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CHARACTERIZING AN ODD $[1, b]$-FACTOR ON THE DISTANCE SIGNLESS LAPLACIAN SPECTRAL RADIUS

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Abstract. Let G be a connected graph of even order n. An odd $[1, b]$-factor of G is a spanning subgraph F of G such that $d_F(v) \in \{1, 3, 5, \cdots, b\}$ for any $v \in V(G)$, where b is positive odd integer. The distance matrix $\mathcal{D}(G)$ of G is a symmetric real matrix with (i, j)-entry being the distance between the vertices v_i and v_j. The distance signless Laplacian matrix $\mathcal{Q}(G)$ of G is defined by $\mathcal{Q}(G) = Tr(G) + \mathcal{D}(G)$, where $Tr(G)$ is the diagonal matrix of the vertex transmissions in G. The largest eigenvalue $\eta_1(G)$ of $\mathcal{Q}(G)$ is called the distance signless Laplacian spectral radius of G. In this paper, we verify sharp upper bounds on the distance signless Laplacian spectral radius to guarantee the existence of an odd $[1, b]$-factor in a graph; we provide some graphs to show that the bounds are optimal.

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1. Introduction

Graphs discussed in this paper are simple and undirected. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. The distance $d_G(v_i, v_j)$ between the vertices v_i and v_j is the length of the shortest path between v_i and v_j in G. The distance matrix $\mathcal{D}(G)$ of G is a symmetric real matrix with (i, j)-entry being $d_G(v_i, v_j)$. The transmission $Tr(v_i)$ of a vertex $v_i \in V(G)$ is defined by the sum of the distances from v_i to all other vertices in G, namely, $Tr(v_i) = \sum_{j=1}^{n} d_G(v_i, v_j)$. Let $Tr(G) = diag(Tr(v_1), Tr(v_2), \cdots, Tr(v_n))$ denote the diagonal matrix of the vertex transmissions in G. The distance signless Laplacian matrix $\mathcal{Q}(G)$ of G is defined by $\mathcal{Q}(G) = Tr(G) + \mathcal{D}(G)$. For an $n \times n$ symmetric real matrix M, let $\rho_1(M)$ denote the spectral radius of M, namely, the largest eigenvalue of M. Let $\eta_1(G) \geq \eta_2(G) \geq \cdots \geq \eta_n(G)$ be the eigenvalues of $\mathcal{Q}(G)$. The largest eigenvalue $\eta_1(G)$ of $\mathcal{Q}(G)$ is called to be the distance signless Laplacian spectral radius of G.

The degree of a vertex v in G, written $d_G(v)$, is the number of vertices adjacent to v. The neighborhood of a vertex v in G, written $N_G(v)$, is the set of vertices adjacent to v. Let S be a vertex subset of G. Set $N_G(S) = \bigcup_{v \in S} N_G(v)$. We denote by $G[S]$ the subgraph of G induced by S, and write $G - S = G[V(G) \setminus S]$. The complement graph $\overline{G}$ of G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The Wiener index $W(G)$ of a connected graph G of order n is defined

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by the sum of all distances in G, that is, $W(G) = \sum_{i<j} d_G(v_i, v_j)$. The complete graph of order n is denoted by K_n.

Let G_1 and G_2 be two disjoint graphs. The union $G_1 \cup G_2$ denotes the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The join $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

A $[1,2]$-factor of G is defined as a spanning subgraph F of G with $d_F(x) \in \{1,2\}$ for any $x \in V(G)$. Let $f : V(G) \to \{1,3,5,\cdots\}$ be an odd integer-valued function. An odd $(1,f)$-factor of G is a spanning subgraph F of G such that $d_F(v) \in \{1,3,5,\cdots,f(x)\}$ for any $v \in V(G)$. Obviously, an odd $(1,f)$-factor is a 1-factor or a perfect matching if $f(v) = 1$ for any $v \in V(G)$. An odd $(1,f)$-factor is called an odd $[1,b]$-factor if $f(v) = b$ for any $v \in V(G)$, where b is an odd integer with $b \geq 1$.

Brouwer and Haemers [1] claimed that a graph G contains a 1-factor if $\mu_1(G) \leq 2\mu_{n-1}(G)$, where $\mu_1(G)$ and $\mu_{n-1}(G)$ are the largest and second smallest Laplacian eigenvalues of G, respectively. Suil [2] claimed two sufficient conditions for a graph to admit a 1-factor in light of its size and adjacency spectral radius, respectively. Liu *et al.* [3] put forward an adjacency spectral radius condition to ensure the existence of a 1-factor in a graph. Liu and Li [4] characterized a graph or a bipartite graph with a 1-factor by virtue of its distance signless Laplacian spectral radius. Lots of researchers presented some sufficient conditions on various parameters to guarantee the existence of $[1,2]$-factors in graphs, such as the neighborhood condition [5], the degree conditions [6–8], the binding number [8,9], the independence number [10,11], the isolated toughness [12–15], and the sun toughness [16]. Zhou [17] derived some sufficient conditions for graphs to possess $[1,2]$-factors. Cui and Kano [18] showed a neighborhood condition for a graph having an odd $[1,b]$-factor. Lu *et al.* [19] characterized a graph with an odd $[1,b]$-factor in terms of its Laplacian eigenvalues and adjacency eigenvalues, respectively. Kim *et al.* [20] established an upper bound for the third largest adjacency eigenvalue in a regular graph to ensure the existence of an odd $[1,b]$-factor. Fan *et al.* [21] showed a lower bound for the adjacency spectral radius to ensure a graph admitting an odd $[1,b]$-factor. More other results on graph factors can be found in [22–32].

In this paper, we establish a connection between the distance signless Laplacian spectral radius of a graph and its odd $[1,b]$-factor, and present a distance signless Laplacian spectral radius condition to ensure the existence of an $[1,b]$-factor in a graph.

Theorem 1.1. *Let G be a connected graph of even order n, and let b be a positive odd integer.*

(i) *For $n \geq b+3$ and $(b,n) \notin \{(1,6),(1,8),(1,10)\}$, if $\eta_1(G) < \theta(n)$, then G has an odd $[1,b]$-factor, where $\theta(n)$ is the largest root of $x^3 - (5n+2b-7)x^2 + (8n^2 + (2b-27)n + 4b(b+1) + 24)x - 4n^3 + 22n^2 - (4b^2 + 8b + 42)n + 6b^2 + 14b + 28 = 0$.*

(ii) *For $(b,n) \in \{(1,6),(1,8),(1,10)\}$, if $\eta_1(G) < 2n - 2 + \frac{\sqrt{2n(n+2)}}{2}$, then G has a 1-factor.*

2. Preliminaries

We first put forward a fundamental result to compare the distance signless Laplacian spectral radius of a graph and its spanning subgraph, which is a corollary of the Perron–Frobenius theorem.

Lemma 2.1 ([33])**.** *Let e be an edge of a graph G such that $G - e$ is still connected. Then $\eta_1(G) < \eta_1(G-e)$.*

Xing *et al.* [34] presented a lower bound on the distance signless Laplacian spectral radius of a graph.

Lemma 2.2 ([34])**.** *Let G be a connected graph of order n. Then*

$$\eta_1(G) \geq \frac{4W(G)}{n}$$

with equality if and only if G is transmission regular.

Lemma 2.3 ([4]). *Let n, q, s and n_i $(i = 1, 2, \cdots, q)$ be positive integers with $n_1 \geq n_2 \geq \cdots \geq n_q \geq 1$ and $n = s + \sum_{i=1}^{q} n_i$. Then*

$$\eta_1(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})) \geq \eta_1(K_s \vee (K_{n-s-q+1} \cup (q-1)K_1))$$

with equality if and only if $K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q}) \cong K_s \vee (K_{n-s-q+1} \cup (q-1)K_1)$.

In what follows, we show the concepts of equitable matrices and equitable partitions.

Definition 2.4 ([35]). Let M be the following $n \times n$ real matrix

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1r} \\ M_{21} & M_{22} & \cdots & M_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rr} \end{pmatrix},$$

whose rows and columns are partitioned into subsets $X_1, X_2, \cdots, X_r$ of $\{1, 2, \cdots, n\}$. Let M_{ij} denote the submatrix (called a block) of M by removing the rows in $\{1, 2, \cdots, n\} - X_i$ and removing the columns in $\{1, 2, \cdots, n\} - X_j$. The quotient matrix $B(M)$ of the matrix M (with respect to the given partition) is the $r \times r$ matrix whose entries are the average row sums of the blocks M_{ij} of M. If every block M_{ij} of M admits constant row sum, then the partition is equitable and $B(M)$ is called an equitable quotient matrix of M.

Lemma 2.5 ([36]). *Let M be a partitioned matrix, and let $B(M)$ be its equitable quotient matrix. Then the eigenvalues of $B(M)$ are eigenvalues of M. Furthermore, if M is a nonnegative matrix, then $\rho_1(B(M)) = \rho_1(M)$, where $\rho_1(B(M))$ denotes the spectral radius of $B(M)$ and $\rho_1(M)$ denotes the spectral radius of M.*

Amahashi [37] characterized a graph with an odd $[1, b]$-factor, which is the main tool in our proof.

Lemma 2.6 ([37]). *Let G be a graph and let b be a positive odd integer. Then G admits an odd $[1, b]$-factor if and only if*

$$o(G - S) \leq b|S|$$

for any $S \subseteq V(G)$, where $o(G - S)$ denotes the number of odd components in $G - S$.

3. The proof of Theorem 1.1

Proof of Theorem 1.1. Let $\varphi(x) = x^3 - (5n + 2b - 7)x^2 + (8n^2 + (2b - 27)n + 4b(b + 1) + 24)x - 4n^3 + 22n^2 - (4b^2 + 8b + 42)n + 6b^2 + 14b + 28$ and let $\theta(n)$ be the largest root of $\varphi(x) = 0$. Suppose to the contrary that G has no odd $[1, b]$-factor. According to Lemma 2.6, we possess

$$o(G - S) > b|S|$$

for some nonempty subset S of $V(G)$. Let $o(G - S) = q$ and $|S| = s$. Since n is even, q and bs admit the same parity, hence

$$q \geq bs + 2. \tag{3.1}$$

In what follows, we choose a connected graph G of order n such that its distance signless Laplacian spectral radius is as small as possible. Combining this with Lemma 2.1, we see that all the components of $G - S$ are odd, the induced subgraph $G[S]$ (resp. each connected component of $G - S$) is a complete graph, and $G \cong G[S] \vee (G - S)$.

Let $G_1, G_2, \cdots, G_q$ be all the components of $G - S$ with $|V(G_i)| = n_i$ $(i = 1, 2, \cdots, q)$ and $n_1 \geq n_2 \geq \cdots \geq n_q \geq 1$. It's clear that $n = s + \sum_{i=1}^{q} n_i$ and $G \cong K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})$. Let $n_2 = n_3 = \cdots = n_q = 1$, then $n_1 = n - s - q + 1$ is odd and we get a new graph $G' \cong K_s \vee (K_{n-s-q+1} \cup (q-1)K_1)$. Note that $o(G' - S) = q \geq bs + 2$ by (3.1) and $n - s - q + 1$ being odd, and G' is a connected graph of order n. Utilizing Lemma 2.3, $\eta_1(G) \geq \eta_1(G')$ with equality if and only if $G \cong G'$. Combining this with the choice of G, we deduce $G \cong G' \cong K_s \vee (K_{n-s-q+1} \cup (q-1)K_1)$ (otherwise, $\eta_1(G) > \eta_1(G')$ by Lemma 2.3, which contradicts the choice of G).

Claim 1. $q = bs + 2$.

Proof. Assume that $q \geq bs + 4$. Recall that $G \cong K_s \vee (K_{n-s-q+1} \cup (q-1)K_1)$. Let $G'' \cong K_s \vee (K_{n-(b+1)s-1} \cup (bs+1)K_1)$. Note that $o(G'' - S) = bs + 2$ and G'' is a connected graph of order n. Denote the vertex set of G'' by $V(G'') = V(K_s) \cup V(K_{n-(b+1)s-1}) \cup V((bs+1)K_1)$. Let X be the Perron vector of $\mathcal{Q}(G'')$ and $X(v)$ is the entry of X corresponding to the vertex $v \in V(G'')$. By virtue of symmetry, it's easy to see that all vertices of $V(K_s)$ admit the same entries in X, all vertices of $V(K_{n-(b+1)s-1})$ have the same entries in X and all vertices of $V((bs+1)K_1)$ possess the same entries in X. Thus, we can suppose $X(u) = x_0$ for any $u \in V(K_s)$, $X(v) = x_1$ for any $v \in V(K_{n-(b+1)s-1})$ and $X(w) = x_2$ for any $w \in V((bs+1)K_1)$. Note that $q \geq bs + 4$ and $n \geq s + q \geq (b+1)s + 4$. Then

$$\begin{aligned}\eta_1(G) - \eta_1(G'') &\geq X^T(\mathcal{Q}(G) - \mathcal{Q}(G''))X \\ &= 3(q - bs - 2)(n - s - q + 1)x_1^2 + (q - bs - 3)(q - bs - 2)x_1^2 \\ &\quad + (q - bs - 2)(n - (b+1)s - 2)x_1^2 > 0,\end{aligned}$$

which implies $\eta_1(G) > \eta_1(G'')$, which contradicts the choice of G. Hence, $q \leq bs + 2$. Combining this with (3.1), we get $q = bs + 2$. Claim 1 is verified. □

Recall that $G \cong K_s \vee (K_{n-s-q+1} \cup (q-1)K_1)$. Then using Claim 1, we deduce $G \cong K_s \vee (K_{n-(b+1)s-1} \cup (bs+1)K_1)$. We easily see that $n - (b+1)s - 1$ is odd. In what follows, we discuss two cases by the value of $n - (b+1)s - 1$.

Case 1. $n - (b+1)s - 1 \geq 3$.

Obviously, $n \geq (b+1)s + 4$. Then the equitable quotient matrix B_1 of the distance signless Laplacian matrix $\mathcal{Q}(G)$ of the graph $G \cong K_s \vee (K_{n-(b+1)s-1} \cup (bs+1)K_1)$ with the partition $\{V(K_s), V(K_{n-(b+1)s-1}), V((bs+1)K_1)\}$ can be expressed as

$$B_1 = \begin{pmatrix} n+s-2 & n-(b+1)s-1 & bs+1 \\ s & 2n-s-2 & 2bs+2 \\ s & 2n-2(b+1)s-2 & 2n+(2b-1)s-2 \end{pmatrix}.$$

According to simple calculation, the characteristic polynomial of the matrix B_1 is

$$\begin{aligned}f_1(x) = x^3 &- (5n + (2b-1)s - 6)x^2 + (8n^2 + (2b-3)sn - 24n + 4b(b+1)s^2 + 8s + 16)x \\ &- 4n^3 + 2sn^2 + 20n^2 - 4b(b+1)s^2 n - (4b+10)sn - 32n - 2b^2 s^3 \\ &+ (8b^2 + 6b)s^2 + (8b+12)s + 16.\end{aligned}$$

If $s = 1$, then $G \cong K_s \vee (K_{n-(b+1)s-1} \cup (bs+1)K_1)$ becomes $G^* \cong K_1 \vee (K_{n-b-2} \cup (b+1)K_1)$ and the polynomial $f_1(x)$ becomes $\varphi(x) = x^3 - (5n + 2b - 7)x^2 + (8n^2 + (2b-27)n + 4b(b+1) + 24)x - 4n^3 + 22n^2 - (4b^2 + 8b + 42)n + 6b^2 + 14b + 28$.

In terms of Lemma 2.5, $\eta_1(G)$ is the largest root of $f_1(x) = 0$ and $\eta_1(G^*) = \theta(n)$ is the largest root of $\varphi(x) = 0$. Applying Lemma 2.2, $s \geq 2$ and $n \geq (b+1)s + 4$, we get

$$\theta(n) = \eta_1(G^*) \geq \frac{4W(G^*)}{n}$$

$$
\begin{aligned}
&= \frac{2n^2 + 2(2b+1)n - 2(b+1)(b+4)}{n} \\
&\geq \frac{2n^2 + 3bn + (b+2)((b+1)s+4) - 2(b+1)(b+4)}{n} \\
&\geq \frac{2n^2 + 3bn + 2(b+2)(b+3) - 2(b+1)(b+4)}{n} \\
&= \frac{2n^2 + 3bn + 4}{n} > 2n + 3b. \qquad (3.2)
\end{aligned}
$$

By plugging the value $\theta(n)$ into x of $f_1(x) - \varphi(x)$, it follows from $\varphi(\theta(n)) = 0$ that

$$
\begin{aligned}
f_1(\theta(n)) &= f_1(\theta(n)) - \varphi(\theta(n)) \\
&= (s-1)\big(-(2b-1)(\theta(n))^2 + ((2b-3)n + 4b(b+1)(s+1) + 8)\theta(n) \\
&\quad + 2n^2 - (4b(b+1)(s+1) + 4b + 10)n - 2b^2\big(s^2+s+1\big) + (8b^2+6b)(s+1) + 8b + 12\big). \qquad (3.3)
\end{aligned}
$$

Let $g(\theta(n)) = -(2b-1)(\theta(n))^2 + ((2b-3)n + 4b(b+1)(s+1)+8)\theta(n) + 2n^2 - (4b(b+1)(s+1)+4b+10)n - 2b^2(s^2+s+1) + (8b^2+6b)(s+1) + 8b + 12$. Then by (3.2) and $n \geq (b+1)s+4$, we see

$$
\frac{(2b-3)n + 4b(b+1)(s+1) + 8}{2(2b-1)} < 2n + 3b < \theta(n).
$$

So when $\theta(n) > 2n+3b$, $g(\theta(n))$ monotonically decreases as $\theta(n)$ increases and

$$
\begin{aligned}
g(\theta(n)) &< g(2n+3b) \\
&= -(2b-1)(2n+3b)^2 + ((2b-3)n + 4b(b+1)(s+1)+8)(2n+3b) \\
&\quad + 2n^2 - (4b(b+1)(s+1)+4b+10)n - 2b^2\big(s^2+s+1\big) + \big(8b^2+6b\big)(s+1) + 8b + 12 \\
&= -4bn^2 + \big(4b^2s + 4bs - 14b^2 + 3b + 6\big)n + 12b^3s - 6b^3 - 2b^2s^2 + 18b^2s + 27b^2 \\
&\quad + 6bs + 38b + 12. \qquad (3.4)
\end{aligned}
$$

Write $p(n) = g(2n+3b) = -4bn^2 + (4b^2s + 4bs - 14b^2 + 3b + 6)n + 12b^3s - 6b^3 - 2b^2s^2 + 18b^2s + 27b^2 + 6bs + 38b + 12$. Then

$$
\frac{4b^2s + 4bs - 14b^2 + 3b + 6}{8b} < (b+1)s + 4 \leq n.
$$

So when $n \geq (b+1)s+4$, $p(n)$ is monotonically decreasing and

$$
\begin{aligned}
p(n) &\leq p((b+1)s+4) \\
&= -4b((b+1)s+4)^2 + (4b^2s + 4bs - 14b^2 + 3b + 6)((b+1)s+4) \\
&\quad + 12b^3s - 6b^3 - 2b^2s^2 + 18b^2s + 27b^2 + 6bs + 38b + 12 \\
&= -2b^2s^2 - \big(2b^3 + 9b^2 + b - 6\big)s - \big(6b^3 + 29b^2 + 14b - 36\big) \\
&< 0. \qquad (3.5)
\end{aligned}
$$

It follows from $s \geq 1$, (3.3)–(3.5) that

$$
f_1(\theta(n)) = (s-1)g(\theta(n)) \leq (s-1)g(2n+3b) = (s-1)p(n) \leq 0,
$$

which implies $\eta_1(G) \geq \theta(n)$, which contradicts $\eta_1(G) < \theta(n)$.

Case 2. $n-(b+1)s-1=1$.

In this case, $n=(b+1)s+2$. Recall that $G\cong K_s\vee(K_{n-(b+1)s-1}\cup(bs+1)K_1)$, $G^*\cong K_1\vee(K_{n-b-2}\cup(b+1)K_1)$ and $\eta_1(G^*)=\theta(n)$ is the largest root of $\varphi(x)=0$. If $s=1$, then $G\cong G^*$, and so $\eta_1(G)=\eta_1(G^*)=\theta(n)$ for $n=b+4$, which contradicts $\eta_1(G)<\theta(n)$. Next, we assume $s\geq 2$.

In terms of Lemma 2.2 and $n=(b+1)s+2\geq 2b+4$, we obtain

$$\begin{aligned}\theta(n)=\eta_1(G^*)&\geq\frac{4W(G^*)}{n}\\&=\frac{2n^2+2(2b+1)n-2(b+1)(b+4)}{n}\\&\geq\frac{2n^2+3bn+(b+2)(2b+4)-2(b+1)(b+4)}{n}\\&=\frac{2n^2+3bn-2b}{n}>2n+3b-1.\end{aligned}\tag{3.6}$$

Recall that $g(\theta(n))=-(2b-1)(\theta(n))^2+((2b-3)n+4b(b+1)(s+1)+8)\theta(n)+2n^2-(4b(b+1)(s+1)+4b+10)n-2b^2(s^2+s+1)+(8b^2+6b)(s+1)+8b+12$. Then from (3.6) and $n=(b+1)s+2$, we deduce

$$\frac{(2b-3)n+4b(b+1)(s+1)+8}{2(2b-1)}<2n+3b-1<\theta(n).$$

So when $\theta(n)>2n+3b-1$, $g(\theta(n))$ monotonically decreases as $\theta(n)$ increases and

$$\begin{aligned}g(\theta(n))&<g(2n+3b-1)\\&=-4bn^2+\big(4b^2s+4bs-14b^2+9b+5\big)n+12b^3s-6b^3-2b^2s^2+14b^2s\\&\quad+35b^2+2bs+26b+5.\end{aligned}\tag{3.7}$$

Set $h(n)=g(2n+3b-1)=-4bn^2+(4b^2s+4bs-14b^2+9b+5)n+12b^3s-6b^3-2b^2s^2+14b^2s+35b^2+2bs+26b+5$. In the following, we consider two subcases by the value of b.

Subcase 2.1. $b\geq 3$.

Note that $s\geq 2$, $b\geq 3$ and $n=(b+1)s+2$. Hence, we infer

$$\begin{aligned}h(n)&=h((b+1)s+2)\\&=-2b^3s-2b^2s^2+b^2s+8bs+5s-6b^3+7b^2+28b+15\\&\leq-2b^3s-3b^2s+8bs+5s-6b^3+7b^2+28b+15\\&\leq-4b^3-6b^2+16b+10-6b^3+7b^2+28b+15\\&=-10b^3+b^2+44b+25\\&\leq-30b^2+b^2+44b+25\\&=-29b^2+44b+25\\&\leq-87b+44b+25\\&=-43b+25<0.\end{aligned}\tag{3.8}$$

By virtue of $s\geq 2$, (3.3), (3.7) and (3.8), we deduce

$$f_1(\theta(n))=(s-1)g(\theta(n))<(s-1)g(2n+3b-1)=(s-1)h(n)<0,$$

which implies $\eta_1(G)>\theta(n)$, which contradicts $\eta_1(G)<\theta(n)$.

Subcase 2.2. $b = 1$.

In this subcase, $n = (b+1)s + 2 = 2s + 2$ and $G \cong K_s \vee (s+2)K_1$. Then the equitable quotient matrix B_2 of the distance signless Laplacian matrix $\mathcal{Q}(G)$ of the graph $G \cong K_s \vee (s+2)K_1$ with the partition $\{V((s+2)K_1), V(K_s)\}$ equals

$$B_2 = \begin{pmatrix} 5s+4 & s \\ s+2 & n+s-2 \end{pmatrix}.$$

The characteristic polynomial of the matrix B_2 can be written as

$$f_2(x) = x^2 - (n + 6s + 2)x + 5sn + 4n + 4s^2 - 8s - 8.$$

Since the partition $\{V((s+2)K_1), V(K_s)\}$ is equitable, it follows from Lemma 2.5 that $\eta_1(G)$ is the largest root of $f_2(x) = 0$, that is, $f_2(\eta_1(G)) = 0$. Recall that $n = 2s + 2$. By simple computation, we get

$$\begin{aligned}\eta_1(G) &= \frac{n + 6s + 2 + \sqrt{(n+6s+2)^2 - 4(5sn + 4n + 4s^2 - 8s - 8)}}{2} \\ &= 2n - 2 + \frac{\sqrt{2n(n+2)}}{2}.\end{aligned}$$

Note that $b = 1$ and $s \geq 2$. Then $\varphi(x) = x^3 - (5n-5)x^2 + (8n^2 - 25n + 32)x - 4n^3 + 22n^2 - 54n + 48$ and $\theta(n)$ is the largest root of $\varphi(x) = 0$, that is, $\varphi(\theta(n)) = 0$. If $s = 2$ and $n = 6$, then $\eta_1(G) = 10 + 2\sqrt{6}$ and $\varphi(\eta_1(G)) = \varphi(10 + 2\sqrt{6}) = -28 - 12\sqrt{6} < 0$, so $\eta_1(G) = 10 + 2\sqrt{6} < \theta(6)$, which contradicts $\eta_1(G) < 10 + 2\sqrt{6}$ for $(b, n) = (1, 6)$. If $s = 3$ and $n = 8$, then $\eta_1(G) = 14 + 2\sqrt{10}$ and $\varphi(\eta_1(G)) = \varphi(14 + 2\sqrt{10}) = -44 - 18\sqrt{10} < 0$, so $\eta_1(G) = 14 + 2\sqrt{10} < \theta(8)$, which contradicts $\eta_1(G) < 14 + 2\sqrt{10}$ for $(b, n) = (1, 8)$. If $s = 4$ and $n = 10$, then $\eta_1(G) = 18 + 2\sqrt{15}$ and $\varphi(\eta_1(G)) = \varphi(18 + 2\sqrt{15}) = -24 - 12\sqrt{15} < 0$, so $\eta_1(G) = 18 + 2\sqrt{15} < \theta(10)$, which contradicts $\eta_1(G) < 18 + 2\sqrt{15}$ for $(b, n) = (1, 10)$. If $s \geq 5$ and $n = 2s + 2 \geq 12$, then the following claim holds.

Claim 2. If $s \geq 5$ and $n = 2s + 2 \geq 12$, then $\eta_1(G) > \theta(n)$.

Proof. Recall that $b = 1$ and $n = 2s + 2$. Then $\varphi(x) = x^3 - (5n-5)x^2 + (8n^2 - 25n + 32)x - 4n^3 + 22n^2 - 54n + 48$ and $\varphi'(x) = 3x^2 - 2(5n-5)x + 8n^2 - 25n + 32$. If $s = 5$ and $n = 12$, then $\eta_1(G) = 22 + 2\sqrt{21}$, $\varphi(\eta_1(G)) = \varphi(22 + 2\sqrt{21}) = 56 > 0$ and $\varphi'(\eta_1(G)) = \varphi'(22 + 2\sqrt{21}) = 168 + 44\sqrt{21} > 0$, so $\eta_1(G) > \theta(12)$. If $s = 6$ and $n = 14$, then $\eta_1(G) = 26 + 4\sqrt{7}$, $\varphi(\eta_1(G)) = \varphi(26 + 4\sqrt{7}) = 220 + 40\sqrt{7} > 0$ and $\varphi'(\eta_1(G)) = \varphi'(26 + 4\sqrt{7}) = 234 + 104\sqrt{7} > 0$, so $\eta_1(G) > \theta(14)$. If $s = 7$ and $n = 16$, then $\eta_1(G) = 42$, $\varphi(\eta_1(G)) = \varphi(42) = 780 > 0$ and $\varphi'(\eta_1(G)) = \varphi'(42) = 672 > 0$, so $\eta_1(G) > \theta(16)$. If $s = 8$ and $n = 18$, then $\eta_1(G) = 34 + 6\sqrt{5}$, $\varphi(\eta_1(G)) = \varphi(34 + 6\sqrt{5}) = 896 + 252\sqrt{5} > 0$ and $\varphi'(\eta_1(G)) = \varphi'(34 + 6\sqrt{5}) = 402 + 204\sqrt{5} > 0$, so $\eta_1(G) > \theta(18)$. If $s \geq 9$ and $n = 2s + 2 \geq 20$, then

$$\begin{aligned} h(n) &= g(2n+2) \\ &= -4n^2 + 8sn - 2s^2 + 28s + 60 \\ &= -4(2s+2)^2 + 8s(2s+2) - 2s^2 + 28s + 60 \\ &= -2(s-3)^2 + 62 < 0. \end{aligned} \tag{3.9}$$

Applying $b = 1$, $s \geq 9$, (3.3), (3.7) and (3.9), we infer

$$f_1(\theta(n)) = (s-1)g(\theta(n)) < (s-1)g(2n + 3b - 1) = (s-1)g(2n+2) = (s-1)h(n) < 0,$$

which yields $\eta_1(G) > \theta(n)$. Claim 2 is verified. □

According to Claim 2 and the hypothesis of Theorem 1.1, we possess $\theta(n) < \eta_1(G) < \theta(n)$ for $b = 1$ and $n \geq 12$, which is a contradiction. This completes the proof of Theorem 1.1. □

4. Extremal graphs

In this section, we create some graphs to claim that the bounds on the distance signless Laplacian spectral radius shown in Theorem 1.1 are sharp, respectively.

For $k \geq 3$, the sequential join $G_1 \vee G_2 \vee \cdots \vee G_k$ denotes the graph with vertex set $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_k)$ and edge set $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k) \cup \{e = x_i x_{i+1} : x_i \in V(G_i), x_{i+1} \in V(G_{i+1}), 1 \leq i \leq k-1\}$.

Theorem 4.1. *Let $\theta(n)$ be the largest root of $x^3-(5n+2b-7)x^2+(8n^2+(2b-27)n+4b(b+1)+24)x-4n^3+22n^2-(4b^2+8b+42)n+6b^2+14b+28=0$, where b is a positive odd integer and $n \geq b+3$ is a positive even integer. For $n \geq b+3$ and $(b,n) \notin \{(1,6),(1,8),(1,10)\}$, we possess $\eta_1(K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}) = \theta(n)$ and $K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}$ contains no odd $[1,b]$-factor. For $(b,n) \in \{(1,6),(1,8),(1,10)\}$, we admit $\eta_1(K_{\frac{n-2}{2}} \vee \overline{K_{\frac{n+2}{2}}}) = 2n-2+\frac{\sqrt{2n(n+2)}}{2}$ and $K_{\frac{n-2}{2}} \vee \overline{K_{\frac{n+2}{2}}}$ contains no 1-factor.*

Proof. Partition the vertex set $V(K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}})$ as $\{V(K_1), V(K_{n-b-2}), V(\overline{K_{b+1}})\}$. Then the equitable quotient matrix of the distance signless Laplacian matrix $\mathcal{Q}(K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}})$ of the graph $K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}$ can be written as

$$B(K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}) = \begin{pmatrix} n-1 & n-b-2 & b+1 \\ 1 & 2n-3 & 2b+2 \\ 1 & 2n-2b-4 & 2n+2b-3 \end{pmatrix}.$$

By simple computation, its characteristic polynomial is $x^3-(5n+2b-7)x^2+(8n^2+(2b-27)n+4b(b+1)+24)x-4n^3+22n^2-(4b^2+8b+42)n+6b^2+14b+28$. Note that the partition is equitable. In light of Lemma 2.5, the largest root $\theta(n)$ of $x^3-(5n+2b-7)x^2+(8n^2+(2b-27)n+4b(b+1)+24)x-4n^3+22n^2-(4b^2+8b+42)n+6b^2+14b+28=0$ equals the distance signless Laplacian spectral radius of the graph $K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}$, that is, $\eta_1(K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}) = \theta(n)$. Set $S = V(K_1)$. Then $o(K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}} - S) = b+2 = b|S|+2 > b|S|$. Applying Lemma 2.6, $K_{n-b-2} \vee K_1 \vee \overline{K_{b+1}}$ contains no odd $[1,b]$-factor.

Partition the vertex set $V(K_{\frac{n-2}{2}} \vee \overline{K_{\frac{n+2}{2}}})$ as $\{V(\overline{K_{\frac{n+2}{2}}}), V(K_{\frac{n-2}{2}})\}$. Then the equitable quotient matrix of the distance signless Laplacian matrix $\mathcal{Q}(K_{\frac{n-2}{2}} \vee \overline{K_{\frac{n+2}{2}}})$ of the graph $K_{\frac{n-2}{2}} \vee \overline{K_{\frac{n+2}{2}}}$ can be written as

$$B\Big(K_{\frac{n-2}{2}} \vee \overline{K_{\frac{n+2}{2}}}\Big) = \begin{pmatrix} \frac{5n}{2}-1 & \frac{n}{2}-1 \\ \frac{n}{2}+1 & \frac{3n}{2}-3 \end{pmatrix}.$$

By a simple calculation, its characteristic polynomial is $x^2-4(n-1)x+\frac{7n^2}{2}-9n+4$. Note that the partition is equitable. By means of Lemma 2.5, the largest root $\theta(n)$ of $x^2-4(n-1)x+\frac{7n^2}{2}-9n+4=0$ equals the distance signless Laplacian spectral radius of the graph $K_{\frac{n-2}{2}} \vee \overline{K_{\frac{n+2}{2}}}$, that is, $\eta_1(K_{\frac{n-2}{2}} \vee \overline{K_{\frac{n+2}{2}}}) = 2n-2+\frac{\sqrt{2n(n+2)}}{2}$. Set $S = V(K_{\frac{n-2}{2}})$. Then $o(K_{\frac{n-2}{2}} \vee \overline{K_{\frac{n+2}{2}}} - S) = \frac{n+2}{2} = \frac{n-2}{2}+2 = |S|+2 = b|S|+2 > b|S|$, where $b=1$. Utilizing Lemma 2.6, $K_{\frac{n-2}{2}} \vee \overline{K_{\frac{n+2}{2}}}$ contains no 1-factor. □

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References

[1] A. Brouwer and W. Haemers, Eigenvalues and perfect matchings. *Linear Algebra App.* **395** (2005) 155–162.
[2] O. Suil, Spectral radius and matchings in graphs. *Linear Algebra App.* **614** (2021) 316–324.
[3] W. Liu, M. Liu and L. Feng, Spectral conditions for graphs to be ß-deficient involving minimum degree. *Linear Multilinear Algebra* **66** (2018) 792–802.

[4] C. Liu and J. Li, Distance signless Laplacian spectral radius and perfect matchings in graphs and bipartite graphs. Preprint arXiv:2104.01288v1.
[5] S. Zhou, Z. Sun and H. Liu, Some sufficient conditions for path-factor uniform graphs. *Aequationes Math.* **97** (2023) 489–500.
[6] K. Ando, Y. Egawa, A. Kaneko, K. Kawarabayashi and H. Matsuda, Path factors in claw-free graphs. *Discrete Math.* **243** (2002) 195–200.
[7] S. Zhou, Degree conditions and path factors with inclusion or exclusion properties. *Bull. Math. Soc. Sci. Math. Roumanie* **66** (2023) 3–14.
[8] S. Zhou and Q. Bian, The existence of path-factor uniform graphs with large connectivity. *RAIRO: Oper. Res.* **56** (2022) 2919–2927.
[9] H. Liu, Binding number for path-factor uniform graphs. *Proc. Roman. Acad. Ser. A: Math. Phys. Tech. Sci. Inf. Sci.* **23** (2022) 25–32.
[10] S. Wang and W. Zhang, Independence number, minimum degree and path-factors in graphs. *Proc. Roman. Acad. Ser. A: Math. Phys. Tech. Sci. Inf. Sci.* **23** (2022) 229–234.
[11] J. Wu, Path-factor critical covered graphs and path-factor uniform graphs. *RAIRO: Oper. Res.* **56** (2022) 4317–4325.
[12] W. Gao, Y. Chen and Y. Wang, Network vulnerability parameter and results on two surfaces. *Int. J. Intell. Syst.* **36** (2021) 4392–4414.
[13] S. Zhou, Some results on path-factor critical avoidable graphs. *Discuss. Math. Graph Theory* **43** (2023) 233–244.
[14] S. Zhou, J. Wu and Y. Xu, Toughness, isolated toughness and path factors in graphs. *Bull. Aust. Math. Soc.* **106** (2022) 195–202.
[15] S. Wang and W. Zhang, Isolated toughness for path factors in networks. *RAIRO: Oper. Res.* **56** (2022) 2613–2619.
[16] S. Zhou, J. Wu and Q. Bian, On path-factor critical deleted (or covered) graphs. *Aequationes Math.* **96** (2022) 795–802.
[17] S. Zhou, Path factors and neighborhoods of independent sets in graphs. *Acta Math. Appl. Sin. Engl. Ser.* **39** (2023) 232–238.
[18] Y. Cui and M. Kano, Some results on odd factors of graphs. *J. Graph Theory* **12** (1988) 327–333.
[19] H. Lu, Z. Wu and X. Yang, Eigenvalues and $[1, n]$-odd factors. *Linear Algebra App.* **433** (2010) 750–757.
[20] S. Kim, O. Suil, J. Park and H. Ree, An odd $[1, b]$-factor in regular graphs from eigenvalues. *Discrete Math.* **343** (2020) 111906.
[21] D. Fan, H. Lin and H. Lu, Spectral radius and $[a, b]$-factors in graphs. *Discrete Math.* **345** (2022) 112892.
[22] S. Wang and W. Zhang, On k-orthogonal factorizations in networks. *RAIRO: Oper. Res.* **55** (2021) 969–977.
[23] S. Wang and W. Zhang, Research on fractional critical covered graphs. *Prob. Inf. Transm.* **56** (2020) 270–277.
[24] M. Axenovich and J. Rollin, Brooks type results for conflict-free colorings and $\{a, b\}$-factors in graphs. *Discrete Math.* **338** (2015) 2295–2301.
[25] N. Haghparast and K. Ozeki, 2-Factors of cubic bipartite graphs. *Discrete Math.* **344** (2021) 112357.
[26] X. Lv, A degree condition for graphs being fractional (a, b, k)-critical covered. *Filomat* **37** (2023) 3315–3320.
[27] S. Zhou and H. Liu, Discussions on orthogonal factorizations in digraphs. *Acta Math. Appl. Sin. Engl. Ser.* **38** (2022) 417–425.
[28] S. Zhou, H. Liu and Y. Xu, A note on fractional ID-$[a, b]$-factor-critical covered graphs. *Discrete Appl. Math.* **319** (2022) 511–516.
[29] S. Zhou, A neighborhood union condition for fractional (a, b, k)-critical covered graphs. *Discrete Appl. Math.* **323** (2022) 343–348.
[30] S. Zhou, A note of generalization of fractional ID-factor-critical graphs. *Fundam. Inf.* **187** (2022) 61–69.
[31] S. Zhou, Remarks on restricted fractional (g, f)-factors in graphs. *Discrete Appl. Math.* DOI: 10.1016/j.dam.2022.07.020.
[32] S. Zhou, J. Wu and H. Liu, Independence number and connectivity for fractional (a, b, k)-critical covered graphs. *RAIRO: Oper. Res.* **56** (2022) 2535–2542.
[33] H. Minc, Nonnegative Matrices. John Wiley & Sons, New York (1988).
[34] R. Xing, B. Zhou and J. Li, On the distance signless Laplacian spectral radius of graphs. *Linear Multilinear Algebra* **62** (2014) 1377–1387.
[35] A. Brouwer and W. Haemers, Spectra of Graphs – Monograph. Springer (2011).
[36] L. You, M. Yang, W. So and W. Xi, On the spectrum of an equitable quotient matrix and its application. *Linear Algebra App.* **577** (2019) 21–40.
[37] A. Amahashi, On factors with all degrees odd. *Graphs Comb.* **1** (1985) 111–114.