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SOME BOUNDS ON SPECTRAL RADIUS OF SIGNLESS LAPLACIAN MATRIX OF k-GRAPHS

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Abstract. For a k-graph $H = (V(H), E(H))$, let $B(H)$ be its incidence matrix, and $Q(H) = B(H)B(H)^{\mathrm{T}}$ be its signless Laplacian matrix, and this name comes from the fact that $Q(H)$ is exactly the well-known signless Laplacian matrix for 2-graph. Define the largest eigenvalue $\rho(H)$ of $Q(H)$ as the spectral radius of H. In this paper, we give some lower and upper bounds on $\rho(H)$ using some structural parameters (such as independent number, maximum degree, minimum degree, diameter, and so on) of H, which extended or improved some known results.

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1. Introduction

The graph spectra theory concentrates on the studies of the connection between structural properties of a graph and the eigenvalues and eigenvectors of a matrix associated with that graph, which is a well-studied and highly applicable subject [2, 5, 9]. Based on the success of this field, many researchers try to expand the corresponding research to hypergraph by using some matrices related to it, for example, see [1,12,19–22]. Because of the integration of pioneering work of Qi [18], Lim [14], and Cooper and Dutle [8], the study of hypergraph by tensors has aroused wide interest of researchers in this field and, consequently, the study of hypergraph through its matrices has been put aside. Hillar and Lim [11] pointed out that most tensor problems are NP-hard, and so are the calculations of the eigenvalues of a tensor. This shows that the application of this theory has some limitations, so we also believe that the study of hypergraphs through matrices remains its place, as Cardoso and Trevisan said in [3].

Let $H = (V(H), E(H))$ be an k-uniform hypergraph (k-graph for simplicity) with vertex set $V(H) = \{v_1, v_2, \ldots, v_n\}$ and hyperedge set $E(H) = \{e_1, e_2, \ldots, e_m\} \subset V(H)^{(k)}$, where $V(H)^{(k)}$ is the family of all k-subsets of $V(H)$. Denote $N_H(v) = \{u : u \in V(H), u, v \in e, e \in E(H)\}$, $E_H(v) = \{e : v \in e, e \in E(H)\}$. Then the degree d_v of vertex v is equal to $|E_H(v)|$, the co-degree of vertices $u, v \in V(H)$ is the number of edges containing both u and v. The maximum, minimum and average degree, respectively, is defined as following:

$$\Delta(H) = \max_{v\in V(H)}\{d_v\}, \delta(H) = \min_{v\in V(H)}\{d_v\}, d(H) = \frac{\sum_{v\in V(H)} d_v}{n}.$$

Keywords. k-graph, incidence matrix, signless Laplacian matrix.

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For two vertices $u, v \in V(H)$, u, v are adjacent if there exists an edge e in H satisfying $u, v \in e$, otherwise non-adjacent. An independent set of H is a set of pairwise non-adjacent vertices of H. The independent number of H is the number of vertices of a maximum independent set of H.

For simplicity, let $\{1, 2, \ldots, n\} = [n]$. An alternating sequence of vertices and edges

$$P = v_1, e_1, v_2, e_2, \ldots, e_l, v_{l+1}$$

is a path of length l, if $v_1, v_2, \ldots, v_{l+1}$ are distinct vertices, $e_1, e_2, \ldots, e_l$ are distinct edges and $v_i, v_{i+1} \in e_i$ for $i \in [l]$. The distance between u and v is the length of a shortest path connecting them. The diameter $D(H)$ of H is the largest distance between any two vertices in $V(H)$. A hypergraph H is connected if there exists a path connecting u and v for any $u, v \in V(H)$. A hypergraph H is h-connected [26] if there exist h pairwise vertex-disjoint paths between u and v for any $u, v \in V(H)$. A hypergraph H is called to be a linear hypergraph if any two edges in H share at most one vertex.

The incidence matrix $B(H)$ of H is defined as $B(H) = (b(v, e))_{|V(H)|\times|E(H)|}$, where

$$b(v, e) = \begin{cases} 1, & \text{if } v \in e, \\ 0, & \text{otherwise.} \end{cases}$$

Define $Q(H) = B(H)B(H)^{\mathrm{T}} = (Q_{ij})$ be the signless Laplacian matrix of H [3] (This name may be used only because of the definition of signless Laplacian matrix of 2-graph.), obviously, Q_{ij} is exactly the co-degree of vertices v_i and v_j for $i \neq j$ and Q_{ii} is exactly the degree of vertex v_i. For an edge $e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(H)$ and $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in R^n$, let $x(e) = x_{i_1} + x_{i_2} + \cdots + x_{i_k}$, then $x^{\mathrm{T}}Q(H)x = X^{\mathrm{T}}B(H)B(H)^{\mathrm{T}}x = \sum_{e\in E(H)}[x(e)]^2$. For a connected k-graph, it is obvious that $Q(H)$ is non-negative, symmetric, Irreducible, and semi-definite positive. Then the spectral radius $\rho(Q(H))$ is the largest eigenvalue of $Q(H)$ and it has unique positive eigenvector $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ satisfying $||x||_2 = 1$. It is obvious that $\rho(Q(H)) = \max_{x\in R^n, ||x||_2=1} x^{\mathrm{T}}Q(H)x = \max_{x\in R^n,||x||_2=1} \sum_{e\in E(H)}[x(e)]^2$. We also call $\rho(Q(H))$ (for simplicity, $\rho(Q(H)) = \rho(H)$) the spectral radius of H and call x the Perron vector of H. Let $P_H(\lambda) = |\lambda I - Q(H)|$ be the characteristic polynomial of H, where I is an identity matrix of order $|V(H)|$.

In [24], Stevanović raised a problem: How small the difference between the maximum degree and the spectral radius of a 2-graph can be when it is an irregular 2-graph? This question is closely related to the structural properties of 2-graphs. Many results corresponding to this problem have been obtained, for example, see [4, 6, 7, 13, 16, 17, 23, 25]. In [3], Cardoso and Trevisan established some elegant relations between the structural properties of H and the algebra properties of $Q(H)$. For example, for a connected k-graph, it is regular if and only if $\rho(H) = kd(H) = k\Delta(H)$. Then for a connected irregular k-graph, it must have $kd(H) < \rho(H) < k\Delta(H)$. Similar to the problem proposed by Stevanović [24], we can also ask the following question:

Question 1. How small the differences $\rho(H) - kd(H)$ and $k\Delta(H) - \rho(H)$ can be when it is an irregular k-graph?

Based on the success of research results on the problem raised by Stevanović [24], in this paper, we will make further efforts to study structural properties of hypergraphs through the above question.

2. Main results

In this section, we first give some upper bounds on $\rho(H)$.

Theorem 2.1. *For a connected k-graph $H = (V(H), E(H))$ with $|V(H)| = n$, let d_{v_i} be the degree of vertex v_i, Q_{ij} be the co-degree of vertices v_i and v_j, b_i be a real positive number, where $i, j \in [n]$. Then*

$$\rho(H) \leq \max_{e\in E(H)} \max_{\{v_i,v_j\}\subseteq e} \frac{d_{v_i} + d_{v_j} + \sqrt{\left(d_{v_i} + d_{v_j}\right)^2 - 4\left(d_{v_i}d_{v_j} - B_iB_j\right)}}{2},$$

where $B_i = \sum_{h=1,h\neq i}^{n} b_i^{-1} b_h Q_{ih}$ for $i \in [n]$.

Proof. Let diagonal matrix $D = diag(b_1, b_2, \dots, b_n)$, then $\rho(D^{-1}Q(H)D) = \rho(Q(H)) = \rho(H)$ and there exists a positive eigenvector $x = (x_{v_1}, x_{v_2}, \dots, x_{v_n})^{\mathrm{T}}$ satisfying

$$D^{-1}Q(H)Dx = \rho(H)x. \tag{2.1}$$

Furthermore, we may assume that $x_{v_i} = 1$ and $x_{v_h} \le 1$ for $h \ne i$. Let $x_{v_j} = \max\{x_{v_p} : \{v_i, v_p\} \in e, e \in E(H)\}$. From (2.1), we have

$$\rho(H) - d_{v_i} = \sum_{h=1, h\ne i}^{n} b_i^{-1} b_h Q_{ih} x_{v_h} \le \sum_{h=1, h\ne i}^{n} b_i^{-1} b_h Q_{ih} x_{v_j} = B_i x_{v_j} \tag{2.2}$$

$$\big(\rho(H) - d_{v_j}\big) x_{v_j} = \sum_{h=1, h\ne j}^{n} b_j^{-1} b_h Q_{jh} x_{v_h} \le \sum_{h=1, h\ne j}^{n} b_j^{-1} b_h Q_{jh} = B_j. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\big(\rho(H) - d_{v_i}\big)\big(\rho(H) - d_{v_j}\big) \le B_i B_j.$$

Then

$$\rho(H) \le \max_{e\in E(H)} \max_{\{v_i, v_j\}\subseteq e} \frac{d_{v_i} + d_{v_j} + \sqrt{\big(d_{v_i} + d_{v_j}\big)^2 - 4\big(d_{v_i} d_{v_j} - B_i B_j\big)}}{2}.$$

□

For a connected linear k-graph H, it has $\sum_{h=1, h\ne i}^{n} Q_{ih} = (k-1)d_{v_i}$. It is easy to obtain the following result by Theorem 2.1.

Corollary 2.2. *For a connected linear k-graph $H = (V(H), E(H))$ with $|V(H)| = n$, let d_{v_i} be the degree of vertex $v_i, i \in [n]$. Then*

$$\rho(H) \le \max_{e\in E(H)} \max_{\{v_i, v_j\}\subseteq e} \frac{d_{v_i} + d_{v_j} + \sqrt{\big(d_{v_i} + d_{v_j}\big)^2 - 4 d_{v_i} d_{v_j} [1 - (k-1)^2]}}{2}.$$

Proof. Let $b_h = 1, h \in [n]$ in Theorem 2.1, we have $B_i = (k-1)d_{v_i}$ and

$$\rho(H) \le \max_{e\in E(H)} \max_{\{v_i, v_j\}\subseteq e} \frac{d_{v_i} + d_{v_j} + \sqrt{\big(d_{v_i} + d_{v_j}\big)^2 - 4 d_{v_i} d_{v_j} [1 - (k-1)^2]}}{2}.$$

□

For a nonnegative matrix $M = (m_{ij})_{n\times n}$, let $r_i(M) = \sum_{j=1}^{n} m_{ij}$. It is well known that $\min_{1\le i\le n} r_i(M) \le \rho(M) \le \max_{1\le i\le n} r_i(M)$. From this, we can obtain an upper bound on $\rho(H)$.

Theorem 2.3. *For a connected linear k-graph $H = (V(H), E(H))$ with $|V(H)| = n$, let $d_1 \ge d_2 \ge \cdots \ge d_n$ be its degree sequence. Then*

$$\rho(H) \le d_1 + (k-1) d_1^{\frac{1}{2}} d_2^{\frac{1}{2}}.$$

Proof. Let $D = diag(b_1, 1, \ldots, 1)$, then

$$D^{-1}Q(H)D = \begin{pmatrix} Q_{11} & b_1^{-1}Q_{12} & \cdots & b_1^{-1}Q_{1n} \\ b_1Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_1Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix}.$$

Note that H is a linear k-graph, let $b_1 = (\frac{d_1}{d_2})^{\frac{1}{2}}$, we have

$$r_1\big(D^{-1}Q(H)D\big) = d_1 + b_1^{-1}\sum_{j=2}^{n} Q_{1j} = d_1 + b_1^{-1}(k-1)d_1 = d_1 + (k-1)d_1^{\frac{1}{2}}d_2^{\frac{1}{2}},$$

and for $j \geq 2$,

$$\begin{aligned} r_j\big(D^{-1}Q(H)D\big) &= b_1Q_{j1} + \sum_{h=2}^{n} Q_{jh} = d_j + (b_1-1)Q_{j1} + \sum_{h=1, h\neq j}^{n} Q_{jh} \\ &= kd_j + (b_1-1)Q_{j1} \leq (k+b_1-1)d_j \leq (k+b_1-1)d_2 = (k-1)d_2 + d_1^{\frac{1}{2}}d_2^{\frac{1}{2}}. \end{aligned}$$

Since

$$\Big[d_1 + (k-1)d_1^{\frac{1}{2}}d_2^{\frac{1}{2}}\Big] - \Big[(k-1)d_2 + d_1^{\frac{1}{2}}d_2^{\frac{1}{2}}\Big] \geq (k-2)d_1^{\frac{1}{2}}d_2^{\frac{1}{2}} + (2-k)d_2 = (k-2)d_2^{\frac{1}{2}}\Big(d_1^{\frac{1}{2}} - d_2^{\frac{1}{2}}\Big) \geq 0,$$

we have $\rho(H) \leq \max_{1\leq i\leq n} r_i(Q(H)) = d_1 + (k-1)d_1^{\frac{1}{2}}d_2^{\frac{1}{2}}$. □

Remark 2.4. For $k = 2$, by Corollary 2.2, $\rho(H) \leq \max_{e\in E(H)} \max_{\{v_i,v_j\}\subseteq e}(d_{v_i} + d_{v_j})$ and $\rho(H) \leq d_1 + d_1^{\frac{1}{2}}d_2^{\frac{1}{2}}$, this is similar to the results in [25]; and if $d_{v_i} = d_{v_j} = \Delta$, $\rho(H) \leq k\Delta$, and this is exactly the upper bound obtained in [3].

In the following theorem, some lower and upper bounds related to Question 1 are obtained.

Theorem 2.5. *For $2 \leq k < n$, let $H = (V(H), E(H))$ be a connected irregular k-graph with order n and size m, then*

$$\rho(H) - kd(H) \geq \sum_{v_i\in S} \frac{d_{v_i}\left(\sqrt{\frac{sd_{v_i}^2}{\sum_{v_i\in S} d_{v_i}^2}} - 1\right)\left(\sqrt{\frac{sd_{v_i}^2}{\sum_{v_i\in S} d_{v_i}^2}} + 2k - 1\right)}{n}$$

for any independent set S of H with order s.

Proof. Let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be a vector in R^n, where

$$x_i = \begin{cases} \frac{a_i}{\sqrt{n}}, & \text{if } v_i \in S, \\ \frac{1}{\sqrt{n}}, & \text{otherwise,} \end{cases}$$

and $\sum_{i\in S} a_i^2 = s$. It is easy to that $\parallel x \parallel_2 = 1$. Let $\bar{S} = V(H) - S$, it is easy to see that $|e \cap S| \leq 1$ for any edge $e \in E(H)$. Then

$$\rho(H) - kd(H) = \rho(H) - \frac{k^2m}{n} \geq \sum_{e\in E(H)} x^2(e) - \frac{k^2m}{n}$$

$$= \sum_{e\in E(H),|e\cap S|=1} x^2(e) + \sum_{e\in E(H),|e\cap S|=0} x^2(e) - \frac{k^2 m}{n}$$
$$= \sum_{e\in E(H),|e\cap S|=1} \left[\frac{a_i+k-1}{\sqrt{n}}\right]^2 + \sum_{e\in E(H),|e\cap S|=0} \left(\frac{k}{\sqrt{n}}\right)^2 - \frac{k^2 m}{n}$$
$$= \sum_{v_i\in S} \frac{d_{v_i}(a_i+k-1)^2}{n} + \left(m - \sum_{i\in S} d_{v_i}\right)\frac{k^2}{n} - \frac{k^2 m}{n}$$
$$= \sum_{v_i\in S} \frac{d_{v_i}(a_i-1)(a_i+2k-1)}{n}.$$

Select $a_i = \sqrt{\frac{sd_{v_i}^2}{\sum_{v_i\in S} d_{v_i}^2}}$, we have

$$\rho(H) - kd(H) \geq \sum_{v_i\in S} \frac{d_{v_i}\left(\sqrt{\frac{sd_{v_i}^2}{\sum_{v_i\in S} d_{v_i}^2}} - 1\right)\left(\sqrt{\frac{sd_{v_i}^2}{\sum_{v_i\in S} d_{v_i}^2}} + 2k - 1\right)}{n}.$$

□

For any $x, y \in R$, it has $(x+y)^2 \geq 4xy$. Then the result in Theorem 2.5 can be simplified as follows.

Corollary 2.6. *For $2 \leq k < n$, let $H = (V(H), E(H))$ be a connected irregular k-graph with order n, size m, and independent number s, respectively. Then*

$$\rho(H) - kd(H) \geq \frac{4(k-1)\sqrt{s\sum_{v_i\in S} d_{v_i}^2} - k^2\sum_{v_i\in S} d_{v_i}}{n}.$$

Proof. By Theorem 2.5, we have

$$\rho(H) - kd(H) \geq \sum_{v_i\in S} \frac{d_{v_i}\left[\sqrt{\frac{sd_{v_i}^2}{\sum_{i\in S} d_{v_i}^2}} + (k-1)\right]^2 - k^2 d_{v_i}}{n}$$
$$\geq \sum_{v_i\in S} \frac{4(k-1)d_{v_i}\sqrt{\frac{sd_{v_i}^2}{\sum_{v_i\in S} d_{v_i}^2}} - k^2 d_{v_i}}{n} = \frac{4(k-1)\sqrt{s\sum_{v_i\in S} d_{v_i}^2} - k^2\sum_{v_i\in S} d_{v_i}}{n}.$$

□

Remark 2.7. For $k = 2$, $Q(H) = D + A(H)$ is exactly the signless Laplacian matrix of a 2-graph, where $A(H)$ (resp. D) is the adjacent matrix (resp. diagonal matrix of vertex degrees) of the 2-graph H. To some extent, Theorem 2.2 and Corollary 2.3 in [17] are special cases for $k = 2$ in our Theorem 2.5 and Corollary 2.6, respectively.

We will first give three lemmas which are useful for obtaining a lower bound on $k\Delta - \rho(H)$.

Lemma 2.8 ([13, 23])**.** *Let a, b, h_1, h_2 be positive numbers, then $a(h_1 - h_2)^2 + bh_2^2 \geq \frac{ab}{a+b}h_1^2$, the equality holds if and only if $h_2 = \frac{ah_1}{a+b}$*

Lemma 2.9. *Let $H=(V(H),E(H))$ be a connected k-graph with maximum degree Δ, spectral radius $\rho(H)$ and Perron vector $x=(x_{v_1},x_{v_2},\ldots,x_{v_n})^{\mathrm{T}}$, respectively, then*

$$k\Delta-\rho(H)=\sum_{v\in V(H)}(k\Delta-kd_v)x_v^2+\sum_{v_i,v_j\in e\in E(H)}\left(x_{v_i}-x_{v_j}\right)^2.$$

Proof. By a direct calculation, we have

$$\begin{aligned}k\Delta-\rho(H)&=k\Delta\sum_{v\in V(H)}x_v^2-x^{\mathrm{T}}Q(H)x\\&=\sum_{v\in V(H)}(k\Delta-kd_v)x_v^2+\sum_{v\in V(H)}kd_vx_v^2-\sum_{e\in E(H)}x(e)^2\\&=\sum_{v\in V(H)}(k\Delta-kd_v)x_v^2+\sum_{e\in E(H)}\left[k\sum_{v\in e}x_v^2-x(e)^2\right]\\&=\sum_{v\in V(H)}(k\Delta-kd_v)x_v^2+\sum_{v_i,v_j\in e\in E(H)}\left(x_{v_i}-x_{v_j}\right)^2.\end{aligned}$$

□

Lemma 2.10. *For a connected k-graph $H=(V(H),E(H))$, let $P=u_0,e_1,u_1,e_2,u_2,\ldots,u_{l-1},e_l,u_l$ be a shortest path from u to w with $u_0=u$ and $u_l=w$, and $x=(x_{v_1},x_{v_2},\ldots,x_{v_n})^{\mathrm{T}}$ be the Perron vector of H, then*

$$\sum_{e\in P}\sum_{\{v_i,v_j\}\subseteq e}\left(x_{v_i}-x_{v_j}\right)^2\geq\frac{k}{2l}(x_u-x_w)^2.$$

Proof. By a direct calculation, we have

$$\begin{aligned}\sum_{e\in P}\sum_{\{v_i,v_j\}\subseteq e}\left(x_{v_i}-x_{v_j}\right)^2&\geq\sum_{i=1}^{l}\left[\left(x_{u_{i-1}}-x_{u_i}\right)^2+\sum_{u_s\in e_i\setminus\{u_{i-1},u_i\}}\left(\left(x_{u_{i-1}}-x_{u_s}\right)^2+\left(x_{u_s}-x_{u_i}\right)^2\right)\right]\\&\geq\sum_{i=1}^{l}\left[\left(x_{u_{i-1}}-x_{u_i}\right)^2+\sum_{u_s\in e_i\setminus\{u_{i-1},u_i\}}\frac{1}{2}\left(x_{u_{i-1}}-x_{u_i}\right)^2\right]\\&=\sum_{i=1}^{l}\left[\left(x_{u_{i-1}}-x_{u_i}\right)^2+\frac{k-2}{2}\left(x_{u_{i-1}}-x_{u_i}\right)^2\right]=\frac{k}{2}\sum_{i=1}^{l}\left(x_{u_{i-1}}-x_{u_i}\right)^2\\&\geq\frac{k}{2l}(x_u-x_w)^2,\end{aligned}$$

where the second and the third inequalities follow from the Cauchy–Schwarz inequality. □

By the above three lemmas, we can obtain a lower bound on $k\Delta-\rho(H)$, which will extend the result in [15].

Theorem 2.11. *For $2\leq k<n$, let $H=(V(H),E(H))$ be a connected irregular k-graph with order n, size m, maximum degree Δ and diameter D, respectively, then*

$$k\Delta-\rho(H)\geq\frac{k\Delta(n\Delta-km)}{2Dkm(n\Delta-km)+n\Delta}.$$

Proof. Let $x = (x_{v_1}, x_{v_2}, \dots, x_{v_n})^{\mathrm{T}}$ be the Perron vector of H, $x_u = \max_{v_i \in V(H)} x_{v_i}$, $x_w = \min_{v_i \in V(H)} x_{v_i}$, $P = u_0, e_1, u_1, e_2, u_2, \dots, u_{l-1}, e_l, u_l$ be a shortest path from u to w with $u_0 = u$ and $u_l = w$. By Lemmas 2.8–2.10, we have

$$\begin{aligned} k\Delta - \rho(H) &= \sum_{v \in V(H)} (k\Delta - kd_v)x_v^2 + \sum_{e=\{v_1,v_2,\dots,v_k\} \in E(H)} \sum_{v_i, v_j \in e} \left(x_{v_i} - x_{v_j}\right)^2 \\ &\geq \sum_{v \in V(H)} (k\Delta - kd_v)x_v^2 + \sum_{e \in P} \sum_{\{v_i, v_j\} \subseteq e} \left(x_{v_i} - x_{v_j}\right)^2 \\ &\geq \left(kn\Delta - k^2 m\right)x_w^2 + \frac{k}{2l}(x_u - x_w)^2 \geq \left(kn\Delta - k^2 m\right)x_w^2 + \frac{k}{2D}(x_u - x_w)^2 \\ &\geq \frac{k(n\Delta - km)}{2D(n\Delta - km) + 1} x_u^2. \end{aligned}$$

Note that

$$\rho(H) = x^{\mathrm{T}} Q(H) x = \sum_{e \in E(H)} [x(e)]^2 \leq \sum_{e \in E(H)} [kx_u]^2 = k^2 m x_u^2,$$

we have

$$x_u^2 \geq \frac{\rho(H)}{k^2 m}.$$

Further by direct calculation, we obtain our result

$$k\Delta - \rho(H) \geq \frac{k\Delta(n\Delta - km)}{2Dkm(n\Delta - km) + n\Delta}.$$

□

Remark 2.12. Since $Q(H) = 2A_\alpha(H) = 2[\alpha D + (1-\alpha)A(H)]$ for $k = 2$ and $\alpha = \frac{1}{2}$. In this case, Theorem 2.11 is exactly consistent with the result in [15].

Now we are going to study another lower bound on $k\Delta - \rho(H)$ according to hypergraph parameters: order, size, connectivity, maximum degree, and minimum degree. Let's first give two lemmas.

Lemma 2.13 ([26]). *If a hypergraph $H = (V(H), E(H))$ is h-connected, then there are h mutual vertex-disjoint paths between each pair of vertices.*

Lemma 2.14. *Let $H = (V(H), E(H))$ be a connected k-graph with maximum degree Δ, minimum degree δ, and Perron vector $x = (x_{v_1}, x_{v_2}, \dots, x_{v_n})^{\mathrm{T}}$, then*

$$\left(\max_{v_i \in V(H)} x_{v_i}\right)^2 \geq \frac{1}{\frac{\delta}{\Delta} + n - 1}.$$

Proof. Without loss of generality, let $d_{v_p} = \Delta$ and $d_{v_q} = \delta$. From $Q(H)x = \rho(H)x$, we have

$$\begin{aligned} \rho(H)x_{v_p} &= (Q(H)x)_p = \sum_{e \in E_{[p]}} x(e) \geq \sum_{e \in E_{[p]}} \left(k \min_{v_i \in V(H)} x_{v_i}\right) = k\Delta \min_{v_i \in V(H)} x_{v_i}, \\ \rho(H)x_{v_q} &= (Q(H)x)_q = \sum_{e \in E_{[p]}} x(e) \leq \sum_{e \in E_{[p]}} \left(k \max_{v_i \in V(H)} x_{v_i}\right) = k\delta \max_{v_i \in V(H)} x_{v_i}. \end{aligned}$$

Then

$$\frac{\Delta}{\delta} \le \frac{\max_{v_i \in V(H)} x_{v_i}}{\min_{v_i \in V(H)} x_{v_i}} \cdot \frac{x_{v_p}}{x_{v_q}} \le \left(\frac{\max_{v_i \in V(H)} x_{v_i}}{\min_{v_i \in V(H)} x_{v_i}} \right)^2.$$

Further together with

$$1 = \sum_{i=1}^{n} x_i^2 \le \left(\min_{v_i \in V(H)} x_{v_i} \right)^2 + (n-1) \left(\max_{v_i \in V(H)} x_{v_i} \right)^2,$$

we have

$$\left(\max_{v_i \in V(H)} x_{v_i} \right)^2 \ge \frac{1}{\frac{\delta}{\Delta} + n - 1}.$$

□

From the above two lemmas, we obtain another lower bound on $k\Delta - \rho(H)$.

Theorem 2.15. *For $2 \le k < n$, let $H = (V(H), E(H))$ be a h-connected irregular linear k-graph with order n, size m, maximum degree Δ, minimum degree δ, then*

$$k\Delta - \rho(H) \ge \frac{kh^2(n\Delta - km)}{(\frac{\delta}{\Delta} + n - 1)[2(n\Delta - km)(m + 2h - \Delta - \delta) + h^2]}.$$

Proof. Let $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^{\mathrm{T}}$ be the Perron vector of H, $x_u = \max_{v_i \in V(H)} x_{v_i}$, and $x_w = \min_{v_i \in V(H)} x_{v_i}$. Since H is a h-connected hypergraph, by Lemma 2.13, there exist at least h mutual vertex-disjoint paths $P_t : u = v_{0,t}, e_{1,t}, v_{1,t}, \ldots, e_{l_t,t}, v_{l_t,t} = w$ between u and w, $t \in [h]$. By Lemma 2.10, we have

$$\sum_{\{v_{i,t}, v_{j,t}\} \subseteq e \in E(P_t)} \left(x_{v_{i,t}} - x_{v_{j,t}} \right)^2 \ge \frac{k}{2l_t} (x_u - x_w)^2.$$

Further, we have

$$\begin{aligned} \sum_{\{v_i, v_j\} \subseteq e \in E(H)} \left(x_{v_i} - x_{v_j} \right)^2 &\ge \sum_{t=1}^{h} \sum_{\{v_{i,t}, v_{j,t}\} \subseteq e \in E(P_t)} \left(x_{v_{i,t}} - x_{v_{j,t}} \right)^2 \\ &\ge \sum_{t=1}^{h} \frac{k}{2l_t} (x_u - x_w)^2 \ge \frac{kh^2}{\sum_{t=1}^{h} 2l_t} (x_u - x_w)^2. \end{aligned}$$

The last inequality follows from the Cauchy–Schwarz inequality. Now from Lemma 2.9, we have

$$\begin{aligned} k\Delta - \rho(H) &= \sum_{v \in V(H)} (k\Delta - kd_v) x_v^2 + \sum_{\{v_i, v_j\} \subseteq e \in E(H)} \left(x_{v_i} - x_{v_j} \right)^2 \\ &\ge \left(kn\Delta - k^2 m \right) x_w^2 + \frac{kh^2}{\sum_{t=1}^{h} 2l_t} (x_u - x_w)^2 \\ &\ge \frac{\left(kn\Delta - k^2 m \right) \frac{kh^2}{\sum_{t=1}^{h} 2l_t}}{(kn\Delta - k^2 m) + \frac{kh^2}{\sum_{t=1}^{h} 2l_t}} x_u^2 = \frac{kh^2 (n\Delta - km)}{2(n\Delta - km) \sum_{t=1}^{h} l_t + h^2} x_u^2. \end{aligned}$$

The last inequality follows from Lemma 2.8. Further by Lemma 2.14, we have

$$k\Delta - \rho(H) \ge \frac{kh^2(n\Delta - km)}{(\frac{\delta}{\Delta} + n - 1) \left[2(n\Delta - km) \sum_{t=1}^{h} l_t + h^2 \right]}.$$

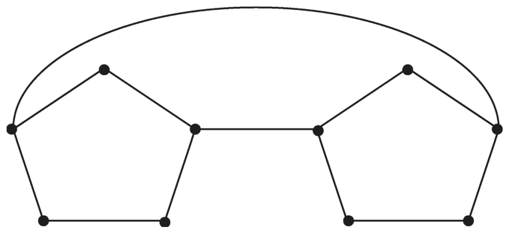

FIGURE 1. The example graph G.

Case 1. If $d_H(u) = \Delta$, we have $\sum_{t=1}^{h} l_t \leq m - (\Delta - h) - (\delta - h) = m + 2h - \Delta - \delta$ since $d_H(v) \geq \delta$. Then

$$k\Delta - \rho(H) \geq \frac{kh^2(n\Delta - km)}{\left(\frac{\delta}{\Delta} + n - 1\right)[2(n\Delta - km)(m + 2h - \Delta - \delta) + h^2]}.$$

Case 2. If $d_H(u) \leq \Delta - 1$, from $Q(H)x = \rho(H)x$, we have

$$\rho(H)x_u = \sum_{e \in E_{[u]}} x(e) \leq \sum_{e \in E_{[u]}} (kx_u) = kd_H(u)x_u \leq k(\Delta - 1)x_u.$$

Further by $(\frac{\delta}{\Delta} + n - 1)[2(n\Delta - km)(m + 2h - \Delta - \delta) + h^2] > 2(n-1)h(n\Delta - km) > h^2(n\Delta - km)$, we have

$$k\Delta - \rho(H) \geq k \geq \frac{kh^2(n\Delta - km)}{\left(\frac{\delta}{\Delta} + n - 1\right)[2(n\Delta - km)(m + 2h - \Delta - \delta) + h^2]}.$$

□

Example. Let G (as shown in Fig. 1) be an irregular 2-graph with $n = 10, m = 12, h = 2, \Delta = 3$, and $\delta = 2$. By Theorem 2.15, we have $k\Delta - \rho(H) \geq 0.044335$, by Theorem 3.2 in [17], $k\Delta - \rho(H) \geq 0.026667$. This show that Theorem 2.15 is better than Theorem 3.2 in [17] in some case. Note that

$$\frac{kh^2(n\Delta - km)}{\left(\frac{\delta}{\Delta} + n - 1\right)[2(n\Delta - km)(m + 2h - \Delta - \delta) + h^2]} > \frac{kh^2(n\Delta - km)}{\left(\frac{\delta}{\Delta} + n - 1\right)[2m(n\Delta - km) + h^2]}$$

and $Q(H) = 2A_\alpha(H) = 2[\alpha D + (1 - \alpha)A(H)]$ for $k = 2$ and $\alpha = \frac{1}{2}$. In this case, we can see that the result in Theorem 2.15 is better the result in [10].

In the following, we will turn to study the lower bound on $kd - \rho(H)$ by hypergraph parameters: order, size and connectivity. Let's first give two useful lemmas.

Lemma 2.16 ([3])**.** *Let $G = (V(G), E(G))$ be a connected k-graph, then $\rho(G)$ is an algebraically simple eigenvalue, with a positive eigenvector.*

Lemma 2.17. *Let $G = (V(G), E(G))$ be a connected k-graph. If H is a proper subhypergraph of G, then $\rho(H) < \rho(G)$.*

Proof. If H is connected spanning proper subhypergraph of G, then $Q(H)$ is irreducible. By Lemma 2.16, there exists a positive principal eigenvector $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^{\mathrm{T}}$ of H. Then $\rho(H) = x^{\mathrm{T}}Q(H)x$ and $\rho(G) \geq x^{\mathrm{T}}Q(G)x$.

If $\rho(G) = x^{\mathrm{T}}Q(G)x$, that is, x is also the eigenvector of $Q(G)$ corresponding to $\rho(G) = \rho(H)$, then $Q(G)x = Q(H)x$. Since H is connected spanning proper subhypergraph of G, there must exist a $u \in V(G)(= V(H))$ such that $E_{[u]}(H) \subset E_{[u]}(G)$, then

$$(Q(G)x)_u = \sum_{e\in E_{[u]}(G)} x(e) > \sum_{e\in E_{[u]}(H)} x(e) = (Q(H)x)_u,$$

a contradiction.

If H is a proper subhypergraph of G, it must be a subhypergraph of a connected spanning proper subhypergraph H_1 of G. Then we have $\rho(G) > \rho(H_1) \geq \rho(H)$. □

Remark 2.18. Obviously, this result is stronger than Proposition 12 in [3].

Now we can deduce the following main result.

Theorem 2.19. *For $2 \leq k < n, h \geq 1$, let $G = (V(G), E(G))$ be a h-connected d-regular k-graph with order n, size m. Then*

$$kd - \rho(H) \geq \begin{cases} \frac{dk^2(h-1)^2}{[2k(m+2h-2d-2)+(h-1)^2](nd-1)}, & \text{if } h \geq 2, \\ \frac{dk^2}{(2k(m-2d+2)+1|)(nd-1)}, & \text{if } h = 1. \end{cases}$$

Proof. By Lemma 2.17, $\max_{H\subset G} \rho(H) = \max_{e\in E(G)}\{\rho(G-e)\}$, so we only need to consider the hypergraphs in $\{G-e, e \in E(G)\}$ for obtaining a lower bound on $kd-\rho(H)$. For convenience, let $e = \{v_1, v_2, \ldots, v_k\}$, $H = G-e$, $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^{\mathrm{T}}$ be the Perron vector of H, $x_{v_1} = \min_{v\in e} x_v, x_u = \min_{v\in V(H)} x_v, x_w = \max_{v\in V(H)} x_v$, respectively. Then $w \neq v_i, i \in [k]$. Otherwise, $w = v_i$ for some $i \in [k]$, it has

$$\rho(H)x_w = \sum_{e\in E_{[w]}} x(e) \leq \sum_{e\in E_{[w]}} kx_w = k(d-1)x_w,$$

so $\rho(H) \leq k(d-1)$. But from

$$\rho(H) \geq \frac{k^2m}{n} = k\frac{\sum_{v\in V(H)} d_H(v)}{n} = k\frac{(n-k)d + k(d-1)}{n} = k\left(d - \frac{k}{n}\right) > k(d-1),$$

a contradiction. Thus $d_H(w) = d$.

Case 1. $h \geq 2$. Let $P_1, P_2, \ldots, P_{h-1}$ be $h-1$ pairwise vertex-disjoint paths from u to w in H. Then

$$\sum_{t=1}^{h-1} |E(P_t)| \leq (m-1) - [d-(h-1)] - [(d-1)-(h-1)] = m + 2h - 2d - 2,$$

since $d_H(u) \geq d-1$.

By Lemmas 2.9, 2.10 and 2.14, we have

$$\begin{aligned} kd - \rho(H) &= \sum_{v\in V(H)} (kd - kd_v)x_v^2 + \sum_{u_i,u_j\in a\in E(H)} \left(x_{u_i} - x_{u_j}\right)^2 \\ &\geq k\left(x_{v_1}^2 + x_{v_2}^2 + \cdots + x_{v_k}^2\right) + \sum_{t=1}^{h-1} \sum_{u_i,u_j\in a\in E(P_t)} \left(x_{u_i} - x_{u_j}\right)^2 \\ &\geq k^2x_u^2 + \sum_{t=1}^{h-1} \frac{k}{2|E(P_t)|}(x_u - x_w)^2 \end{aligned}$$

$$\begin{aligned}
&\geq k^2x_u^2+\frac{k(h-1)^2}{2\sum_{t=1}^{h-1}|E(P_t)|}(x_u-x_w)^2\\
&\geq k^2x_u^2+\frac{k(h-1)^2}{2(m+2h-2d-2)}(x_u-x_w)^2\\
&\geq \frac{k^2(h-1)^2}{2k(m+2h-2d-2)+(h-1)^2}x_w^2\\
&\geq \frac{dk^2(h-1)^2}{[2k(m+2h-2d-2)+(h-1)^2](nd-1)}.
\end{aligned}$$

Case 2. $h=1$.

Subcase 2.1. H is connected. Let P be path connecting u and w in H,

$$|E(P)|\leq (m-1)-(d-1)-[(d-1)-1]=m-2d+2.$$

By Lemmas 2.9, 2.10 and 2.14, we have

$$\begin{aligned}
kd-\rho(H)&\geq k\big(x_{v_1}^2+x_{v_2}^2+\cdots+x_{v_k}^2\big)+\sum_{u_i,u_j\in a\in E(P)}\big(x_{u_i}-x_{u_j}\big)^2\\
&\geq k^2x_u^2+\frac{k}{2|E(P)|}(x_u-x_w)^2\\
&\geq \frac{k^2}{2k|E(P)|+1}x_w^2\geq \frac{k^2}{2k(m-2d+2)+1}x_w^2\\
&\geq \frac{dk^2}{(2k(m-2d+2)+1|)(nd-1)}.
\end{aligned}$$

Subcase 2.2. H is not connected. Assume that $H=H_1\cup H_2\cup\cdots\cup H_b, b\geq 2$ and $\rho(H)=\rho(H_1)$. Let $y=(y_1,y_2,\ldots,y_{|V(H_1)|})^{\mathrm{T}}$ be the positive principal eigenvector of H_1, $y_u=\min_{v\in V(H_1)}y_v, y_w=\max_{v\in V(H_1)}y_v$. Similar to the proof of Case 2.1, we have

$$\begin{aligned}
kd-\rho(H)&\geq \frac{k^2}{2k|E(P)|+1}y_w^2\geq \frac{k^2}{2k(m-2d+2)+1}y_w^2\\
&\geq \frac{dk^2}{(2k(m-2d+2)+1|)(|V(H_1)|d-1)}>\frac{dk^2}{(2k(m-2d+2)+1|)(nd-1)}.
\end{aligned}$$

This completes the proof. □

3. Conclusions

Since the definition of adjacency tensor of hypergraph proposed by Cooper and Dutle [8], the study of hypergraph by tensors has aroused wide interest of researchers. But due to their computational complexity and theoretical difficulty, the study of hypergraphs through matrices remains its place. In this paper, we obtain some lower and upper bounds on the spectral radius of signless Laplacian matrix of hypergraph by some structural parameters such as independent number, maximum degree, minimum degree, diameter, connectivity, and so on. However, it is very difficult to characterize the corresponding extremal structures. Obviously, a natural problem is how to improve these lower and upper bounds and characterize their corresponding extremal hypergraphs.

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