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A NEW LOWER BOUND FOR THE INDEPENDENT DOMINATION NUMBER OF A TREE

Abel Cabrera-Martinez*

Abstract. A set D of vertices in a graph G is an independent dominating set of G if D is an independent set and every vertex not in D is adjacent to a vertex in D. The independent domination number of G, denoted by $i(G)$, is the minimum cardinality among all independent dominating sets of G. In this paper we show that if T is a nontrivial tree, then $i(T) \geq \frac{n(T)+\gamma(T)-l(T)+2}{4}$, where $n(T)$, $\gamma(T)$ and $l(T)$ represent the order, the domination number and the number of leaves of T, respectively. In addition, we characterize the trees achieving this new lower bound.

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1. Introduction

Let G be a simple graph of order $n(G) = |V(G)|$. Given a vertex $v \in V(G)$, the *degree* of v in G, denoted by $deg(v)$, is the cardinality of the *open neighbourhood* of v, *i.e.*, $deg(v) = |N(v)|$. If $deg(v) = 1$, then v is a *leaf* vertex. The set of leaves is denoted by $\mathcal{L}(G)$ and $l(G)$ represents the number of leaves of G, *i.e.*, $l(G) = |\mathcal{L}(G)|$.

A set $D \subseteq V(G)$ of vertices is said to be a *dominating set* of G if $N(v) \cap D \neq \emptyset$ for every vertex $v \in V(G) \setminus D$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of G. A $\gamma(G)$-*set* is a dominating set of G of cardinality $\gamma(G)$. Dominating sets and their variants in graphs have been interesting topics in graph theory. In the books [10,11], the authors expose some of the best known and studied varieties of dominating sets as well as some theoretical results and practical applications. In this article we will refer to one of these variants: the independent dominating sets of a graph.

A dominating set D of a graph G is said to be an *independent dominating set* of G if the subgraph induced by D is isomorphic to an empty graph. The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality among all independent dominating sets of G. An $i(G)$-*set* is an independent dominating set of G of cardinality $i(G)$. In general, the problem of computing this parameter is NP-hard [8]. The idea of obtaining an independent dominating set arose in chessboard problems. In particular, the goal was to find "*the minimum number of mutually non-attacking queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens*" [9]. For more information on independent domination see the excellent survey [9]. In addition, we suggest the recent papers [1,2,14–16].

Keywords. Independent domination number, Domination number, Trees.

Universidad de Córdoba, Departamento de Matemáticas, Campus de Rabanales, 14071 Córdoba, Spain.
*Corresponding author: acmartinez@uco.es

The aim of this paper is providing a new improved lower bound on this parameter in trees. One can find in the literature several works related to the same subject, but associated with other domination parameters. For instance, to just name a few of them, see [5, 6, 12, 17]. Let us first consider some of the known bounds for the independent domination number in trees. In 1992, Favaron [7] proved that $i(T) \leq (n(T) + l(T))/3$ for any nontrivial tree T. Thirty years later, Cabrera-Martínez [3] improved the previous bound, although for this he considered an additional parameter. In particular, he proved that $i(T) \leq (n(T) + \gamma(T) + l(T))/4$. Moreover, Lemańska [13] proved that $\gamma(T) \geq (n(T) - l(T) + 2)/3$. Since $i(T) \geq \gamma(T)$, it follows that the value $(n(T) - l(T) + 2)/3$ represents a lower bound for $i(T)$.

In summary, the following inequality chain shows the previously established lower and upper bounds for the independent domination number of a nontrivial tree T.

$$\frac{n(T) - l(T) + 2}{3} \leq i(T) \leq \frac{n(T) + \gamma(T) + l(T)}{4} \leq \frac{n(T) + l(T)}{3}. \tag{1}$$

Analysing carefully the inequality chain (1), it is observed that the bounds given by Favaron and Lemańska have a certain symmetry. In this sense, and taking into consideration the improved upper bound given by Cabrera-Martínez, it is natural to consider the following problem.

Problem 1.1. Give a lower bound for $i(T)$ in terms of $n(T)$, $\gamma(T)$ and $l(T)$ for any nontrivial tree T, which improves the bound $i(T) \geq (n(T) - l(T) + 2)/3$.

In this paper we give a solution to Problem 1.1. The statement of our main result follows.

Theorem 1.2. *If T is a nontrivial tree, then*

$$i(T) \geq \frac{n(T) + \gamma(T) - l(T) + 2}{4}.$$

The proof of the lower bound given in Theorem 1.2 is deferred to Section 2. The rest of the paper is organized as follows. In Subsection 1.1 we give some necessary terminology and tools needed to develop the proof of the main result. Finally, in Section 3 we characterize the trees achieving this new lower bound.

1.1. Some necessary terminology and tools

We begin this short subsection with some necessary terminology and notation. Let G be a nontrivial connected graph. Given a set $D \subseteq V(G)$, its *open neighbourhood* is $N(D) = \cup_{v \in D} N(v)$. Moreover, let $\mathcal{S}(G) = N(\mathcal{L}(G))$ and $\mathcal{SS}(G) = N(\mathcal{S}(G)) \setminus (\mathcal{L}(G) \cup \mathcal{S}(G))$. The set $\mathcal{S}(G)$ is generally called *support set of* G. As usual, let $G - D$ be the graph with vertex set $V(G - D) = V(G) \setminus D$ and edge set $E(G - D) = \{uv \in E(G) : u, v \in V(G) \setminus D\}$.

A *tree* is a connected and acyclic graph. A *rooted tree* is a tree with a distinguished special vertex r, called the root. Given two different vertices u and v in a tree T, the *distance* $d(u, v)$ equals the minimum length of a (u, v)-path in T. Moreover, by attaching a path P to a vertex v in T we mean adding the path P and joining v to a leaf of P.

Finally, we consider the following useful lemma.

Lemma 1.3. *[4] The following statements hold for any nontrivial tree T.*

(i) *If T is obtained from any nontrivial tree T' by attaching a path P_2 to any vertex $u \in \mathcal{S}(T') \cup \mathcal{SS}(T')$, then $\gamma(T) = \gamma(T') + 1$.*

(ii) *If T is obtained from any nontrivial tree T' by attaching a path P_3 to any vertex $u \in V(T')$, then $\gamma(T) = \gamma(T') + 1$.*

2. Proof of Theorem 1.2

In this section, we provide the following proof for the Theorem 1.2.

Proof of Theorem 1.2. We proceed by induction on the order of T. If T is a tree of order $n(T) \in \{2, 3, 4\}$, then it is easy to check that $i(T) \geq (n(T) + \gamma(T) - l(T) + 2)/4$. These particular cases establish the base cases. Let T be a tree of order $n(T) \geq 5$ and assume that $i(T') \geq (n(T') + \gamma(T') - l(T') + 2)/4$ for each tree T' of order $n(T') < n(T)$.

We next show that $i(T) \geq (n(T)+\gamma(T)-l(T)+2)/4$. For this purpose, let $v_1 \cdots v_d v_{d+1}$ be a diametrical path in T. From now on we assume that T is a rooted tree with root v_1. Notice that $v_d \in \mathcal{S}(T)$ and $v_{d+1} \in \mathcal{L}(T)$. We next analyse the following three cases, considering that D is an $i(T)$-set such that $|D \cap \mathcal{L}(T)|$ is minimum.

Case 1. $\deg(v_d) \geq 3$. Let $T' = T - \{v_{d+1}\}$. In this case, we have that $i(T) \geq i(T')$. Moreover, it is easy to check that $l(T) = l(T') + 1$ and $\gamma(T) = \gamma(T')$. Hence, by inequalities above and the induction hypothesis we obtain that

$$\begin{aligned} i(T) \geq i(T') &\geq \frac{n(T') + \gamma(T') - l(T') + 2}{4} \\ &= \frac{n(T) - 1 + \gamma(T) - (l(T) - 1) + 2}{4} \\ &= \frac{n(T) + \gamma(T) - l(T) + 2}{4}, \end{aligned}$$

as required.

Case 2. $\deg(v_d) = 2$ and $\deg(v_{d-1}) \geq 3$. In this case, we have that $v_{d-1} \in \mathcal{S}(T) \cup \mathcal{SS}(T)$. Let $T' = T - \{v_{d+1}, v_d\}$. Moreover, notice that $l(T) = l(T') + 1$ and by Lemma 1.3 we have that $\gamma(T) = \gamma(T') + 1$. Now, we observe that $|D \cap \{v_{d+1}, v_d\}| = 1$ and as $v_{d-1} \in \mathcal{S}(T) \cup \mathcal{SS}(T)$ and $|D \cap \mathcal{L}(T)|$ is minimum, it is easy to deduce that $D \cap V(T')$ is an independent dominating set of T'. So, $i(T') \leq |D \cap V(T')| = i(T) - 1$. Hence, by inequalities above and the induction hypothesis we obtain that

$$\begin{aligned} i(T) \geq i(T') + 1 &\geq \frac{n(T') + \gamma(T') - l(T') + 2}{4} + 1 \\ &= \frac{n(T) - 2 + (\gamma(T) - 1) - (l(T) - 1) + 2}{4} + 1 \\ &> \frac{n(T) + \gamma(T) - l(T) + 2}{4}, \end{aligned}$$

as required.

Case 3. $\deg(v_d) = \deg(v_{d-1}) = 2$. Let $T' = T - \{v_{d-1}, v_d, v_{d+1}\}$. Suppose that $v_d \notin D$. By the minimality of $|D \cap \mathcal{L}(T)|$, it follows that $v_{d-1} \in D$, which leads to $v_{d-2} \notin D$ and $v_{d+1} \in D$. If $|N(v_{d-2}) \cap D| \geq 2$, then $D' = (D \setminus \{v_{d-1}, v_{d+1}\}) \cup \{v_d\}$ is an independent dominating set of T such that $|D'| < |D| = i(T)$, a contradiction. Hence, $N(v_{d-2}) \cap D = \{v_{d-1}\}$. Now, we observe that $D'' = (D \setminus \{v_{d-1}, v_{d+1}\}) \cup \{v_{d-2}, v_d\}$ is an $i(T)$-set with $|D'' \cap \mathcal{L}(T)| < |D \cap \mathcal{L}(T)|$, a contradiction too. Therefore $v_d \in D$, and as a consequence, $v_{d-1}, v_{d+1} \notin D$. This implies that $D \setminus \{v_d\}$ is an independent dominating set of T'. Hence, $i(T') \leq |D \setminus \{v_d\}| = i(T) - 1$. In addition, by Lemma 1.3 we have that $\gamma(T) = \gamma(T') + 1$, and it is easy to see that $l(T') \leq l(T)$. Hence, by inequalities above and the induction hypothesis we obtain that

$$\begin{aligned} i(T) \geq i(T') + 1 &\geq \frac{n(T') + \gamma(T') - l(T') + 2}{4} + 1 \\ &= \frac{n(T) - 3 + (\gamma(T) - 1) - l(T) + 2}{4} + 1 \end{aligned}$$

$$= \frac{n(T) + \gamma(T) - l(T) + 2}{4},$$

as required.

Hence, from the three previous cases, the proof is complete. □

3. Trees T with $i(T) = \frac{n(T)+\gamma(T)-l(T)+2}{4}$

As previously exposed, Lemańska [13] proved that $\gamma(T) \geq (n(T) - l(T) + 2)/3$. In addition, she characterized the family $\mathcal{R}$ of trees which satisfy the equality above. In particular, a tree $T \in \mathcal{R}$ if $d(x, y) \equiv 2 \pmod 3$ for any different leaves $x, y \in \mathcal{L}(T)$.

Theorem 3.1. *[13] If T is a tree of order $n(T) \geq 3$, then $\gamma(T) = \frac{n(T)-l(T)+2}{3}$ if and only if $T \in \mathcal{R}$.*

In addition, she provided the following lemma, which describes an useful property on trees belonging to $\mathcal{R}$.

Lemma 3.2. *[13] Let T be a tree belonging to $\mathcal{R}$. If D is a $\gamma(T)$-set such that $D \subseteq \mathcal{S}(T)$, then $d(x, y) \equiv 0 \pmod 3$ for any different vertices $x, y \in D$.*

From the previous lemma, we deduce that if D is a $\gamma(T)$-set which satisfies Lemma 3.2, then D is also an independent dominating set of T. Hence, $i(T) \leq |D| = (n(T) - l(T) + 2)/3$ whenever $T \in \mathcal{R}$. Therefore, by Theorem 1.2 and the fact that $(n(T) - l(T) + 2)/3 \leq (n(T) + \gamma(T) - l(T) + 2)/4$, we obtain the next result.

Lemma 3.3. *Let T be a tree of order $n(T) \geq 3$. If $T \in \mathcal{R}$, then $i(T) = \frac{n(T)+\gamma(T)-l(T)+2}{4}$.*

In the light of Lemma 3.3, it is natural to ask whether the bound given in Theorem 1.2 is reached if and only if T belongs to $\mathcal{R}$. The following theorem answers the previous question in the affirmative.

Theorem 3.4. *Let T be a tree of order $n(T) \geq 3$. Then $i(T) = \frac{n(T)+\gamma(T)-l(T)+2}{4}$ if and only if $T \in \mathcal{R}$.*

Proof. By Lemma 3.3 we have that if $T \in \mathcal{R}$, then $i(T) = (n(T) + \gamma(T) - l(T) + 2)/4$. Now, we suppose that $T \notin \mathcal{R}$. We only need to prove that $i(T) > (n(T) + \gamma(T) - l(T) + 2)/4$. We proceed by induction on the order of T. Observe that $n(T) \geq 4$. If $n(T) = 4$, then T is the path P_4, and it is straightforward that $i(T) > (n(T) + \gamma(T) - l(T) + 2)/4$. This particular case establish the base case. From now on we assume that $i(T') > (n(T') + \gamma(T') - l(T') + 2)/4$ for each tree $T' \notin \mathcal{R}$ of order $n(T') < n(T)$. Let $v_1 \cdots v_d v_{d+1}$ be a diametrical path in T, and assume that T is a rooted tree with root v_1. Notice that $v_d \in \mathcal{S}(T)$ and $v_{d+1} \in \mathcal{L}(T)$. We next analyse the following three cases.

Case 1. $\deg(v_d) \geq 3$. Let $T' = T - \{v_{d+1}\}$. In this case, we have that $i(T) \geq i(T')$. Moreover, it is easy to check that $l(T) = l(T') + 1$ and $\gamma(T) = \gamma(T')$. Now, we observe that if $T' \in \mathcal{R}$, then $T \in \mathcal{R}$, a contradiction. Hence, $T' \notin \mathcal{R}$ and by induction hypothesis it follows that $i(T') > (n(T') + \gamma(T') - l(T') + 2)/4$. Therefore, by the previous inequalities we obtain that

$$\begin{aligned} i(T) \geq i(T') &> \frac{n(T') + \gamma(T') - l(T') + 2}{4} \\ &= \frac{n(T) - 1 + \gamma(T) - (l(T) - 1) + 2}{4} \\ &= \frac{n(T) + \gamma(T) - l(T) + 2}{4}, \end{aligned}$$

as required.

Case 2. $\deg(v_d) = 2$ and $\deg(v_{d-1}) \geq 3$. In this case, we proceed in the same way as in Case 2 of the proof of Theorem 1.2, obtaining that $i(T) > (n(T) + \gamma(T) - l(T) + 2)/4$, as required.

Case 3. $\deg(v_d) = \deg(v_{d-1}) = 2$. Let $T' = T - \{v_{d-1}, v_d, v_{d+1}\}$. Proceeding in an analogous way to Case 3 of the proof of Theorem 1.2, it is obtained that $i(T) \geq i(T') + 1$ and $\gamma(T) = \gamma(T') + 1$. Moreover, it is easy to deduce that $l(T') \leq l(T) \leq l(T') + 1$. In such a sense, we analyse the following two subcases.

Subcase 3.1. $l(T) = l(T') + 1$. From the previous inequalities and by Theorem 1.2 we obtain that

$$\begin{aligned} i(T) \geq i(T') + 1 &\geq \frac{n(T') + \gamma(T') - l(T') + 2}{4} + 1 \\ &= \frac{n(T) - 3 + (\gamma(T) - 1) - (l(T) - 1) + 2}{4} + 1 \\ &> \frac{n(T) + \gamma(T) - l(T) + 2}{4}, \end{aligned}$$

as required.

Subcase 3.2. $l(T) = l(T')$. In this subcase, it follows that $v_{d-2} \in \mathcal{L}(T')$. If T is the path P_5, then we are done. Hence, we assume that $n(T') \geq 3$. Now, we observe that if $T' \in \mathcal{R}$, then $d(x, v_{d-2}) \equiv 2 \pmod 3$ for any vertex $x \in \mathcal{L}(T') \setminus \{v_{d-2}\}$. So $d(x, v_{d+1}) \equiv 2 \pmod 3$ for any vertex $x \in \mathcal{L}(T) \setminus \{v_{d+1}\}$ because $d(v_{d-2}, v_{d+1}) = 3$, which implies that $T \in \mathcal{R}$, a contradiction. Therefore, $T' \notin \mathcal{R}$ and by induction hypothesis it follows that $i(T') > (n(T') + \gamma(T') - l(T') + 2)/4$. Thus, by the previous inequalities we obtain that

$$\begin{aligned} i(T) \geq i(T') + 1 &> \frac{n(T') + \gamma(T') - l(T') + 2}{4} + 1 \\ &= \frac{n(T) - 3 + (\gamma(T) - 1) - l(T) + 2}{4} + 1 \\ &= \frac{n(T) + \gamma(T) - l(T) + 2}{4}, \end{aligned}$$

as required.

Hence, from the three previous cases, the proof is complete. □

As an immediate consequence of the previous results, it follows that for any tree T of order at least three, $i(T) = \frac{n(T)+\gamma(T)-l(T)+2}{4} = \frac{n(T)-l(T)+2}{3}$ if only if $\gamma(T) = \frac{n(T)-l(T)+2}{3}$, which is equivalent to $T \in \mathcal{R}$.

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