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PROXIMAL ALGORITHM WITH QUASIDISTANCES FOR MULTIOBJECTIVE QUASICONVEX MINIMIZATION IN RIEMANNIAN MANIFOLDS

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Abstract. We introduce a proximal algorithm using quasidistances for multiobjective minimization problems with quasiconvex functions defined in arbitrary Riemannian manifolds. The reason of using quasidistances instead of the classical Riemannian distance comes from the applications in economy, computer science and behavioral sciences, where the quasidistances represent a non symmetric measure. Under some appropriate assumptions on the problem and using tools of Riemannian geometry we prove that accumulation points of the sequence generated by the algorithm satisfy the critical condition of Pareto-Clarke. If the functions are convex then these points are Pareto efficient solutions.

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1. Introduction

Martinet [23] introduced the Proximal Point Method (PPM) to solve minimization models with convex functions in Hilbert spaces. This method was investigated by Rockafellar [34] and various researchers to find zeroes of monotone operators, solution points of variational inequalities, complementary problems, equilibrium problems, multiobjective minimization problems, vector optimization problems and bi-level optimization problems, between others.

Bonnel *et al.* [13] studied the PPM to find optimal points of vector minimization problems with convex objective functions and proved the convergence of the iterates generated by both exact and inexact algorithms. Extensions and variants of the Bonnel *et al*, approach were introduced in [2, 8, 14, 15, 17, 29, 33, 46].

More generally, the PPM has been studied to solve mono and multiobjective optimization problems in Hadamard manifolds as we can verify, for example, in the following papers [6, 10, 11, 16, 21, 27, 28, 30, 44] and

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references therein. Observe that Riemannian manifolds are natural spaces where we can obtain several applications, see for example [1, 19, 22].

However, it has been observed that to cover more applications it is need to substitute in the proximal regularization function the Riemannian distance by using a quasidistance. This has been done in recent years due to a close relationship between the PPM and the variational rationality model of Soubeyran [37–43] and Attouch and Soubeyran [3]. We may find works using this methodology with applications in economics, social and behavioral sciences, see [5, 19, 25, 35].

The first paper about proximal method and quasidistances to solve minimization problems was introduced by Moreno *et al.* [25] where it was proved the convergence to a limiting critical point of a sequence generated by an exact PPM using the condition of Kurdyka-Lojasiewicz (KL). After that, Rocha *et al.* [33] presented an exact PPM with quasidistances for multiobjective minimization (in the Euclidean space) and proved without the KL condition but under the convexity of the objective functions, weak results of convergence (all cluster points are Pareto efficient solutions). According to our review, researches on PPMs using quasidistances for multiobjective problems on arbitrary Riemannian manifolds has not yet been developed. Previous works have focused in working with PPMs on Hadamard manifolds associated with the Riemannian distance or using other distances, see the papers [10, 11, 16, 30, 44].

The main interest of our work is to analyse mathematically the properties of a Proximal Point Algorithm (PPA) using quasidistances to find Pareto-Clarke critical points or Pareto efficient solutions of multiobjective minimization problems where the objectives are quasiconvex functions defined in arbitrary Riemannian manifolds. Quasiconvex functions is a class of nonconvex functions which has a strong relation with convex preferences in economy, that is, diversification of the consumption. We can also find quasiconvex functions in fractional optimization, location theory and surrogative optimization. On the other hand, our interest in inexact algorithms is because exact solutions of the regularized subproblems of the PPA are very hard to find, see [34], and thus inexact versions are more recommended to obtain approximate solutions of the proximal subproblems.

The introduced algorithm generates a sequence of points on the Riemannian manifold such that each point of that sequence is an approximation of a Clarke critical point of the proximal regularized function using quasidistance instead of Riemannian distances. It worthwhile to mention that, from our point of view, the algorithm and the results presented in this paper are new inclusive in the Euclidian space. It is proved that accumulation points are Pareto-Clarke critical points and Pareto efficient solutions for the convex case (more than weak efficient solutions find in the literature). Also, we give an application to economy following the methodology of the worthwhile to change of the Variational Principle where we found an interesting interpretation of equilibrium with respect to profits of each function.

The paper is distributed a follow: Section 2 gives basic tools on Riemannian manifolds and multiobjective optimization. Section 3 presents subdifferential theory on Clarke subdiffeential. Section 4 presents quasidistances in Riemannian manifolds. Section 6 introduces the proximal algorithm and presents the convergence results. In Section 7, an application of the algorithm in consumer demand is presented. Finally, Section 8, gives some conclusions, as also, some future research.

2. Riemannian geometry tools

We will introduce notations and basic results that we are going to use in the paper. For interested reader of these concepts and more properties we recommend the following references [20, 32, 36, 45]. We also introduce definitions of quasiconvexity, normal cone and Pareto efficient solutions for multiobjective minimization problems on Riemannian manifolds.

Along the paper M is a real finite dimensional and complete Riemannian manifold. T_xM denotes the tangent space of M at x. Given $v, w \in T_xM$ the inner product of v and w at $x \in M$ is defined by $\langle v, w\rangle := g(v, w)$, where g is the Riemannian metric at x. The norm of $v \in T_xM$ is defined by $\|v\| := \langle v, v\rangle^{1/2}$. We denote the Riemannian distance between two points x and y by $d(x, y)$ and the exponential map $\exp_x : T_xM \to M$ is defined by $\exp_x(v) = \gamma(1)$, where γ is a geodesic starting at x and with velocity $v \in T_xM$.

The normal cone of a totally convex set X at a point $x \in X$ is defined by $\mathcal{N}_X(x) = \{w \in T_xM : \langle w, \gamma'(0)\rangle \le 0\}$, where γ is a geodesic starting at x. If $X = \{x\}$, then we define $\mathcal{N}_X(x) = T_xM$.

We use the notation $F = (F_1, \dots, F_m) : M \to \mathbb{R}^m$ to mean that F is a multifunction where each $F_i : M \to \mathbb{R}$. We say that F is quasiconvex, if each F_i is quasiconvex in M. We refer to Section 2 of [30], for definitions of locally Lipschitz, continuous, differentiable and continuously differentiable. We use also the notation $F(y) \preceq F(x)$ ($F(y) \prec F(x)$) to mean $F_i(y) \le F_i(x)$ ($F_i(y) \le F_i(x)$ respectively) for all $i = 1, 2, \dots, m$.

F is quasiconvex in M if each function F_i, where $i = 1, 2, \dots, m$, is a quasiconvex function in M, that is, if for all $x, y \in M$, $t \in [0, 1]$, it holds that

$$F_i(\gamma(t)) \le \max\{F_i(x), F_i(y)\},$$

for each geodesic $\gamma : [0, 1] \to \mathbb{R}$, so that $\gamma(0) = x$ and $\gamma(1) = y$.

F is convex in M if each function F_i, where $i = 1, 2, \dots, m$, is a convex function in M, that is, if for all $x, y \in M$, $t \in [0, 1]$, it holds that

$$F_i(\gamma(t)) \le tF_i(y) + (1 - t)F_i(x),$$

for each geodesic $\gamma : [0, 1] \to \mathbb{R}$, so that $\gamma(0) = x$ and $\gamma(1) = y$.

The optimization model to be considered is

$$\min\{F(x) : x \in M\}, \tag{1}$$

where $F(x) = (F_1(x), F_2(x), \dots, F_m(x))$ and each $F_i : M \to \mathbb{R}$. We will use also the concepts of Pareto efficient solution and weak Pareto efficient solution of the problem (1) which can be find in Section 2 of [30].

3. Fréchet and Clarke subdifferentials

Let M be a Riemanninan manifold and $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function with $\mathrm{dom}(f) \neq \emptyset$. The Fréchet subdifferential of f at a point $x \in \mathrm{dom}(f)$, denoted by $\partial^F f(x)$, is defined

$$\partial^F f(x) = \left\{d\phi_x : \phi \in C^1(M, \mathbb{R}), f - \phi \text{ has a local minimum at } x\right\},$$

where $d\phi_x$ is the differential of ϕ at x. If $x \notin \mathrm{dom}(f)$ then we define $\partial^F f(x) = \emptyset$. Azagra *et al.* proved in Theorem 4.3 of [4] that this definition is equivalent to the following: For each chart $h : U \subset M \to \mathbb{R}^n$ with $x \in U$, if $\zeta_\phi = d\phi_x \circ d(h^{-1})_{h(x)}$ the following inequality is satisfied:

$$\liminf_{v \to 0} \frac{1}{\|v\|} \left[\left(foh^{-1}\right)(h(x) + v) - f(x) - \langle \zeta, v\rangle\right] \ge 0,$$

where ζ is the unique element of $T_\circ(T_xM) \equiv T_xM$ obtained from the Riez lemma which satisfies $\zeta_\phi(v) = \langle \zeta, v\rangle$. Now, considering the chart $h(\cdot) = \exp_x^{-1}(\cdot)$ defined on $B(x, r_x)$ where $B(x, r_x) = \exp_x(B(0, r_x))$ is a normal ball and using the fact that $d(\exp_x)_0$ is the identity, in the above inequality we obtain

$$\liminf_{v \to 0} \frac{1}{\|v\|} [f(\exp_x v) - f(x) - \langle \zeta, v\rangle] \ge 0. \tag{2}$$

Thus, equation (2) implies that

$$\partial^F f(x) = \partial^F (f \circ \exp_x)(0_x). \tag{3}$$

Note further that $\partial^F f(x)$ may be empty, but if f attains a local minimum at x, then

$$0 \in \partial^F f(x). \tag{4}$$

A point $x \in M$ satisfying the above inclusion is called Fréchet critical point.

Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper function with effective domain denoted by $\text{dom}(f)$. Consider also that f is locally Lipschitz. The Clarke's directional derivate of f at $x \in M$ in the direction $v \in T_xM$, introduced by Motreanu and Pavel [26] is defined by

$$f^\circ(x, v) := \limsup_{\substack{u\to x\\ t\searrow 0}} \frac{f \circ \varphi^{-1}(\varphi(u) + td\varphi_x(v)) - f \circ \varphi^{-1}(\varphi(u))}{t}, \tag{5}$$

where (φ, U) is a chart at x. Observe that the above definition is equivalent to $f^\circ(x, v) = (f\circ\varphi^{-1})^\circ(\varphi(x), d\varphi_x(v))$ and as this definition is independent of charts it gives

$$f^\circ(x, w) = (f \circ \exp_x)^\circ(0_x, w). \tag{6}$$

Of fact, taking $U = B(x, r_x)$ where where $B(x, r_x) = \exp_x(B(0, r_x))$ is a normal ball and the chart $\varphi = \exp_x^{-1}$. Then, define the curve $\alpha(t) = \exp_x(tv)$, where $t < \frac{r_x}{\|v\|}$ we obtain that $\alpha(0) = x$ and $\alpha'(0) = d(\exp_x)_{0_x}(v) = v$. Thus $d\varphi_x(v) = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_x \circ \alpha)(0) = v$.

The Clarke subdifferential of a function f at $x \in M$, is a subset of $T_xM^* \simeq T_xM$, given by

$$\partial^\circ f(x) = \{s \in T_xM : \langle s, v\rangle \le f^\circ(x, v),\ \forall\, v \in T_xM\}.$$

Observe that $\partial^\circ f(x)$ is a nonempty set, as also, closed and convex.

Remark 1. We recall that if $f : M \to \mathbb{R}\cup\{+\infty\}$ is a proper locally Lipschitz function then, $\partial^F f(x) \subseteq \partial^\circ f(x)$ for all $x \in \text{dom}(f)$. It is due from (3) and (6) and by Bolte *et al.* ([12], inequality (7)).

Due to above remark, the following result may be proved easily.

Lemma 1. *Let $f_i : M \to \mathbb{R} \cup \{+\infty\}, i \in \{1, 2, \dots, m\}$, be locally Lipschitz functions. Then*

$$\partial^\circ\left(\sum_{i=1}^m f_i\right)(x) \subset \sum_{i=1}^m \partial^\circ f_i(x),$$
$$\partial^\circ(\alpha f_i)(x) = \alpha\partial^\circ f_i(x).$$

Remark 2. It is worth mentioning that Bento *et al.* [7] introduced an equivalent definition of Clarke subdifferential on Hadamard manifolds given in an intrinsic way.

Definition 1. Given $\epsilon > 0$ and $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper locally Lipschitz, the ϵ-Clarke subdifferential of f at x, denoted by $\partial^o_\epsilon f(x)$ is defined as

$$\partial^o_\epsilon f(x) = \{w \in T_xM :\ \langle w, d\rangle \le f^o(x, d) + \epsilon, \quad \forall d \in T_xM\}$$

where $f^\circ(x, v)$ is given by (5) and (φ, U) is a chart at x.

Definition 2. Let $F := (F_1, \dots, F_m) : M \to \mathbb{R}^m$ be a locally Lipschitz function in M. The point $x \in M$ is a Pareto-Clarke critical point (or satisfies the critical condition of Pareto-Clarke) of F if, for any $d \in T_xM$, there exists $j_0 \in \{1, 2, \dots, m\}$ such that $F^\circ_{j_0}(x, d) \ge 0$.

4. Quasidistances on Riemannian manifolds

Definition 3. Let $\mathbb{R}_+$ be the set of the nonnegative real numbers. A function $q : M\times M \to \mathbb{R}_+$ is a quasidistance if

(1) $q(x, y) = 0 = q(y, x)$, iff $x = y$.

(2) $q(x,w) \leq q(x,y) + q(y,w)$, for all $x, y, w \in M$.

Observe from Definition 3, if q is also symmetric then q is a distance.

Example 1 (Motivated from [18])**.** Given a partial order $\leq$, defined in M. It can be proved that the function $q : M \times M \to \mathbf{R}_+$ defined by

$$q(x,z) = \begin{cases} 0 & \text{if } x \leq z, \\ 1 & \text{otherwise.} \end{cases}$$

is a quasidistance on M.

Some particular examples are

- Consider the Riemannian manifold $M_1 = (\mathbf{R}^{\mathrm{n}}_{++}, X^{-2})$, where $\mathbf{R}^{\mathrm{n}}_{++}$ is the n-dimensional positive orthant and $X^{-2} = \mathrm{diag}(1/x_1^2, \ldots, 1/x_n^2)$ is the matrix that represent the metric $\langle v, w\rangle_x := v^T X^{-2} w$, for all $v, w \in \mathbf{R}^{\mathrm{n}}$. If we define $x \leq y$ as $x_i \leq y_i$, for each $i = 1, \ldots, n$ then q is a quasidistance on M_1.
- Let S^n_{++} be the set of the symmetric positive definite matrices and let the Riemannian manifold $M_2 = (S^n_{++}, G(x))$, where $G(x) = \mathrm{Hess}(-\ln\det(x))$ (Hess denotes the hessian and $\det(x)$ is the determinant of the matrix x). If we define the partial order $x \leq y$ means that $y - x \in S^n_+$ then q is a quasidistance on M_2.

Example 2. Let $\mathbf{R}_{++}$ the set of positive real numbers and $a_i, b_i > 0$, for $i = 1, \ldots, n$. Define on $\mathbf{R}_{++} \times \mathbf{R}_{++}$ the function

$$q_i(x_i, y_i) = \begin{cases} b_i \ln\left(\frac{x_i}{y_i}\right) & \text{if } x_i - y_i > 0, \\ a_i \ln\left(\frac{y_i}{x_i}\right) & \text{if } x_i - y_i \leq 0, \end{cases}$$

for each $i = 1, \ldots, n$. It can be proved that q_i defined as above is a quasidistance in $\mathbf{R}_{++}$. Thus it may be verified that $\bar{q}(x,y) = \sum_{i=1}^n q_i(x_i, y_i)$ is a quasidistance in $\mathbf{R}^n_{++}$. It is easy to show that, for each $\overline{y} \in \mathbf{R}^n_{++}$ it gives

$$q(x, \overline{y}) = \sum_{i=1}^n q_i(x_i, \overline{y}_i) = \sum_{i=1}^n \max\left\{ b_i \ln\left(\frac{x_i}{\overline{y}_i}\right), a_i \ln\left(\frac{\overline{y}_i}{x_i}\right)\right\}, \quad x \in \mathbf{R}^n_{++}.$$

As the functions $\ln(\frac{x_i}{\bar{y}_i})$ and $\ln(\frac{\bar{y}_i}{x_i})$ are geodesically linear on the manifold $M = (\mathbb{R}^n_{++}, X^{-2})$, then $q(\cdot, \overline{y})$ and $q(\overline{y}, \cdot)$ are convex on M_1.

Let us now give some conditions on the quasidistance q: There are positive constants α and β :

$$q(x,y) \leq \beta d(x,y), \qquad \forall\, x, y \in M \tag{7}$$

$$\alpha d(x,y) \leq q(x,y), \qquad \forall\, x, y \in M. \tag{8}$$

We can verify that the quasidistances of Example 2 verifies the conditions (7) and (8) with $\beta = n \max_{i=1,\ldots,n}\{a_i, b_i\}$ and $\alpha = \min_{i=1,\ldots,n}\{a_i, b_i\}$ respectively.

The proofs of the following propositions are adaptations from Propositions 3.6 and 3.7 of Moreno *et al.* [25] to Riemannian manifolds.

Proposition 1. *Let $q : M \times M \to \mathbb{R}_+$ be a quasidistance satisfying the condition (7). Then, for an arbitrary $x \in M$, $q(x, \cdot)$ and $q(\cdot, x)$ are Lipschitz functions.*

Proposition 2. *Let $x \in M$. If q verifies (7) then $q^2(x, \cdot)$ and $q^2(\cdot, x)$ are locally Lipschitz functions.*

5. Coercivity

Definition 4. Let M be a finite dimensional and complete Riemannian manifold and a subset $X \subset M$. We say that a sequence $\{x^k\}$ in X is critical (in relation to the set X) if: there exists $\hat{x} \in M$ such that $\lim_{k\to+\infty} d(x^k, \hat{x}) = +\infty \ \vee \ \lim_{k\to+\infty} x^k = \bar{x} \in \bar{X}\backslash X$, where the notation $\bar{x} \in \bar{X}\backslash X$ means $\bar{x} \in \bar{X}$ and $\bar{x} \notin X$.

The function $f : X \subset M \to \mathbb{R}$ is called *coercive in* X if for all critical sequence $\{x^k\}$ we have:

$$\limsup_{k\to+\infty} f(x^k) = +\infty,$$

where $\limsup_{k\to+\infty} f(x^k) := \inf_{n\in\mathbb{N}} \sup_{k\geq n} f(x^k)$.

Remark 3. If M is compact, the condition $\lim_{k\to+\infty} d(x^k, \hat{x}) = +\infty$ in Definition 4 does not work. If M is non compact then from [20], page 153, M contains a ray starting from x for each x and thus that condition makes sense.

Proposition 3. *Let M be a finite dimensional and complete Riemannian manifold and $f : X \subseteq M \to \mathbb{R}$. If f is coercive and lower semicontinuous in a nonempty set X, then there exists a minimum point of f in X.*

Proof. Suppose that there exists $\alpha \in \mathbb{R}$ such that $L_f(\alpha) := \{x \in X : f(x) \leq \alpha\} \neq \emptyset$ (it is possible because we can take some $\tilde{x} \in X$ and define $\alpha = f(\tilde{x})$). We affirm that $L_f(\alpha)$ is bounded. By contradiction, suppose that M is not compact and $L_f(\alpha)$ does not bounded, then there exists $\{x^k\} \subset L_f(\alpha)$ such that

$$\lim_{k\to+\infty} d\big(x^k, \bar{x}\big) = +\infty, \text{ for some } \bar{x} \in L_f(\alpha).$$

Due to $x^k \in L_f(\alpha)$, it gives $f(x^k) \leq \alpha$, taking limsup as $k \to +\infty$ and using the coercivity of f we obtain $\limsup_{k\to+\infty} f(x^k) = +\infty \leq \alpha$, which is a contradiction. Therefore, $L_f(\alpha)$ is bounded.

We prove now that $L_f(\alpha)$ is closed. Let $\{x^k\} \subset L_f(\alpha)$ be a sequence such that $x^k \to \bar{x}$ and suppose that $\bar{x} \notin L_f(\alpha)$, that is, $\bar{x} \notin X$ or $f(\bar{x}) > \alpha$.

If $\bar{x} \notin X$ then $\bar{x} \in \bar{X}\backslash X$, that is, $\{x^k\}$ is critical and so

$$+\infty = \limsup_{k\to+\infty} f(x^k) \leq \alpha,$$

which is a contradiction.

If $f(\bar{x}) > \alpha$. By the lower semicontinuity of f we have

$$\alpha < f(\bar{x}) \leq \liminf_{k\to+\infty} f\big(x^k\big) \leq \alpha,$$

which is a contradiction. Therefore, $\bar{x} \in L_f(\alpha)$ and thus $L_f(\alpha)$ is closed.

Finally, as M is a complete Riemannian manifold, using the Hopf and Rinow theorem, see Do Carmo [20] Theorem 2.8, we have that $L_f(\alpha)$ is compact and using Weierstrass theorem we obtain the desired result. □

6. The Algorithm

Consider the optimization problem:

$$\min\{F(x) : x \in M\}, \tag{9}$$

where $F : M \to \mathbb{R}^m$ is a vector function (see the notation given in (1)). We impose the following assumptions:

(A1) $q : M \times M \to \mathbb{R}_+$ is a quasidistance holds (7) and (8).
(A2) F is quasiconvex on M.
(A3) F is locally Lipschitz on M.
(A4) $0 \preceq F(x)$, for each $x \in M$.

Assumption (A1) is to work with quasidistances not so asymmetric. Assumptions (A2) and (A3) imply that given an arbitrary point $y \in M$, the set $\{x \in M : F(x) \preceq F(y)\}$ is totally convex (see Udriste [45], pages 59 and 98) and closed. The condition (A4) may be always obtained because both problems $\min\{F(x) : \ x \in M\}$ and $\min\{e^{F(x)} : x \in M\}$, where $e^{F(x)} = (e^{F_1(x)}, \dots, e^{F_m(x)})$, are equivalents. Note also that, if F is quasiconvex then $e^{F(\cdot)}$ is also quasiconvex.

Proximal point algorithm with quasidistance

Let $\mathbb{R}^m_+$ be the m-dimensional nonnegative orthant and M a Riemannian manifold. The algorithm is given by the following steps

(1) Choose $x^0 \in M$, $\{z^k\} \subset \mathbb{R}^m_+ \setminus \{0\}$ a bounded sequence such that $\|z^k\| = 1$, $\{\lambda_k\} \subset \mathbb{R}_+$ a sequence satisfying $c_1 < \lambda_k < c_2$, for some positive numbers c_1, c_2 and $\{\varepsilon_k\} \subset \mathbb{R}_+$ is a sequence such that $\lim_{k\to\infty} \varepsilon_k = 0$.
(2) Let x^k be a known point. If x^k satisfies the critical condition of Pareto-Clarke, then finish. Otherwise, continue to the step 3.
(3) Find $x^{k+1} \in \Omega_k$ and $e^{k+1} \in T_{x^{k+1}}M$, such that:

$$e^{k+1} \in \partial^o_{\varepsilon_k} h_k\big(x^{k+1}\big) + \lambda_k \partial^o\big(q^2\big(\cdot, x^k\big)\big)\big(x^{k+1}\big) + \mathcal{N}_{\Omega_k}\big(x^{k+1}\big), \tag{10}$$

$$q^2\big(x^{k+1}, x^k\big) \le \bar{c}\ \big\|F\big(x^{k+1}\big) - F\big(x^k\big)\big\| \tag{11}$$

where $\Omega_k = \{x \in M : \ F(x) \preceq F(x^k)\}$, $h_k(\cdot) = \langle F(\cdot), z^k\rangle$, $\partial^o_{\varepsilon_k} h_k(\cdot)$ is the ε_k-Clarke subdifferential of the function $h_k(\cdot)$ introduced in Definition 1, $\mathcal{N}_{\Omega_k}(x)$ is the normal cone at x related to Ω_k and $\bar{c} \in \mathbb{R}_{++}$.
(4) If $x^{k+1} = x^k$ then finish. Otherwise, make $k \leftarrow k + 1$ and go back to step 2.

Proposition 4. *Under assumptions (A1)–(A4), $\{(x^k, e^k)\}$ exists.*

Proof. For induction on k. It is true for $k = 0$ because x^0 is given and e^0 may be arbitrary. Suppose that the proposition is true to $j = 0, 1, 2, \dots, k$. We will prove that it is true to $j = k + 1$. Let the function $\varphi^k(\cdot) = h_k(\cdot) + \frac{\lambda_k}{2} q^2(\cdot, x^k)$ and consider

$$\min\big\{\varphi^k(x) : x \in \Omega_k\big\}. \tag{12}$$

If M is compact, due to Ω_k is closed then there exists a minimum point of (12). Suppose that M is non compact, we prove that $\varphi^k(\cdot)$ is coercive in Ω_k. In fact, due to Ω_k is closed, from Definition 4 consider a sequence $\{y_l\} \subset \Omega_k$ such that $\lim_{l\to+\infty} d(y_l, \hat{y}) = +\infty$, for some $\hat{y} \in M$. As $h_k(\cdot) \ge 0$, $\lambda_k > c_1$, and from the triangular inequality of q and (8) we have

$$\varphi^k(y_l) \ge \frac{c_1}{2} q^2\big(y_l, x^k\big),$$

and

$$q\big(y_l, x^k\big) \ge q(y_l, \hat{y}) - q\big(x^k, \hat{y}\big) \ge \alpha d(y_l, \hat{y}) - q\big(x^k, \hat{y}\big).$$

Due to x^k and $\hat{y}$ are fixed then taking lim sup when $l \to +\infty$ we have that $\limsup_{l\to+\infty} q(y_l, x^k) = +\infty$ and therefore

$$\limsup_{l\to+\infty} \varphi^k(y_l) = +\infty.$$

Thus $\varphi^k(\cdot)$ is coercive in Ω_k. Then, from Proposition 3 we have that there exists a minimizer of (12) which we denote by x^{k+1}. From (4) we obtain

$$0 \in \partial^F\big(\varphi^k + \delta_{\Omega_k}(\cdot)\big)\big(x^{k+1}\big) = \partial^F\big(\varphi^k o \exp_{x^{k+1}}(\cdot) + \delta_{\Omega_k} o \exp_{x^{k+1}}(\cdot)\big)(0),$$

where the equality is due to (3). Using the fact that $\varphi^k o \exp_{x^{k+1}}(\cdot)$ is locally Lipschitz, $\delta_{\Omega_k} o \exp_{x^{k+1}}(\cdot)$ is lower semicontinuous and Theorem 2.33 of Mordukhovich [24] we obtain that

$$0 \in \partial^{\text{Lim}}\big(\varphi^k o \exp_{x^{k+1}}(\cdot)\big)(0) + \partial^{\text{Lim}}(\delta_{\Omega_k} o \exp_{x^{k+1}}(\cdot))(0)$$

where $\partial^{\rm Lim}$ denotes the limiting subdifferential. From Remark 2.5.1 of [2], Remark 1, Lemma 1 and since $\delta_{\Omega_k}(\cdot)$ is convex we obtain

$$0 \in \partial^o\big(\langle F(\cdot), z^k\rangle\big)\big(x^{k+1}\big) + \lambda_k \partial^o\big(q^2\big(\cdot, x^k\big)\big)\big(x^{k+1}\big) + N_{\Omega_k} x^{k+1},$$

that is, $\{(x^{k+1}, e^{k+1})\}$ satisfies the condition (10) with $e^{k+1} = 0$.

Also, as $x^{k+1} \in \operatorname{argmin}\{\varphi^k(x) : x \in \Omega_k\}$ and as $q(x^k, x^k) = 0$ we obtain

$$h_k\big(x^{k+1}\big) + \frac{\lambda_k}{2}\, q^2\big(x^{k+1}, x^k\big) \le h_k\big(x^k\big).$$

Then, as $0 < c_1 < \lambda_k$, $\|z^k\| = 1$, it gives

$$0 \le q^2\big(x^{k+1}, x^k\big) \le \frac{2}{c_1}\big\|F\big(x^k\big) - F\big(x^{k+1}\big)\big\|.$$

Therefore $\{(x^{k+1}, e^{k+1})\}$ satisfies (10) and (11). □

We need to introduce a condition on the starting point x^0 which will be important in the convergence analysis.

(A5) $\Omega_0 = \{x \in M : F(x) \preceq F(x^0)\}$ is bounded;

The above condition may be obtained assuming, for example, that F_{i_0} is coercive for some $i_0 \in \{1, 2, \ldots, m\}$.

Proposition 5. *Under assumptions (A1)–(A5), the sequence $\{x^k\}$ generated by the algorithm satisfies:*

(a) *$\{x^k\}$ is bounded.*
(b) *$\{F_i(x^k)\}$ is a non increasing convergent sequence for each $i = 1, \ldots, m$.*
(c) *$\{F(x^{k+1}) - F(x^k)\}$ converges to the zero vector.*
(d) *The quasidistance between two consecutive iterations converges to zero, that is, $\{q(x^{k+1}, x^k)\}$ converges o zero.*
(e) *The Riemannian distance between two consecutive iterations converges to zero, that is, $\{d(x^{k+1}, x^k)\}$ converges to zero.*
(f) *If $\{x^{k_j}\} \to \bar{x}$ we obtain $\{x^{k_j+1}\} \to \bar{x}$.*

Proof. (a) Since $\Omega_{k+1} \subset \Omega_k$, we have $x^k \in \Omega_{k-1} \subset \Omega_0$, for all k. Then, from assumption (A5), $\{x^k\}$ is bounded.
(b) As for each i $F_i(x^{k+1}) \le F_i(x^k)$ and from assumption $(A4)$ we obtain that $\{F(x^k)\}$ is convergent.
(c) It is immediate from (b).
(d) By the condition (11) and the previous item.
(e) It is immediate from assumption (A1). and the previous item.
(f) Use the triangular inequality and apply (e).

□

Theorem 1. *Under assumptions (A1)–(A5) and if one of the below conditions on the error is satisfied:*

(i) *The error norm is bounded from above by $\eta_{k+1} d(x^{k+1}, x^k)$, that is,*

$$\big\|e^{k+1}\big\| \le \eta_{k+1} q\big(x^{k+1}, x^k\big) \tag{13}$$

where $\{\eta_{k+1}\}$ is a bounded positive real sequence.

(ii) *The error norm is bounded from above by an exogenous sequence*

$$\big\|e^{k+1}\big\| \le a_{k+1}, \tag{14}$$

with $\sum_{k=0}^{+\infty} a_k < +\infty$, *then, all accumulation points of $\{x^k\}$ satisfy the critical condition of Pareto-Clarke of F. If $x^{k_0+1} = x^{k_0}$ for some k_0, then x^{k_0} is also a Pareto-Clarke critical point.*

Proof. First, Suppose $x^{k+1} \neq x^k$ for all k. By Proposition 5(a), there are $\hat{x} \in M$ and $\{x^{k_j}\}_{j\in\mathbb{N}}$, such that $\lim_{j\to\infty} x^{k_j} = \hat{x}$. By contradiction, suppose that $\hat{x}$ is not a Pareto-Clarke critical point, then from Definition 2 there is $\hat{v} \in T_{\hat{x}}M$:

$$F_i^o(\hat{x}, \hat{v}) < 0, \tag{15}$$

$\forall i \in \{1, \cdots, m\}$. Thus, there exits $\delta > 0$ satisfying $F_i(\exp_{\hat{x}}(\lambda\hat{v})) < F_i(\hat{x})$, for all $\lambda \in (0, \delta]$ and from Proposition 5(b), $\exp_{\hat{x}}(\lambda\hat{v}) \in \Omega_k$.

Taking $p_{k+1} \in \mathcal{N}_{\Omega_k}(x^{k+1})$, $\zeta^{k+1} \in \partial^o(q(\cdot, x^k))(x^{k+1})$ in (10) and using Definition 1, it gives

$$\left\langle e^{k+1} - \lambda_k q\left(x^{k+1}, x^k\right)\zeta^{k+1} - p_{k+1}, v\right\rangle \leq h_k^\circ\left(x^{k+1}, v\right) + \epsilon_k, \quad \forall v \in T_{x^{k+1}}M,$$

where $h_k^\circ(x^{k+1}, w)$ represents the Clarke directional derivative. For simplicity of notation consider $k = k_j$ and let $r_{\hat{x}} > 0$ such that $\exp_{\hat{x}} : B(0, r_{\hat{x}}) \to M$ is a diffeomorphism with $B(\hat{x}, r_{\hat{x}}) := \exp_{\hat{x}}(B(0, r_{\hat{x}}))$. Then taking $\lambda \in (0, \delta) \cap (0, r_{\hat{x}})$ and considering k sufficient large such that $x^{k+1} \in B(\hat{x}, r_{\hat{x}})$ (due to Prop. 5(f)), we can define $v^k = \exp^{-1}_{x^{k+1}}(\exp_{\hat{x}}(\lambda\hat{v}))$ and using definition of normal cone we have

$$\left\langle e^{k+1}, v^k\right\rangle - \lambda_k q\left(x^{k+1}, x^k\right)\left\langle \zeta^{k+1}, v^k\right\rangle \leq \left(\sum_{i=1}^m z_i^k F_i(\cdot)\right)^\circ \left(x^{k+1}, v^k\right) + \epsilon_k, \tag{16}$$

where z_i^k are the components z^k. Applying a similar version of (3), (4) and (5) from [31] but on Riemannian manifolds (see Lem. 2 of that paper), it gives

$$\left\langle e^{k+1}, v^k\right\rangle - \lambda_k q\left(x^{k+1}, x^k\right)\left\langle \zeta^{k+1}, v^k\right\rangle \leq \sum_{i=1}^m F_i^\circ\left(x^{k+1}, z_i^k v^k\right) + \epsilon_k. \tag{17}$$

Using (13) or (14), and considering that $v^k \to \lambda\hat{v}$, $0 < \lambda_k < c_2$, $\{\zeta^k\}$ is bounded and Proposition 5(d), (e), taking lim sup we have

$$0 \leq \limsup_{k\to+\infty} \sum_{i=1}^m F_i^o\left(x^{k+1}, z_i^k v^k\right) = \sum_{i=1}^m \limsup_{k\to+\infty} F_i^o\left(x^{k+1}, z_i^k v^k\right).$$

Since F_i° is upper semicontinuous and from Proposition 5(f), we have $\limsup_{k\to+\infty} F_i^\circ(x^{k+1}, z_i^k v^k) \leq F_i^\circ(\hat{x}, \overline{z}_i\lambda\hat{v})$; so the above expression becomes

$$0 \leq \sum_{i=1}^m F_i^\circ(\hat{x}, \overline{z}_i\lambda\hat{v}),$$

and thus

$$0 \leq \overline{z}^1 F_1^o(\hat{x}, \hat{v}) + \cdots + \overline{z}^m F_m^o(\hat{x}, \hat{v}). \tag{18}$$

As $\|\overline{z}\| = 1$, then from (18) there exists at least $j_0 \in \{1, 2, \ldots, m\}$ such that $F_{j_0}^\circ(\hat{x}, \hat{v}) \geq 0$, and thus it contradicts (15).

The proof when $x^{k_0+1} = x^{k_0}$ for some k_0 is very similar. □

Corollary 1. *If the assumption (A2) in the previous theorem is replaced by the assumption of convexity of F, then all points of accumulation of the sequence generated by the algorithm are weak Pareto efficient solutions of the problem (9). In the case that $x^{k_0+1} = x^{k_0}$ for some k_0, we have that x^{k_0} is a weak Pareto efficient solution of the problem (9).*

If F is convex, then it is possible redefine the algorithm to obtain a strong result (all accumulation points are Pareto efficient solutions). For that, we consider the following assumptions:

(CA1) $q : M \times M \to \mathbb{R}_+$ is a quasidistance satisfying the properties (7) and (8);
(CA2) F is convex on M
(CA3) $\Omega_0 = \{x \in M : F(x) \preceq F(x^0)\}$ is bounded.
(CA4) $0 \preceq F(x), \forall\, x \in M$.

We use the classical convex ϵ-subdifferential for a convex function f:

$$\partial_\epsilon f(x) = \{g \in T_xM : f(y) \geq f(x) + \langle g, \gamma'(0)\rangle - \epsilon, \forall y \in M\} \tag{19}$$

where $\gamma : [0,1] \longrightarrow M$ is a geodesic linking x to y and $\gamma'(0)$ the covariant derivative of γ.

Convex proximal point algorithm with quasidistance

(1) Choose $x^0 \in M$, $\{z^k\} \subset \mathbb{R}^m_{++}$ a bounded sequence such that $\|z^k\| = 1$ and $\{z^k\} \to \bar{z}$ with $\bar{z} \in \mathbb{R}^m_{++}$ $\{\lambda_k\} \subset \mathbb{R}$ a sequence satisfying $c_1 < \lambda_k < c_2$, for some positive numbers c_1, c_2 and $\{\varepsilon_k\} \subset \mathbb{R}_+$ is a sequence such that $\lim_{k\to\infty} \varepsilon_k = 0$.
(2) Given x^k. If x^k satisfies the critical condition of Pareto-Clarke then finish the algorithm. Otherwise, continue to the step 3.
(3) Find $x^{k+1} \in \Omega_k$ and $e^{k+1} \in T_{x^{k+1}}M$:

$$e^{k+1} \in \partial_{\varepsilon_k} h_k\big(x^{k+1}\big) + \lambda_k \partial\big(q^2\big(\cdot, x^k\big)\big)\big(x^{k+1}\big) + \mathcal{N}_{\Omega_k}\big(x^{k+1}\big), \tag{20}$$

$$q^2\big(x^{k+1}, x^k\big) \leq \bar{c}\, \big\|F\big(x^{k+1}\big) - F\big(x^k\big)\big\| \tag{21}$$

where $\mathcal{N}_{\Omega_k}(\cdot)$ is the normal cone, $\partial_{\varepsilon_k} h_k(x)$ is the ε_k-subdifferential defined in (19), $h_k(\cdot) = \langle F(\cdot), z^k\rangle$ and $\bar{c} \in \mathbb{R}_{++}$.
(4) If $x^{k+1} = x^k$ then stop. Otherwise, make $k \leftarrow k+1$ and got back to step 2.

Remark 4. The well definition of the Convex Proximal Point Algorithm with Quasidistance, that is, the existence of $\{(x^k, e^k)\}$ under assumptions (CA1), (CA2) and (CA4), is guaranteed under similar arguments as in the proof of Proposition 4.

Theorem 2. *Under assumptions (CA1)–(CA4) and assume one of the conditions given in* (13) *and* (14)*. Then, all accumulation points of the sequence generated by the convex algorithm are Pareto efficient solutions of the problem* (9)*. In the case that $x^{k_0+1} = x^{k_0}$ for some k_0, then x^{k_0} is a Pareto efficient solution of* (9).

Proof. Suppose that $x^{k+1} \neq x^k$ for all k, then, By Proposition 5(a), there are $x^* \in M$ and $\{x^{k_l}\}$ such that $\lim_{l\to\infty} x^{k_l} = x^*$. By contradiction, suppose that there exists $\bar{x} \in M$:

$$F(\bar{x}) \preceq F(x^*) \text{ and } F_{j_0}(\bar{x}) < F_{j_0}(x^*) \tag{22}$$

for some $j_0 \in \{1, \ldots, m\}$. As each component of $\bar{z}$ is strictly positive we obtain:

$$\langle F(\bar{x}), \bar{z}\rangle < \langle F(x^*), \bar{z}\rangle \tag{23}$$

and from (22), $\bar{x} \in \Omega_k$.

By (20) and (19), there are $\zeta^{k_l+1} \in \partial(q(\cdot, x^{k_l}))(x^{k_l+1})$ and $v^{k_l+1} \in \mathcal{N}_{\Omega^{k_l}}(x^{k_l+1})$, such that

$$\big\langle F(\bar{x}), z^{k_l}\big\rangle \geq \big\langle F\big(x^{k_l+1}\big), z^{k_l}\big\rangle + \big\langle e^{k_l+1}, \gamma'_{k_l}(0)\big\rangle - \lambda_{k_l} q\big(x^{k_l+1}, x^{k_l}\big)\big\langle \zeta^{k_l+1}, \gamma'_{k_l}(0)\big\rangle - \varepsilon_{k_l},$$

where $\gamma'_{k_l}(0)$ is the direction of the geodesic γ_{k_l} linking x^{k_l+1} and $\bar{x}$.

As $\langle F(x^{k_l+1}), z^{k_l}\rangle \geq \langle F(x^*), z^{k_l}\rangle$, then, the above inequality implies

$$\big\langle F(\bar{x}), z^{k_l}\big\rangle \geq \big\langle F(x^*), z^{k_l}\big\rangle + \big\langle e^{k_l+1}, \gamma'_{k_l}(0)\big\rangle - \lambda_{k_l} q\big(x^{k_l+1}, x^{k_l}\big)\big\langle \zeta^{k_l+1}, \gamma'_{k_l}(0)\big\rangle - \varepsilon_{k_l}. \tag{24}$$

Taking $l \to +\infty$ and using the assumptions we have $\langle F(\bar{x}), \bar{z}\rangle \geq \langle F(x^*), \bar{z}\rangle$, which contradicts (23).

If $x^{k_0+1} = x^{k_0}$ for some k_0 the proof is similar to the first case. □

7. Application to consumer demand

We examine an application of the proposed algorithm to the Variational Rationality model (based on worthwhile-to-change exploration-explotation) of Soubeyran [37–43] and Attouch and Souberyan [3] for consumer demand in microeconomy in Riemannian manifold. Our application extends the application of [30] considering now Riemannian manifolds (sphere, Grasman manifolds, Symmetric positive matrices, ...) and quasidistances instead of only Hadamard manifolds and Riemannian distances respectively.

Consider M as a Riemannian manifold and $\preceq_i$ be binary relations which are reflexive and transitive in M (know as relations of preference in economy). For each $i = 1, \cdots, m$, let $F_i : M \to \mathbb{R}$ be real functions of scalar values. We say that F_i is an utility function of the preference $\preceq_i$, if the condition $y \preceq_i x$ is equivalent to $F_i(y) \leq F_i(x)$, for all x and y in M. Observe also that the diversification of the preferences $\preceq_i$ are related to the quasiconcavity of the utility functions F_i, see Section 6 of [6] for a detailed explanation.

Given an economic agent (company, government, state, institution, person, etc.) and m different preferences $\preceq_i$. Consider the problem of the agent to maximize his satisfaction regarding all these preferences around the manifold M. This process is not easy since in general there is no single consumption that maximizes all preferences. Thus the agent must consider a different type of optimality called Pareto efficient solution.

On the other hand, to find a Pareto efficient solution is not immediate, so an algorithmic procedure is needed. There are several algorithms that can be used to obtain an efficient point, these are generally iterative where from a given consumption (initial point) a sequence of consumptions (points) are generated such that in a certain iteration an approximate solution of the problem is founded.

However, since the solution is not known and if we consider the Variational Rationaly approach, we must consider at each stage the cost of changing from one point (state) to another new point (new state), see Moreno *et al.* [25]. The cost of change has been represented in several papers by Euclidean distance, Bregman distances, proximal distances, Riemannian distances, quasidistances in Eudidean spaces. In this application we associate with the square of a quasidistance defined on the manifold because in the applications the cost to move of a certain point x to another y is not the same that the cot from y to x.

We will introduce a new criterion to move from one state to another state, which achieves individual objectives as well as group goals, so that it is worthwhile to move from one state to another. Let x^k be a present consumer good and the agent initially wish to improve his preferences, that is, to improve the value (profit) of each function, F_i, for $i = 1, \cdots, m$. So, the agent explores around the set

$$\Omega_k = \left\{x \in M : (-F)(x) \preceq (-F)\left(x^k\right)\right\}.$$

The change from x^k to x^{k+1} has a cost which motivated from Moreno *et al.* [25] we define as $\frac{\beta_k}{2} q^2(x^{k+1}, x^k)$ where β_k is certain positive real number and $q(\cdot, \cdot)$ denote the quasidistance from x^{k+1} to x^k. The best situation that can happen is when each profit is greater than or equal to the exchange cost, that is, when for each i we have

$$F_i\left(x^{k+1}\right) - F_i\left(x^k\right) \geq \frac{\beta_k}{2} q^2\left(x^{k+1}, x^k\right). \tag{25}$$

However, it may happen that a benefit does not meet the inequality (25), that is, may be exists i_0 such that $F_{i_0}(x^{k+1}) - F_{i_0}(x^k) < \frac{\beta_k}{2} q^2(x^{k+1}, x^k)$. To overcome this situation, a reasonable option to the agent is to consider the interest of the group, that is, the change from x^k to x^{k+1} is worthwhile if the sum of weights of all $F_i(x^{k+1}) - F_i(x^k)$ is higher than some positive parameter $\frac{\beta_k}{2}$ of the cost to change $q^2(x^{k+1}, x^k)$, that is, that the new point x^{k+1} should satisfy the following inequality

$$\left(F_1\left(x^{k+1}\right) - F_1\left(x^k\right)\right) z_1^k + \cdots + \left(F_m\left(x^{k+1}\right) - F_m\left(x^k\right)\right) z_m^k \geq \frac{\beta_k}{2} q^2\left(x^{k+1}, x^k\right) \tag{26}$$

where $z_1^k + z_2^k \ldots + z_m^k = 1$. Thus, our application is related to the decision of an agent to obtain individual profits with the condition that the weighted sum of all the gains is greater than the cost of move of a stage to other. Observe that the weighted z_i^k change in each stage (possibly from the motivation of the agent).

Defining $z^k = (z_1^k, z_2^k, \ldots, z_m^k) \in \mathbb{R}_+^m$ we obtain that the relation (26) becomes

$$h_k\left(x^{k+1}\right) \geq h_k\left(x^k\right) + \frac{\beta_k}{2} q^2\left(x^{k+1}, x^k\right) \tag{27}$$

where $h(,) = \langle F(\cdot), z^k \rangle$. We can verify that a sufficient condition to obtain x^{k+1} satisfying (27) is for solving

$$\min\left\{ -h_k(x) + \frac{\beta_k}{2} q^2(x, x^k) : x \in \Omega_k \right\}. \tag{28}$$

In addition, applying the optimality conditions we obtain that the point x^{k+1} solving the above problem jointly with $e^{k+1} = 0$ satisfy the conditions (10) and (11) of the proposed algorithm. That is, an algorithm considering the iteration (28) is a particular case of our algorithm. Furthermore, from (28) we have

$$\left\langle F(x^*), z^k \right\rangle - \left\langle F\left(x^{k+1}\right), z^k \right\rangle \leq \frac{\beta_k}{2} q^2\left(x^*, x^k\right)$$

where x^* is a Pareto efficient solution (assuming that such point exists).

Assuming the assumptions (A1)–(A5) and taking $\liminf$ when k go to $+\infty$ and using the result that $\{d(x^*, x^k)\}$ is bounded (see Prop. 5, part a)), we obtain

$$\liminf_{k\to\infty} \sum_{i=1}^{m} \left(F_i(x^*) - F_i\left(x^{k+1}\right)\right) z_i^k \leq \left(\liminf_{k\to\infty} \beta_k/2\right)\left(\lim_{k\to\infty} q^2\left(x^*, x^k\right)\right). \tag{29}$$

Let $\hat{x}$ be an accumulation point of $\{x^k\}$, then we have the following results:

(1) From Theorem 1, $\hat{x}$ is a Pareto-Clarke critical point the problem.
(2) If $\{\beta_k\}$ go to zero, then from (29), condition (7) and part (a) and (f) of Proposition 5, then $\hat{x}$ is a Pareto efficient solution of the problem.
(3) If assumptions (CA1)–(CA4) are satisfied, then from Theorem 2, $\hat{x}$ is a Pareto efficient solution of the problem.

8. Conclusions

– The convergence results obtained in this paper are weak (each accumulation point of the sequence is a Pareto-Clarke critical point or a Pareto efficient solution when we assume convexity). We believe that this weak result is due that the quasidistance doesn't have the same geometric properties than a Riemannian distance. Also, an arbitrary Riemannian manifold doesn't have the same geometric properties than a Hadamard manifold. For example, in the paper [30] it was proved the strong convergence of a proximal point algorithm to Pareto-Clarke critical point in Hadamard manifolds. So, it should be desirable to research appropriate conditions on the quasidistance (or a generalized distance), on the Riemannian manifold (possibly with bounded sectional curvature) and on the objective functions (may be satisfying the condition of Kurdyka-Lojasiewicz) to obtain global convergence results. Observe that in the scalar minimization problem, Bento *et al.* [9] obtained the global convergence of an abstract descent algorithm using the condition of Kurdyka-Lojasiewicz.
– Observe that an advantage of our algorithm with respect to other proposals is that we can adjust the algorithm to obtain that each accumulation point let be a Pareto efficient solution (and not only a weak Pareto solution) for the convex case, see Theorem 2.
– A future research is the extension of the algorithm using more general distances to cover more applications, see for example [44]. Another research is to retire the Assumption (A2) (quasiconvexity of the function F) and extend the results of the paper to solve arbitrary locally Lipschitz functions on Riemannian manifolds, see the paper [11]. Another research is the extension of the paper to solve quasiconvex vector optimization problems, see [10].

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