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GRAPHS WITH UNIQUE MINIMUM VERTEX-EDGE DOMINATING SETS

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Abstract. A vertex u of a graph $G = (V, E)$, *ve*-dominates every edge incident to u, as well as every edge adjacent to these incident edges. A set $S \subseteq V$ is a vertex-edge dominating set (or a *ved-set* for short) if every edge of E is *ve*-dominated by at least one vertex of S. The vertex-edge domination number is the minimum cardinality of a *ved-set* in G. In this paper, we investigate the graphs having unique minimum *ved-sets* that we will call UVED-graphs. We start by giving some basic properties of UVED-graphs. For the class of trees, we establish two equivalent conditions characterizing UVED-trees which we subsequently complete by providing a constructive characterization.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The *order* of a graph G denoted by n is the number of its vertices $|V|$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v of G is $d_G(v) = |N_G(v)|$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. If P is a path in G with endvertices x and y such that x is a leaf and y is a vertex of degree two in G and is the only vertex of P adjacent to some vertex z of $V(G) - V(P)$, then z is said to be adjacent to P. A *trivial graph* is a graph containing only one vertex. A *star* of order $n \geq 2$, denoted $K_{1,n-1}$, is a tree with one distinguished vertex of degree $n - 1$, and the remaining vertices are all of degree one and are adjacent only to the distinguished vertex. A *subdivided star* is obtained from a star $K_{1,p}$ $(p \geq 2)$ by subdividing each edge by exactly one vertex. A *path* joining two vertices x and y is called a (x, y)-path. The *distance* $d_G(x, y)$ between two vertices x and y in a connected graph G is the number of edges in a shortest (x, y)-path. The *diameter* of a connected graph G, denoted diam(G), is the maximum distance between two vertices. A *diametral path* of a graph G is a shortest path whose length is equal to diam(G). In Section 4, we consider *rooted trees* distinguished by one vertex r called the *root*. For a vertex $v \neq r$ in a rooted tree T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $w \neq v$ such that the unique (r, w)-path contains v. The *maximal subtree* at v denoted by T_v is the subtree of T induced by v and all its descendants. The *depth* of v is

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the largest distance from v to a descendant of v. If $G = (V, E)$ and $v \in V$, then the subgraph obtained from G by removing v and all edges incident with v is denoted by $G - v$. Also, if $e \in E$, then the spanning graph of G resulting from the removal of e will be denoted by $G - e$.

A vertex $u \in V$ is said to *ve-dominate* an edge $vw \in E$ if: (i) $u = v$ or $u = w$, that is, u is incident to vw or (ii) uv or uw is an edge in G, that is u is incident to an edge that is adjacent to vw. In other words, a vertex u *ve*-dominates the edges incident to vertices in $N[u]$. A set $S \subseteq V$ is a *vertex-edge dominating set* (or simply a *ved-set*) if for all edges $e \in E$, there exists a vertex $v \in S$ such that v *ve*-dominates e. The property for a subset of V to be *ved-set* is super-hereditary, that is, every superset of a *ve*-dominating set is a *ve*-dominating set. A *ved-set* S is minimal if, for every vertex $v \in S$, $S - \{v\}$ is not *ved-set* in G. The minimum cardinality of a *ved-set* of G is called the *vertex-edge domination number* $\gamma_{ve}(G)$. A *ved-set* of G with cardinality $\gamma_{ve}(G)$ is called a $\gamma_{ve}(G)$-set. It is noteworthy that if D is a $\gamma_{ve}(G)$-set, then the set $V(G) - N[D]$ is independent, that is no two vertices of $V(G) - N[D]$ are adjacent. Also, a *ve*-dominating set of a graph G need not to contain vertices of degree 0 nor to *ve*-dominate them, and therefore the graphs considered in this paper are assumed to be without isolated vertices. The concept of vertex-edge domination was introduced by Peters [13], and further studied in [2, 10–12, 15].

Our goal in this paper is to investigate the graphs having unique minimum *ved-sets* and call such a graph a UVED-graph. We start by giving some basic properties of UVED-graphs. Restricted to the class of trees, we establish two equivalent conditions that characterize UVED-trees. Moreover, a constructive characterization of UVED-trees is also provided. It is worth noting that Gunther *et al.* [8] were the first to study such graphs with respect to the domination number and who also gave a characterization of the trees having unique minimum dominating sets. Subsequently, the graphs having a unique set for some domination parameters have been studied for several classes of graphs (for example, see [1, 3–7, 9, 14]).

2. Preliminary results

Before presenting the main results of this section, we give some additional definitions. A vertex $v \in S \subseteq V$ has a *private edge* $e = uw \in E$ (with respect to a set S), if: (i) v is incident to e or v is adjacent to either u or w, and (ii) for all vertices $x \in S - \{v\}$, x is not incident to e and x is not adjacent to either u or w. In other words, v *ve*-dominates the edge e and no other vertex in S *ve*-dominates e. Let $\mathrm{pe}(v, S)$ denote the set of private edges of v with respect to S.

The following result on the minimality of *ved-sets* was given in [2].

Proposition 1 ([2])**.** *Let S be a ved-set of a nontrivial connected graph G. Then S is a minimal ved-set if and only if every vertex $v \in S$ has at least one private edge with respect to S.*

Since every $\gamma_{ve}(G)$-set S is minimal, by Proposition 1, every vertex $v \in S$ has at least one private edge. But such a private edge is incident with either v or a vertex adjacent to v. Consequently, as defined in [2], a vertex v in a $\gamma_{ve}(G)$-set is said to be of *Type*-1 *vertex* if all its private edges are incident with v, and is of *Type*-2 *vertex* for otherwise. Therefore, we can consider $\mathrm{pe}_1(v, S)$ as the set of all private edges of v with respect to S that are incident with v and let $\mathrm{pe}_2(v, S) = \mathrm{pe}\,(v, S) - \mathrm{pe}_1(v, S)$. Moreover, for every vertex $v \in S$, let $A_S(v)$ be the set of vertices in $N(v)$ that are incident with edges in $\mathrm{pe}_2(v, S)$. Clearly a vertex v in a $\gamma_{ve}(G)$-set S is of type-2 if and only if $A_S(v) \neq \emptyset$.

The first result we establish gives a necessary condition for UVED-graphs.

Proposition 2. *If G is a graph with a unique $\gamma_{ve}(G)$-set D, then for every vertex $v \in D$, $|A_D(v)| \geq 2$.*

Proof. Assume for a contradiction that D contains a type-1 vertex v. Since D is minimal and v is of type-1, all private edges of v with respect to D are incident with v. Hence all edges (if any) incident with $N(v)$ but $\mathrm{pe}_1(v, D)$ are *ve*-dominated by $D - \{v\}$. Therefore for any vertex $x \in N(v)$, the set $\{x\} \cup D - \{v\}$ is a second $\gamma_{ve}(G)$-set, a contradiction. Hence no vertex of D is of type-1, that is for every vertex $v \in D$, $|A_D(v)| \geq 1$. Now if $|A_D(v)| = 1$ for some vertex $v \in D$, then $A_D(v) \cup D - \{v\}$ is a second $\gamma_{ve}(G)$-set, a contradiction, which completes the proof. □

Note that the converse of Proposition 2 is not true as it can be seen by considering the complete bipartite graph $K_{2,3}$ with partite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$. Then each $\{y_i\}$ is a $\gamma_{ve}(G)$-set with $A_{\{y_i\}}(y_i) = \{x_1, x_2\}$, and thus each $\{y_i\}$ satisfies the property of Proposition 2, however G does not have a single $\gamma_{ve}(G)$-set but three.

Proposition 3. *Let D be a $\gamma_{ve}(G)$-set of a graph G. If $\gamma_{ve}(G - v) > \gamma_{ve}(G)$ for every vertex $v \in D$, then D is the unique $\gamma_{ve}(G)$-set.*

Proof. Assume that D is not the unique $\gamma_{ve}(G)$-set, and let D' be a $\gamma_{ve}(G)$-set different from D. Let x be a vertex of $D - D'$. Since D' *ve*-dominates G, it also *ve*-dominates $G - x$. Therefore, $|D'| \geq \gamma_{ve}(G - x)$ and since $\gamma_{ve}(G - x) > \gamma_{ve}(G)$, we obtain that $|D'| > \gamma_{ve}(G)$, which leads to a contradiction. □

We note that the converse of Proposition 3 is not true as it can be seen by the graph G obtained from a cycle C_6 whose vertices are labeled in order $x_1, x_2, \ldots, x_6$ by adding a new vertex y and edges yx_1, yx_4, x_4x_2 and x_4x_6. One can easily see that G has a unique $\gamma_{ve}(G)$-set $D = \{x_4\}$ while x_1 *ve*-dominates all edges in $G - x_4$.

In the following we will see the effect of the deletion of any vertex on the *ve*-domination number of a graph G having a unique $\gamma_{ve}(G)$-set.

Proposition 4. *Let G be a graph with a unique $\gamma_{ve}(G)$-set D. Then for every $x \notin D$, $\gamma_{ve}(G - x) = \gamma_{ve}(G)$.*

Proof. We first note that removing any vertex $x \notin D$ also leads to the removal of all edges incident with x, and thus D remains a *ved-set* of $G - x$. Therefore, $\gamma_{ve}(G - x) \leq \gamma_{ve}(G)$ for every $x \notin D$. To get equality as desired, let us assume that $\gamma_{ve}(G - x) < \gamma_{ve}(G)$ for some vertex $x \notin D$. Hence all edges in $G - x$ are *ve*-dominated by a set of vertices D' with $|D'| < \gamma_{ve}(G)$. But then $D' \cup \{x\}$ is a $\gamma_{ve}(G)$-set different from D which contradicts the uniqueness of D. Consequently, $\gamma_{ve}(G - x) = \gamma_{ve}(G)$ for every $x \notin D$. □

Proposition 5. *Let G be a graph with a unique $\gamma_{ve}(G)$-set D. Then for every $x \in D$, $\gamma_{ve}(G - x) \geq \gamma_{ve}(G)$.*

Proof. Notice first that the uniqueness of D implies that every vertex in D is incident with at least two edges. Now, suppose to the contrary that $\gamma_{ve}(G - x) < \gamma_{ve}(G)$ for some vertex $x \in D$. Since all edges in $G - x$ are *ve*-dominated by a set of vertices D' with $|D'| < \gamma_{ve}(G)$, it follows that for any neighbor y of x in G, $D' \cup \{y\}$ is a $\gamma_{ve}(G)$-set different from D which contradicts the uniqueness of D. Therefore, $\gamma_{ve}(G - x) \geq \gamma_{ve}(G)$ for every $x \in D$. □

3. Equivalent conditions for UVED-trees

Our aim in this section is to show that the two conditions given in Propositions 2 and 3 characterize unique minimum *ved-sets* in trees. Since we are dealing with trees, the following observation is essential.

Observation 6. Every UVED-tree T has diameter at least four.

Proof. If T has diameter one or two, then any vertex of T forms a minimum *ved-set* of T, and if T has diameter three, then any support vertex forms a minimum *ved-set* of T. Any case contradicts the fact that T is a UVED-tree. □

As a consequence of Observation 6, the path on five vertices is the smallest UVED-tree. Now we are ready to state and prove the main result of this section.

Theorem 7. *Let T be a tree of order at least 5. Then the following conditions are equivalent:*

(i) *T has the unique $\gamma_{ve}(T)$-set D.*
(ii) *T has a $\gamma_{ve}(T)$-set D such that for every vertex v in D, $|A_D(v)| \geq 2$.*
(iii) *T has a $\gamma_{ve}(T)$-set D for which $\gamma_{ve}(T - v) > \gamma_{ve}(T)$ for every vertex $v \in D$.*

Proof. (i) ⇒ (ii) follows from Proposition 2, and (iii) ⇒ (i) follows from Proposition 3. To complete the proof we will show that (i) ⇒ (iii) and (ii) ⇒ (i).

(i) ⇒ (iii): Assume that T has the unique $\gamma_{ve}(T)$-set D, and let us show that $\gamma_{ve}(T-v) > \gamma_{ve}(T)$ for every vertex $v \in D$.

By Proposition 2, every vertex $v \in D$ satisfies $|A_D(v)| \geq 2$. We fix some vertex $x \in D$, and let $A_D(x) = \{x_1, x_2, \ldots, x_p\}$ and $B_D(x) = N(x) - A_D(x) = \{y_1, y_2, \ldots, y_t\}$, where $p \geq 2$ and $t \geq 0$. For each x_i, let T_{x_i} be the the component of $T-x$ containing x_i and let T_{y_i} be defined similarly for each y_i. Moreover, let $D_{x_i} = D \cap V(T_{x_i})$ and $D_{y_i} = D \cap V(T_{y_i})$. Of course,

$$\gamma_{ve}(T) = |D| = 1 + \sum_{i=1}^{p} |D_{x_i}| + \sum_{i=1}^{t} |D_{y_i}|. \tag{1}$$

Let $x_i \in A_D(x)$ for some i. Assume first that T_{x_i} is a tree such that $D_{x_i} = \emptyset$. Then T_{x_i} must be a star centered at x_i. In this case, it is quite clear that $\gamma_{ve}(T_{x_i}) = 1 > |D_{x_i}| = 0$. Hence we can assume that T_{x_i} is a tree for which $D_{x_i} \neq \emptyset$, that is T_{x_i} is not a star. Note that every vertex $z \in D_{x_i}$ satisfies $\left|A_{D_{x_i}}(z)\right| \geq 2$ but D_{x_i} does not *ve*-dominate all edges of T_{x_i} since at least one edge incident with x_i belonged to $\mathrm{pe}_2(x, D)$. Moreover, $\gamma_{ve}(T_{x_i}) \geq |D_{x_i}|$ (otherwise for any $\gamma_{ve}(T_{x_i})$-set, say D'_{x_i}, the set $(D - D_{x_i}) \cup D'_{x_i}$ would be a *ved-set* of T smaller than D). We claim that $\gamma_{ve}(T_{x_i}) \geq |D_{x_i}| + 1$. Suppose to the contrary that $\gamma_{ve}(T_{x_i}) = |D_{x_i}|$, and let D'_{x_i} be a $\gamma_{ve}(T_{x_i})$-set. As mentioned above, since at least one edge incident with x_i belonged to $\mathrm{pe}_2(x, D)$, it follows that $D'_{x_i} \neq D_{x_i}$. But then $(D - D_{x_i}) \cup D'_{x_i}$ would be a second $\gamma_{ve}(T)$-set, contradicting the uniqueness of D. Hence in either case, $\gamma_{ve}(T_{x_i}) \geq |D_{x_i}| + 1$ for every $i \in \{1, \ldots, p\}$.

Now, let $y_i \in B_D(x)$ for some i. If $V(T_{y_i}) = \{y_i\}$, then $\gamma_{ve}(T_{y_i}) = 0 = |D_{y_i}|$. If $V(T_{y_i}) \neq \{y_i\}$, then using the same argument as above we can see that $\gamma_{ve}(T_{y_i}) \geq |D_{y_i}|$. Consequently and by using (1) we get

$$\begin{aligned}
\gamma_{ve}(T-x) &= \sum_{i=1}^{p} \gamma_{ve}(T_{x_i}) + \sum_{i=1}^{t} \gamma_{ve}(T_{y_i}) \\
&\geq \sum_{i=1}^{p} (|D_{x_i}| + 1) + \sum_{i=1}^{t} |D_{y_i}| \\
&\geq \gamma_{ve}(T) + p - 1 > \gamma_{ve}(T)
\end{aligned}$$

since $p \geq 2$.

(ii) ⇒ (i): Let T be a tree having a $\gamma_{ve}(T)$-set D such that every vertex v in D, $|A_D(v)| \geq 2$. To prove the uniqueness of D, we will use an induction on the order n of T. By Observation 6, the base case is the path P_5. Let T be a tree of order n and assume that the result is true for every tree of order less than n. Let T be a tree having a $\gamma_{ve}(T)$-set D such that for every vertex v in D, $|A_D(v)| \geq 2$. Assume first that D contains two adjacent vertices x and y. Let T_x and T_y be the components of $T - xy$ containing x and y, respectively. Note that since $xy \in E(T)$, no edge incident with x or y is a private edge of any vertex of D. Also since $|A_D(v)| \geq 2$ for every $v \in D$, each of T_x and T_y has order at least 5. Let $D_x = D \cap V(T_x)$ and $D_y = D \cap V(T_y)$. If D_x is not a $\gamma_{ve}(T_x)$-set, then for any $\gamma_{ve}(T_x)$-set S_x, $S_x \cup D_y$ would be a *ved-set* of T of size $|S_x \cup D_y| < |D_x| \cup |D_y| = |D|$, a contradiction. Hence D_x is a $\gamma_{ve}(T_x)$-set and likewise D_y is a $\gamma_{ve}(T_y)$-set. In addition, each $v \in D_x$ (resp. D_y) satisfies $|A_{D_x}(v)| \geq 2$ (resp. $\left|A_{D_y}(v)\right| \geq 2$). By the induction hypothesis on T_x and T_y leads that D_x is a unique $\gamma_{ve}(T_x)$-set and D_y is a unique $\gamma_{ve}(T_y)$-set, respectively. To show that D is the unique $\gamma_{ve}(T)$-set, suppose that T has a second $\gamma_{ve}(T)$-set D', and let $D' \cap V(T_x) = D'_x$ and $D' \cap V(T_y) = D'_y$. If $x, y \in D'$, then the uniqueness of D_x and D_y implies that $D'_x = D_x$ and $D'_y = D_y$. Thus $D' = D$ contradicts $D' \neq D$. Assume now that $x, y \notin D'$. Then D'_x is a *ved-set* of T_x but not a $\gamma_{ve}(T_x)$-set because of the uniqueness of D_x. Hence $|D'_x| \geq |D_x| + 1$. Likewise, D'_y is a *ved-set* of T_y but not a $\gamma_{ve}(T_y)$-set because of the uniqueness of D_y, and thus $\left|D'_y\right| \geq |D_y| + 1$. It follows that

$$|D'| = |D'_x| + \left|D'_y\right| \geq |D_x| + |D_y| + 2 > |D|,$$

a contradiction. Finally, we can assume, without loss of generality, that $x \notin D'$ and $y \in D'$. Thus $x \notin D'_x$ and $y \in D'_y$. Then D'_y is a *ved-set* of T_y and so $\left|D'_y\right| \geq |D_y|$, since D_y is a $\gamma_{ve}(T_y)$-set. Notice that D'_x can be or not a *ved-set* of T_x and this depends on whether or not the vertex x belongs or not to $A_{D'}(y)$. Consequently, the uniqueness of D_x with the fact that $x \in D_x$ imply the following: if D'_x is a *ved-set* of T_x, then $|D'_x| \geq |D_x| + 1$, while if D'_x is not a *ved-set* of T_x, then by item (iii) (already proved) $\gamma_{ve}(T_x - x) > \gamma_{ve}(T_x)$, that is $|D'_x| \geq |D_x| + 1$. In either case, we obtain that

$$|D'| = |D'_x| + \left|D'_y\right| > |D_x| + |D_y| = |D|,$$

a contradiction. Therefore, D is the unique $\gamma_{ve}(T)$-set.

From now on, we can assume that no two vertices of D are adjacent. Since the result is true when $|D| = 1$, we can therefore assume that $|D| \geq 2$. Hence the distance between any two vertices of D is at least 2. Moreover there must exist in D two vertices, say u and w at distance at most 4, for otherwise some edge of T is not *ve*-dominated. Three possible situations are to be considered.

Case 1. $d_T(u, w) = 3$.

Let x and y be two adjacent vertices such that $x \in N(u)$ and $y \in N(w)$. Hence $x, y \notin D$ and the edge xy is not a private edge of neither u nor v. Let T_x and T_y be the components of $T - xy$ containing x and y, respectively. Each component of $T - xy$ has order at least five, since each vertex v of D has $|A_D(v)| \geq 2$. Let $D_x = D \cap V(T_x)$ and $D_y = D \cap V(T_y)$. Then D_x is a *ved-set* of T_x and likewise D_y is a *ved-set* of T_y. Moreover, if T_x has a *ved-set* D^*_x smaller than D_x, then $D^*_x \cup D_y$ would be a *ved-set* of T smaller than D, a contradiction. Hence D_x is a $\gamma_{ve}(T_x)$-set and similarly D_y is a $\gamma_{ve}(T_y)$-set. In addition, each vertex $v \in D_x$ (resp. D_y) keeps in T_x (resp. T_y) the property $|A_{D_x}(v)| \geq 2$ (resp. $\left|A_{D_y}(v)\right| \geq 2$). By the induction hypothesis on T_x, D_x is a unique $\gamma_{ve}(T_x)$-set and D_y is a unique $\gamma_{ve}(T_y)$-set. To show that D is the unique $\gamma_{ve}(T)$-set, let us assume that D' is a second $\gamma_{ve}(T)$-set. Let $D'_x = D' \cap V(T_x)$ and $D'_y = D' \cap V(T_y)$. Suppose that $x \in D'_x$. Then from the uniqueness of D_x with the fact that $x \notin D_x$ we deduce that $|D'_x| \geq |D_x| + 1$. Now, if D'_y is a *ved-set* of T_y, then $\left|D'_y\right| \geq |D_y|$, while if D'_y is not a *ved-set* of T_y, then as x *ve*-dominates all edges incident with y, D'_y is a *ved-set* of $T - y$. But for the latter case, the uniqueness of D_y yields by Proposition 5, $\left|D'_y\right| \geq \gamma_{ve}(T_y - y) \geq \gamma_{ve}(T_y) = |D_y|$. Hence in either case, we have $\left|D'_y\right| \geq |D_y|$. Therefore, $|D'| = |D'_x| + \left|D'_y\right| \geq |D_x| + 1 + |D_y| > |D|$, a contradiction. Hence $x \notin D'_x$ and by symmetry $y \notin D'_y$. Since D' is assumed to be different from D, we must have either $D'_x \neq D_x$ or $D'_y \neq D_y$. Without loss of generality, let $D'_x \neq D_x$. The uniqueness of D_x leads again to the fact $|D'_x| \geq |D_x| + 1$. Consequently, $|D'| = |D'_x| + \left|D'_y\right| \geq |D_x| + 1 + |D_y| = |D| + 1$, a contradiction, which completes this case.

Case 2. $d_T(u, w) = 4$.

We will assume in addition that the (u, w)-path contains no other vertex of D, for otherwise each of u and v will be at distance 2 from some vertex in D, and this case will be considered subsequently. Hence let $utxyw$ be the such a path joining the vertices u and w. Observe that $x, y \notin D$, $N(y) \cap D \neq \emptyset$ and the edge xy is *ve*-dominated by w. We can assume further that $N(x) \cap D = \emptyset$, for otherwise for any vertex $z \in N(x) \cap D$, we have $d_T(z, w) = 3$ which leads us to the previous case. Now, consider the subtrees T_x and T_y of T defined as follows: T_x is the component of $T - xy$ containing x and T_y is the component containing y obtained from T by removing all edges incident with x except xy. Note that x belongs to each of T_x and T_y, and x is a leaf in T_y. Let $D_x = D \cap V(T_x)$ and $D_y = D \cap V(T_y)$. Then D_x (resp. D_y) is a *ved-set* of T_x (resp. T_y). Moreover, if T_x has a *ved-set* D^* smaller than D_x, then $D^* \cup D_y$ would be a *ved-set* of T smaller than D, a contradiction. Hence D_x is a $\gamma_{ve}(T_x)$-set. A similar argument shows that D_y is also a $\gamma_{ve}(T_y)$-set. On the other hand, each vertex $v \in D_x$ (resp. D_y) keeps in T_x (resp. T_y) the property $|A_{D_x}(v)| \geq 2$ (resp. $\left|A_{D_y}(v)\right| \geq 2$). By the induction hypothesis on T_x, D_x is a unique $\gamma_{ve}(T_x)$-set and similarly D_y is a unique $\gamma_{ve}(T_y)$-set. Notice that $x \notin D_x$ and $N(x) \cap D_x = \emptyset$. To show that D is the unique $\gamma_{ve}(T)$-set, let us assume that T has a second $\gamma_{ve}(T)$-set D'. Let $D' \cap V(T_x) = D'_x$ and $D' \cap V(T_y) = D'_y$.

Suppose that x or any neighbor of x except y belongs to D', and thus particularly to D'_x. Then D'_x is a *ved-set* of T_x, and since the edge xy has been chosen such that $x \notin D$ and $N(x) \cap D = \emptyset$, we deduce from the fact D_x is a unique $\gamma_{ve}(T_x)$-set that $|D'_x| \geq |D_x|+1$. Moreover, D'_y *ve*-dominates all edges of T_y except possibly the edge xy. More precisely:

(a) $x \in D'$ or $y \in D'$ and thus $x \in D'_y$ or $y \in D'_y$. Then using the fact that neither x nor y belongs to D, and thus to D_y, we deduce from the uniqueness of D_y that $\left|D'_y\right| \geq |D_y|+1$.
(b) $x \notin D'$ and $y \notin D'$ but $N(y) \cap D' \neq \emptyset$, and thus $N(y) \cap D'_y \neq \emptyset$. Then D'_y is a *ved-set* of T_y and so $\left|D'_y\right| \geq |D_y|$.
(c) $x \notin D'$ and $y \notin D'$ and $N(y) \cap D' \neq \emptyset$, and thus $N(y) \cap D'_y \neq \emptyset$. Then D'_y *ve*-dominates all edges of T_y except xy, that is all edges of $T_y - x$. Hence $\gamma_{ve}(T_y - x) \leq \left|D'_y\right|$. Now, since D_y is a unique $\gamma_{ve}(T_y)$-set and $x \notin D_y$, by Proposition 4, $\gamma_{ve}(T_y - x) = \gamma_{ve}(T_y)$, and therefore $\left|D'_y\right| \geq |D_y|$.

Note that the vertex x can be counted twice if (a) occurs, once in D'_x and once in D'_y. But regardless the situation (a), (b) or (c) that occurs, we have $|D'| = |D'_x| + \left|D'_y\right| \geq |D_x|+1+|D_y| > |D|$ yielding a contradiction.

In the following, we can suppose that neither x nor any neighbor of x other than y is in D'_x. Then D'_y is a *ved-set* of T' and so $\left|D'_y\right| \geq |D_y|$. However, because of the uniqueness of D_y with the fact that $y \notin D_y$, the inequality $\left|D'_y\right| \geq |D_y|$ becomes strict if $y \in D'_y$. Consider the two cases depending on whether y belongs to D or not.

Firstly, suppose that $y \in D'_y$. Then $\left|D'_y\right| \geq |D_y|+1$. Since all edges incident with x are *ve*-dominated by y, we conclude that D'_x *ve*-dominates all edges of $T_x - x$, and thus $\gamma_{ve}(T_x - x) \leq |D'_x|$. Using the uniqueness of D_x and the fact that $x \notin D_x$, Proposition 4 implies that $\gamma_{ve}(T_x - x) = \gamma_{ve}(T_x)$ and so $|D'_x| \geq |D_x|$. Therefore $|D'| = |D'_x| + \left|D'_y\right| \geq |D_x| + |D_y| + 1 > |D|$, a contradiction.

Secondly, suppose that $y \notin D'_y$. Recall the assumption that neither x nor any neighbor of x other than y is in D'_x. This assumption with the fact $y \notin D'_y$ imply that D'_x is a *ved-set* of T_x, and the edge xy cannot be *ve*-dominated by no vertex of D'_x. It follows from this fact that D'_y is a *ved-set* of T_y. Since D' is assumed to be a $\gamma_{ve}(T)$-set different from D, we must have either $D'_x \neq D_x$ or $D'_y \neq D_y$. The uniqueness of D_x and D_y leads to have $|D'_x| \geq |D_x|+1$ or $\left|D'_y\right| \geq |D_y|+1$. In either case, we obtain $|D'| = |D'_x| + \left|D'_y\right| > |D_x| + |D_y| = |D|$, a contradiction.

Case 3. $d_T(u,w) = 2$.

Let $x \notin D$ be a be a common neighbor of u and w. Observe that no edge incident with x is a private edge of any vertex of D and so x belongs to no $A_D(v)$ for any $v \in D$. To avoid the previous case, we can assume that every neighbor of x not in D (if any) is a leaf. Let $N(x) \cap D = \{u_1, u_2, \ldots, u_t\}$, where $t \geq 2$. Delete all edges xu_i of T and let T_{u_i} denote the component containing u_i and let T_x be the remaining component containing x. Note that T_x is either a tree of order one or a star of order at least two. Also each T_{u_i} has order at least five (since each vertex v of D satisfies $|A_D(v)| \geq 2$). Let $D_i = D \cap V(T_{u_i})$ for each $i \in \{1, ldots, t\}$, and notice that $D_x = D \cap V(T_x) = \emptyset$. Then each D_i is a *ved-set* of T_{u_i}. Moreover, if T_{u_i} has a *ved-set* D^*_i smaller than D_i, then $D^*_i \cup (D - D_i)$ would be a *ved-set* of T smaller than D, a contradiction. Hence each D_i is a $\gamma_{ve}(T_i)$-set containing in particular u_i. On the other hand, each vertex $v \in D_i$ keeps in T_{u_i} the property $|A_{D_i}(v)| \geq 2$. By the induction hypothesis on each T_{u_i}, D_i is a unique $\gamma_{ve}(T_{u_i})$-set. It remains to show that D is the unique $\gamma_{ve}(T)$-set. For this purpose, assume that T has a second $\gamma_{ve}(T)$-set D'. Let $D'_i = D' \cap V(T_{u_i})$ for each $i \in \{1, ldots, t\}$ and $D'_x = D' \cap V(T_x)$. Since $|D'_x| \leq 1$, we will assume that $x \in D'_x$ if $D'_x \neq \emptyset$.

Suppose first that $x \in D'_x$. If $u_i \in D'_i$, then D'_i is a *ved-set* of T_{u_i}, and thus $|D'_i| \geq |D_i|$. The minimality of D' implies that for some index j, $u_j \notin D'_j$ (otherwise $D' - \{x\}$ would be a *ved-set* of T). Now if D'_j ve-dominates T_{u_j}, then $\left|D'_j\right| \geq |D_j|$. If D'_j does not ve-dominates T_{u_j}, then it is a *ved-set* for $T_{u_j} - u_j$ and thus the uniqueness of D_j yields by Proposition 5, $\left|D'_j\right| \geq \gamma_{ve}(T_{u_j} - u_j) \geq \gamma_{ve}(T_{u_j}) = |D_j|$. Therefore in either case, we can assume that $|D'_i| \geq |D_i|$ for every $i \in \{1, \ldots, t\}$. It follows that

$$|D'| = 1 + \sum_{i=1}^{t} |D'_i| \geq 1 + \sum_{i=1}^{t} |D_i| = 1 + |D|,$$

a contradiction. Hence we can assume that $x \notin D'_x$, and so $D'_x = \emptyset$. Since each D'_i is a ved-set of T_{u_i}, we have $|D'_i| \geq |D_i|$ for every i. Moreover, since D' is assumed to be different from D, we must have $D'_j \neq D_j$ for some j. The uniqueness of D_j implies that $\left|D'_j\right| \geq |D_j| + 1$. It follows that

$$|D'| = \sum_{i=1}^{t} |D'_i| \geq 1 + \sum_{i=1}^{t} |D_i| = 1 + |D|,$$

a contradiction too. Therefore D is the unique $\gamma_{ve}(T)$-set and the proof is complete. □

4. Constructive characterization of UVED-trees

Our aim in this section is to provide a constructive characterization of the UVED-trees. For this purpose, we introduce the family $\mathcal{H}$ of trees $T = T_p$ that can be obtained as follows. Let T_1 be the path P_5 with center vertex x. If $p \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the operations defined below. Moreover, for each tree T_i three sets will be defined, namely $R_0(T_i)$, $R_1(T_i)$ and $R_2(T_i)$, starting with $R_0(T_1) = \{x\}$, $R_1(T_1) = N(x)$ and $R_2(T_1) = V(T_1) - N[x]$. For ease of presentation and if no confusion arises, we will simply write R_0^i, R_1^i and R_2^i instead of $R_0(T_i)$, $R_1(T_i)$ and $R_2(T_i)$, respectively.

- **Operation** $\mathcal{O}_1$: Assume w is a vertex of $R_0^i \cup R_1^i$. Then T_{i+1} is obtained from T_i by adding a new vertex v and the edge wv. Let $R_0^{i+1} = R_0^i, R_1^{i+1} = R_1^i$ and either $R_2^{i+1} = R_2^i \cup \{v\}$ if $w \in R_1^i$ or $R_2^{i+1} = R_2^i$ if $w \in R_0^i$.
- **Operation** $\mathcal{O}_2$: Assume w is a vertex of R_0^i adjacent to two paths P_2. Then T_{i+1} is obtained from T_i by adding a path $P_2 : uv$ and the edge wv. Let $R_0^{i+1} = R_0^i, R_1^{i+1} = R_1^i \cup \{v\}$ and $R_2^{i+1} = R_2^i \cup \{u\}$.
- **Operation** $\mathcal{O}_3$: Assume w is a vertex of R_2^i. Then T_{i+1} is obtained from T_i by adding a path $P_4 : abcd$ and the edge wa. Let $R_0^{i+1} = R_0^i \cup \{b\}, R_1^{i+1} = R_1^i \cup \{a, c\}$ and $R_2^{i+1} = R_2^i \cup \{d\}$.
- **Operation** $\mathcal{O}_4$: Assume w is either a leaf or vertex of R_1^i. Then T_{i+1} is obtained from T_i by adding a path $P_5 : abcde$ and the edge wd. Let $R_0^{i+1} = R_0^i \cup \{c\}, R_1^{i+1} = R_1^i \cup \{b, d\}$ and $R_2^{i+1} = R_2^i \cup \{a, e\}$.
- **Operation** $\mathcal{O}_5$: Assume w is a vertex of R_0^i. Then T_{i+1} is obtained from T_i by adding a path $P_5 : abcde$ and the edge wc. Let $R_0^{i+1} = R_0^i \cup \{c\}, R_1^{i+1} = R_1^i \cup \{b, d\}$ and $R_2^{i+1} = R_2^i \cup \{a, e\}$.
- **Operation** $\mathcal{O}_6$: Assume w is a vertex of T_i. Then T_{i+1} is obtained from T_i by adding $k \geq 1$ disjoint paths $P_5 : a_j b_j c_j d_j e_j$ and a new vertex y and the edges yw, yc_j for every j. Let $R_0^{i+1} = R_0^i \cup \{c_1, ..., c_k\}, R_1^{i+1} = R_1^i \cup \{y, b_1, d_1, ..., b_k, d_k\}$ and $R_2^{i+1} = R_2^i \cup \{a_1, e_1, ..., a_k, e_k\}$.

We note that from the way a tree $T \in \mathcal{H}$ is constructed, the set $R_0(T)$ is a *ved-set* of T. Also the sets $R_0(T), R_1(T)$ and $R_2(T)$ are disjoint. Moreover, every support vertex in T belongs to $R_0(T) \cup R_1(T)$, while every leaf in T belongs to $R_2(T)$ except for those added by Operation $\mathcal{O}_1$ and attached to the vertices of R_0^i which do not belong to the set R_j for any $j \in \{0, 1, 2\}$. Finally, for all i, no vertex in R_2^i has a neighbor in R_0^i.

Observation 8. If T is a tree of diameter at least two, then there exist a $\gamma_{ve}(T)$-set which contains no leaves.

In the rest of the paper, we shall prove the following. But for the sake of presentation we will split the proof into two propositions.

Theorem 9. *A tree T is a UVED-tree if and only if $T \in \mathcal{H}$.*

Proposition 10. *If $T \in \mathcal{H}$, then $R_0(T)$ is the unique $\gamma_{ve}(T)$-set.*

Proof. Let $T \in \mathcal{H}$. Then T can be obtained from a sequence $T_1, T_2, \ldots, T_p$ $(p \geq 1)$ of trees, where T_1 is the path P_5, $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the six operations defined. We use the terminology of the construction for sets R_0^i, R_1^i and R_2^i as well as the same designation of the vertices used in the definition of each operation. If $p = 1$, then $T = P_5$ and $R_0(T_1)$ is the unique $\gamma_{ve}(T)$-set. This establishes our base case.

Let $p \geq 2$ and assume that the result holds for all trees $T \in \mathcal{H}$ that can be constructed from a sequence of length at most $p-1$, and let $T' = T_{p-1}$. Applying our inductive hypothesis to $T' \in \mathcal{H}$ shows that $R_0(T')$ is the unique $\gamma_{ve}(T')$-set. If T is obtained from T' using Operation $\mathcal{O}_1$, then $\gamma_{ve}(T) = \gamma_{ve}(T')$ and $R_0(T') = R_0(T)$ is the unique $\gamma_{ve}(T)$-set. Hence let us examine the following five cases.

Case 1. T is obtained from T' using Operation $\mathcal{O}_2$.

Since $w \in R_0(T')$, $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. The equality $\gamma_{ve}(T) = \gamma_{ve}(T')$ follows from the fact that T has a $\gamma_{ve}(T)$-set that contains w and neither u nor v. Now if T has a $\gamma_{ve}(T)$-set S containing u or v, then $w \notin S$ and thus replacing u or v in S by some neighbor of w in T' provides a $\gamma_{ve}(T')$-set different from $R_0(T')$, a contradiction. Hence no $\gamma_{ve}(T)$-set contains u or v and so w belongs to every $\gamma_{ve}(T)$-set. Since $R_0(T')$ is the unique $\gamma_{ve}(T')$-set, $R_0(T) = R_0(T')$ is also the unique $\gamma_{ve}(T)$-set.

Case 2. T is obtained from T' using Operation $\mathcal{O}_3$.

Since $R_0(T') \cup \{b\}$ is a *ved-set* of T, $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. The equality $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$ follows from the fact that there is a $\gamma_{ve}(T)$-set that contains b and no vertex of $\{a, c, d\}$. Hence $R_0(T) = R_0(T') \cup \{b\}$ is a $\gamma_{ve}(T)$-set. Now, suppose that D is a second $\gamma_{ve}(T)$-set different from $R_0(T)$. Then $|D \cap \{a, b, c, d\}| \in \{1, 2\}$. If $|D \cap \{a, b, c, d\}| = 2$, then certainly the set $D_1 = \{b, w\} \cup (D - \{a, b, c, d\})$ is also a $\gamma_{ve}(T)$-set. In this case, $D_1 \cap V(T')$ is a $\gamma_{ve}(T')$-set different from $R_0(T')$, since it contains w, which contradicts the uniqueness of $R_0(T')$. Hence $|D \cap \{a, b, c, d\}| = 1$. If $b \in D$, then obviously $D - \{b\} = D \cap V(T')$ is a $\gamma_{ve}(T')$-set. But since $b \in R_0(T) \cap D$, it follows that $D \cap V(T') \neq R_0(T')$ yielding a second $\gamma_{ve}(T')$-set for T', a contradiction. Thus let $b \notin D$. Then $a \notin D$, else the edge cd is not *ve*- dominated by a, and thus $\{c, d\} \cap D \neq \emptyset$. Whatever the vertex of $\{c, d\}$ belonging to D, w or one of its neighbors in T' must be in D to *ve*-dominates the edge aw. Since neither w nor its neighbors in T' are in $R_0(T')$, we deduce that $D \cap V(T')$ is a $\gamma_{ve}(T')$-set different from $R_0(T')$, a contradiction. Therefore $R_0(T)$ is the unique $\gamma_{ve}(T)$-set.

Case 3. T is obtained from T' using Operation $\mathcal{O}_4$.

The inequality $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$ is obtained from the fact that $R_0(T') \cup \{c\}$ is a *ved-set* of T, and the equality $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$ follows from the fact that there is a $\gamma_{ve}(T)$-set that contains c and no vertex of $\{a, b, d, e\}$. Hence $R_0(T) = R_0(T') \cup \{c\}$ is a $\gamma_{ve}(T)$-set. Now, let D be a second $\gamma_{ve}(T)$-set different from $R_0(T)$. Then $|D \cap \{a, b, c, d, e\}| \in \{1, 2\}$. If $|D \cap \{a, b, c, d, e\}| = 2$, then the set $D_1 = \{c, w\} \cup (D - \{a, b, c, d, e\})$ is also a $\gamma_{ve}(T)$-set. In this case, $D_1 \cap V(T')$ is a $\gamma_{ve}(T')$-set different from $R_0(T')$, since it contains w, contradicting the uniqueness of $R_0(T')$. Hence $|D \cap \{a, b, c, d, e\}| = 1$. If $c \in D$, then since $c \in R_0(T) \cap D$, we must have $D \cap V(T') \neq R_0(T')$, which leads again to a contradiction with the uniqueness of $R_0(T')$. Therefore, let $c \notin D$. Then to *ve*-dominate the edge ab, we must have $\{a, b\} \cap D \neq \emptyset$, and thus $w \in D$ to *ve*-dominate the edge de. It follows that $D \cap V(T')$ is a $\gamma_{ve}(T')$-set different from $R_0(T')$, since it contains w, a contradiction. Consequently, $R_0(T)$ is the unique $\gamma_{ve}(T)$-set.

Case 4. *T is obtained from T' using Operation $\mathcal{O}_5$.*

The equality $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$ follows from the fact that $R_0(T) = R_0(T') \cup \{c\}$ is a *ved-set* of T. To show the equality, suppose that $\gamma_{ve}(T) < \gamma_{ve}(T') + 1$, and let S be a $\gamma_{ve}(T)$-set that contains c and neither a, b, d nor e. Notice that such a set exist. If $N_{T'}[w] \cap S \neq \emptyset$, then $S - \{c\}$ is a *ved-set* of T' and thus by our assumption we get

$$\gamma_{ve}(T') \leq |S| - 1 = \gamma_{ve}(T) - 1 < \gamma_{ve}(T'),$$

a contradiction. Hence $N_{T'}[w] \cap S = \emptyset$. Let w^* be a neighbor of w in T'. Then the set $S' = \{w^*\} \cup S - \{c\}$ is a *ved-set* of T' and thus by our assumption we get

$$\gamma_{ve}(T') \leq |S'| = |S| = \gamma_{ve}(T) < \gamma_{ve}(T') + 1,$$

which leads to the fact that S' is a $\gamma_{ve}(T')$-set containing w^* but not w, contradicting $R_0(T')$ is the unique $\gamma_{ve}(T')$-set. Therefore $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$ and thus $R_0(T) = R_0(T') \cup \{c\}$ is a $\gamma_{ve}(T)$-set.

To show that $R_0(T)$ is the unique $\gamma_{ve}(T)$-set, let D be a second $\gamma_{ve}(T)$-set. As seen in the previous cases, if $|D \cap \{a,b,c,d,e\}| \geq 2$, then we can construct a $\gamma_{ve}(T')$-set different from $R_0(T')$ which leads to a contradiction. Hence $|D \cap \{a,b,c,d,e\}| = 1$. To *ve*-dominate the edges ab and de, we must have $c \in D$. Now, if $w \in D$, then since $c \in R_0(T) \cap D$, we must have $D \cap V(T') \neq R_0(T')$ implying that $D \cap V(T')$ is a $\gamma_{ve}(T')$-set different from $R_0(T')$, a contradiction. Hence we assume that $w \notin D$. Since c *ve*- dominates all edges of T' incident with w, we deduce that $D \cap V(T')$ is a *ved-set* of $T' - w$, and thus $|D \cap V(T')| \geq \gamma_{ve}(T' - w)$. Now since T' is a UVED-tree, by Theorem 7, $\gamma_{ve}(T'-w) > \gamma_{ve}(T')$. Consequently, $|D| = 1 + |D \cap V(T')| > 1 + \gamma_{ve}(T') = \gamma_{ve}(T)$, a contradiction. Therefore $R_0(T)$ is the unique $\gamma_{ve}(T)$-set.

Case 5. T is obtained from T' using Operation $\mathcal{O}_6$.

The equality $\gamma_{ve}(T) \leq \gamma_{ve}(T') + k$ follows from the fact that $R_0(T) = R_0(T') \cup \{c_1, \dots, c_k\}$ is a *ved-set* of T. On the other hand, since T has a $\gamma_{ve}(T)$-set that contains each c_j and neither y nor any a_j, b_j, d_j and e_j, such a set by removing $c_1, \dots, c_k$ is a *ved-set* of T' and so $\gamma_{ve}(T') \leq \gamma_{ve}(T') - k$. Consequently, $\gamma_{ve}(T) = \gamma_{ve}(T') + k$ and $R_0(T) = R_0(T') \cup \{c_1, \dots, c_k\}$ is a $\gamma_{ve}(T)$-set.

To show that $R_0(T)$ is the unique $\gamma_{ve}(T)$-set, let D be a second $\gamma_{ve}(T)$-set. Then $|D \cap \{a_j, b_j, c_j, d_j, e_j\}| \geq 1$ for every j. Once again, as we have already seen in the previous cases, if $|D \cap \{a_j, b_j, c_j, d_j, e_j\}| \geq 2$ for some j, then we can easily construct a $\gamma_{ve}(T')$-set different from $R_0(T')$. Hence $|D \cap \{a_j, b_j, c_j, d_j, e_j\}| = 1$ for every j, which implies that every c_j is in D (to *ve*-dominates the edges a_jb_j and d_je_j). Now if $y \notin D$, then certainly $D \cap V(T')$ is a second $\gamma_{ve}(T')$-set different from $R_0(T')$, a contradiction. Hence we assume that $y \in D$. Then replace y in D by w if $w \notin R_0(T')$. Further if $w \in R_0(T')$, then we have to replace y in D by a neighbor of w which is not in $R_0(T')$. In either case, the resulting set D by deleting $c_1, \dots, c_k$ would be a second $\gamma_{ve}(T')$-set, a contradiction too. Therefore $R_0(T)$ is the unique $\gamma_{ve}(T)$-set.

Through all the previous cases, we conclude that $R_0(T)$ is the unique $\gamma_{ve}(T)$-set and T is UVED-tree. □

Proposition 11. *If T is a UVED-tree, then $T \in \mathcal{H}$.*

Proof. We will proceed by induction on the order n of T. Since there is no UVED-tree T with diameter at most three (by Obs. 6), let $\text{diam}(T) \geq 4$. The smallest UVED-tree with diameter at least four is the path P_5 and $P_5 \in \mathcal{H}$. Let $n \geq 6$, and assume that any UVED-tree T' of order $n' < n$ is in $\mathcal{H}$. Let T be a UVED-tree of order n with the unique $\gamma_{ve}(T)$-set D.

If any support vertex, say u, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to u. Then u is still a support vertex in T', and since every *ved-set* of T' is a *ved-set* of T, we deduce that D is the unique $\gamma_{ve}(T')$-set. Hence T' is a UVED-tree of order $n' \geq 5$. Applying our inductive hypotheses to T', we have that $T' \in \mathcal{H}$ and thus $u \in R_0(T') \cup R_1(T')$. Since T can be formed from T' by using Operation $\mathcal{O}_1$, $T \in \mathcal{H}$. Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

Let $v_1v_2 \dots v_k$ ($k \geq 5$) be a diametral path in T chosen so that $d_T(v_3)$ is as large as possible. Root T at v_k. Clearly, $d_T(v_2) = 2$. Since there is a $\gamma_{ve}(T)$-set that contains v_3, the uniqueness of D implies that $v_3 \in D$. Hence $v_1, v_2 \notin D$. If v_3 is a support vertex with leaf neighbor z, then let $T' = T - z$. The unicity of D implies that $z \notin D$ and thus D remains a *ved-set* of T'. Moreover, every $\gamma_{ve}(T')$-set D' containing v_3 is a *ved-set* of T. Hence $\gamma_{ve}(T') = \gamma_{ve}(T)$. Now if $D' \neq D$, then D' would be a second $\gamma_{ve}(T)$-set, a contradiction. Therefore $D' = D$ and thus D is the unique $\gamma_{ve}(T')$-set. Applying our inductive hypothesis, we have that $T' \in \mathcal{H}$. Consequently, $D = R_0(T')$ and $v_3 \in R_0(T')$. Therefore, T can be obtained from T' using Operation $\mathcal{O}_1$, and so $T \in \mathcal{H}$. In the following, we can assume that v_3 is not a support vertex, that is, every child of v_3 is a support vertex of degree exactly two. Consider the following two cases.

Case 1. $d_T(v_3) = 2$.

Since $d_T(v_3) = d_T(v_2) = 2$, Theorem 7-(ii) implies that $v_4 \in A_D(v_3)$, and thus $D \cap N(v_4) = \{v_3\}$. It follows that any child of v_4 except v_3 (if any) has to be a leaf. Since T has no support vertex with at least two leaves, we conclude that $d_T(v_4) = 2$ or v_4 has degree three and exactly one leaf. Now, let $T' = T - T_{v_4}$. Note that if

T' is a trivial tree, then $n = 6$ and vertex v_4 has to be a support vertex with two leaves which contradicts our earlier assumption. Hence we can assume that T' is a nontrivial tree. First assume that $d_T(v_4) = 2$. Then, since $|A_D(v_3)| \geq 2$ (by Thm. 7), we deduce that $\text{pe}_2(v_3, D) = \{v_1v_2, v_4v_5\}$, and thus $N_T[v_5] \cap D = \emptyset$. It is easy to see that $\gamma_{ve}(T) = \gamma_{ve}(T')+1$ and thus $D \cap V(T')$ is a $\gamma_{ve}(T')$-set. If T' has a second $\gamma_{ve}(T')$-set, then such a set plus v_3 would be a $\gamma_{ve}(T)$-set different from D, a contradiction. Hence $D \cap V(T')$ is the unique $\gamma_{ve}(T')$-set. By the inductive hypothesis, we have that $T' \in \mathcal{H}$ with $v_5 \in R_2(T')$ (since $N_{T'}[v_5] \cap D = \emptyset$). Thus, T can be obtained from T' using Operation $\mathcal{O}_3$, and so $T \in \mathcal{H}$. Second, assume that v_4 is a support vertex of degree 3, and let w be the leaf neighbor of v_4. Then $wv_4 \in \text{pe}_2(v_3, D)$. As before one can see that $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$ and the unicity of D implies that $D \cap V(T')$ is the unique $\gamma_{ve}(T')$-set. By the inductive hypothesis, we have that $T' \in \mathcal{H}$. If $v_4v_5 \in \text{pe}_2(v_3, D)$, then $N_T[v_5] \cap D = \emptyset$ and thus $v_5 \in R_2(T')$. In that case, let T'' be the tree obtained from T' using Operation $\mathcal{O}_3$. Then $T'' \in \mathcal{H}$ with $v_4 \in R_1(T'')$. Now since T can be obtained from T'' using Operation $\mathcal{O}_1$, we have $T \in \mathcal{H}$. Assume now that $v_4v_5 \notin \text{pe}_2(v_3, D)$. Since the edge v_4w must belong to $\text{pe}_2(v_3, D)$, we deduce that $v_5 \notin D$. Thus $N_T(v_5) \cap D \neq \emptyset$ (since v_4v_5 is not a private edge of v_3) and $v_5 \in R_1(T')$. In that case, T can be obtained from T' using Operation $\mathcal{O}_4$, and hence $T \in \mathcal{H}$. Therefore, in either case, $T \in \mathcal{H}$.

Case 2. $d_T(v_3) \geq 3$.

First assume that $d_T(v_3) \geq 4$. Note in this case that all children of v_3 are support vertices of degree two, that is T_{v_3} is a subdivided star of order $2d_T(v_3) - 1$. Let $T' = T - \{v_1, v_2\}$. Since $v_1, v_2 \notin D$, D is a *ved-set* of T and so $\gamma_{ve}(T') \leq \gamma_{ve}(T')$. The equality follows from the fact that any $\gamma_{ve}(T')$-set containing v_3 is a *ved-set* of T. Now assume that $D \cap V(T')$ is not the unique $\gamma_{ve}(T')$-set, and let D' be a second $\gamma_{ve}(T')$-set. If $v_3 \in D'$, then D' would be a second $\gamma_{ve}(T)$-set which leads to a contradiction. Hence $v_3 \notin D$. Since T'_3 is a subdivided star of order at least 5, we deduce that $\left|D' \cap V(T'_{v_3})\right| \geq 2$. Then for any $y \in D' \cap V(T'_{v_3})$, the set $\{v_3\} \cup D' - \{y\}$ is a $\gamma_{ve}(T)$-set different from D, a contradiction. Therefore $D \cap V(T')$ is the unique $\gamma_{ve}(T')$-set. Applying our inductive hypothesis, we have that $T' \in \mathcal{H}$ with $v_3 \in R_0(T')$. Since T is obtained from T' using Operation $\mathcal{O}_2$, we deduce that $T \in \mathcal{H}$. From now on, we can assume that $d_T(v_3) = 3$. Let v'_2 be the second child of v_3 and v'_1 be the leaf neighbor of v'_2.

Suppose that $v_4 \in D$. Then $d_T(v_4) \geq 3$ (for otherwise, replacing v_4 with v_5 in D provides a second $\gamma_{ve}(T)$-set). Let $T' = T - T_{v_3}$. Since $D - \{v_3\}$ is a *ved-set* of T', we obtain that $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. Also, if D' is a $\gamma_{ve}(T')$-set, then $D' \cup \{v_3\}$ is a *ved-set* of T, implying $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Therefore $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$. As above, one can see that the uniqueness of D leads that $D \cap V(T')$ is the unique $\gamma_{ve}(T')$-set. Applying our inductive hypothesis, we have that $T' \in \mathcal{H}$ with $v_4 \in R_0(T')$. Since T is obtained from T' using Operation $\mathcal{O}_5$, we deduce that $T \in \mathcal{H}$. In the following, we may assume that $v_4 \notin D$.

Assume that $d_T(v_4) = 2$, and let $T' = T - T_{v_4}$. If T' is trivial, then T is a subdivided star of order exactly 7 and thus $T \in \mathcal{H}$ since it can be obtained from $T_1 = P_5$ by using one or more times Operation $\mathcal{O}_2$. Hence we can assume that T' is a nontrivial tree. It is easy to see $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$ and the unicity of D implies that $D \cap V(T')$ is the unique $\gamma_{ve}(T')$-set. Applying our inductive hypothesis, we have that $T' \in \mathcal{H}$. Now since T can be obtained from T' using Operation $\mathcal{O}_6$, we conclude that $T \in \mathcal{H}$.

Assume now that $d_T(v_4) \geq 3$. Let y be any child of v_4 different from v_3. Then y cannot have a depth one, for otherwise v_4 may belong to D to *ve*-dominate the edges incident with y which contradicts our assumption that $v_4 \notin D$. Hence y is either a leaf or, according to the situations treated above, T_y is a subdivided star centered at y. Notice that if y is not a leaf, then the uniqueness of D implies that $y \in D$ which yields that $d_T(y) \geq 3$ (for otherwise replacing y in D with its child provides a second $\gamma_{ve}(T)$-set). In that case, by our choice of v_3, $3 = d_T(v_3) \geq d_T(y) \geq 3$, and thus $d_T(y) = 3$, that is T_y is a path P_5 centered at y. Let $T' = T - T_{v_4}$ and recall that v_4 can also have a child which is a leaf. Note that T' cannot be trivial for otherwise we would have either v_4 as a support vertex with two leaves or a longer shortest path between two vertices in T_{v_4}. Hence we can assume that T' has order at least two. Let $C'(v_4)$ be the set of children of v_4 of degree three, and let $t = |C'(v_4)|$. Then $d_T(v_4) - 2 \leq t \leq d_T(v_4) - 1$, and if the lower bound is achieved then v_4 is a support vertex. It is easy to see $\gamma_{ve}(T) = \gamma_{ve}(T') + t$. Moreover, if $D \cap V(T')$ is not the unique $\gamma_{ve}(T')$-set, then any other $\gamma_{ve}(T')$-set plus $C'(v_4)$ would be a $\gamma_{ve}(T)$-set different from D, a contradiction. Hence $D \cap V(T')$ is the unique $\gamma_{ve}(T')$-set.

Applying our inductive hypothesis, we have that $T' \in \mathcal{H}$. Now since T can be obtained from T' using Operation $\mathcal{O}_6$ and possibly Operation $\mathcal{O}_1$ (if v_4 is a support vertex), it follows that $T \in \mathcal{H}$. □

According to Propositions 10 and 11, we have proven Theorem 9.

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