\mathbf{O}\big((1+2\kappa)\sqrt{n}(\log n)\log\frac{n}{\epsilon}\big)$ iteration complexity for large update methods. For some other related **IPMs** solving $P_*(\kappa)$-HLCP, we refer to [1,2,9,11,13]. Moreover, Wang *et al.* [22] presented a feasible interior-point method IPM for the Cartesian $P_*(\kappa)$-linear complementarity problem over symmetric cones ($P_*(\kappa)$-SCLCP) that is based on the classical logarithmic barrier function. The method uses Nesterov-Todd search directions and full step updates of iterates. With the appropriate choice of parameters the algorithm generates a sequence of iterates in the small neighbourhood of the central path which implies global convergence of the method. Because linear complementarity problems are closely related to linear programming problems, Wang *et al.* [23] proved that the full-Newton step to the central path is local quadratically convergent and the proposed algorithm has polynomial iteration complexity.

Furthermore, Wang *et al.* [24] considered a full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-Linear Complementarity Problems ($P_*(\kappa)$-LCPs). The perturbed complementarity equation $xs = \mu e$ is transformed by using a strictly increasing function, *i.e.*, replacing $xs = \mu e$ by $\varphi(xs) = \varphi(\mu e)$ with $\varphi(t) = \sqrt{t}, \varphi(t) = t$, and the proposed interior-point algorithm is based on that algebraic equivalent transformation. Furthermore, established the currently best known iteration bound for $P_*(\kappa)$-LCPs, namely, $\mathbf{O}\big((1+2\kappa)\sqrt{n}(\log n)\log\frac{n}{\epsilon}\big)$ iteration complexity for large update methods.

Besides, Wang *et al.* [5,15] analyzed IPMs for $P_*(\kappa)$-LCPs based on a new eligible parametric kernel function. The concept eligible kernel functions introduced by Bai *et al.* [3] and based on these kernel functions the best known complexity bounds were obtained. In recent years several new kernel functions have been proposed. An attractive kernel function that has trigonometric in its barrier term was first suggested by El ghami *et al.* [8], in 2005.

In 2016, Bouafia *et al.* [4], we proposed the first with trigonometric barrier terms for interior point methods in linear optimization problem (**LO**). We generalized and improved the complexity bound based on a new kernel function with trigonometric barrier terms obtained in [8,18,19]. We obtained the best known complexity results for large and small-update methods.

In 2017, El ghami *et al.* [6], extended the results obtained for kernel function based **IPMs** in [4] for **LO** to $P_*(\kappa)$-LCPs.

Recently, in 2021, Hazzam *et al.* [10], extended and improved the kernel function proposed by Bouafia *et al.* [4], for **LO** to $P_*(\kappa)$-HLCP.

Motivated by these works, we introduce new classes of eligible kernel functions, which are different from known kernel functions in [1, 2, 7, 11, 13] and have the multi parametric kernel function, with parameters $p_i > 0$ for all $i \in 1, 2, \ldots, m$, are presented. We define $\psi_M(t)$ as follows:

$$\psi_M(t) = 2t^2 - 2 - 2\log t + \frac{\sum_{i=1}^m t^{-p_i}}{a} + \frac{t^{-a}}{a} - \frac{m+1}{a},\ p_i > 0; \quad a = \sum_{i=1}^m p_i.$$

The proposed parametric kernel function is logarithmic barrier term, and propose a complexity analysis of the **IPMs** for $P_*(\kappa)$-HLCP based on these kernel functions. We show that these algorithms have

$$\mathbf{O}\Big((1+2\kappa)\sqrt{n}(\log n)\log\frac{n}{\epsilon}\Big) \text{ iteration complexity.}$$

and

$$\mathbf{O}\Big((1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon}\Big) \text{ iteration complexity.}$$

for large- and small update methods, respectively, which are currently the best known iteration bounds for such methods.

The paper is organized as follows. In Section 2, we propose some basic concepts and a generic interior point algorithm for $P_*(\kappa)$-HLCP. In Section 3, we introduce new classes of eligible kernel functions and their technical properties. Section 4, is devoted to describe the proximity reduction during an inner iteration and we analyze the algorithm and give a default step size. We derive the framework for analyzing the iteration bounds and the complexity results of the algorithms based on these kernel functions in Section 5. We provide some numerical results with other functions to illustrate the efficiency of the proposed algorithm are given in Section 6. Finally, Concluding remarks showing the added value of our work.

We use the following notations throughout the paper. $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the set of n-dimensional nonnegative vectors and positive vectors, respectively. For $x,\ y \in \mathbb{R}^n$, $x_{\min}$ and xs denote the smallest component of the vector x and the componentwise product of the vector x and y, respectively. We denote by $X = diag(x)$ the $n \times n$ diagonal matrix with the components of the vector $x \in \mathbb{R}^n$ are the diagonal entries, finally e denotes the n-dimensional vector of ones. And throughout the paper, $\|\|$ denotes the 2-norm of a vector.

2. The generic interior-point algorithm for $P_*(\kappa)$-HLCP

The main idea of **IPMs** is to replace the second equation in (1) by the parameterized nonlinear equation $xs = \mu e$ with $\mu > 0$ and $e = (1, 1, \ldots, 1)^T$. This leads to the following system

$$\begin{aligned} -Mx + Ny = q, &\quad (x, y) \geq 0,\\ xy = \mu e. \end{aligned} \tag{2}$$

Without loss of generality, we assume that $P_*(\kappa)$-HLCP satisfy the interior-point condition **IPC**, *i.e.*, there exist $(x^0,\ y^0)$ such that

$$-Mx^0 + Ny^0 = q,\ x^0 > 0,\ y^0 > 0. \tag{3}$$

The parameterized system (3) has a unique solution for each $\mu > 0$. This solution is denoted as $(x(\mu), y(\mu))$ and we call $(x(\mu), y(\mu))$ the μ-center of $P_*(\kappa)$-HLCP. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of $P_*(\kappa)$-HLCP. If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition $xy = 0$, it yields the solution for $P_*(\kappa)$-HLCP. Using Newton's method to the system (3), we have

$$\begin{aligned} -M\Delta x + N\Delta y &= 0,\\ y\Delta x + x\Delta y &= \mu e - xs. \end{aligned} \tag{4}$$

Note that this system has a unique solution. Now we can derive the new point as

$$x_+ = x + \alpha\Delta x, \quad y_+ = y + \alpha\Delta y. \tag{5}$$

where the step size α satisfies $0 < \alpha \leq 1$.

Now, we introduce the scaled vector v, that $v = \sqrt{\frac{xy}{\mu}}$. System (5) can be rewritten as follows:

$$\begin{aligned} -M\Delta x + N\Delta y &= 0,\\ y\Delta x + x\Delta y &= \mu v(v^{-1} - v), \end{aligned} \tag{6}$$

equivalent

$$\begin{aligned} -M\Delta x + N\Delta y &= 0,\\ y\Delta x + x\Delta y &= -\mu v\nabla\Psi(v), \end{aligned} \tag{7}$$

where the logarithmic barrier function $\Psi(v):\ \mathbb{R}^n_{++} \to \mathbb{R}_+$ is defined as follows:

$$\Psi(v) = \Psi(x, y; \mu) = \sum_{i=1}^{n} \psi(v_i), \tag{8}$$

$$\psi(v_i) = \frac{v_i^2 - 1}{2} - \log v_i. \tag{9}$$

Now, we introduce the scaled search directions d_x and d_y as follows:

$$d_x = \frac{v\Delta x}{x}, \quad d_y = \frac{v\Delta y}{y}. \tag{10}$$

Then we have the scaled Newton system (7) as follows:

$$-\overline{M} d_x + \overline{N} d_y = 0, \quad d_x + d_y = -\nabla\Psi(v), \tag{11}$$

where $\overline{M} = \mathrm{DMD}$, $\overline{N} = \mathrm{DND}$ with $D = X^{\frac{1}{2}} Y^{\frac{1}{2}}$, $X = \mathrm{diag}(x)$, $Y = \mathrm{diag}(y)$. We use $\Psi(v)$ as the proximity function to measure the distance between the current iterate and the μ-center for given $\mu > 0$. We also define the norm-based proximity measure, $\delta(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$, as follows

$$\delta(v) = \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\|d_x + d_y\|.$$

It is clear from the above description that the closeness of (x, y) to $(x(\mu), y(\mu))$ is measured by the value of $\Psi(v)$, with $\tau > 0$ as a threshold value.In all of the paper, based of on a new kernel function, we replace $\psi(t)$ by a new kernel function $\psi_M(t)$ and $\Psi(v)$ by a new barrier function $\Psi_M(v)$. If $\Psi_M(v) \leq \tau$, then we start a new outer iteration by performing a μ-update, otherwise we enter an inner iteration by computing the search directions at the current iterates with respect to the current value of μ and apply (5) to get new iterates. If necessary, we repeat the procedure until we find iterates that are in the neighborhood of $(x(\mu), y(\mu))$. Then μ is again reduced by the factor $1 - \theta$ with $0 < \theta < 1$, and we apply Newton's method targeting the new μ-centers, and so on. This process is repeated until μ is small enough, say until $n\mu < \epsilon$; at this stage we have found an ϵ-approximate solution of $P_*(\kappa)$-HLCP. The parameters τ, θ and the step size α should be chosen in such a way that the algorithm is optimized in the sense that the number of iterations required by algorithm is as small as possible. In the theoretical analysis the step size α is usually given a value that depends on the closeness of the current iterates to the μ-center. The generic form of this algorithm is as follows in Figure 1.

Generic interior-point algorithm for $P_*(\kappa)$-HLCP

Input:
 A proximity the function $\Psi_M(v)$;
 a threshold parameter $\tau > 1$;
 an accuracy parameter $\epsilon > 0$;
 a fixed barrier update parameter $\theta, 0 < \theta < 1$;
begin
 $x = e$; $y = e$; $\mu = 1$; $v = e$.
 while $n\mu \geq \epsilon$ **do**
 begin (outer iteration)
  $\mu = (1 - \theta)\,\mu$;
  while $\Psi_M\,(x, y; \mu) > \tau$ **do**
  begin (inner iteration)
   Solve the system (7), $\Psi\,(v)$ replaced by $\Psi_M\,(v)$ to obtain $(\Delta x, \Delta y)$;
   Choose a suitable step size α;
   $x = x + \alpha\Delta x$;
   $y = y + \alpha\Delta y$;
    $v = \sqrt{\frac{xy}{\mu}}$;
  end (inner iteration)
 end (outer iteration)
end.

FIGURE 1. Generic algorithm.

3. Properties of the new parametric kernel function and the barrier function

3.1. Properties of the new kernel function

In this section we investigate some properties of the new kernel function multi barrier terms which are essential to our complexity analysis. We call $\psi(t) : \mathbb{R}_{++} \to \mathbb{R}_+$ a kernel function if ψ is twice differentiable and satisfies the following conditions:

$$\begin{aligned} \psi'(1) &= \psi(1) = 0, \\ \psi''(t) &> 0, \\ \lim_{t\downarrow 0^+} \psi(t) &= \lim_{t\to+\infty} \psi(t) = +\infty. \end{aligned}$$

Now, we define a new function $\psi_M(t)$ as follows:

$$\psi_M(t) = 2t^2 - 2 - 2\log t + \frac{\sum_{i=1}^m t^{-p_i}}{a} + \frac{t^{-a}}{a} - \frac{m+1}{a},\ p_i > 0;\ a = \sum_{i=1}^m p_i. \tag{12}$$

For convenience of reference we put

$$b^2 = \sum_{i=1}^m p_i^2 \tag{13}$$

and we gives the first three derivatives with respect to t as follows:

$$\begin{aligned} \psi_M'(t) &= 4t - \frac{2}{t} - \frac{1}{a}\left(\sum_{i=1}^m p_i t^{-p_i-1} + a t^{-a-1}\right), \\ \psi_M''(t) &= 4 + \frac{2}{t^2} + \left(\sum_{i=1}^m \frac{p_i(p_i+1)}{a} t^{-p_i-2} + (a+1)t^{-a-2}\right), \\ \psi_M'''(t) &= -\left(\frac{4}{t^3} + \sum_{i=1}^m \frac{p_i(p_i+1)(p_i+2)}{a} t^{-p_i-3} + (a^2+3a+2)t^{-a-3}\right). \end{aligned} \tag{14}$$

The next lemma serves to prove that the new kernel function (12) is eligible.

Lemma 1. *Let $\psi_M(t)$ be as defined in (12) and $t > 0$. Then,*

$$\psi_M''(t) > 1, \tag{15}$$

$$\psi_M'''(t) < 0, \tag{16}$$

$$t\psi_M''(t) - \psi_M'(t) > 0, \tag{17}$$

$$t\psi_M''(t) + \psi_M'(t) > 0, \tag{18}$$

$$\psi_M''(t)\psi_M'(\beta t) - \beta\psi_M'(t)\psi_M''(\beta t) > 0, \qquad t > 1, \beta > 1. \tag{19}$$

Proof. For (15)) the second derivative of $\psi_M(t)$ is given in (12), using (14), we have

$$\psi_M''(t) > 4 \quad \text{for all } t > 0,$$

which implies that (15). For (16), the sign of $\psi_M'''(t)$, using (14), we have

$$\psi_M'''(t) < 0, \qquad t > 0,$$

which implies that (16). For (17), we again use (14),

$$t\psi_M''(t) - \psi_M'(t) = \frac{4}{t} + \frac{1}{a}\left(\sum_{i=1}^{m} p_i(p_i+2)t^{-p_i-1} + \left(a^2+2a\right)t^{-a-1}\right) > 0,$$

which proves (17). For (18), using (14), we have

$$t\psi_M''(t) + \psi_M'(t) = 8t + \frac{1}{a}\left(\sum_{i=1}^{m} p_i^2 t^{-p_i-1} + a^2 t^{-a-1}\right) > 0,$$

which proves (18). For (19), using Lemma 2.4 in [3], (16) and (17), we have the result. This completes the proof. □

Lemma 2. *For $\psi_M(t)$, we have*

$$2(t-1)^2 \le \psi_M(t) \le \frac{1}{8}[\psi_M'(t)]^2, \qquad t > 0. \tag{20}$$

$$\psi_M(t) \le \left[\frac{b^2+a^2+8a}{2a}\right](t-1)^2, \qquad t > 1. \tag{21}$$

Proof. For (20), using (15) and (16), we have

$$\begin{aligned}
\psi_M(t) &= \int_1^t \int_1^x \psi_M''(y)\,\mathrm{d}y\,\mathrm{d}x \ge \int_1^t \int_1^x 4\,\mathrm{d}y\,\mathrm{d}x = 2(t-1)^2\\
\psi_M(t) &= \int_1^t \int_1^x \psi_M''(y)\,\mathrm{d}y\,\mathrm{d}x\\
&\le \int_1^t \int_1^x \psi_M''(y)\frac{\psi_M''(x)}{4}\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{1}{4}\int_1^t \psi_M''(x)\psi_M'(x)\,\mathrm{d}x\\
&= \frac{1}{4}\int_1^t \psi_M'(x)d\psi_M'(x)\\
&= \frac{1}{8}[\psi_M'(t)]^2.
\end{aligned}$$

For (21), since

$$\psi_M(1) = \psi_M'(1) = 0, \psi_M'''(t) < 0, \qquad \psi_M''(1) = 6 + \frac{b^2+a^2+2a}{a}$$

and by using Taylor's Theorem, we have

$$\begin{aligned}
\psi_M(t) &= \psi_M(1) + \psi_M'(1)(t-1) + \frac{1}{2}\psi_M''(1)(t-1)^2 + \frac{1}{6}\psi_M'''(\xi)(\xi-1)^3\\
&= \frac{1}{2}\psi_M''(1)(t-1)^2 + \frac{1}{6}\psi_M'''(\xi)(\xi-1)^3\\
&\le \frac{1}{2}\psi_M''(1)(t-1)^2\\
&= \frac{8a+b^2+a^2}{2a}(t-1)^2
\end{aligned}$$

for some ξ, $1 \le \xi \le t$. This completes the proof. □

Let $\varrho : [0, +\infty[\to [1, +\infty[$ be the inverse function of $\psi_M(t)$ for $t \geq 1$ and $\rho : [0, +\infty[\to]0, 1]$ be the inverse function of $\frac{-1}{2}\psi'_M(t)$ for all $t \in]0, 1]$. Then we have the following lemma.

Lemma 3. *For $\psi_M(t)$, we have*

$$1 + \sqrt{\frac{2a}{8a + b^2 + a^2} s} \leq \varrho(s) \leq 1 + \sqrt{s}, \qquad s \geq 0, \tag{22}$$

$$\rho(z) \geq \frac{1}{(2z + 4)^{\frac{1}{a+1}}}, \qquad z \geq 0. \tag{23}$$

Proof. For (22), let

$$s = \psi_M(t), \ t \geq 1, \quad i.e., \ \varrho(s) = t, \ t \geq 1.$$

By (20), and $\log t \leq t$ for all $t \geq 1$ we have

$$\psi_M(t) \geq 2t^2 - 2t - 2 \log t \geq (t - 1)^2,$$

we have

$$s \geq (t - 1)^2, \quad t \geq 1.$$

which implies that

$$t = \varrho(s) \leq 1 + \sqrt{s}.$$

By (21), we have

$$s = \psi_M(t) \leq \frac{8a + b^2 + a^2}{2a}(t - 1)^2,$$

so

$$t = \varrho(s) \geq 1 + \sqrt{\frac{2a}{8a + b^2 + a^2} s}.$$

For (23), let

$$z = \frac{-1}{2}\psi'_M(t), \quad t \in]0, 1] \Leftrightarrow 2z = -\psi'_M(t), \quad t \in]0, 1].$$

By the definition of

$$\rho : \rho(z) = t, t \in]0, 1].$$

By the definition of $\psi'_M(t)$, we have

$$2z = -\left[4t - \frac{2}{t} - \frac{1}{a}\left(\sum_{i=1}^{m} p_i t^{-p_i - 1} + a t^{-a-1}\right)\right], \quad \text{for } t \in]0, 1]$$

this implies

$$\begin{aligned} 2z &\geq -4t + t^{-a-1} \\ &\geq -4 + t^{-a-1}, \end{aligned}$$

which implies that

$$\rho(z) = t \geq \frac{1}{(2z + 4)^{\frac{1}{a+1}}}, \quad z \geq 0.$$

This completes the proof. $\square$

Lemma 4. *Let $\varrho : [0, +\infty[\to [1, +\infty[$ be the inverse function of $\psi_M(t)$ for $t \geq 1$. Then we have*

$$\Psi_M(\beta v) \leq n\psi_M\left(\beta\varrho\left(\frac{\Psi_M(v)}{n}\right)\right), \quad v \in \mathbb{R}_{++}, \ \beta \geq 1.$$

Proof. Using Lemma 2, (19), and Theorem 3.2 in [3], we can get the result. This completes the proof. □

Lemma 5. *Let $0 \leq \theta < 1$, $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Psi_M(v) \leq \tau$, then we have*

$$\Psi_M(v_+) \leq \frac{2\theta n + 2\tau + 4\sqrt{\tau n}}{(1-\theta)}.$$

Proof. Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\varrho\left(\frac{\Psi_M(v)}{n}\right) \geq 1$, then $\frac{\varrho\left(\frac{\Psi_M(v)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. And for $t \geq 1$, we have

$$\psi_M(t) \leq 2(t^2 - 1).$$

Using Lemma 4 with $\beta = \frac{1}{\sqrt{1-\theta}}$, (22), and $\Psi_M(v) \leq \tau$, we have

$$\begin{aligned}
\Psi_M(v_+) &\leq n\psi_M\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi_M(v)}{n}\right)\right) \\
&\leq 2n\left(\left[\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi_M(v)}{n}\right)\right]^2 - 1\right) \\
&= \frac{2n}{(1-\theta)}\left(\left[\varrho\left(\frac{\Psi_M(v)}{n}\right)\right]^2 - (1-\theta)\right) \\
&\leq \frac{2n}{(1-\theta)}\left(\left[1 + \sqrt{\frac{\Psi_M(v)}{n}}\right]^2 - (1-\theta)\right) \\
&= \frac{2n}{(1-\theta)}\left(\left[1 + \frac{\Psi_M(v)}{n} + 2\sqrt{\frac{\Psi_M(v)}{n}}\right] - 1 + \theta\right) \\
&\leq \frac{n}{(1-\theta)}\left(2\theta + \frac{2\tau}{n} + 4\sqrt{\frac{\tau}{n}}\right) \\
&= \frac{2\theta n + 2\tau + 4\sqrt{\tau n}}{(1-\theta)}.
\end{aligned}$$

This completes the proof. □

Denote

$$(\Psi_M)_0 = \frac{2\theta n + 2\tau + 4\sqrt{\tau n}}{(1-\theta)} = L(n, \theta, \tau); \tag{24}$$

then $(\Psi_M)_0$ is an upper bound for $\Psi_M(v_+)$ during the process of the algorithm.

4. Analysis of the algorithm

4.1. The compute a feasible step size

In this section, we compute a default step size α and the resulting decrease of the barrier function. After a damped step we have

$$x_+ = x + \alpha\Delta x,$$

$$y_+ = y + \alpha\Delta y.$$

Using (10), we have

$$x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}(v + \alpha d_x),$$
$$y_+ = y\left(e + \alpha\frac{\Delta y}{y}\right) = s\left(e + \alpha\frac{d_y}{v}\right) = \frac{y}{v}(v + \alpha d_y).$$

So, we have

$$v_+ = \sqrt{\frac{x_+ y_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_y)}.$$

Define, for $\alpha > 0$,

$$f(\alpha) = \Psi_M(v_+) - \Psi_M(v).$$

Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed μ. By (18), we have

$$\Psi_M(v_+) = \Psi_M\left(\sqrt{(v + \alpha d_x)(v + \alpha d_y)}\right) \le \frac{1}{2}(\Psi_M(v + \alpha d_x) + \Psi_M(v + \alpha d_y)).$$

Therefore, we have $f(\alpha) \le f_1(\alpha)$, where

$$f_1(\alpha) = \frac{1}{2}(\Psi_M(v + \alpha d_x) + \Psi_M(v + \alpha d_y)) - \Psi_M(v).$$

Obviously, $f(0) = f_1(0) = 0$. Taking the first two derivatives of $f_1(\alpha)$ with respect to α, we have

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi_M'(v_i + \alpha d_{xi})d_{xi} + \psi_M'(v_i + \alpha d_{yi})d_{yi}),$$
$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi_M''(v_i + \alpha d_{xi})d_{xi}^2 + \psi_M''(v_i + \alpha d_{yi})d_{yi}^2\right),$$

Using (8) and (11), we have

$$f_1'(0) = \frac{1}{2}\nabla\Psi_M(v)^t(d_x + d_y) = -\frac{1}{2}\nabla\Psi_M(v)^t\nabla\Psi_M(v) = -2\delta(v)^2. \tag{25}$$

For convenience, we denote

$$\delta = \delta(v), \quad \Psi_M = \Psi_M(v).$$

Lemma 6. *Let $\delta(v)$ be as defined in (11). Then we have*

$$\delta(v) \ge \sqrt{\frac{1}{2}\Psi_M(v)}. \tag{26}$$

Proof. Using (20), we have

$$\Psi_M(v) = \sum_{i=1}^{n}\psi_M(v_i) \le \sum_{i=1}^{n}\frac{1}{8}(\psi_M'(v_i))^2 = \frac{1}{8}\|\nabla\Psi_M(v)\|^2 = \frac{1}{2}\delta(v)^2,$$

so

$$\delta(v) \ge \sqrt{2\Psi_M(v)}.$$

□

Remark 1. *Throughout the paper, we assume that $\tau \geq 1$. Using Lemma 6 and the assumption that $\Psi_M(v) \geq \tau$, we have*

$$\delta(v) \geq \sqrt{2}.$$

In the sequel we use the following notation

$$v_{\min} = \min_{i \in I} v_i, \quad \sigma_+ = \sum_{i \in I_+} d_{xi} d_{yi}, \quad \sigma_- = -\sum_{i \in I_-} d_{xi} d_{yi}. \tag{27}$$

Since M and N is $P_*(\kappa)$ and $-M\Delta x + N\Delta y = 0$ from the scaled Newton-system, we obtain

$$(1+4\kappa) \sum_{i \in I_+} \Delta_{xi} \Delta_{yi} + \sum_{i \in I_-} \Delta_{xi} \Delta_{yi} \geq 0 \tag{28}$$

where $\kappa \geq 0, I_+ = \{i \in I : \Delta_{xi}\Delta_{yi} \geq 0\}, I_- = \{i \in I : \Delta_{xi}\Delta_{yi} < 0\},$ and $I = \{1, 2, \ldots, n\}$

From (10), we have

$$\mathrm{d}x\,\mathrm{d}y = \frac{\Delta x \Delta y}{\mu}.$$

Thus we can rewrite (28) as

$$(1+4\kappa) \sum_{i \in I_+} d_{xi} d_{yi} + \sum_{i \in I_-} d_{xi} d_{yi} \geq 0. \tag{29}$$

In order to facilitate discussion, we denote $\delta := \delta(v)$.

The following lemma gives an upper bound of σ_+ and σ_-.

From Lemmas 3.1–3.4 in [21]. We have the following Lemmas 7–10.

Lemma 7. *One has*

$$\sigma_+ \leq \delta^2 \quad \text{and} \quad \sigma_- \leq (1+4\kappa)\delta^2.$$

Proof. We have

$$\sigma_+ = \sum_{i \in I_+} d_{xi} d_{yi} \leq \frac{1}{4} \sum_{i \in I_+} (d_{xi} + d_{yi})^2 \leq \frac{1}{4} \sum_{i \in I} (d_{xi} + d_{yi})^2 = \frac{1}{4} \|d_x + d_y\|^2 = \delta^2.$$

On the other hand, from (27) and (29), we get

$$(1+4\kappa)\sigma_+ - \sigma_- \geq 0.$$

Then

$$\sigma_- \leq (1+4\kappa)\delta^2.$$

This completes the proof of the lemma. □

Lemma 8. *One has*

$$\|d_x\| \leq 2\sqrt{1+2\kappa}\,\delta \quad \text{and} \quad \|d_y\| \leq 2\sqrt{1+2\kappa}\,\delta.$$

Proof. From Lemma 7, we have

$$\begin{aligned} 4\delta^2 &= \|d_x + d_y\|^2 \\ &= \sum_{i \in I} (d_{xi} + d_{yi})^2 \\ &= \sum_{i \in I} \left(d_{xi}^2 + d_{yi}^2\right) + 2(\sigma_+ - \sigma_-) \end{aligned}$$

$$\geq \sum_{i\in I}\bigl(d_{xi}^2+d_{yi}^2\bigr)+2\left(\frac{\sigma_-}{1+4\kappa}-\sigma_-\right),$$

thus

$$\sum_{i\in I}\bigl(d_{xi}^2+d_{yi}^2\bigr)\leq 4\delta^2+\frac{8\kappa}{1+4\kappa}\sigma_-\leq 4(1+2\kappa)\delta^2.$$

This completes the proof of the lemma. □

Lemma 9. *One has*

$$f_1''(\alpha)\leq 2(1+2\kappa)\delta^2\psi_M''\bigl(v_{\min}-2\alpha\sqrt{1+2\kappa}\delta\bigr).$$

Proof. From Lemma 8, for all $i\in I$ we have

$$v_i+\alpha d_{xi}\geq v_{\min}-2\alpha\sqrt{1+2\kappa}\delta \quad\text{and}\quad v_i+\alpha d_{yi}\geq v_{\min}-2\alpha\sqrt{1+2\kappa}\delta.$$

Since $\psi_M''(t)$ is monotonically decreasing for all $t>0$, from (16), we get

$$\begin{aligned} f_1''(\alpha)&\leq \frac{1}{2}\psi_M''\bigl(v_{\min}-2\alpha\sqrt{1+2\kappa}\delta\bigr)\sum_{i\in I}\bigl(d_{xi}^2+d_{yi}^2\bigr)\\ &\leq 2(1+2\kappa)\delta^2\psi_M''\bigl(v_{\min}-2\alpha\sqrt{1+2\kappa}\delta\bigr).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 10. *If the step size α satisfies the inequality*

$$\psi_M'(v_{\min})-\psi_M'\bigl(v_{\min}-2\alpha\sqrt{1+2\kappa}\delta\bigr)\leq\frac{2\delta}{\sqrt{1+2\kappa}}, \tag{30}$$

we have

$$f_1'(\alpha)\leq 0.$$

Proof. From (25) and Lemma 9, we have

$$\begin{aligned} f_1'(\alpha)&=f_1'(0)+\int_0^\alpha f_1''(\xi)\,\mathrm{d}\xi\\ &\leq -2\delta^2+2(1+2\kappa)\delta^2\int_0^\alpha \psi_M''\bigl(v_{\min}-2\delta\sqrt{1+2\kappa}\xi\bigr)\,\mathrm{d}\xi\\ &=-2\delta^2-\sqrt{1+2\kappa}\delta\int_0^\alpha \psi_M''\bigl(v_{\min}-2\delta\sqrt{1+2\kappa}\xi\bigr)d\bigl(v_{\min}-2\delta\sqrt{1+2\kappa}\xi\bigr)\\ &=-2\delta^2+\sqrt{1+2\kappa}\delta\bigl[\psi_M'(v_{\min})-\psi_M'\bigl(v_{\min}-2\alpha\sqrt{1+2\kappa}\delta\bigr)\bigr]\\ &\leq -2\delta^2+\sqrt{1+2\kappa}\delta\frac{2\delta}{\sqrt{1+2\kappa}}\\ &\leq 0.\end{aligned}$$

This completes the proof of the lemma. □

From Lemmas 5.6–5.7 in [14], we have the following Lemmas 11–12 gives an upper bound of (α) in terms of δ and $\psi_M''(t)$

Lemma 11. *The largest step size $\overline{\alpha}$ satisfying (30) is given by*

$$\overline{\alpha}=\frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta)-\rho\left(\left(1+\frac{1}{2\sqrt{1+2\kappa}}\right)\delta\right)\right).$$

Lemma 12. *Let $\overline{\alpha}$ be as defined in Lemma 11. Then*

$$\overline{\alpha} \geq \frac{1}{(1+2\kappa)\psi_M''(\rho(2\delta))}.$$

Lemma 13. *Let ρ and $\overline{\alpha}$ be as defined in Lemma 12. If $\psi_M(v) \geq \tau \geq 1$, then we have*

$$\overline{\alpha} \geq \frac{a(4\delta+4)^{-\frac{a+2}{a+1}}}{(1+2\kappa)(b^2+a^2+5a)}.$$

Proof. Using Lemma 12, the definition of $\psi_M''(t)$,and (23) we have

$$\begin{aligned}
\overline{\alpha} &\geq \frac{1}{(1+2\kappa)\psi_M''(\rho(2\delta))} \\
&\geq \frac{1}{(1+2\kappa)\psi_M''\Big(\frac{1}{(4\delta+4)^{\frac{1}{a+1}}}\Big)} \\
&\geq \frac{1}{(1+2\kappa)\Big(4+2(4\delta+4)^{\frac{2}{a+1}}+\Big(\sum_{i=1}^m \frac{p_i(p_i+1)}{a}(4\delta+4)^{\frac{p_i+2}{a+1}}+(a+1)(4\delta+4)^{\frac{a+2}{a+1}}\Big)\Big)} \\
&\geq \frac{1}{(1+2\kappa)\Big(4+(4\delta+4)^{\frac{a+2}{a+1}}\Big(2+\Big(\sum_{i=1}^m \frac{p_i(p_i+1)}{a}+(a+1)\Big)\Big)\Big)} \\
&= \frac{1}{(1+2\kappa)\Big(4+(4\delta+4)^{\frac{a+2}{a+1}}\frac{b^2+a^2+4a}{a}\Big)} \\
&\geq \frac{1}{(1+2\kappa)(4\delta+4)^{\frac{a+2}{a+1}}\big(1+\frac{b^2+a^2+4a}{a}\big)} \\
&= \frac{a(4\delta+4)^{-\frac{a+2}{a+1}}}{(1+2\kappa)(b^2+a^2+5a)}.
\end{aligned}$$

This completes the proof. □

Denoting

$$\widetilde{\alpha} = \frac{a(4\delta+4)^{-\frac{a+2}{a+1}}}{(1+2\kappa)(b^2+a^2+5a)}, \tag{31}$$

we have that $\widetilde{\alpha}$ is the default step size and that $\widetilde{\alpha} \leq \overline{\alpha}$. From Lemma 1.3.3 in [17], we can get the following lemma.

Lemma 14. *Suppose that $h(t)$ is a twice differentiable convex function with*

$$h(0) = 0, \quad h'(0) > 0,$$

that $h(t)$ attains its global minimum at $t^* > 0$ and that $h''(t)$ is increasing with respect to t. Then, for any $t \in [0, t^*]$, we have

$$h(t) \leq \frac{th'(0)}{2}.$$

Let the univariate function h be such that

$$h(0) = f_1(0) = 0,\ h'(0) = f_1'(0) = -2\delta^2,\ h''(\alpha) = \ 2(1+2\kappa)\delta^2\psi_M''\big(v_{\min} - 2\alpha\delta\sqrt{1+2\kappa}\big).$$

Thus, we may conclude that

$$f(\alpha) \le f_1(\alpha) \le \frac{th'(0)}{2} = -\alpha\delta^2 \quad \text{for all } \alpha \in]0, \overline{\alpha}]$$

Lemma 15. *Let $\widetilde{\alpha}$ be the default step size as defined in (31) and let*

$$\Psi_M(v) \ge 1.$$

Then

$$f(\widetilde{\alpha}) \le -\frac{a}{64(1+2\kappa)(b^2+a^2+5a)}[\Psi_M(v)]^{\frac{a}{2(a+1)}}. \tag{32}$$

Proof. Using Lemma 4.5 in [3] and, if the step size α satisfies $\alpha \le \overline{\alpha}$, then $f(\alpha) \le -\alpha\delta^2$. So, for $\widetilde{\alpha} \le \overline{\alpha}$, we have

$$\begin{aligned}
f(\widetilde{\alpha}) &\le -\widetilde{\alpha}\delta^2 \\
&= -\frac{a(4\delta+4)^{-\frac{a+2}{a+1}}}{(1+2\kappa)(b^2+a^2+5a)}\delta^2 \\
&\le -\frac{a(4\delta+4\delta)^{-\frac{a+2}{a+1}}}{(1+2\kappa)(b^2+a^2+5a)}\delta^2 \\
&= -\frac{a(8\delta)^{-\frac{a+2}{a+1}}}{(1+2\kappa)(b^2+a^2+5a)}\delta^2 \\
&= -\frac{a(8)^{-\frac{a+2}{a+1}}}{(1+2\kappa)(b^2+a^2+5a)}\delta^{\frac{a}{a+1}} \\
&\le -\frac{a}{64(1+2\kappa)(b^2+a^2+5a)}\delta^{\frac{a}{a+1}} \\
&\le -\frac{a}{64(1+2\kappa)(b^2+a^2+5a)}[\Psi_M(v)]^{\frac{a}{2(a+1)}}.
\end{aligned}$$

This completes the proof. □

5. Complexity of the algorithm

After the update of μ to $(1-\theta)\mu$, we have

$$\Psi_M(v_+) \le (\Psi_M)_0 = \frac{2\theta n + 2\tau + 4\sqrt{\tau n}}{(1-\theta)} = L(n,\theta,\tau).$$

We need to count how many inner iterations are required to return to the situation where $\Psi_M(v) \le \tau$. We denote the value of $\Psi_M(v)$ after the μ update as $(\Psi_M)_0$; the subsequent values in the same outer iteration are denoted as $(\Psi_M)_k$, $k = 1, 2, \ldots, K$, where K denotes the total number of inner iterations in the outer iteration. The decrease in each inner iteration is given by (32). In [3] we can find the appropriate values of β and $\gamma \in]0, 1]$:

$$\beta = \frac{a}{64(1+2\kappa)(b^2+a^2+5a)}, \quad \gamma = 1 - \frac{a}{2a+2} = \frac{a+2}{2a+2}.$$

Lemma 16. *Let K be the total number of inner iterations in the outer iteration. Then we have*

$$K \le K \le \frac{(\Psi_M)_0^\gamma}{\beta\gamma} = \frac{64(1+2\kappa)(b^2+a^2+5a)(2a+2)}{(a+2)a}(\Psi_M)_0^{\frac{a+2}{2a+2}}.$$

Proof. By Lemma 1.3.2 in [17], we have

$$K \leq \frac{(\Psi_M)_0^{\gamma}}{\beta\gamma} = \frac{64(1+2\kappa)\big(b^2+a^2+5a\big)(2a+2)}{(a+2)a}(\Psi_M)_0^{\frac{a+2}{2a+2}}.$$

This completes the proof. □

5.1. Total iteration bound

The number of outer iterations is bounded above by $\frac{\log\frac{n}{\epsilon}}{\theta}$(see [20] Lemma **II.17**, p. **116**). Through multiplying the number of outer iterations by the number of inner iterations, we get an upper bound for the total number of iterations, namely,

$$\frac{(\Psi_M)_0^{\gamma}}{\beta\gamma} = \frac{64(1+2\kappa)(b^2+a^2+5a)(2a+2)}{(a+2)a}(\Psi_M)_0^{\frac{a+2}{2a+2}}\frac{\log\frac{n}{\epsilon}}{\theta}. \tag{33}$$

For large-update methods with $\tau = \mathbf{O}(n)$ and $\theta = \theta(1)$, and as $b^2 \leq a^2$ we have

$$\mathbf{O}\Big((1+2\kappa)an^{\frac{a+2}{2a+2}}\log\frac{n}{\epsilon}\Big) \text{ iteration complexity.}$$

With a special choice of the parameter $a = (\frac{\log n}{2} - 1)$, which minimizes the iteration bound. Then the iteration bound has

$$\mathbf{O}\Big((1+2\kappa)\sqrt{n}(\log n)\log\frac{n}{\epsilon}\Big) \quad \text{iteration complexity.}$$

It is the best known iteration bound for $P_*(\kappa)$-HLCP. In case of a small-update methods, we have $\tau = \mathbf{O}(1)$ and $\theta = \theta\Big(\frac{1}{\sqrt{n}}\Big)$. Substitution of these values into (33) does not give the best possible bound. A better bound is obtained as follows. By (21), with

$$\psi_M(t) \leq \left[\frac{b^2+a^2+8a}{2a}\right](t-1)^2, \quad t > 1,$$

we have

$$\begin{aligned}
\Psi_M(v_+) &\leq n\psi_M\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Psi_M(v)}{n}\right)\right)\\
&\leq n\left[\frac{b^2+a^2+8a}{2a}\right]\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Psi_M(v)}{n}\right)-1\right)^2\\
&\leq \frac{n(a+4)}{1-\theta}\left(\sigma\left(\frac{\Psi_M(v)}{n}\right)-\sqrt{1-\theta}\right)^2\\
&\leq \frac{n(a+4)}{1-\theta}\left(\left(1+\sqrt{\frac{\Psi_M(v)}{n}}\right)-\sqrt{1-\theta}\right)^2\\
&= \frac{n(a+4)}{1-\theta}\left(\left(1-\sqrt{1-\theta}\right)+\sqrt{\frac{\Psi_M(v)}{n}}\right)^2\\
&\leq \frac{n(a+4)}{1-\theta}\left(\theta+\sqrt{\frac{\tau}{n}}\right)^2\\
&= \frac{a+4}{1-\theta}\big(\theta\sqrt{n}+\sqrt{\tau}\big)^2 = (\Psi_M)_0.
\end{aligned}$$

where we also used that $1-\sqrt{1-\theta}=\frac{\theta}{1+\sqrt{1-\theta}}\leq\theta$ and $\Psi_M(v)\leq\tau$,using this upper bound for $(\Psi_M)_0$, we get the following iteration bound:

$$\frac{64(1+2\kappa)(b^2+a^2+5a)(2a+2)}{(a+2)a}(\Psi_M)_0^{\frac{a+2}{2a+2}}\frac{\log\frac{n}{\epsilon}}{\theta}.$$

Note the now $(\Psi_M)_0=\mathbf{O}(1)$, for $a=\theta(1)$, and we obtained the iteration bound becomes

$$\mathbf{O}\Big((1+2\kappa)\sqrt{n}\log\frac{n}{\epsilon}\Big)\text{ iteration complexity.}$$

It is the best known iteration bound for $P_*(\kappa)$-HLCP.

6. Numerical tests

In this section, we present some numerical results. Our numerical example serves to demonstrate the influence of the $a=\sum_{i=1}^m p_i$ and $p_i>0$; on the number of iterations. To prove the effectiveness of our new kernel function

$$\psi_M(t)=2t^2-2-2\log t+\frac{\sum_{i=1}^m t^{-p_i}}{a}+\frac{t^{-a}}{a}-\frac{m+1}{a},\quad p_i>0;\ a=\sum_{i=1}^m p_i.$$

and evaluate its effect on the behavior of the algorithm, we conducted numerical tests. To be solved, we used the Software Dev Pascal. We have taken $\epsilon=10^{-4}$, $\mu^0=1$, $\theta=0.5$, $\tau=m^2n$.

We consider the following example for all $i=1,\ldots,n$ and for all $j=1,\ldots,n$

$$\begin{aligned}
M(i,j)&=\begin{cases}-1 & \text{if } i=j,\\ -2 & \text{if } i<j,\end{cases}\\
N(i,j)&=-I_n(i,j)=\begin{cases}-1 & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases}\\
q(i)&=1.
\end{aligned}$$

The exact solution is $x^*(i)=0$, $y^*(i)=1$, for $i=1,\ldots,n-1$ and $x^*(n)=1, y^*(n)=0$.

The interior-point condition **IPC**, $x^0(i)=2$, $y^0(i)=4(n-i)+1$, for $i=1,\ldots,n$. Thus we should modify the system (7), $\Psi(v)$ replaced by $\Psi_M(v)$ as follows

$$\begin{aligned}
-M\Delta x+N\Delta y&=0,\\
y\Delta x+x\Delta y&=-\mu v\nabla\Psi_M(v),
\end{aligned}\tag{34}$$

The new search directions Δx and Δy are obtained by solving (34). It should be noted that the step size selected during each inner iteration is small enough for analyzing the algorithm, while in practice the step size during each inner iteration should be large enough for the efficiency of the algorithm. Then the step sizes in the primal space α_P and the dual space α_D during each inner iteration in this experiment are chosen according to the following strategy (see *e.g.*, [9,16,20])

First, compute the maximum allowable step sizes in the primal space $\alpha_P^{\max}$ and the dual space $\alpha_D^{\max}$ to the boundary given by

$$\begin{aligned}
\alpha_P^{\max}&=\frac{1}{\max_{i=1,\ldots,n}\Big\{1;\ \frac{-\Delta x_i}{x_i}\Big\}},\\
\alpha_D^{\max}&=\frac{1}{\max_{i=1,\ldots,n}\Big\{1;\ \frac{-\Delta y_i}{y_i}\Big\}}.
\end{aligned}$$

Then the maximum allowable step sizes are slightly reduced by a fixed factor $0 < \alpha_0 < 1$ (we choose a $\alpha_0 = 0.60$) to prevent hitting the boundary, *i.e.*,

$$\alpha_P = \alpha_0 \alpha_P^{\max},$$
$$\alpha_D = \alpha_0 \alpha_D^{\max}.$$

In the table of results, n: is the number of constraints, $\left(\texttt{Med} = \frac{\text{Total iteration}}{\text{Outer iteration}}\right)$ represent the median value of the number of internal iterations using the function ψ_M and `CPU time(s)` used by the algorithm to solve the test problem with different values for parameters $m, p_i, i = 1, \ldots, m$. We summarize this numerical study in Tables 1–4.

TABLE 1. Comparison of examples for $m = 10,\ p_i = 1$.

Examples	Total iteration	Outer iteration	`Med`	`CPU time(s)`
$n = 5$	62	29	2.14	0.01
$n = 10$	214	32	6.69	0.02
$n = 20$	881	35	25.17	0.20
$n = 50$	6503	39	166.74	12.86

TABLE 2. Comparison of examples for $m = 10,\ p_i = \frac{1}{i}\cdot$

Examples	Total iteration	Outer iteration	`Med`	`CPU time(s)`
$n = 5$	19	29	0.66	0.00
$n = 10$	34	32	1.06	0.01
$n = 20$	42	35	1.20	0.02
$n = 50$	7055	38	185.55	13.77

TABLE 3. Comparison of examples for $m = n, p_i = \frac{\sqrt{\log n}}{m}\cdot$

Examples	Total iteration	Outer iteration	`Med`	`CPU time(s)`
$n = 5$	23	27	0.85	0.00
$n = 10$	280	32	8.75	0.05
$n = 20$	685	37	18.51	0.17
$n = 50$	1829	44	41.57	4.16

TABLE 4. Comparison of examples for $m = n, p_i = \frac{\log n}{m}\cdot$

Examples	Total iteration	Outer iteration	`Med`	`CPU time(s)`
$n = 5$	21	27	0.78	0.00
$n = 10$	22	32	0.69	0.02
$n = 20$	40	37	1.08	0.03
$n = 50$	1760	44	40.00	3.93

7. Concluding remarks

We have proposed a class of polynomial interior-point algorithms for $P_*(\kappa)$-HLCP based on a new and efficient multi parametric kernel function with logarithmic barrier term. And we developed some new analysis tools that can be used in complexity analysis of the algorithms. Motivated by this, we obtain for large-update methods with $\tau = \mathbf{O}(n)$ and $\theta = \theta(1)$, and as $b^2 \leq a^2$ we have

$$\mathbf{O}\Big((1+2\kappa)a^2 n^{\frac{a+2}{2a+2}} \log \frac{n}{\epsilon}\Big) \quad \text{iteration complexity.}$$

With a special choice of the parameter $a = \frac{\log n}{2} - 1$, which minimizes the iteration bound. Then the iteration bound has

$$\mathbf{O}\Big((1+2\kappa)\sqrt{n}(\log n) \log \frac{n}{\epsilon}\Big) \quad \text{iteration complexity.}$$

It is the best known iteration bound for $P_*(\kappa)$-HLCP. And we obtain for small-update methods with

$$\frac{64(1+2\kappa)(b^2+a^2+5a)(2a+2)}{(a+2)a} (\Psi_M)_0^{\frac{a+2}{2a+2}} \frac{\log \frac{n}{\epsilon}}{\theta}.$$

Note the now $(\Psi_M)_0 = \mathbf{O}(1)$, for $a = \theta(1)$, and we obtained the iteration bound becomes

$$\mathbf{O}\Big((1+2\kappa)\sqrt{n} \log \frac{n}{\epsilon}\Big) \text{ iteration complexity.}$$

It is the best known iteration bound for $P_*(\kappa)$-HLCP. Moreover, the numerical results were presented to illustrate the advantage of our kernel function. These results are an important contribution for improving the computational complexity of the problem under study.

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