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SEIDEL SPECTRA OF SOME VARIANTS OF CORONA OPERATIONS

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Abstract. The Seidel spectrum of a graph is defined as the multiset of all eigenvalues of its Seidel matrix. Recently, there has been a renewed interest in studying Seidel spectrum of graphs and some achievements have been made in this regard. In this paper, we determine the Seidel spectra of some variants of corona operations, such as edge corona, subdivision-vertex corona, subdivision-vertex neighbourhood corona of two graphs and so on. The corresponding Seidel eigenvectors are also described completely. Applying the obtained results, we give some sufficient and necessary conditions for edge corona, subdivision-vertex corona and subdivision-vertex neighbourhood corona of two graphs to be Seidel integral.

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1. Introduction

Seidel matrices were introduced by van Lint and Seidel [44] to study equiangular line systems in Euclidean spaces, and then there has been a renewed interest in recent years, for example, see [8, 12, 20, 21, 40] and cited references therein. Up till now, some achievements have been made in studying Seidel spectrum of graphs. For example, Berman *et al.* [10] gave some sufficient conditions for a complete multipartite graph to be determined, up to Seidel switching, by its Seidel spectrum. In [4], some basic properties of the Seidel spectrum of graphs were obtained. In [40], Seidel matrices with exactly three distinct Seidel eigenvalues of order $n \leq 23$ were discussed and classified. Furthermore, Greaves [19] gave some sufficient and necessary conditions for the graphs with exactly three distinct Seidel eigenvalues to be regular. The Seidel spectrum and energy of chain graphs were studied by Mandal *et al.* [34]. Recently, Tian *et al.* [42] studied the change of Seidel energy of tripartite Turán graph due to edge deletion. For more information about the Seidel spectrum of graphs and their properties, readers may refer to [3, 11, 18, 22, 27, 30, 33, 35, 38, 39] and the cited references therein.

It is worth observing that the adjacency spectra and (signless) Laplacian spectra of some graph operations have been researched extensively in recent years. More relevant results and details in this direction can be found in [1, 6, 7, 9, 14, 15, 26, 31, 32, 41, 43]. However, only a few results have been obtained on Seidel spectra of graph operations. For example, Haemers and Oboudi [23] used equitable matrix partitions to obtain the Seidel characteristic polynomial of the join of regular graphs. Adiga *et al.* [2] determined completely the Seidel spectra,

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along with the corresponding eigenvectors, of corona and neighbourhood corona of two graphs. Wang *et al.* [45] gave the Seidel characteristic polynomial of complete multipartite graphs and some bounds on the smallest Seidel eigenvalue of compete multipartite graphs.

In addition to characterizing some spectra of graph operations, integral graphs have also been studied for more than half a century [24], and many integral graphs have been characterized in various kinds of graphs, for example, see [5,17,28] and the cited references therein. It has been found that integral graphs play an important role in the quantum state transfer in quantum spin networks in [13,29,43]. Likewise, there has also been very little research on Seidel integral graphs in the literature so far. In 2012, Lv *et al.* [33] presented two conditions for the complete tripartite graphs and specific complete multipartite graphs to be Seidel integral. Along this line, Wang *et al.* [45] gave some sufficient and necessary conditions for complete multipartite graphs to be Seidel integral and constructed infinitely many new classes of Seidel integral graphs. In further discussion they also proposed two open questions about the Seidel integral graphs, which have been partly answered by Pokorný, Híc and Stevanović [37] and Pokorný [36].

Motivated by above some results, we shall focus on determining the Seidel spectra of some variants of corona operations of two graphs. At the same time, we also construct some family of Seidel integral graphs. This paper is organized as follows. In Section 2, we mainly recall some definitions and basic knowledge, which will be used in the following sections. In Section 3, we characterize the Seidel spectra of some variants of corona operations, such as edge corona, subdivision-vertex corona, subdivision-vertex neighbourhood corona of two graphs. Furthermore, the corresponding Seidel eigenvectors are also described completely. Applying the results obtained above, we also give some sufficient and necessary conditions for edge corona, subdivision-vertex corona and subdivision-vertex neighbourhood corona of two graphs to be Seidel integral. Finally, some numerical examples are provided to explain the validity of our results.

2. Preliminaries

Throughout this paper we only consider finite simple connected graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The *adjacency matrix* of G is the $n \times n$ matrix $A(G) = (a_{i,j})$ with entries $a_{i,j} = 1$ if vertex v_i is adjacent to vertex v_j and 0, otherwise. The *adjacency spectrum* of G is denoted by $\mathrm{Sp}(A(G)) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all eigenvalues of $A(G)$. Let $X_1, X_2, \ldots, X_n$ be orthogonal eigenvectors of $A(G)$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. The *signless Laplacian matrix* of G is defined as the matrix $Q(G) = D(G)+A(G)$, where $D(G)$ is the degree diagonal matrix of G. Clearly, if G is an r-regular graph, then $Q(G) = rI_n + A(G)$ and $Q(G)\mathbf{1}_n = 2r\mathbf{1}_n$, where I_n and $\mathbf{1}_n$ denote the identity matrix of order n and column vector of size n having all its entries as 1. For more review related to the adjacency spectrum and signless Laplacian spectrum of graphs, we refer readers to [11,16] and the cited references therein.

The *Seidel matrix* $S(G) = (s_{i,j})$ of a graph G is the matrix with the diagonal entries 0, $s_{i,j} = -1$ if vertex v_i is adjacent to vertex v_j and otherwise 1. All eigenvalues of $S(G)$ are called the *Seidel eigenvalues* of G. The multiset of all eigenvalues of $S(G)$ is called the *Seidel spectrum* of G. We denote the Seidel spectrum of G by $\mathrm{Sp}(S(G)) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all Seidel eigenvalues of G. It is easy to see that $S(G) = J_n - I_n - 2A(G)$, where J_n denotes the square matrix of order n with all entries 1. So the Seidel spectrum of G is completely determined by its adjacency spectrum whenever G is a regular graph on n vertices. A graph is called *Seidel integral* if its Seidel spectrum consists entirely of integer. Let K_n and $K_{s,t}$ be the complete graph of order n and complete bipartite graph of order $s+t$, respectively. It is well known that K_n and $K_{s,t}$ are Seidel integral.

Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, m_1 and m_2 edges, respectively. Here we recall the definitions of some variants of corona operations, such as edge corona, subdivision-vertex corona, subdivision-vertex neighbourhood corona of two graphs.

- The *edge corona* [26] of G_1 and G_2, denoted by $G_1 \diamond G_2$, is the graph obtained by taking one copy of G_1 and m_1 copies of G_2, and joining two end vertices of the i-th edge of G_1 to every vertex in the i-th copy of G_2.

– The *subdivision-vertex corona* [32] $G_1^{\mathbb{S}} \odot G_2$ of G_1 and G_2 is the graph obtained by taking one copy of the subdivision graph $\mathbb{S}(G_1)$ of G_1 and n_1 copies of G_2, and joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2, where the *subdivision graph* $\mathbb{S}(G_1)$ of G_1 is the graph obtained by inserting a new vertex into every edge of G_1.
– The *subdivision-vertex neighbourhood corona* [31] $G_1^{\mathbb{S}} \boxdot G_2$ of G_1 and G_2 is the graph obtained by taking one copy of the subdivision graph $\mathbb{S}(G_1)$ of G_1 and n_1 copies of G_2, and joining the neighbours of the i-th vertex of G_1 to every vertex in the i-th copy of G_2.

In the following, we recall some basic knowledge, which will be used in the sequel. The adjacency matrix of the subdivision graph $\mathbb{S}(G_1)$ of G_1 can be written as

$$A(\mathbb{S}(G_1)) = \begin{pmatrix} 0 & M(G_1) \\ M^{\mathrm{T}}(G_1) & 0 \end{pmatrix},$$

where $M(G_1)$ is the vertex-edge incidence matrix of G_1 and $M^{\mathrm{T}}(G_1)$ is the transpose of $M(G_1)$. Assume that ξ_i and ζ_i are singular vector pairs of $M(G_1)$ corresponding to the singular value s_i for $i = 1, 2, \ldots, n_1$, equivalently, $M(G_1)\xi_i = s_i\zeta_i$ and $M^{\mathrm{T}}(G_1)\zeta_i = s_i\xi_i$. Let η_j be an orthogonal vector such that $M(G_1)\eta_j = 0$ for $j = 1, 2, \ldots, m_1 - n_1$. It was proved [9] that, for $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, m_1 - n_1$,

$$\begin{pmatrix} \pm\zeta_i \\ \xi_i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0_n \\ \eta_j \end{pmatrix}$$

are the orthogonal eigenvectors of $A(\mathbb{S}(G_1))$ corresponding to eigenvalues $\pm s_i$ and 0, respectively.

Note that $M(G_1)M^{\mathrm{T}}(G_1)\zeta_i = {s_i}^2\zeta_i$ and $M^{\mathrm{T}}(G_1)M(G_1)\xi_i = {s_i}^2\xi_i$ for $i = 1, 2, \ldots, n_1$. If G_1 is an r-regular graph, then $M(G_1)M^{\mathrm{T}}(G_1) = A(G_1) + rI_{n_1} = Q(G_1)$ and $M^{\mathrm{T}}(G_1)M(G_1) = A(\mathcal{L}(G_1)) + 2I_{m_1}$, where $\mathcal{L}(G_1)$ denotes the line graph of G_1. This implies that $s_i^2 = \lambda_i + r$, where λ_i is the adjacency spectrum of G_1 for $i = 1, 2, \ldots, n_1$. Furthermore, $\zeta_1 = \mathbf{1}_{n_1}, \zeta_2, \ldots, \zeta_{n_1}$ are the orthogonal eigenvectors of $Q(G_1)$ corresponding to the respective eigenvalue s_i^2 for $i = 1, 2, \ldots, n_1$. It is easy to see that $\xi_1 = \mathbf{1}_{m_1}, \xi_2, \ldots, \xi_{n_1}$ are the orthogonal eigenvectors of $A(\mathcal{L}(G_1)) + 2I_{m_1}$ corresponding to the respective eigenvalue s_i^2 for $i = 1, 2, \ldots, n_1$, and $\eta_1, \eta_2, \ldots, \eta_{m_1 - n_1}$ are the rest orthogonal eigenvectors of $A(\mathcal{L}(G_1)) + 2I_{m_1}$ corresponding to the eigenvalue 0.

We also recall the concept of Kronecker product of two matrices. Let $A = (a_{i,j})$ be an $n \times m$ matrix and $B = (b_{i,j})$ be a $p \times q$ matrix. The *Kronecker product* $A \otimes B$ of A and B is the $np \times mq$ matrix defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix}.$$

As we all know, $(A \otimes B)(C \otimes D) = AC \otimes BD$. For more properties of Kronecker product can be found in [25].

3. Seidel spectra of some variants of coronae

3.1. The Seidel spectrum of the edge corona

Let $G_1 = (V(G_1), E(G_1))$ be an r_1-regular graph of order n_1 with m_1 edges, where the vertex set $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and edge set $E(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$. Also let $G_2(V(G_2), E(G_2))$ be an r_2-regular graph of order n_2 with m_2 edges and vertex set $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. Let $u_1^i, u_2^i, \ldots, u_{n_2}^i$ denote the vertices of the i-th copy of G_2 for $i = 1, 2, \ldots, m_1$. We denote $P_j = \{u_j^1, u_j^2, \ldots, u_j^{m_1}\}$ for $j = 1, 2, \ldots, n_2$, then the vertex set of $G_1 \diamond G_2$ may be divided into $V(G_1) \cup P_1 \cup P_2 \cup \cdots \cup P_{n_2}$. With respect to this partition of $V(G_1 \diamond G_2)$, the Seidel matrix of $G_1 \diamond G_2$ can be written as

$$S(G_1 \diamond G_2) = \begin{pmatrix} J_{n_1} - I_{n_1} - 2A(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes (J_{n_1 \times m_1} - 2M(G_1)) \\ \mathbf{1}_{n_2} \otimes \left(J_{m_1 \times n_1} - 2M^{\mathrm{T}}(G_1)\right) & J_{n_2} \otimes J_{m_1} - (2A(G_2) + I_{n_2}) \otimes I_{m_1} \end{pmatrix},$$

where $M(G_1)$ is the vertex-edge incident matrix of G_1.

Next, we shall determine the Seidel spectrum of $G_1 \diamond G_2$ under some special conditions on G_1 and G_2.

Theorem 3.1. *For $i = 1, 2$, let G_i be an r_i-regular graph of order n_i with m_i edges. Assume that $Sp(A(G_1)) = \{\lambda_1 = r_1, \lambda_2, \ldots, \lambda_{n_1}\}$ and $Sp(A(G_2)) = \{\mu_1 = r_2, \mu_2, \ldots, \mu_{n_2}\}$. Then the Seidel spectrum of $G_1 \diamond G_2$ consists precisely of the following:*

(i) *The eigenvalue $-2\mu_j - 1$ with multiplicity m_1 for every non-maximum eigenvalue μ_j $(j = 2, \ldots, n_2)$;*
(ii) *The eigenvalue $-2r_2 - 1$ with multiplicity $m_1 - n_1$ (if possible);*
(iii) *The eigenvalues $-1 - r_2 - \lambda_i \pm \sqrt{(r_2 - \lambda_i)^2 + 4s_i^2 n_2}$ for $i = 2, 3, \ldots, n_1$;*
(iv) *The eigenvalues $-1 - 2r_2 + n_2 m_1$ and $3 - 2r_1$ if $n_1 = 4$, otherwise $\frac{-b \pm \sqrt{b^2 - 4c}}{2}$, where $b = 2 + 2r_2 + 2r_1 - n_1 - n_2 m_1$ and $c = 2r_2 + 4r_1 r_2 - 2n_1 r_2 + 1 + 2r_1 - n_1 + 7m_1 n_2 - 2r_1 m_1 n_2 - 8r_1 n_2$.*

Proof. (i) Let $Y_1, Y_2, \ldots, Y_{n_2}$ be the orthogonal eigenvectors of $A(G_2)$ corresponding to the eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n_2}$, respectively. And let $c_{m_1}^i$ be a column vector whose i-th entry is 1 and all other entries are 0 for $i = 1, 2, \ldots, m_1$. Since G_2 is regular, then $Y_1 = \mathbf{1}_{n_2}$ and $J_{n_2} Y_j = 0$ for $j = 2, 3, \ldots, n_2$. Thus, for $j = 2, 3, \ldots, n_2$,

$$
\begin{aligned}
S(G_1 \diamond G_2)\begin{pmatrix} 0 \\ Y_j \otimes c_{m_1}^i \end{pmatrix} &= \begin{pmatrix} J_{n_1} - I_{n_1} - 2A(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes (J_{n_1 \times m_1} - 2M(G_1)) \\ \mathbf{1}_{n_2}^{\mathrm{T}} \otimes \left(J_{m_1 \times n_1} - 2M^{\mathrm{T}}(G_1)\right) & J_{n_2} \otimes J_{m_1} - (2A(G_2) + I_{n_2}) \otimes I_{m_1} \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 \\ Y_j \otimes c_{m_1}^i \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1}_{n_2}^{\mathrm{T}} Y_j \otimes (J_{n_1 \times m_1} - 2M(G_1)) c_{m_1}^i \\ J_{n_2} Y_j \otimes J_{m_1} c_{m_1}^i - (2A(G_2) + I_{n_2}) Y_j \otimes I_{m_1} c_{m_1}^i \end{pmatrix} \\
&= -(2\mu_j + 1)\begin{pmatrix} 0 \\ Y_j \otimes c_{m_1}^i \end{pmatrix}.
\end{aligned}
$$

Hence, $-(2\mu_j + 1)$ is the eigenvalue of $S(G_1 \diamond G_2)$ with multiplicity m_1, corresponding to the eigenvector $\begin{pmatrix} 0 \\ Y_j \otimes c_{m_1}^i \end{pmatrix}$.

(ii) Let η_j be an orthogonal vector such that $M(G_1)\eta_j = 0$ for $j = 1, 2, \ldots, m_1 - n_1$. As described in Section 2, $\mathbf{1}_{m_1}^{\mathrm{T}} \eta_j = 0$ for $j = 1, 2, \ldots, m_1 - n_1$, then

$$
\begin{aligned}
S(G_1 \diamond G_2)\begin{pmatrix} 0 \\ \mathbf{1}_{n_2} \otimes \eta_j \end{pmatrix} &= \begin{pmatrix} J_{n_1} - I_{n_1} - 2A(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes (J_{n_1 \times m_1} - 2M(G_1)) \\ \mathbf{1}_{n_2}^{\mathrm{T}} \otimes \left(J_{m_1 \times n_1} - 2M^{\mathrm{T}}(G_1)\right) & J_{n_2} \otimes J_{m_1} - (2A(G_2) + I_{n_2}) \otimes I_{m_1} \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 \\ \mathbf{1}_{n_2} \otimes \eta_j \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1}_{n_2}^{\mathrm{T}} \mathbf{1}_{n_2} \otimes (J_{n_1 \times m_1} - 2M(G_1)) \eta_j \\ J_{n_2} \mathbf{1}_{n_2} \otimes J_{m_1} \eta_j - (2A(G_2) + I_{n_2}) \mathbf{1}_{n_2} \otimes I_{m_1} \eta_j \end{pmatrix} \\
&= -(2r_2 + 1)\begin{pmatrix} 0 \\ \mathbf{1}_{n_2} \otimes \eta_j \end{pmatrix}.
\end{aligned}
$$

Thus, $-(2r_2 + 1)$ is the eigenvalue of $S(G_1 \diamond G_2)$, corresponding to the eigenvector $\begin{pmatrix} 0 \\ \mathbf{1}_{n_2} \otimes \eta_j \end{pmatrix}$ for $j = 1, 2, \ldots, m_1 - n_1$.

(iii) Let ξ_i and ζ_i be singular vector pairs of $M(G_1)$ corresponding to the singular value s_i for $i = 1, 2, \ldots, n_1$, that is, $M(G_1)\xi_i = s_i \zeta_i$ and $M^{\mathrm{T}}(G_1)\zeta_i = s_i \xi_i$. Without loss of generality, assume that $\xi_1 = \mathbf{1}_{m_1}$ and $\zeta_1 = \mathbf{1}_{n_1}$. For $i = 2, 3, \ldots, n_1$, we have $J\zeta_i = 0$ and $M(G_1)M^{\mathrm{T}}(G_1)\zeta_i = Q(G_1)\zeta_i = (A(G_1) + r_1)\zeta_i$, which implies that

$$A(G_1)\zeta_i = M(G_1)M^{\mathrm{T}}(G_1)\zeta_i - r_1 \zeta_i = \left(s_i^2 - r_1\right)\zeta_i. \tag{1}$$

Let k_1 and k_2 be two unknown scalars to be determined. By a simple calculation, one has

$$
\begin{aligned}
&S(G_1 \diamond G_2)\begin{pmatrix} k_1\zeta_i \\ k_2\mathbf{1}_{n_2}\otimes\xi_i \end{pmatrix}\\
&= \begin{pmatrix} J_{n_1}-I_{n_1}-2A(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}}\otimes(J_{n_1\times m_1}-2M(G_1)) \\ \mathbf{1}_{n_2}\otimes\big(J_{m_1\times n_1}-2M^{\mathrm{T}}(G_1)\big) & J_{n_2}\otimes J_{m_1}-(2A(G_2)+I_{n_2})\otimes I_{m_1} \end{pmatrix}\begin{pmatrix} k_1\zeta_i \\ k_2\mathbf{1}_{n_2}\otimes\xi_i \end{pmatrix}\\
&= \begin{pmatrix} (J_{n_1}-I_{n_1}-2A(G_1))k_1\zeta_i + k_2\mathbf{1}_{n_2}^{\mathrm{T}}\mathbf{1}_{n_2}\otimes(J_{n_1\times m_1}-2M(G_1))\xi_i \\ \mathbf{1}_{n_2}\otimes\big(J_{m_1\times n_1}-2M^{\mathrm{T}}(G_1)\big)k_1\zeta_i + k_2J_{n_2}\mathbf{1}_{n_2}\otimes J_{m_1}\xi_i - k_2(2A(G_2)+I_{n_2})\mathbf{1}_{n_2}\otimes I_{m_1}\xi_i \end{pmatrix}\\
&= \begin{pmatrix} k_1\big(-1-2(s_i^2-r_1)\big)\zeta_i + k_2n_2(-2s_i)\zeta_i \\ k_1\mathbf{1}_{n_2}\otimes(-2s_i\xi_i) - k_2(2r_2+1)\mathbf{1}_{n_2}\otimes\xi_i \end{pmatrix} = \begin{pmatrix} (-k_1-2\lambda_ik_1-2k_2s_in_2)\zeta_i \\ (-k_2(1+2r_2)-2s_ik_1)\mathbf{1}_{n_2}\otimes\xi_i \end{pmatrix}\\
&= \alpha\begin{pmatrix} k_1\zeta_i \\ k_2\mathbf{1}_{n_2}\otimes\xi_i \end{pmatrix}.
\end{aligned}
$$

According to the above equation, we can find that

$$
-k_1-2\lambda_ik_1-2k_2s_in_2=\alpha k_1,\quad -k_2-2k_2r_2-2s_ik_1=\alpha k_2. \tag{2}
$$

Now we consider the following two cases:

Case 1. G_1 is a bipartite graph. In this case, $-r_1$ is also an eigenvalue of $A(G_1)$. Bear in mind that G_1 is connected. It follows from (1) that there exists a unique index $i_0 \in \{2,3,\ldots,n_1\}$ such that $s_{i_0}=0$. Notice that k_1 and k_2 cannot be zero simultaneously. Now we discuss (2) for this singular value $s_{i_0}=0$. If $k_1=0$ and $k_2 \in \mathbb{R}\setminus 0$, then $\alpha=-1-2r_2$; If $k_1\in\mathbb{R}\setminus 0$ and $k_2=0$, then $\alpha=-1+2r_1$; If $k_1\neq 0$ and $k_2\neq 0$, then $r_2=\lambda_{i_0}=-r_1$, a contradiction.

In what following, we consider any nonzero singular value s_i for $i\in\{2,3,\ldots,n_1\}\setminus\{i_0\}$. At this time, we have $k_1\neq 0$ and $k_2\neq 0$. Without loss of generality, set $k_2=1$. It follows from (2) that

$$
\alpha=-1-r_2-\lambda_i\pm\sqrt{(r_2-\lambda_i)^2+4s_i^2n_2},\quad k_1=\frac{\lambda_i-r_2\mp\sqrt{(r_2-\lambda_i)^2+4s_i^2n_2}}{2s_i}.
$$

Case 2. G_1 is a non-bipartite graph. In this case, $s_i\neq 0$ for $i=2,3,\ldots,n_1$. Similar to Case 1, we easily get that $-1-r_2-\lambda_i\pm\sqrt{(r_2-\lambda_i)^2+4s_i^2n_2}$ are the eigenvalues of $S(G_1\diamond G_2)$ for any $i=2,3,\ldots,n_1$.

Above all, $-1-r_2-\lambda_i\pm\sqrt{(r_2-\lambda_i)^2+4s_i^2n_2}$ are the eigenvalues of $S(G_1\diamond G_2)$, corresponding to the eigenvector $\begin{pmatrix} k_1\zeta_i \\ k_2\mathbf{1}_{n_2}\otimes\xi_i \end{pmatrix}$ for $i=2,3,\cdots n_1$.

(iv) Let k_1 and k_2 be two unknown scalars to be determined. Then

$$
\begin{aligned}
&S(G_1 \diamond G_2)\begin{pmatrix} k_1\mathbf{1}_{n_1} \\ k_2\mathbf{1}_{n_2}\otimes\mathbf{1}_{m_1} \end{pmatrix}\\
&= \begin{pmatrix} J_{n_1}-I_{n_1}-2A(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}}\otimes(J_{n_1\times m_1}-2M(G_1)) \\ \mathbf{1}_{n_2}\otimes\big(J_{m_1\times n_1}-2M^{\mathrm{T}}(G_1)\big) & J_{n_2}\otimes J_{m_1}-(2A(G_2)+I_{n_2})\otimes I_{m_1} \end{pmatrix}\begin{pmatrix} k_1\mathbf{1}_{n_1} \\ k_2\mathbf{1}_{n_2}\otimes\mathbf{1}_{m_1} \end{pmatrix}\\
&= \begin{pmatrix} (J_{n_1}-I_{n_1}-2A(G_1))k_1\mathbf{1}_{n_1} + k_2\mathbf{1}_{n_2}^{\mathrm{T}}\mathbf{1}_{n_2}\otimes(J_{n_1\times m_1}-2M(G_1))\mathbf{1}_{m_1} \\ \mathbf{1}_{n_2}\otimes\big(J_{m_1\times n_1}-2M^{\mathrm{T}}(G_1)\big)k_1\mathbf{1}_{n_1} + k_2J_{n_2}\mathbf{1}_{n_2}\otimes J_{m_1}\mathbf{1}_{m_1} - k_2(2A(G_2)+I_{n_2})\mathbf{1}_{n_2}\otimes I_{m_1}\mathbf{1}_{m_1} \end{pmatrix}\\
&= \begin{pmatrix} k_1(n_1-1-2r_1)\mathbf{1}_{n_1} + k_2n_2(m_1-2r_1)\mathbf{1}_{n_1} \\ k_1\mathbf{1}_{n_2}\otimes(n_1-4)\mathbf{1}_{m_1} + (k_2n_2m_1-k_2(2r_2+1))\mathbf{1}_{n_2}\otimes\mathbf{1}_{m_1} \end{pmatrix}\\
&= \begin{pmatrix} (k_1(n_1-1-2r_1)+k_2n_2(m_1-2r_1))\mathbf{1}_{n_1} \\ (-1k_2-2r_2k_2-4k_1+k_1n_1+k_2n_2m_1)\mathbf{1}_{n_2}\otimes\mathbf{1}_{m_1} \end{pmatrix}\\
&= \alpha\begin{pmatrix} k_1\mathbf{1}_{n_1} \\ k_2\mathbf{1}_{n_2}\otimes\mathbf{1}_{m_1} \end{pmatrix},
\end{aligned}
$$

where α is an eigenvalue of $S(G_1 \diamond G_2)$ to be determined. From the previous equation, we can find that

$$\alpha k_1 = k_1(n_1 - 1 - 2r_1) + k_2 n_2(m_1 - 2r_1),$$
$$\alpha k_2 = -4k_1 + k_1 n_1 + k_2(n_2 m_1 - 1 - 2r_2).$$

Now we consider the following two cases:

Case 1. $n_1 = 4$. In this case, we have $m_1 = 2r_1$ because G_1 is r_1-regular. Thus the equations above become the following

$$\alpha k_1 = k_1(3 - 2r_1),$$
$$\alpha k_2 = k_2(-1 - 2r_2 + n_2 m_1).$$

Notice that k_1 and k_2 cannot be zero simultaneously. Hence, $\alpha = -1 - 2r_2 + n_2 m_1$ whenever $k_1 = 0$ and $k_2 \in \mathbb{R} \setminus 0$, and $\alpha = 3 - 2r_1$ whenever $k_1 \in \mathbb{R} \setminus 0$, $k_2 = 0$.

Case 2. $n_1 \neq 4$. In this case, we have $n_1 - 4 \neq 0$. The above equations mean

$$k_1 = k_2\left(\frac{\alpha + 2r_2 + 1 - n_2 m_1}{n_1 - 4}\right),$$
$$\left(\alpha^2 + b\alpha + c\right)k_2 = 0,$$

where

$$b = 2 + 2r_2 + 2r_1 - n_1 - n_2 m_1,$$
$$c = 2r_2 + 4r_1 r_2 - 2n_1 r_2 + 1 + 2r_1 - n_1 + 7m_1 n_2 - 2r_1 m_1 n_2 - 8r_1 n_2.$$

By a simple calculation, one has

$$b^2 - 4c = (2r_1 - 2r_2 - n_1 + n_2 m_1)^2 + 4n_2(m_1 - 2r_1)(n_1 - 4).$$

Obviously, for any regular graph G_1, we have $b^2 - 4c > 0$ for $n_1 > 4$. It is easy to see that $b^2 - 4c > 0$ for $1 \leq n_1 \leq 3$. Therefore, $\frac{-b \pm \sqrt{b^2 - 4c}}{2}$ are Seidel eigenvalues of $G_1 \diamond G_2$ for $n_1 \neq 4$.

From what has been discussed above, we have $-1 - 2r_2 + n_2 m_1$ and $3 - 2r_1$ are Seidel eigenvalues of $G_1 \diamond G_2$ for $n_1 = 4$, otherwise $\frac{-b \pm \sqrt{b^2 - 4c}}{2}$, along with corresponding to the eigenvector $\begin{pmatrix} k_1 \mathbf{1}_{n_1} \\ k_2 \mathbf{1}_{n_2} \otimes \mathbf{1}_{m_1} \end{pmatrix}$. □

Remark 3.2. Let G_1 is an r_1-regular graph of order n_1 with m_1 edges. We can find that, if $m_1 - n_1 \geq 0$, then all Seidel eigenvalues of $G_1 \diamond G_2$ are completely determined by Theorem 3.1. Remark that, if $m_1 - n_1 < 0$, then G_1 must be a tree. This implies $G_1 = K_2$ as G_1 is r_1-regular. At this time, it is easy to see that $\alpha = -1 - 2r_2$ is not the Seidel eigenvalue of $G_1 \diamond G_2$. Hence, our results still hold in Theorem 3.1.

Example 1. Consider the edge corona of $K_2 \diamond G_2$, where G_2 is an r_2-regular graph of order n_2 with m_2 edges. Assume that $\mathrm{Sp}(A(G_2)) = \{\mu_1 = r_2, \mu_2, \ldots, \mu_{n_2}\}$. Applying Theorem 3.1, we have

(i) For every eigenvalue μ_j ($j = 2, 3, \ldots, n_2$) of G_2, $-2\mu_j - 1$ is the Seidel eigenvalue of $K_2 \diamond G_2$ with multiplicity m_1.

(ii) For the eigenvalue -1 of G_1, 1 is the Seidel eigenvalue of $K_2 \diamond G_2$ corresponding to the eigenvector $\begin{pmatrix} \zeta_2 \\ \mathbf{0}_{n_2} \end{pmatrix}$. Noting that $-1 - 2r_2$ is not the Seidel eigenvalue of $K_2 \diamond G_2$.

(iii) $-1 - r_2 + \frac{n_2}{2} \pm \frac{\sqrt{(n_2 - 2r_2)^2 + 8n_2}}{2}$ are also the Seidel eigenvalues of $K_2 \diamond G_2$.

Example 2. Let us consider two complete graphs $G_1 = K_4$ and $G_2 = K_3$. Then $\mathrm{Sp}(A(G_1)) = \{3, -1^{(3)}\}$ and $\mathrm{Sp}(A(G_2)) = \{2, -1^{(2)}\}$, where $x^{(y)}$ means that the multiplicity of x is y. Applying Theorem 3.1, we obtain that the Seidel spectrum of $K_4 \diamond K_3$ consists of:

(i) For the eigenvalue -1 of G_2, $1^{(12)}$ are the Seidel eigenvalues of $K_4 \diamond K_3$;
(ii) For the eigenvalue 2 of G_2, $-5^{(2)}$ are the Seidel eigenvalues of $K_4 \diamond K_3$;
(iii) For the eigenvalue -1 of G_1, $-2 \pm \sqrt{33}^{(3)}$ are the Seidel eigenvalues of $K_4 \diamond K_3$;
(iv) 13 and -3 are also the Seidel eigenvalues of $K_4 \diamond K_3$.

On the other hand, according to the computation of MATLAB, the Seidel spectrum of $K_4 \diamond K_3$ is also $\mathrm{Sp}(S(K_4 \diamond K_3)) = \{-2 - \sqrt{33}^{(3)}, -5^{(2)}, -3, 1^{(12)}, -2 + \sqrt{33}^{(3)}, 13\}$.

Corollary 3.3. *Let G_1 be an r_1-regular graph of order n_1 with m_1 edges, G_2 be an r_2-regular graph of order n_2 with m_2 edges. Then $G_1 \diamond G_2$ is Seidel integral if and only if*

(i) *The graph G_2 is adjacency integral;*
(ii) $-1 - r_2 - \lambda_i \pm \sqrt{(r_2 - \lambda_i)^2 + 4s_i^2 n_2}$ *are integers for any* $i = 2, 3, \ldots, n_1$;
(iii) $\frac{-b \pm \sqrt{b^2 - 4c}}{2}$ *are all integers, where* $b = 2 + 2r_2 + 2r_1 - n_1 - n_2 m_1$ *and* $c = 2r_2 + 4r_1 r_2 - 2n_1 r_2 + 1 + 2r_1 - n_1 + 7m_1 n_2 - 2r_1 m_1 n_2 - 8r_1 n_2$.

Example 3. Consider the edge corona of $K_2 \diamond K_n$ for any positive integer n. According to Corollary 3.3, we get that $K_2 \diamond K_n$ is Seidel integral. In fact, since $K_2 \diamond K_n = K_{n+2}$, then the Seidel spectra of $K_2 \diamond K_n$ is $\mathrm{Sp}(S(K_2 \diamond K_n)) = \{-1 - n, 1^{(n+1)}\}$.

3.2. The Seidel spectrum of the subdivision-vertex corona

Let $G_1 = (V(G_1), E(G_1))$ be an r_1-regular graph of order n_1 with m_1 edges, where the vertex set $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and the edge set $E(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$. Also let $G_2 = (V(G_2), E(G_2))$ be an r_2-regular graph of order n_2 with m_2 edges, where $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. Let $u_1^i, u_2^i, \ldots, u_{n_2}^i$ denote the vertices of the i-th copy of G_2, for $i = 1, 2, \ldots, n_1$. Denote $P_j = u_j^1, u_j^2, \ldots, u_j^{n_1}$, for $j = 1, 2, \ldots, n_2$. Let $I(G_1)$ denote the set of inserted new vertices of $\mathbb{S}(G_1)$. Then the vertex set of $G_1^{\mathbb{S}} \odot G_2$ may be partitioned into $V(G_1) \cup I(G_1) \cup P_1 \cup P_2 \cup \cdots \cup P_{n_2}$. With respect to this partition of $V(G_1^{\mathbb{S}} \odot G_2)$, the Seidel matrix of $G_1^{\mathbb{S}} \odot G_2$ may be written as

$$S\big(G_1^{\mathbb{S}} \odot G_2\big) = \begin{pmatrix} J_{n_1} - I_{n_1} & J_{n_1 \times m_1} - 2M(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes (J_{n_1} - 2I_{n_1}) \\ J_{m_1 \times n_1} - 2M^{\mathrm{T}}(G_1) & J_{m_1} - I_{m_1} & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes J_{m_1 \times n_1} \\ \mathbf{1}_{n_2} \otimes (J_{n_1} - 2I_{n_1}) & \mathbf{1}_{n_2} \otimes J_{n_1 \times m_1} & J_{n_2} \otimes J_{n_1} - (2A(G_2) + I_{n_2}) \otimes I_{n_1} \end{pmatrix}.$$

Theorem 3.4. *For $i = 1, 2$, let G_i be an r_i-regular graph of order n_i with m_i edges. Assume that $Sp(A(G_1)) = \{\lambda_1 = r_1, \lambda_2, \ldots, \lambda_{n_1}\}$ and $Sp(A(G_2)) = \{\mu_1 = r_2, \mu_2, \ldots, \mu_{n_2}\}$. Then the Seidel spectrum of $G_1^{\mathbb{S}} \odot G_2$ consists precisely of the following:*

(i) *The eigenvalue -1 with multiplicity $m_1 - n_1$ (if possible);*
(ii) *The eigenvalue $-2\mu_j - 1$ with multiplicity n_1 for every non-maximum eigenvalue μ_j ($j = 2, \ldots, n_2$);*
(iii) *The roots of polynomial* $f(\alpha) = \alpha^3 + (3 + 2r_2)\alpha^2 + (4r_2 + 3 - 4s_i^2 - 4n_2)\alpha + 1 + 2r_2 - 4n_2 - 4s_i^2(2r_1 + 1)$ *for* $i = 2, 3, \ldots, n_1$;
(iv) *The roots α's of the following system with respect to the variables α, k_1, k_2, and k_3. Notice that k_1, k_2 and k_3 cannot be zero simultaneously.*

$$\begin{cases} k_1(n_1 - 1 - \alpha) + k_2(m_1 - 2r_1) + k_3 n_2(n_1 - 2) = 0, \\ k_1(n_1 - 4) + k_2(m_1 - 1 - \alpha) + k_3 n_1 n_2 = 0, \\ k_1(n_1 - 2) + k_2 m_1 + k_3(n_1 n_2 - 2r_2 - 1 - \alpha) = 0. \end{cases}$$

Proof. (i) Let η_j be an orthogonal vector of $M(G_1)$ such that $M(G_1)\eta_j = 0$ and $\mathbf{1}_{m_1}^{\mathrm{T}}\eta_j = 0$, for $j = 1, 2, \ldots, m_1 - n_1$. Then

$$S\big(G_1^{\mathbb{S}} \odot G_2\big)\begin{pmatrix} 0 \\ \eta_j \\ 0 \end{pmatrix} = \begin{pmatrix} (J_{n_1\times m_1} - 2M(G_1))\eta_j \\ (J_{m_1} - I_{m_1})\eta_j \\ (\mathbf{1}_{n_2} \otimes J_{n_1\times m_1})\eta_j \end{pmatrix} = \begin{pmatrix} 0 \\ -\eta_j \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ \eta_j \\ 0 \end{pmatrix},$$

which implies that $\begin{pmatrix} 0 \\ \eta_j \\ 0 \end{pmatrix}$ is an eigenvector of $S(G_1^{\mathbb{S}} \odot G_2)$ corresponding to the eigenvalue -1 with multiplicity $m_1 - n_1$ for $j = 1, 2, \ldots, m_1 - n_1$.

(ii) Let $Y_1 = \mathbf{1}_{n_2}$ and $\{Y_j\}_{j=2}^{n_2}$ be the orthogonal eigenvectors of $A(G_2)$, where Y_j corresponds to the eigenvalue μ_j. Let $c_{n_1}^i$ be an $n_1 \times 1$ matrix whose i-th entry is 1 and other entries are 0, where $i = 1, 2, \ldots, n_1$. Then, for $j = 2, 3, \ldots, n_2$,

$$\begin{aligned} S\big(G_1^{\mathbb{S}} \odot G_2\big)\begin{pmatrix} 0 \\ 0 \\ Y_j \otimes c_{n_1}^i \end{pmatrix} &= \begin{pmatrix} \mathbf{1}_{n_2}^{\mathrm{T}} Y_j \otimes (J_{n_1} - 2I_{n_1})c_{n_1}^i \\ \mathbf{1}_{n_2}^{\mathrm{T}} Y_j \otimes J_{m_1\times n_1} c_{n_1}^i \\ J_{n_2} Y_j \otimes J_{n_1} c_{n_1}^i - (2A(G_2) + I_{n_2})Y_j \otimes I_{n_1} c_{n_1}^i \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ -(2\mu_j + 1)Y_j \otimes c_{n_1}^i \end{pmatrix} = -(2\mu_j + 1)\begin{pmatrix} 0 \\ 0 \\ Y_j \otimes c_{n_1}^i \end{pmatrix}. \end{aligned}$$

Thus, $-2\mu_j - 1 \in \mathrm{Sp}(S(G_1^{\mathbb{S}} \odot G_2))$ with multiplicity n_1, for $j = 2, 3, \ldots, n_2$. At the same time, the vector $\begin{pmatrix} 0 \\ 0 \\ Y_j \otimes c_{n_1}^i \end{pmatrix}$ is an eigenvector of $S(G_1^{\mathbb{S}} \odot G_2)$ corresponding to the eigenvalue $-2\mu_j - 1$ for $j = 2, 3, \ldots, n_2$.

(iii) Let ζ_i and ξ_i be singular vector pairs of $M(G_1)$ corresponding to the singular value s_i, for $i = 1, 2, \ldots, n_1$. That is, $M(G_1)\xi_i = s_i\zeta_i$ and $M^{\mathrm{T}}(G_1)\zeta_i = s_i\xi_i$. As described in Section 2, $\zeta_1 = \mathbf{1}_{n_1}$ and $\{\zeta_i\}_{i=2}^{n_1}$ are orthogonal eigenvectors of $Q(G_1)$ corresponding to the respective eigenvalue s_i^2 for $i = 1, 2, \ldots, n_1$. Thus ${\zeta_1}^{\mathrm{T}}\zeta_i = 0$. Similarly, $\xi_1\xi_i = 0$ for $i = 2, \ldots, n_1$. All these imply that $J_{n_1}\zeta_i = 0$ and $J_{m_1}\xi_i = 0$ for $i = 2, \ldots, n_1$.

Let k_1, k_2 and k_3 be three unknown scalars to be determined. Then, for $i = 2, 3, \ldots, n_1$,

$$\begin{aligned} &S\big(G_1^{\mathbb{S}} \odot G_2\big)\begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ k_3\mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix} \\ &= \begin{pmatrix} k_1(J_{n_1} - I_{n_1})\zeta_i + k_2(J_{n_1\times m_1} - 2M(G_1))\xi_i + k_3 n_2(J_{n_1} - 2I_{n_1})\zeta_i \\ k_1\big(J_{m_1\times n_1} - 2M^{\mathrm{T}}(G_1)\big)\zeta_i + k_2(J_{m_1} - I_{n_1})\xi_i + k_3 n_2 J_{m_1\times n_1}\zeta_i \\ k_1\mathbf{1}_{n_2} \otimes (J_{n_1} - 2I_{n_1})\zeta_i + k_2\mathbf{1}_{n_2} \otimes J_{n_1\times m_1}\xi_i + k_3 J_{n_2}\mathbf{1}_{n_2} \otimes J_{n_1}\zeta_i - k_3(2A(G_2) + I_{n_2})\mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix} \\ &= \begin{pmatrix} k_1 J_{n_1}\zeta_i - k_1 I_{n_1}\zeta_i + k_2 J_{n_1\times m_1}\xi_i - 2k_2 M(G_1)\xi_i + k_3 n_2 J_{n_1}\zeta_i - 2k_3 n_2 I_{n_1}\zeta_i \\ k_1 J_{m_1\times n_1}\zeta_i - 2k_1 M^{\mathrm{T}}(G_1)\zeta_i + k_2 J_{m_1}\xi_i - k_2\xi_i + k_3 n_2 J_{m_1\times n_1}\zeta_i \\ k_1\mathbf{1}_{n_2} \otimes J_{n_1}\zeta_i - 2k_1\mathbf{1}_{n_2} \otimes \zeta_i + k_2\mathbf{1}_{n_2} \otimes J_{n_1\times m_1}\xi_i + k_3 J_{n_2}\mathbf{1}_{n_2} \otimes J_{n_1}\zeta_i - k_3(2r_2 + 1)\mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix} \\ &= \begin{pmatrix} -k_1\zeta_i - 2k_2 s_i\zeta_i - 2k_3 n_2\zeta_i \\ -2k_1 s_i\xi_i - k_2\xi_i \\ -2k_1\mathbf{1}_{n_2} \otimes \zeta_i - k_3(2r_2 + 1)\mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix} \\ &= \alpha\begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ k_3\mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}. \end{aligned}$$

Thus, we have

$$\begin{cases}\alpha k_1 + k_1 + 2k_2 s_i + 2k_3 n_2 = 0,\\ \alpha k_2 + 2k_1 s_i + k_2 = 0,\\ \alpha k_3 + 2k_1 + 2r_2 k_3 + k_3 = 0.\end{cases} \tag{3}$$

Now we consider the following two cases:

Case 1. G_1 is a bipartite graph. In this case, $-r_1$ is also an eigenvalue of $A(G_1)$. Bear in mind that G_1 is connected. It follows from (1) that there exists a unique index $i_0 \in \{2, 3, \ldots, n_1\}$ such that $s_{i_0} = 0$. Notice that k_1, k_2 and k_3 cannot be zero simultaneously. Next we discuss (3) for this singular value $s_{i_0} = 0$.

If $k_3 \neq 0$, without loss of generally, set $k_3 = 1$. By the above equation (3), we have

$$\begin{cases}(\alpha+1)k_1 + 2n_2 = 0,\\ (\alpha+1)k_2 = 0,\\ (\alpha+1) + 2k_1 + 2r_2 = 0.\end{cases}$$

Since G_2 is a non-empty graph, then $\alpha \neq -1$ and $k_2 = 0$. It follows that

$$\alpha^2 + (2r_2+2)\alpha + 2r_2 - 4n_2 + 1 = 0.$$

Therefore, $\begin{pmatrix} -\frac{\alpha+2r_2+1}{2}\zeta_{i_0} \\ 0 \\ \mathbf{1}_{n_2} \otimes \zeta_{i_0} \end{pmatrix}$ is an eigenvector of $S(G_1^{\mathbb{S}} \odot G_2)$ corresponding to the eigenvalue α, where α's are the roots of $\alpha^2 + (2r_2+2)\alpha + 2r_2 - 4n_2 + 1 = 0$.

If $k_3 = 0$, then $k_1 = 0$, $\alpha = -1$ and $k_2 \in \mathbb{R} \setminus 0$. At this time, it follows that $\begin{pmatrix} 0 \\ \xi_{i_0} \\ 0 \end{pmatrix}$ is an eigenvector of $S(G_1^{\mathbb{S}} \odot G_2)$ corresponding to the eigenvalue -1, whenever $\xi_{i_0} \neq 0$.

Case 2. G_1 is a non-bipartite graph. In this case, $s_i \neq 0$ for $i = 2, 3, \ldots, n_1$. Similar to the Case 1, we discuss (3) for any singular value $s_i \neq 0$. We easily get that the roots of

$$\alpha^3 + (3+2r_2)\alpha^2 + \big(4r_2 + 3 - 4s_i^2 - 4n_2\big)\alpha + 1 + 2r_2 - 4n_2 - 4s_i^2(2r_1+1) = 0 \tag{4}$$

are the eigenvalues of $S(G_1^{\mathbb{S}} \odot G_2)$. By the above equation, it follows that $\begin{pmatrix} -\frac{\alpha+2r_2+1}{2}\zeta_i \\ \frac{(\alpha+2r_1+1)s_i}{\alpha+1}\xi_i \\ \mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}$ is an eigenvector of $S(G_1^{\mathbb{S}} \odot G_2)$ corresponding to the eigenvalue α, where α's are the roots of the above equation (4).

Above all, the roots of $\alpha^3 + (3+2r_2)\alpha^2 + (4r_2 + 3 - 4s_i^2 - 4n_2)\alpha + 1 + 2r_2 - 4n_2 - 4s_i^2(2r_1+1) = 0$ are the eigenvalues of $S(G_1^{\mathbb{S}} \odot G_2)$ corresponding to the eigenvector $\begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ k_3\mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}$ for $i = 2, 3, \ldots, n_1$.

(iv) Let k_1, k_2 and k_3 be three unknown scalars to be determined. By some calculations, we have

$$\begin{aligned} &S\big(G_1^{\mathbb{S}} \odot G_2\big)\begin{pmatrix} k_1\mathbf{1}_{n_1} \\ k_2\mathbf{1}_{m_1} \\ k_3\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix} \\ &= \begin{pmatrix} J_{n_1} - I_{n_1} & J_{n_1\times m_1} - 2M(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes (J_{n_1} - 2I_{n_1}) \\ J_{m_1\times n_1} - 2M^{\mathrm{T}}(G_1) & J_{m_1} - I_{m_1} & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes J_{m_1\times n_1} \\ \mathbf{1}_{n_2} \otimes (J_{n_1} - 2I_{n_1}) & \mathbf{1}_{n_2} \otimes J_{n_1\times m_1} & J_{n_2} \otimes J_{n_1} - (2A(G_2) + I_{n_2}) \otimes I_{n_1} \end{pmatrix}\begin{pmatrix} k_1\mathbf{1}_{n_1} \\ k_2\mathbf{1}_{m_1} \\ k_3\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} k_1 J_{n_1} \mathbf{1}_{n_1} - k_1 \mathbf{1}_{n_1} + k_2 J_{n_1 \times m_1} \mathbf{1}_{m_1} - 2k_2 M(G_1) \mathbf{1}_{m_1} + k_3 n_2 (J_{n_1} - 2I_{n_1}) \mathbf{1}_{n_1} \\ k_1 J_{m_1 \times n_1} \mathbf{1}_{n_1} - 2k_1 M^{\mathrm{T}}(G_1) \mathbf{1}_{n_1} + k_2 J_{m_1} \mathbf{1}_{m_1} - k_2 \mathbf{1}_{m_1} + k_3 n_2 J_{m_1 \times n_1} \mathbf{1}_{n_1} \\ (k_1 + n_2) \mathbf{1}_{n_2} \otimes J_{n_1} \mathbf{1}_{n_1} - 2k_1 \mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} + k_2 \mathbf{1}_{n_2} \otimes J_{n_1 \times m_1} \mathbf{1}_{m_1} - k_3 (2A(G_2) + I_{n_2}) \mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix}$$

$$= \begin{pmatrix} (k_1 n_1 - k_1 + k_2 m_1 - 2k_2 r_1 + k_3 (n_1 n_2 - 2n_2)) \mathbf{1}_{n_1} \\ (k_1 n_1 - 4k_1 + k_2 m_1 - k_2 + k_3 n_1 n_2) \mathbf{1}_{m_1} \\ (k_1 n_1 + k_3 n_1 n_2 - 2k_1 + k_2 m_1 - k_3 (2r_2 + 1)) \mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix}$$

$$= \alpha \begin{pmatrix} k_1 \mathbf{1}_{n_1} \\ k_2 \mathbf{1}_{m_1} \\ k_3 \mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix}.$$

Thus, one gets

$$\begin{cases} k_1(n_1 - 1 - \alpha) + k_2(m_1 - 2r_1) + k_3 n_2 (n_1 - 2) = 0, \\ k_1(n_1 - 4) + k_2(m_1 - 1 - \alpha) + k_3 n_1 n_2 = 0, \\ k_1(n_1 - 2) + k_2 m_1 + k_3(n_1 n_2 - 2r_2 - 1 - \alpha) = 0. \end{cases}$$

Notice that k_1, k_2 and k_3 cannot be zero simultaneously. Now, plugging directly the parameters of G_1 and G_2 into the system of these equations, we may solve α, k_1, k_2 and k_3. Therefore, $\begin{pmatrix} k_1 \mathbf{1}_{n_1} \\ k_2 \mathbf{1}_{m_1} \\ k_3 \mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix}$ is an eigenvector of $S(G_1^{\mathbb{S}} \odot G_2)$ corresponding to the eigenvalue α of $S(G_1^{\mathbb{S}} \odot G_2)$.

□

Remark 3.5. Let G_1 be an r_1-regular graph of order n_1 with m_1 edges. We can find that, if $m_1 - n_1 \geq 0$, then all Seidel eigenvalues of $G_1^{\mathbb{S}} \odot G_2$ are completely determined by Theorem 3.4. However, if $m_1 - n_1 < 0$, then G_1 must be a tree as G_1 is connected. Since G_1 is regular, then $G_1 = K_2$. So, $\xi_1 = \mathbf{1}_1$ and $\xi_2 = \mathbf{0}_1$. At this time, $\begin{pmatrix} 0 \\ \xi_2 \\ 0 \end{pmatrix} = \mathbf{0}_{3+2n_2}$ is not the Seidel eigenvector of $G_1^{\mathbb{S}} \odot G_2$. Hence, the multiplicity of the Seidel eigenvalue $\alpha = -1$ will be reduced by one.

Example 4. Consider two graphs $G_1 = K_4$ and $G_2 = K_3$. We have $\mathrm{Sp}(A(G_1)) = \{3, -1^{(3)}\}$ and $\mathrm{Sp}(A(G_2)) = \{2, -1^{(2)}\}$. Applying Theorem 3.4, the Seidel spectrum of ${K_4}^{\mathbb{S}} \odot K_3$ consists of:

(a) $-1^{(2)}$ are the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \odot K_3$;

(b) For the eigenvalue -1 of G_2 with multiplicity 2, $1^{(8)}$ are the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \odot K_3$;

(c) $3^{(3)}$, $-3^{(3)}$ and $-7^{(3)}$ are the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \odot K_3$. Indeed, noting that $s_2^2 = s_3^2 = s_4^2 = 2$ for $G_1 = K_4$. From the (iii) of Theorem 3.4, we have $(\alpha - 3)(\alpha^2 + 10\alpha + 21) = 0$. Hence, $\alpha = 3, -3, -7$ are the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \odot K_3$;

(d) 15.1114, 3.3087 and -3.4201 (Here the eigenvalues are accurate to four decimal places) are also the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \odot K_3$. Indeed, for $G_1 = K_4, G_2 = K_3$, we see that $n_1 = 4$, $m_1 = 6$, $r_1 = 3$ and $n_2 = 3$, $m_2 = 3$, $r_2 = 2$. Applying the (iv) of Theorem 3.4, we have

$$\begin{cases} k_1(3 - \alpha) + 6k_3 = 0, \\ k_2(5 - \alpha) + 12k_3 = 0, \\ 2k_1 + 6k_2 + (7 - \alpha)k_3 = 0. \end{cases}$$

Notice that k_1, k_2 and k_3 cannot be zero simultaneously. By solving the system of these equations, we get that the roots of $-\alpha^3 + 15\alpha^2 + 13\alpha - 171 = 0$ are the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \odot K_3$.

Above all, $\mathrm{Sp}(S({K_4}^{\mathbb{S}} \odot K_3)) = \{-7^{(3)}, -3.4201, -3^{(3)}, -1^{(2)}, 1^{(8)}, 3^{(3)}, 3.3087, 15.1114\}$. On the other hand, according to the computation of MATLAB, we may get the same results.

Corollary 3.6. *Let G_1 be an r_1-regular graph of order n_1 with m_1 edges, G_2 be an r_2-regular graph of order n_2 with m_2 edges. Then $S(G_1^{\mathbb{S}} \odot G_2)$ is Seidel integral if and only if*

(i) *All adjacency eigenvalues of G_2 are integral;*
(ii) *All the roots of the equation $\alpha^3 + (3+2r_2)\alpha^2 + (4r_2 + 3 - 4s_i^2 - 4n_2)\alpha + 1 + 2r_2 - 4n_2 - 4s_i^2(2r_1+1) = 0$ are integers for $i = 2, 3, \ldots, n_1$;*
(iii) *The roots α's of the following system of equations are integers. Notice that k_1, k_2 and k_3 cannot be zero simultaneously.*

$$\begin{cases} k_1(n_1 - 1 - \alpha) + k_2(m_1 - 2r_1) + k_3 n_2 (n_1 - 2) = 0, \\ k_1(n_1 - 4) + k_2(m_1 - 1 - \alpha) + k_3 n_1 n_2 = 0, \\ k_1(n_1 - 2) + k_2 m_1 + k_3(n_1 n_2 - 2r_2 - 1 - \alpha) = 0. \end{cases}$$

Example 5. Given two graphs $G_1 = K_2$ and $G_2 = K_2$. We have $\mathrm{Sp}(A(G_1)) = \mathrm{Sp}(A(G_2)) = \{1, -1\}$. Applying Theorem 3.4, the Seidel spectrum of ${K_2}^{\mathbb{S}} \odot K_2$ consists of:

(a) For the eigenvalue -1 of G_2, $1^{(2)}$ are the Seidel eigenvalues of ${K_2}^{\mathbb{S}} \odot K_2$;
(b) 1 and -5 are the Seidel eigenvalues of ${K_2}^{\mathbb{S}} \odot K_2$. Indeed, noting that $s_2 = 0$ for $G_1 = K_2$. From the (iii) of Theorem 3.4, we have $(\alpha+1)(\alpha^2 + 4\alpha - 5) = 0$. But, it follows from Remark 3.5 that the multiplicity of $\alpha = -1$ is reduced by one. So -1 is not Seidel eigenvalue of ${K_2}^{\mathbb{S}} \odot K_2$. Hence, $\alpha^2 + 4\alpha - 5 = 0$, which implies that 1 and -5 are the Seidel eigenvalues of ${K_2}^{\mathbb{S}} \odot K_2$.
(c) 1, -2 and 3 are also the Seidel eigenvalues of ${K_2}^{\mathbb{S}} \odot K_2$. Indeed, for $G_1 = G_2 = K_2$, we know that $n_1 = 2$, $m_1 = 1$, $r_1 = 1$ and $n_2 = 2$, $m_2 = 1$, $r_2 = 1$. Applying the (iv) of Theorem 3.4, we have

$$\begin{cases} k_1(1-\alpha) - k_2 = 0, \\ -2k_1 - \alpha k_2 + 4k_3 = 0, \\ k_2 + (1-\alpha)k_3 = 0. \end{cases}$$

Notice that k_1, k_2 and k_3 cannot be zero simultaneously. By solving the system of these equations, we have

$$\begin{cases} \alpha = 1, \\ k_1 = 2k_3, \\ k_2 = 0, \\ k_3 \neq 0, \end{cases} \begin{cases} \alpha = -2, \\ k_1 = -k_3, \\ k_2 \neq 0, \\ k_2 = -3k_3, \end{cases} \begin{cases} \alpha = 3, \\ k_1 = -k_3, \\ k_2 \neq 0, \\ k_2 = 2k_3. \end{cases}$$

Above all, $\mathrm{Sp}(S({K_2}^{\mathbb{S}} \odot K_2)) = \{-5, -2, 1^{(4)}, 3\}$. On the other hand, according to the computation of MATLAB, we may get the same results. Clearly, ${K_2}^{\mathbb{S}} \odot K_2$ is also Seidel integral.

3.3. Seidel Spectrum of the subdivision vertex neighbourhood corona

In this section, we shall consider to determine the Seidel spectra of ${G_1}^{\mathbb{S}} \boxdot G_2$. Let $G_1 = (V(G_1), E(G_1))$ be an r_1-regular graph of order n_1 with m_1 edges, where the vertex set $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and the edge set $E(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$. Also let $G_2 = (V(G_2), E(G_2))$ be an r_2-regular graph of order n_2 with m_2 edges, where $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. Let $u_1^i, u_2^i, \ldots, u_{n_2}^i$ denote the vertices of the i-th copy of G_2, for $i = 1, 2, \ldots, n_1$. Denote $P_j = \{u_j^1, u_j^2, \ldots, u_j^{n_1}\}$ for $j = 1, 2, \ldots, n_2$. We also denote the set of inserted new vertices of $S(G_1)$ by $I(G_1)$. Then the vertex set of $G_1^{\mathbb{S}} \boxdot G_2$ has a partition as below: $V(G_1) \cup I(G_1) \cup P_1 \cup P_2 \cup \cdots \cup P_{n_2}$. Thus the Seidel matrix of ${G_1}^{\mathbb{S}} \boxdot G_2$ can be written as follows:

$$S\Big({G_1}^{\mathbb{S}} \boxdot G_2\Big) = \begin{pmatrix} J_{n_1} - I_{n_1} & J_{n_1 \times m_1} - 2M(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes J_{n_1} \\ J_{m_1 \times n_1} - 2M^{\mathrm{T}}(G_1) & J_{m_1} - I_{m_1} & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes \big(J_{m_1 \times n_1} - 2M^{\mathrm{T}}(G_1)\big) \\ \mathbf{1}_{n_2} \otimes J_{n_1} & \mathbf{1}_{n_2} \otimes (J_{n_1 \times m_1} - 2M(G_1)) & J_{n_2} \otimes J_{n_1} - (I_{n_2} + 2A(G_2)) \otimes I_{n_1} \end{pmatrix}.$$

Theorem 3.7. *Let G_1 and G_2 be two graphs as described above. Also let $Sp(A(G_1)) = \{\lambda_1 = r_1, \lambda_2, \ldots, \lambda_{n_1}\}$, $Sp(A(G_2)) = \{\mu_1 = r_2, \mu_2, \ldots, \mu_{n_2}\}$ and the eigenvalue μ_j corresponds to the orthogonal eigenvector Y_j for $j = 1, 2, \ldots, n_2$. The Seidel spectrum of ${G_1}^{\mathbb{S}} \boxdot G_2$ consists of the following:*

(i) *-1 is the eigenvalue of $S({G_1}^{\mathbb{S}} \boxdot G_2)$, with multiplicity $m_1 - n_1$ (if possible);*
(ii) *$-2\mu_j - 1$ is the eigenvalue of $S({G_1}^{\mathbb{S}} \boxdot G_2)$ with multiplicity n_1, for $j = 2, 3, \ldots, n_2$;*
(iii) *The roots of polynomial $f(\alpha) = \alpha^3 + (3+2r_2)\alpha^2 + (3+4r_2-4n_2s_i^2-4s_i^2)\alpha + (1+2r_2-4n_2s_i^2-4s_i^2-8r_2s_i^2)$ are the eigenvalues of $S({G_1}^{\mathbb{S}} \boxdot G_2)$ for $i = 2, 3, \ldots, n_1$;*
(iv) *All the remaining Seidel eigenvalues satisfy the following system of equations (noting that k_1, k_2 and k_3 cannot be zero simultaneously):*

$$\begin{cases} k_1(n_1 - 1 - \alpha) + k_2(m_1 - 2r_1) + k_3 n_1 n_2 = 0, \\ k_1(n_1 - 4) + k_2(m_1 - 1 - \alpha) + k_3(n_2 n_1 - 4n_2) = 0, \\ k_1 n_1 + k_2(m_1 - 2r_1) + k_3(n_2 n_1 - 1 - 2r_2 - \alpha) = 0. \end{cases}$$

Proof. (i) Let $\{Y_j\}_{j=1}^{n_2}$, $\{\zeta_i\}_{i=1}^{n_1}$, $\{\xi_i\}_{i=1}^{n_1}$, $\{\eta_j\}_{j=1}^{m_1-n_1}$ and $c_{n_1}^i$ be described as these in the proof of Theorem 3.4. Since $\mathbf{1}_{n_1}^{\mathrm{T}}\eta_j = 0$ for $j = 1, \ldots, m_1 - n_1$, then

$$\begin{aligned} &S\Big({G_1}^{\mathbb{S}} \boxdot G_2\Big)\begin{pmatrix} 0 \\ \eta_j \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} J_{n_1} - I_{n_1} & J_{n_1\times m_1} - 2M(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes J_{n_1} \\ J_{m_1\times n_1} - 2M^{\mathrm{T}}(G_1) & J_{m_1} - I_{m_1} & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes \big(J_{m_1\times n_1} - 2M^{\mathrm{T}}(G_1)\big) \\ \mathbf{1}_{n_2} \otimes J_{n_1} & \mathbf{1}_{n_2} \otimes (J_{n_1\times m_1} - 2M(G_1)) & J_{n_2} \otimes J_{n_1} - (I_{n_2} + 2A(G_2)) \otimes I_{n_1} \end{pmatrix}\begin{pmatrix} 0 \\ \eta_j \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (J_{n_1\times m_1} - 2M(G_1))\eta_j \\ (J_{m_1} - I_{m_1})\eta_j \\ (\mathbf{1}_{n_2} \otimes (J_{n_1\times m_1} - 2M(G_1)))\eta_j \end{pmatrix} = -\begin{pmatrix} 0 \\ \eta_j \\ 0 \end{pmatrix}. \end{aligned}$$

Thus, -1 is the eigenvalue of $S({G_1}^{\mathbb{S}} \boxdot G_2)$ with multiplicity $m_1 - n_1$, corresponding to the eigenvector $\begin{pmatrix} 0 \\ \eta_j \\ 0 \end{pmatrix}$.

(ii) Observe that $Y_1 = \mathbf{1}_{n_2}$ and $Y_1^{\mathrm{T}} Y_j = 0$ for $j = 2, 3, \ldots, n_2$. This implies that $J_{n_2} Y_j = 0$. Now, for $j = 2, 3, \ldots, n_2$,

$$\begin{aligned} &S\Big({G_1}^{\mathbb{S}} \boxdot G_2\Big)\begin{pmatrix} 0 \\ 0 \\ Y_j \otimes c_{n_1}^i \end{pmatrix} \\ &= \begin{pmatrix} J_{n_1} - I_{n_1} & J_{n_1\times m_1} - 2M(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes J_{n_1} \\ J_{m_1\times n_1} - 2M^{\mathrm{T}}(G_1) & J_{m_1} - I_{m_1} & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes (J_{m_1\times n_1} - 2M^{\mathrm{T}}(G_1)) \\ \mathbf{1}_{n_2} \otimes J_{n_1} & \mathbf{1}_{n_2} \otimes (J_{n_1\times m_1} - 2M(G_1)) & J_{n_2} \otimes J_{n_1} - (I_{n_2} + 2A(G_2)) \otimes I_{n_1} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ Y_j \otimes c_{n_1}^i \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1}_{n_2}^{\mathrm{T}} Y_j \otimes J_{n_1} c_{n_1}^i \\ \mathbf{1}_{n_2}^{\mathrm{T}} Y_j \otimes \big(J - 2M^{\mathrm{T}}(G_1)\big) c_{n_1}^i \\ J_{n_2} Y_j \otimes J_{n_1} c_{n_1}^i - (I + 2A(G_2)) Y_j \otimes I_{n_1} c_{n_1}^i \end{pmatrix} \\ &= -(1 + 2\mu_j)\begin{pmatrix} 0 \\ 0 \\ Y_j \otimes c_{n_1}^i \end{pmatrix}. \end{aligned}$$

Hence, $-2\mu_j - 1$ is the eigenvalue of $S(G_1{}^{\mathbb{S}} \boxdot G_2)$ with multiplicity n_1, corresponding to the eigenvector $\begin{pmatrix} 0 \\ 0 \\ Y_j \otimes c_{n_1}^i \end{pmatrix}$ for $j = 2, 3, \ldots, n_2$.

(iii) As we all know $\mathbf{1}_{n_1}^{\mathrm{T}} \zeta_i = 0$ and $\mathbf{1}_{m_1}^{\mathrm{T}} \xi_i = 0$, then $J_{n_1}\zeta_i = 0$, $J_{m_1}\xi_i = 0$. Let k_1, k_2 and k_3 be unknown scalars to be determined. Then we have, for $i = 2, 3, \ldots, n_1$,

$$
\begin{aligned}
&S\Big(G_1{}^{\mathbb{S}} \boxdot G_2\Big) \begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ k_3\mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix} \\
&= \begin{pmatrix} J_{n_1} - I_{n_1} & J_{n_1 \times m_1} - 2M(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes J_{n_1} \\ J_{m_1 \times n_1} - 2M^{\mathrm{T}}(G_1) & J_{m_1} - I_{m_1} & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes \big(J_{m_1 \times n_1} - 2M^{\mathrm{T}}(G_1)\big) \\ \mathbf{1}_{n_2} \otimes J_{n_1} & \mathbf{1}_{n_2} \otimes (J_{n_1 \times m_1} - 2M(G_1)) & J_{n_2} \otimes J_{n_1} - (I_{n_2} + 2A(G_2)) \otimes I_{n_1} \end{pmatrix} \begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ k_3\mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix} \\
&= \begin{pmatrix} (J_{n_1} - I_{n_1})k_1\zeta_i + (J_{n_1 \times m_1} - 2M(G_1))k_2\xi_i + k_3\mathbf{1}_{n_2}^{\mathrm{T}}\mathbf{1}_{n_2} \otimes J_{n_1}\zeta_i \\ \big(J_{m_1 \times n_1} - 2M^{\mathrm{T}}(G_1)\big)k_1\zeta_i + (J_{m_1} - I_{m_1})k_2\xi_i + k_3\mathbf{1}_{n_2}^{\mathrm{T}}\mathbf{1}_{n_2} \otimes \big(J_{m_1 \times n_1} - 2M^{\mathrm{T}}(G_1)\big)\zeta_i \\ \mathbf{1}_{n_2} \otimes J_{n_1}k_1\zeta_i + \mathbf{1}_{n_2} \otimes (J_{n_1 \times m_1} - 2M(G_1))k_2\xi_i + k_3J_{n_2}\mathbf{1}_{n_2} \otimes J_{n_1}\zeta_i - k_3(I_{n_2} + 2A(G_2))\mathbf{1}_{n_2} \otimes I_{n_1}\zeta_i \end{pmatrix} \\
&= \begin{pmatrix} -k_1\zeta_i - 2k_2s_i\zeta_i \\ -2k_1s_i\xi_i - k_2\xi_i - 2k_3s_in_2\xi_i \\ -2k_2s_i\mathbf{1}_{n_2} \otimes \zeta_i - k_3(\mathbf{1} + 2r_2)1_{n_2} \otimes \zeta_i \end{pmatrix} \\
&= \alpha \begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ k_3\mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}.
\end{aligned}
$$

From this equation, we find

$$
\begin{cases} -k_1 - 2k_2s_i = \alpha k_1, \\ -2k_1s_i - k_2 - 2k_3s_in_2 = \alpha k_2, \\ -2k_2s_i - k_3 - 2k_3r_2 = \alpha k_3. \end{cases} \tag{5}
$$

Now we consider the following two cases:

Case 1. G_1 is a bipartite graph. In this case, $-r_1$ is also the eigenvalue of $A(G_1)$. Noting that G_1 is connected. It follows from (1) that there exists a unique index $i_0 \in \{2, 3, \ldots, n_1\}$ such that $s_{i_0} = 0$. Notice that k_1, k_2 and k_3 cannot be zero simultaneously. Now we discuss (5) for $s_{i_0} = 0$. In this time, the equation (5) becomes

$$
\begin{cases} (\alpha + 1)k_1 = 0, \\ (\alpha + 1)k_2 = 0, \\ (\alpha + 1 + 2r_2)k_3 = 0. \end{cases}
$$

If $\alpha = -1$, then $k_3 = 0$, k_1 and k_2 cannot be zero simultaneously. Otherwise, $\alpha \neq -1$, then $k_1 = k_2 = 0$ and $k_3 \neq 0$, which implies that $\alpha = -2r_2 - 1$. It follows that $\begin{pmatrix} \zeta_{i_0} \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \xi_{i_0} \\ 0 \end{pmatrix}$ are the eigenvectors of $S(G_1^{\mathbb{S}} \odot G_2)$ corresponding to the eigenvalue -1. $\begin{pmatrix} 0 \\ 0 \\ \mathbf{1}_{n_2} \otimes \zeta_{i_0} \end{pmatrix}$ is the eigenvector of $S(G_1^{\mathbb{S}} \odot G_2)$ corresponding to the eigenvalue $-2r_2 - 1$.

In what following, we consider any nonzero singular value s_i for $i \in \{2, 3, \ldots, n_1\} \setminus \{i_0\}$. At this time, we have $k_1 \neq 0$, $k_2 \neq 0$ and $k_3 \neq 0$. Without loss of generality, set $k_3 = 1$. From (5), we obtain

$$
k_1 = -n_2 + \frac{(\alpha + 1)(1 + 2r_2 + \alpha)}{4s_i^2}, \quad k_2 = -\frac{1 + 2r_2 + \alpha}{2s_i},
$$

and

$$\alpha^3 + (3 + 2r_2)\alpha^2 + \big(3 + 4r_2 - 4n_2 s_i^2 - 4s_i^2\big)\alpha + \big(1 + 2r_2 - 4n_2 s_i^2 - 4s_i^2 - 8r_2 s_i^2\big) = 0. \tag{6}$$

Hence, every root of $\alpha^3 + (3 + 2r_2)\alpha^2 + (3 + 4r_2 - 4n_2 s_i^2 - 4s_i^2)\alpha + (1 + 2r_2 - 4n_2 s_i^2 - 4s_i^2 - 8r_2 s_i^2) = 0$ is an eigenvalue of $S({G_1}^{\mathbb{S}} \boxdot G_2)$, corresponding to the eigenvector $\begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ \mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}$.

Case 2. G_1 is a non-bipartite graph. In this case, $s_i \neq 0$ for $i = 2, 3, \ldots, n_1$. Similar to Case 1, every root of $\alpha^3 + (3 + 2r_2)\alpha^2 + (3 + 4r_2 - 4n_2 s_i^2 - 4s_i^2)\alpha + (1 + 2r_2 - 4n_2 s_i^2 - 4s_i^2 - 8r_2 s_i^2) = 0$ is an eigenvalue of $S({G_1}^{\mathbb{S}} \boxdot G_2)$, corresponding to the eigenvector $\begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ \mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}$.

(iv) Let k_1, k_2 and k_3 be unknown scalars to be determined. By a tedious calculation, we have

$$
\begin{aligned}
&S\Big({G_1}^{\mathbb{S}} \boxdot G_2\Big)\begin{pmatrix} k_1\mathbf{1}_{n_1} \\ k_2\mathbf{1}_{m_1} \\ k_3\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix} \\
&= \begin{pmatrix} J - I_{n_1} & J - 2M(G_1) & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes J_{n_1} \\ J - 2M^{\mathrm{T}}(G_1) & J - I_{m_1} & \mathbf{1}_{n_2}^{\mathrm{T}} \otimes \big(J - 2M^{\mathrm{T}}(G_1)\big) \\ \mathbf{1}_{n_2} \otimes J_{n_1} & \mathbf{1}_{n_2} \otimes (J - 2M(G_1)) & J_{n_2} \otimes J_{n_1} - (I + 2A(G_2)) \otimes I_{n_1} \end{pmatrix}\begin{pmatrix} k_1\mathbf{1}_{n_1} \\ k_2\mathbf{1}_{m_1} \\ k_3\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix} \\
&= \begin{pmatrix} (J - I_{n_1})k_1\mathbf{1}_{n_1} + (J - 2M(G_1))k_2\mathbf{1}_{m_1} + k_3\mathbf{1}_{n_2}^{\mathrm{T}}\mathbf{1}_{n_2} \otimes J_{n_1}\mathbf{1}_{n_1} \\ \big(J - 2M^{\mathrm{T}}(G_1)\big)k_1\mathbf{1}_{n_1} + (J - I_{m_1})k_2\mathbf{1}_{m_1} + k_3\mathbf{1}_{n_2}^{\mathrm{T}}\mathbf{1}_{n_2} \otimes \big(J - 2M^{\mathrm{T}}(G_1)\big)\mathbf{1}_{n_1} \\ \mathbf{1}_{n_2} \otimes J_{n_1}k_1\mathbf{1}_{n_1} + \mathbf{1}_{n_2} \otimes (J - 2M(G_1))k_2\mathbf{1}_{m_1} + k_3J_{n_2}\mathbf{1}_{n_2} \otimes J_{n_1}\mathbf{1}_{n_1} - k_3(I + 2A(G_2))\mathbf{1}_{n_2} \otimes I_{n_1}\mathbf{1}_{n_1} \end{pmatrix} \\
&= \begin{pmatrix} k_1(n_1 - 1)\mathbf{1}_{n_1} + k_2(m_1 - 2r_1)\mathbf{1}_{n_1} + k_3n_1n_2\mathbf{1}_{n_1} \\ k_1(n_1 - 4)\mathbf{1}_{m_1} + k_2(m_1 - 1)\mathbf{1}_{m_1} + k_3n_2(n_1 - 4)\mathbf{1}_{m_1} \\ k_1n_1\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} + k_2(m_1 - 2r_1)\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} + k_3n_2n_1\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} - k_3(\mathbf{1} + 2r_2)\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix} \\
&= \begin{pmatrix} (k_1n_1 - k_1 + k_2m_1 - 2k_2r_1 + k_3n_1n_2)\mathbf{1}_{n_1} \\ (k_1n_1 - 4k_1 + k_2m_1 - k_2 + k_3n_2n_1 - k_34n_2)\mathbf{1}_{m_1} \\ (k_1n_1 + k_2m_1 - 2k_2r_1 + k_3(n_2n_1 - 1 - 2r_2))\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix} \\
&= \alpha\begin{pmatrix} k_1\mathbf{1}_{n_1} \\ k_2\mathbf{1}_{m_1} \\ k_3\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix}.
\end{aligned}
$$

This equation means that

$$\begin{cases} k_1(n_1 - 1 - \alpha) + k_2(m_1 - 2r_1) + k_3n_1n_2 = 0, \\ k_1(n_1 - 4) + k_2(m_1 - 1 - \alpha) + k_3(n_2n_1 - 4n_2) = 0, \\ k_1n_1 + k_2(m_1 - 2r_1) + k_3(n_2n_1 - 1 - 2r_2 - \alpha) = 0. \end{cases}$$

Now we can directly substitute the parameters of G_1 and G_2 into the system of the equations to solve α, k_1, k_2 and k_3. Therefore, every root α of the above system is the eigenvalue of $S({G_1}^{\mathbb{S}} \boxdot G_2)$, corresponding to the eigenvector $\begin{pmatrix} k_1\mathbf{1}_{n_1} \\ k_2\mathbf{1}_{m_1} \\ k_3\mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1} \end{pmatrix}$.

□

Remark 3.8. Let G_1 be an r_1-regular graph of order n_1 with m_1 edges. Then, similar exactly to the discussion of Remark 3.5, we can find that, if $m_1 - n_1 \geq 0$, then all Seidel eigenvalues of ${G_1}^{\mathbb{S}} \boxdot G_2$ are completely determined by Theorem 3.7. But, if $m_1 - n_1 < 0$, then the multiplicity of the Seidel eigenvalue $\alpha = -1$ will be reduced by one.

Example 6. Consider two graphs $G_1 = K_4$ and $G_2 = K_3$. We have $\mathrm{Sp}(A(G_1)) = \{3, -1^{(3)}\}$ and $\mathrm{Sp}(A(G_2)) = \{2, -1^{(2)}\}$. Applying Theorem 3.7, the Seidel spectrum of ${K_4}^{\mathbb{S}} \boxdot K_3$ consists of:

(a) $-1^{(2)}$ are the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \boxdot K_3$;
(b) For the eigenvalue -1 of G_2 with multiplicity 2, $1^{(8)}$ are the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \boxdot K_3$;
(c) $3.5589^{(3)}$, $-1.9187^{(3)}$ and $-8.6402^{(3)}$ are the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \boxdot K_3$. Indeed, noting that $s_2^2 = s_3^2 = s_4^2 = 2$ for $G_1 = K_4$. From the (iii) of Theorem 3.7, we have $\alpha^3 + 7\alpha^2 - 21\alpha - 59 = 0$. Hence, the values of α are the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \boxdot K_3$.
(d) $5 + 2\sqrt{13}$, 5 and $5 - 2\sqrt{13}$ are also the Seidel eigenvalues of ${K_4}^{\mathbb{S}} \boxdot K_3$. Indeed, for $G_1 = K_4, G_2 = K_3$, we all know that $n_1 = 4$, $m_1 = 6$, $r_1 = 3$ and $n_2 = 3$, $m_2 = 3$, $r_2 = 2$. Applying the (iv) of Theorem 3.7, we have

$$\begin{cases} k_1(3-\alpha) + 12k_3 = 0, \\ k_2(5-\alpha) = 0, \\ 4k_1 + (7-\alpha)k_3 = 0. \end{cases}$$

Notice that k_1, k_2 and k_3 cannot be zero simultaneously. By solving the system of this equations, we have

$$\begin{cases} \alpha = 5, \\ k_2 \neq 0, \\ k_1 = 0, \\ k_3 = 0, \end{cases} \qquad \begin{cases} \alpha = 5 \pm 2\sqrt{13}, \\ k_2 = 0, \\ (\alpha - 3)k_1 = 12k_3. \end{cases}$$

Above all, $\mathrm{Sp}(S({K_4}^{\mathbb{S}} \boxdot K_3)) = \{-8.6402^{(3)}, -1.9187^{(3)}, 5 - 2\sqrt{13}, -1^{(2)}, 1^{(8)}, 3.5589^{(3)}, 5, 5 + 2\sqrt{13}\}$. On the other hand, according to the computation of MATLAB, we may get the same results.

Corollary 3.9. *Let G_1 be an r_1-regular graph of order n_1 with m_1 edges, and G_2 be an r_2-regular graph of order n_2 with m_2 edges. Then, ${G_1}^{\mathbb{S}} \boxdot G_2$ is Seidel integral if and only if*

(i) *The adjacency spectrum of G_2 is integral;*
(ii) *For $i \in \{2, 3, \dots, n_1\}$. All the roots of $\alpha^3 + (3 + 2r_2)\alpha^2 + (3 + 4r_2 - 4n_2 s_i^2 - 4s_i^2)\alpha + (1 + 2r_2 - 4n_2 s_i^2 - 4s_i^2 - 8r_2 s_i^2) = 0$ are integers;*
(iii) *We can directly substitute the parameters of G_1 and G_2 into the following system of equations to get α. At that time, all α's are integers.*

$$\begin{cases} k_1(n_1 - 1 - \alpha) + k_2(m_1 - 2r_1) + k_3 n_1 n_2 = 0, \\ k_1(n_1 - 4) + k_2(m_1 - 1 - \alpha) + k_3(n_2 n_1 - 4n_2) = 0, \\ k_1 n_1 + k_2(m_1 - 2r_1) + k_3(n_2 n_1 - 1 - 2r_2 - \alpha) = 0. \end{cases}$$

Example 7. Let us consider two graphs $G_1 = K_2$ and $G_2 = K_2$. We have $\mathrm{Sp}(A(G_1)) = \mathrm{Sp}(A(G_2)) = \{1, -1\}$. Applying Theorem 3.7, the Seidel spectrum of ${K_2}^{\mathbb{S}} \boxdot K_2$ consists of:

(i) For the eigenvalue -1 of G_2, $1^{(2)}$ are the Seidel eigenvalues of ${K_2}^{\mathbb{S}} \boxdot K_2$;
(ii) -1 and -3 are the Seidel eigenvalues of ${K_2}^{\mathbb{S}} \boxdot K_2$. Indeed, noting that $s_2 = 0$ for $G_1 = K_2$. From the (iii) of Theorem 3.7, we have $(\alpha + 1)^2(\alpha + 3) = 0$. But, it follows from Remark 3.8 that the multiplicity of $\alpha = -1$ is reduced by one. This implies that -1 and -3 are the Seidel eigenvalues of ${K_2}^{\mathbb{S}} \boxdot K_2$.
(iii) 5, -2 and -1 are also the Seidel eigenvalues of ${K_2}^{\mathbb{S}} \boxdot K_2$. Indeed, applying the (iv) of Theorem 3.7, we have

$$\begin{cases} (1-\alpha)k_1 - k_2 + 4k_3 = 0, \\ -2k_1 - \alpha k_2 - 4k_3 = 0, \\ 2k_1 - k_2 + (1-\alpha)k_3 = 0. \end{cases}$$

Notice that k_1, k_2 and k_3 cannot be zero simultaneously. By solving the system of these equations, we have

$$\begin{cases} \alpha = 5, \\ k_1 = \frac{4}{3}k_3, \\ k_2 = -\frac{4}{3}k_3, \\ k_3 \neq 0, \end{cases} \qquad \begin{cases} \alpha = -2, \\ k_1 = -k_3, \\ k_2 = k_3, \\ k_3 \neq 0, \end{cases} \qquad \begin{cases} \alpha = -1, \\ k_1 \neq 0, \\ k_2 = 2k_1, \\ k_3 = 0. \end{cases}$$

So 5, -2 and -1 are the Seidel eigenvalues of ${K_2}^{\mathbb{S}} \boxdot K_2$.

Above all, we get that the Seidel spectrum of ${K_2}^{\mathbb{S}} \boxdot K_2$ is $\{-3, -2, -1^{(2)}, 1^{(2)}, 5\}$. Therefore, ${K_2}^{\mathbb{S}} \odot K_2$ is Seidel integral.

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