DEGREE CONDITIONS FOR THE EXISTENCE OF A \( \{P_2, P_3\}\)-FACTOR IN A GRAPH

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Abstract. A subgraph of a graph \( G \) is spanning if the subgraph covers all vertices of \( G \). A path-factor of a graph \( G \) is a spanning subgraph \( H \) of \( G \) such that every component of \( H \) is a path. In this article, we prove that (i) a connected graph \( G \) with \( \delta(G) \geq 5 \) admits a \( \{P_2, P_3\}\)-factor if \( G \) satisfies \( \delta(G) > \frac{2\omega(G) - 4}{3} \); (ii) a connected graph \( G \) of order \( n \) with \( n \geq 7 \) has a \( \{P_2, P_3\}\)-factor if \( G \) satisfies \( \max\{d_G(x), d_G(y)\} \geq \frac{3n}{4} \) for any two nonadjacent vertices \( x \) and \( y \) of \( G \).

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1. Introduction

In this article, we deal only with finite and undirected graphs that have neither loops nor multiple edges. Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). For \( x \in V(G) \), we denote by \( d_G(x) \) the degree of \( x \) in \( G \). Let \( \delta(G) \) and \( \alpha(G) \) denote minimum degree and independence number of a graph \( G \), respectively. We denote by \( \omega(G) \) the number of connected components in \( G \), and by \( i(G) \) the number of isolated vertices in \( G \). The order of a graph \( G \) is the number \( n = |V(G)| \) of its vertices and its size is the number \( q = |E(G)| \) of its edges. For any \( X \subseteq V(G) \), \( G[X] \) denotes the subgraph of \( G \) induced by \( X \) and \( G - X \) denotes the subgraph derived from \( G \) by removing the vertices in \( X \) and the edges incident to vertices in \( X \). For disjoint sets \( X, Y \subseteq V(G) \), we use \( E_G(X, Y) \) to denote the set of edges of \( G \) joining a vertex in \( X \) and a vertex in \( Y \), and write \( e_G(X, Y) = |E_G(X, Y)| \). Let \( G_1 \) and \( G_2 \) be two disjoint graphs. The union \( G_1 \cup G_2 \) denotes the graph with vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \). The join \( G_1 \vee G_2 \) is the graph with vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\} \). For a graph \( G \) and an integer \( k \geq 2 \), we use \( kG \) to denote the disjoint union of \( k \) copies of \( G \). The path and the complete graph of order \( n \) are denoted by \( P_n \) and \( K_n \), respectively.

A subgraph of a graph \( G \) is spanning if the subgraph covers all vertices of \( G \). For a set \( \mathcal{H} \) of connected graphs, a spanning subgraph \( H \) of a graph \( G \) is called an \( \mathcal{H} \)-factor of \( G \) if every component of \( H \) is isomorphic to a member of \( \mathcal{H} \). An \( \mathcal{H} \)-factor is also referred as a component factor. A path-factor of a graph \( G \) is a spanning

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subgraph $H$ of $G$ such that every component of $H$ is a path. Note that a perfect matching can be regarded as a $\{P_2\}$-factor. Let $d \geq 2$ be an integer. A $\{P_d, P_{d+1}, \ldots\}$-factor is simply denoted by a $P_{\geq d}$-factor.


For a graph $G$ and an integer $i \geq 1$, let $\mathcal{C}_i(G)$ denote the set of components of order $i$ in $G$, and set $c_i(G) = |\mathcal{C}_i(G)|$. Obviously, $c_1(G)$ is the number of isolated vertices in $G$ (that is, $c_1(G) = i(G)$).

Egawa and Furuya [2] obtained a sufficient condition for a graph to admit a $\{P_2, P_5\}$-factor.

**Theorem 1.1 (2).** Let $G$ be a graph. If $G$ satisfies

$$c_1(G - X) + \frac{2}{3}c_3(G - X) \leq \frac{4}{3}|X| + \frac{1}{3},$$

for any vertex subset $X$ of $G$, then $G$ admits a $\{P_2, P_5\}$-factor.

In this article, we also study the existence of $\{P_2, P_5\}$-factors in graphs, and derive two sufficient conditions for graphs to possess $\{P_2, P_5\}$-factors with respect to independence number or degree, respectively. Our main results are the following.

**Theorem 1.2.** A connected graph $G$ with $\delta(G) \geq 5$ admits a $\{P_2, P_5\}$-factor if $G$ satisfies

$$\delta(G) > \frac{3\alpha(G) - 1}{4}.$$  

**Theorem 1.3.** Let $G$ be a connected graph of order $n$ with $n \geq 7$. If $G$ satisfies

$$\max\{d_G(x), d_G(y)\} \geq \frac{3n}{7}$$

for any two nonadjacent vertices $x$ and $y$ of $G$, then $G$ has a $\{P_2, P_5\}$-factor.

2. The Proof of Theorem 1.2

**Proof of Theorem 1.2.** Suppose, to the contrary, that $G$ has no $\{P_2, P_5\}$-factor. Then in terms of Theorem 1.1, we possess

$$c_1(G - X) + \frac{2}{3}c_3(G - X) > \frac{4}{3}|X| + \frac{1}{3}$$ (2.1)

for some vertex subset $X$ of $G$. 
Claim 1. $X \neq \emptyset$.

Proof. If $X = \emptyset$, then it follows from (2.1) that

$$c_1(G) + c_3(G) \geq c_1(G) + \frac{2}{3} c_3(G) > \frac{1}{3}.$$  

By virtue of the integrity of $c_1(G) + c_3(G)$, we infer

$$c_1(G) + c_3(G) \geq 1. \quad (2.2)$$

Note that $G$ is a connected graph. Then we obtain $c_1(G) = i(G) = 0$ and $\omega(G) = 1$. Combining these with (2.2), we have $1 \leq c_3(G) \leq \omega(G) = 1$, which implies $c_3(G) = \omega(G) = 1$. Thus, we deduce $\delta(G) \leq 2$, which contradicts $\delta(G) \geq 5$. This completes the proof of Claim 1.

Using (2.1) and Claim 1, we get

$$c_1(G - X) + c_3(G - X) \geq c_1(G - X) + \frac{2}{3} c_3(G - X) > \frac{4}{3} |X| + \frac{1}{3} \geq \frac{5}{3}. \quad (2.3)$$

In what follows, we consider two cases by the value of $c_1(G - X)$.

Case 1. $c_1(G - X) = 0$.

By means of (2.3), we have $c_3(G - X) > \frac{5}{3}$. Thus, we may choose $v \in V(C_3(G - X))$, and so $d_{G - X}(v) \leq 2$. Then we infer $\delta(G) \leq d_{G}(v) = d_{G - X}(v) + |X| = |X| + 2$. Combining this with (2.1), $\delta(G) \geq 5$ and $c_1(G - X) = 0$, we deduce

$$\alpha(G) \geq c_3(G - X) = \frac{3}{2} \left( c_1(G - X) + \frac{2}{3} c_3(G - X) \right) \geq \frac{3}{2} \left( \frac{4}{3} |X| + \frac{2}{3} \right) = 2|X| + 1$$

$$\geq 2(\delta(G) - 2) + 1 = 2\delta(G) - 3 = \frac{4}{3} \delta(G) + \frac{2}{3} \delta(G) - 3$$

$$\geq \frac{4}{3} \delta(G) + \frac{10}{3} - 3 = \frac{4}{3} \delta(G) + \frac{1}{3},$$

which implies

$$\delta(G) \leq \frac{3\alpha(G) - 1}{4},$$

which contradicts that $\delta(G) > \frac{3\alpha(G) - 1}{4}$.

Case 2. $c_1(G - X) \neq 0$.

Since $c_1(G - X) \neq 0$, we may choose an isolated vertex $v$ of $G - X$. Thus, we see $d_{G - X}(v) = 0$ and $\delta(G) \leq d_{G}(v) = d_{G - X}(v) + |X| = |X|$. It follows from (2.1), $c_1(G - X) \neq 0$ and $\delta(G) \leq |X|$ that

$$\alpha(G) \geq c_1(G - X) + c_3(G - X) \geq c_1(G - X) + \frac{2}{3} c_3(G - X)$$

$$\geq \frac{4}{3} |X| + \frac{1}{3} \geq \frac{4}{3} \delta(G) + \frac{1}{3},$$

which implies

$$\delta(G) < \frac{3\alpha(G) - 1}{4},$$

which contradicts that $\delta(G) > \frac{3\alpha(G) - 1}{4}$. This completes the proof of Theorem 1.2.
3. The proof of Theorem 1.3

Proof of Theorem 1.3. Suppose, to the contrary, that $G$ has no $\{P_2, P_3\}$-factor. Then it follows from Theorem 1.1 that

$$c_1(G - X) + \frac{2}{3}c_3(G - X) > \frac{4}{3}|X| + \frac{1}{3}$$  \hspace{1cm} (3.1)

for some vertex subset $X$ of $G$.

Claim 2. $X \neq \emptyset$.

Proof. Assume that $X = \emptyset$. Then by (3.1), we obtain

$$c_1(G) + c_3(G) \geq c_1(G) + \frac{2}{3}c_3(G) > \frac{1}{3},$$

According to the integrity of $c_1(G) + c_3(G)$, we admit

$$c_1(G) + c_3(G) \geq 1.$$  \hspace{1cm} (3.2)

Since $G$ is a connected graph, we deduce $c_1(G) = 0$ and $\omega(G) = 1$. Combining these with (3.2), we derive

$$1 \leq c_1(G) + c_3(G) = c_3(G) \leq \omega(G) = 1,$$

which implies $c_3(G) = \omega(G) = 1$. Thus, we see $n = |V(G)| = 3$, which contradicts $n \geq 7$. This concludes the proof of Claim 2. \qed

The following proof will be divided into three cases by the value of $c_1(G - X)$, and derive a contradiction in every case.

Case 1. $c_1(G - X) = 0$.

By virtue of (3.1) and Claim 2, we get

$$\frac{2}{3}c_3(G - X) = c_1(G - X) + \frac{2}{3}c_3(G - X) > \frac{4}{3}|X| + \frac{1}{3} \geq \frac{5}{3},$$

namely,

$$c_3(G - X) > 2.$$  

Let $G_1, G_2, \ldots, G_t$ be the components of $G - X$ with $|V(G_i)| = 3$ for $1 \leq i \leq t$, where $t > 2$ is an integer. We select $x_i \in V(G_i)$ and $x_j \in V(G_j)$, where $i \neq j$. Obviously, $x_ix_j \notin E(G)$, $d_{G-X}(x_i) \leq 2$ and $d_{G-X}(x_j) \leq 2$.

In terms of the condition of Theorem 1.3, we see

$$\frac{3n}{7} \leq \max\{d_G(x_i), d_G(x_j)\} \leq \max\{d_{G-X}(x_i) + |X|, d_{G-X}(x_j) + |X|\} \leq |X| + 2,$$

which implies

$$|X| \geq \frac{3n}{7} - 2.$$  \hspace{1cm} (3.3)

It follows from (3.1) and (3.3) that

$$n \geq |X| + 3c_3(G - X) = |X| + \frac{9}{2}\left(c_1(G - X) + \frac{2}{3}c_3(G - X)\right)$$
\[
> |X| + \frac{9}{2} \left( \frac{4}{3} |X| + \frac{1}{3} \right) = 7|X| + \frac{3}{2}
\]
\[
\geq 7 \left( \frac{3n}{7} - 2 \right) + \frac{3}{2} > 3n - 14,
\]
that is,
\[n < 7,
\]
which contradicts \(n \geq 7\).

**Case 2.** \(c_1(G - X) = 1\).

By means of (3.1) and \(c_1(G - X) = 1\), we deduce
\[
\frac{2}{3} c_3(G - X) > \frac{4}{3} |X| + \frac{1}{3} - c_1(G - X) = \frac{4}{3} |X| - \frac{2}{3},
\]
that is,
\[c_3(G - X) > 2|X| - 1.
\]
In view of Claim 1 and the integrity of \(c_3(G - X)\) and \(|X|\), we infer
\[c_3(G - X) \geq 2|X| \geq 2.
\]
Let \(G_1\) be the component of \(G - X\) with \(|V(G_1)| = 1\) and let \(G_2\) be the component of \(G - X\) with \(|V(G_2)| = 3\). We choose \(x_1 \in V(G_1)\) and \(x_2 \in V(G_2)\). Clearly, \(x_1 x_2 \in E(G)\), \(d_{G - X}(x_1) = 0\) and \(d_{G - X}(x_2) \leq 2\). According to the condition of Theorem 1.3, we obtain
\[\frac{3n}{7} \leq \max\{d_G(x_1), d_G(x_2)\} \leq \max\{d_{G - X}(x_1) + |X|, d_{G - X}(x_2) + |X|\} \leq |X| + 2,
\]
which implies
\[|X| \geq \frac{3n}{7} - 2.
\]
Using (3.4), (3.5), \(n \geq 7\) and \(c_1(G - X) = 1\), we deduce
\[n \geq |X| + c_1(G - X) + 3c_3(G - X) \geq |X| + 1 + 3 \times 2|X|
\]
\[= 7|X| + 1 \geq 7 \left( \frac{3n}{7} - 2 \right) + 1 = 3n - 13
\]
\[\geq n + 1,
\]
which is a contradiction.

**Case 3.** \(c_1(G - X) \geq 2\).

Let \(G_i\) be the component of \(G - X\) with \(|V(G_i)| = 1\) for \(i = 1, 2\). For \(x_i \in V(G_i)\), it is obvious that \(x_1 x_2 \notin E(G)\) and \(d_{G - X}(x_1) = d_{G - X}(x_2) = 0\). In terms of the condition of Theorem 1.3, we have
\[\frac{3n}{7} \leq \max\{d_G(x_1), d_G(x_2)\} \leq \max\{d_{G - X}(x_1) + |X|, d_{G - X}(x_2) + |X|\} = |X|.
\]
It follows from (3.1) and (3.6) that
\[n \geq |X| + c_1(G - X) + 3c_3(G - X) \geq |X| + c_1(G - X) + 2 \frac{2}{3} c_3(G - X)
\]
\[> |X| + \frac{4}{3} |X| + \frac{1}{3} = \frac{7}{3} |X| + \frac{1}{3}
\]
\[\geq \frac{7}{3} \times \frac{3n}{7} + \frac{1}{3} = n + \frac{1}{3} > n,
\]
which is a contradiction. This completes the proof of Theorem 1.3.
4. CONCLUDING REMARK

This paper investigates the existence of a \( \{P_2, P_3\} \)-factor in a graph, and puts forward two degree conditions for a graph to possess a \( \{P_2, P_3\} \)-factor. But we do not know whether the bounds on degree conditions in Theorems 1.2 and 1.3 are best possible or not, respectively. Naturally, we pose the following two conjectures:

**Conjecture 1.** A connected graph \( G \) with \( \delta(G) \geq 5 \) admits a \( \{P_2, P_3\} \)-factor if \( G \) satisfies

\[
\delta(G) \geq \frac{3\alpha(G) - 1}{4}.
\]

**Conjecture 2.** Let \( G \) be a connected graph of order \( n \) with \( n \geq 7 \). If \( G \) satisfies

\[
\max\{d_G(x), d_G(y)\} \geq \frac{3n - 1}{7}
\]

for any two nonadjacent vertices \( x \) and \( y \) of \( G \), then \( G \) has a \( \{P_2, P_3\} \)-factor.

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REFERENCES


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