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THE PROBABILISTIC HARSANYI POWER SOLUTIONS FOR PROBABILISTIC GRAPH GAMES

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Abstract. This paper analyzes the probabilistic Harsanyi power solutions (PHPSs) for probabilistic graph games (PGGs), which distribute the Harsanyi dividends proportional to weights determined by a probabilistic power measure for probabilistic graph structure. The probabilistic power measure considers the role of players in all possible deterministic graphs, which can reflect the powers of players more effectively. Three axiomatic systems of the PHPSs on PGGs and cycle-free probabilistic graph games (CFPGGs) are provided to show the rationality of the PHPSs, and their independence is analyzed.

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1. Introduction

A cooperative game with transferable utility (TU-game) describes a situation in which several actors can obtain certain transferable payoffs by means of cooperation. In the classical model of cooperative TU-games, it is generally assumed that cooperation among any subset of players has no restriction, and each player in any coalition can obtain a payoff by their cooperation. It is believed that any coalition can form. However, due to political, economic, and/or other reasons, many situations require certain limitations on cooperation. For this reason, Myerson [19] introduced graph games based on graph theory, in which players can cooperate only if they are connected in the graph. To measure the payoffs of players, the author defined the Myerson value and offered its axiomatic system. Different from Myerson [20], Meessen [19] defined the position value for graph games by considering the significance of the links, which is subsequently developed by Borm *et al.* [4]. Later, Alonso-meijide and Fiestras-Janeiro [2] explored the Banzhaf value for graph games. More solutions for graph games can be referred to [9, 16–18, 21, 31].

Inspired by the finding that the channels of communication between players might not be predetermined and set, Calvo *et al.* [5] introduced the probabilistic graph games (PGGs) by assuming that each pair of nodes has a given probability of direct communication. However, each link formation probability is supposed to be independent of the other. The authors defined the Myerson value and offered its axiomatic system for this type of games by extending Myerson's work [20]. By dropping the premise of independence, Gómez *et al.* [11] expanded

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the model developed by Calvo *et al.* [15] to a more general situation, which is referred to as (generalized) PGGs. Also, they proposed the probabilistic Myerson value and provided an axiomatization of the value. Following the work of Gómez *et al.* [11], Ghintran *et al.* [9] defined the probabilistic position value by extending the position value [19] for graph games to such generalized PGGs. Shi and Shan [25] proposed the probabilistic Banzhaf value for PGGs by extending the graph Banzhaf value [2].

In the current paper, we argue that the powers of players in the probabilistic graph affect the distribution of the dividends in the game. van den Brink *et al.* [30] also noted this in graph games and defined the class of Harsanyi power solutions for graph games that are obtained by applying the Harsanyi solutions to graph games. Following this line of thought in van den Brink *et al.* [30], Algaba *et al.* [1] proposed the Harsanyi power solutions for games on union stable systems. Zou and Zhang [32] defined the Harsanyi power solutions for games with restricted cooperation in which a set system specifies the cooperation among players. The Harsanyi solutions are proposed as solutions for TU-games in Vasil'ev [27, 28]. Also, Derks *et al.* [6–8] referred to a Harsanyi solution as a sharing value. A Harsanyi solution distributes the Harsanyi dividend [13, 14] among the players of each coalition in accordance with a selected sharing mechanism. The Shapley value [24] divides each coalition's dividend among its players evenly and is a well-known Harsanyi solution.

However, the Harsanyi power solutions proposed by van den Brink *et al.* [30] only considered the players' powers in a specific graph, which is not rational. Consider the scenario where R is a natural gas producer while N and G are consumers, and there is no common border between any of the three countries. They are attempting to construct a pipeline to move the gas from R to N and G. Think about the connections that may be made in two different ways. One is a pair of underwater direct lines connecting R to N and G, respectively. The other travels *via* a fourth country, P, which shares borders with all three countries, R, N, and G. Two different graphs should result from the framework as characterized: $\{RG, RN\}$ and $\{RP, PN, PG\}$. Clearly, country P plays no role in $\{RG, RN\}$ but plays an essential role in $\{RP, PN, PG\}$. Therefore, the power of country P can't be accurately represented by either the power of country P in graph $\{RG, RN\}$ or $\{RP, PN, PG\}$.

Thus, to consider the role of players in all possible deterministic graphs when sharing the dividends in the game, this paper proposes the PHPSs by extending the Harsanyi power solutions to the setting of generalized probabilistic communication situations. The innovation of the PHPSs is that we relate sharing systems to some probabilistic power measure for a probabilistic graph, which is a function that assigns a non-negative real number to every node in the probabilistic graph and considers the role of players in all possible deterministic graphs. In addition, to show the rationality of the PHPSs, a novel axiom of probabilistic decomposable property is proposed, which establishes the relationship between solutions of probabilistic graph games and that of deterministic graph games.

The main contents of this paper include the following:

(i) A particular power measure based on the graph power measure is constructed, called probabilistic power measure, to measure the power of participation of each player in PGGs;
(ii) The PHPSs are proposed by distributing the Harsanyi dividends over the players in the corresponding coalition in proportion to the probabilistic power measure;
(iii) An axiomatic system of the PHPSs on the class of all PGGs is proposed by employing the axioms of probabilistic component efficiency, probabilistic additivity, the probabilistic σ^p-communication ability property, the probabilistic inessential link property, probabilistic connectedness, and the probabilistic decomposable property;
(iv) Two axiomatic systems of the PHPSs on cycle-free probabilistic graph games (CFPGGs) are given by replacing the probabilistic inessential link property and probabilistic connectedness with the probabilistic superfluous link property.

The paper is structured as follows. Section 2 recalls the main concepts of cooperative TU-games, graph games, and PGGs, which will use in the following sections. Section 3 introduces the PHPS for PGGs. Section 4 offers an axiomatic system of the PHPSs in the class of all PGGs. Section 5 provides two axiomatic systems of the PHPSs on the subclass of CFPGGs. Section 6 gives the concluding remarks.

TABLE 1. The symbols and their descriptions.

Symbol	Description
$i, j, \ldots$	Players
N	The set of players
$S, T, \ldots$	The subsets of N
v	A cooperative game
v^γ	A graph game
v^p	A probabilistic graph game
r^γ	A link game
r^p	A probabilistic link game
$G(N)$	The set of all zero-normalized games on N
$CG(N)$	The set of all graph games on N
$CG_{cf}(N)$	The set of all cycle-free graph games on N
$PG(N)$	The set of all PGGs on N
$PG_{cf}(N)$	The set of all CFPGGs on N
$\Delta_v(T)$	The Harsanyi dividend of coalition T in the game $v \in G(N)$
$\Delta_{v^\gamma}(T)$	The Harsanyi dividend of coalition T in the game $v^\gamma \in \mathrm{CG}(N)$
$\Delta_{v^p}(T)$	The Harsanyi dividend of coalition T in the game $v^p \in \mathrm{PG}(N)$
σ	The graph power measure for any graph (N, γ)
σ^p	The probabilistic graph power measure for any probabilistic graph (N, p)
$Sh(N,\, v)$	The Shapley value for $v \in G(N)$
$My(N, v, \gamma)$	The Myerson value for $v^\gamma \in \mathrm{CG}(N)$
$\pi(N, v, \gamma)$	The position value for $v^\gamma \in \mathrm{CG}(N)$
$My(N, v, p)$	The probabilistic Myerson value for $v^p \in \mathrm{PG}(N)$
$\pi(N, v, p)$	The probabilistic Myerson value for $v^p \in \mathrm{PG}(N)$

2. Preliminaries

In this section, we introduce the main concepts of cooperative TU-games, graph games, and PGGs, which will use in the following sections.

For convenience, the symbols used in this paper are listed in Table 1.

2.1. Cooperative TU-games

Let $N = \{1, 2, \ldots, n\}$ be the player set. The cardinality of any set $S \subseteq N$ is denoted by $|S|$. A cooperative game with transferable utilities or simply a TU-game is a pair $(N,\, v)$, where v is the characteristic function $v : 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$. We can simply represent a TU-game by v only when N is fixed. For a game (N, v), if $v(\{i\}) = 0$ for all $i \in N$ then we call this game (N, v) zero-normalization. Suppose that any game is zero-normalized across the entire paper. The collection of all zero-normalized games on player set N is denoted by $G(N)$. For convenience, we commonly use $S\backslash i$ instead of $S\backslash\{i\}$, $v(i)$ instead of $v(\{i\})$, and $S \cup i$ instead of $S \cup \{i\}$. A solution f over $G(N)$ is a mapping that associates to every $(N, v) \in G(N)$ a vector $f(N, v) \in \mathbb{R}^N$. For any $i \in N$, $f_i(N, v) \in \mathbb{R}^N$ represents the payoff of player i in the game (N, v). To show the payoffs of players, Shapley [24] proposed the Shapley value as follows:

$$Sh_i(N, v) = \sum_{S \subseteq N\backslash i} \frac{|S|!(|N| - |S| - 1)!}{|N|!}(v(S \cup i) - v(S)), \qquad i \in N.$$

For any $T \subseteq N$, u^T is called the unanimity game if $u^T(S) = \begin{cases} 1, T \subseteq S \\ 0, \text{otherwise} \end{cases}$. Moreover, for any $v \in G(N)$, we have $v = \sum_{T \subseteq N, T \neq \varnothing} \Delta_v(T) u_T$, where $\Delta_v(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$ for any $S \subseteq N$. Also, the coefficient $\Delta_v(T)$ is called the Harsanyi dividend [13].

Vasil'ev [27, 28] introduced the Harsanyi solutions, which are also called sharing values [6]. First, a sharing system on N is $w = (w^S)_{S \subseteq N, S \neq \varnothing}$, where w^S is an $|S|$-dimensional vector assigning a non-negative share $w_i^S \geq 0$ to each player $i \in S$ such that $\sum_{j \in S} w_j^S = 1$ for all $S \subseteq N \backslash \varnothing$. We use W^N to represent the collection of all sharing systems on N. For a game (N, v) and a sharing system $w \in W^N$, the corresponding Harsanyi solution $h^w(N, v) \in R^n$ is defined as:

$$h_i^w(N, v) = \sum_{S \subseteq N, i \in S} w_i^S \Delta_v(S), \qquad i \in N.$$

Obviously, the Shapley value is a Harsanyi solution that assigns to any game (N, v) the Harsanyi payoff vector $h^w(N, v)$ with the sharing system w defined by $w_i^S = \frac{1}{|S|}$ for all $S \subseteq N$ such that $i \in S$.

2.2. Graph games

A (*communication*) *graph* is an ordered pair (N, γ), where the nodes in N represent the players, and γ represents a graph on N. A graph on N is a set of unordered pairs of players in N. We call these unordered pairs links, and the link between i and j is denoted by $[i, j]$. If $L^N = \{[i, j] | i, j \in N, i \neq j\}$, then we say that the graph (N, L^N) is complete. When no confusion occurs, we just write a link $[i, j]$ as ij. The *neighborhood* of i in graph (N, γ) is denoted as $R_{(N,\gamma)}(i) = \{j \in N \backslash i | ij \in \gamma\}$. The number of nodes in $R_{(N,\gamma)}(i)$ is called the *degree* of node i in (N, γ). For graph (N, γ), if $d_i(N, \gamma) = \left|R_{(N,\gamma)}(i)\right|$, then $d(N, \gamma) \in \mathbb{R}^n$ is referred to as the degree vector. The collection of *non-isolated* nodes in (N, γ) is denoted by $D(N, \gamma) = \{i \in N | R_{(N,\gamma)}(i) \neq \varnothing\}$. If $\gamma|_S = \{ij \in \gamma | i \in S, j \in S\}$ for $S \subseteq N$, then we say that $(S, \gamma|_S)$ is the subgraph of (N, γ) on S. A sequence of different players $(i_1, \ldots, i_k)$ is a path if $i_l i_{l+1} \in \gamma$ for $l = 1, \ldots, k-1$. Suppose $S \subseteq N$, $\gamma \subseteq L^N$, and given $i, j \in S$, i and j are connected in S with respect to γ if $i = j$ or there exists a path $(i_1, \ldots, i_k)$ with $i_1 = i$ and $i_k = j$. The coalition $S \subseteq N$ is connected if $|S| = 1$ or any two players in S are connected for $(S, \gamma|_S)$. A coalition $C \subseteq N$ is a component if C is connected, and $C \cup i$ is not connected for any $i \in N \backslash C$. We denote the set of all components of N in γ by N/γ. Additionally, we use the symbols S/γ and $(S/\gamma)_i$ to represent the set of all components of S in the subgraph $\gamma|_S$ and the element of S/γ containing i. If $k \geq 3$, $(i_1, \ldots, i_k)$ is a path, $i_k i_{k+1} \in \gamma$ and $i_{k+1} = i_1$, then we call the sequence $(i_1, \ldots, i_k)$ a cycle. When the graph (N, γ) has no cycle, we say it is *cycle-free*. The set of all graphs on N is denoted by $C(N)$, and the set of all cycle-free graphs on N is denoted by $C_{cf}(N)$.

A *graph game* is an ordered triplet (N, v, γ), where $(N, v) \in G(N)$ is a TU-game and $(N, \gamma) \in C(N)$ is a graph. $(S, v|_S, \gamma|_S)$ means (N, v, γ) limited to S, where $S \in 2^N \backslash \varnothing$. By $CG(N)$ and $CG_{cf}(N)$, we denote the sets of graph games and cycle-free graph games on N, respectively. A solution for $CG(N)$ is a mapping $f: CG(N) \to \mathbb{R}^N$. For each $i \in N$, $f_i(N, v, \gamma)$ is the payoff of player i.

According to Myerson [20], only connected players in the communication structure can collaborate and achieve their shared payoff. Myerson [20] proposed the definition of graph restricted games, where $v^\gamma(S) = \sum_{T \in S/\gamma} v(T)$ for all $S \subseteq N$. According to Myerson [20], the Myerson value for $(N, v, \gamma) \in CG(N)$ is defined as:

$$My_i(N, v, \gamma) = Sh_i(N, v^\gamma), \qquad i \in N.$$

Later, Meessen [19] called v^γ the *point game*, and presented the concept of *link game* (γ, r^γ), where $r^\gamma(E) = v^E(N)$ for any $E \subseteq \gamma$. Further, Meessen [19] gave the definition of position value for $CG(N)$: $\pi_i(N, v, \gamma) = \sum_{l \in \gamma_i} Sh_l(\gamma, r^\gamma)/2$, $i \in N$, where $\gamma_i = \{l \in \gamma | i \in l\}$.

Herings *et al.* [16] proposed the average tree value for $CG_{cf}(N)$:

$$AT_i(S, \gamma) = \sum_{S \in C, i \in S} \frac{1 + \sum_{h \in N \backslash S, ih \in \gamma} \left|C^{ih}\right|}{|S| + \sum_{j \in S} \sum_{h \in N \backslash S, jh \in \gamma} \left|C^{jh}\right|} \Delta_{v^\gamma}(S), \qquad i \in N \tag{2.1}$$

where C^{ih} is the component of graph $(N, \gamma\backslash ih)$ containing player h, and C^{jh} is the component of graph $(N, \gamma\backslash jh)$ containing player h. Additionally, some scholars [3,12,22] proved that $\Delta_{v^\gamma}(S) = 0$ for any unconnected coalition $S \subseteq N$.

2.3. Probabilistic graph games

According to Calvo *et al.* [5], a probabilistic graph is described as a pair (N, p), where p is a function: $p : \{ij|i, j \in N, i \neq j\} \to [0, 1]$. For each $ij \in L^N$, $p(ij)$ represents link ij's realization probability. Later, Gómez *et al.* [11] further improved the model in [5] by removing the independence of the probabilities of links. Let $N = \{1, 2, \ldots, n\}$ be a finite set of nodes. A (generalized) probabilistic graph is a pair (N, p), where $p : \{\gamma|\gamma \subseteq L^N\} \to [0, 1]$ with $\sum_{\gamma\subseteq L^N} p(\gamma) = 1$ is a probabilistic map on the set of all possible graphs that assigns each graph $\gamma \subseteq L^N$ a probability to measure the formation of this graph. Let $\mathcal{S}_p = \{\gamma|p(\gamma) > 0, \gamma \subseteq L^N\}$ be the set of graphs with probability greater than 0, and $L_p = \bigcup_{\gamma\in\mathcal{S}_p} \gamma$ be the graph consisting of the union of graphs with probability larger than 0. For example, considering a probabilistic graph (N, p) on $N = \{1, 2, 3\}$, where $p(\gamma_1) = 0.3$, $p(\gamma_2) = 0.7$ such that $\gamma_1 = \{12\}$ and $\gamma_2 = \{13\}$, then $L_p = \bigcup_{\gamma\in\mathcal{S}_p} \gamma = \{12, 13\}$. The collection of all probabilistic graphs on N is denoted by $P(N)$. For any $\xi \subseteq L^N$, $p|_\xi$ is called the probabilistic subgraph for $(N, p) \in P(N)$ if

$$p|_\xi(\gamma) = \begin{cases} \sum_{\eta\subseteq L^N, \eta\cap\xi=\gamma} p(\eta) = \sum_{\delta\subseteq L^N\backslash\xi} p(\gamma \cup \delta), & \text{if } \gamma \subseteq \xi, \\ 0, & \text{otherwise.} \end{cases}$$

There are $2^{\binom{n}{2}}$ probabilistic subgraphs for $(N, p) \in P(N)$. We denote $p|_\xi \subseteq p$ if $p|_\xi$ is the probabilistic subgraph of p. In particular, the probabilistic subgraphs $(N, p|_{L^N\backslash\xi})$ and $(N, p|_{L^N\backslash l})$ are denoted by $(N, p|_{-\xi})$ and $(N, p|_{-l})$, respectively.

In a probabilistic graph (N, p), two nodes i and j are directly connected in (N, p) if there is $\gamma \in \mathcal{S}_p$ such that $ij \in \gamma$. Moreover, two nodes i and j are connected with respect to (N, p) when there is a sequence $(i = i_1, \ldots, i_k = j)(k > 1)$ such that any two neighboring nodes are directly connected with respect to (N, p). The idea of connectivity in (N, p) induces a partition of N called the (probabilistic connected) component. The collection of all (probabilistic connected) components is denoted by N/p. Clearly, $N/p = N/L_p$. If L_p doesn't contain any cycle, then the probabilistic graph (N, p) is said to be *cycle-free*. The set of all cycle-free probabilistic graphs on N is expressed as $P_{cf}(N)$.

A triple (N, v, p) consisting of a probabilistic graph (N, p) and a cooperative TU-game (N, v) is referred to as a probabilistic graph game. We denote the collection of all PGGs on N by $PG(N)$, and the set of all CFPGGs on N by $PG_{cf}(N)$. A solution f on $PG(N)$ is a function that assigns an n-dimensional vector to each $(N, v, p) \in PG(N)$. For any $(N, v, p) \in PG(N)$, the probabilistic graph-restricted game (N, v^p) is defined as

$$v^p(S) = \sum_{\gamma\subseteq L^N} p(\gamma)v^\gamma(S) = \sum_{\gamma\subseteq L^S} p|_{L^S}(\gamma)v^\gamma(S), \qquad S \subseteq N. \tag{2.2}$$

If there is a graph $\gamma \subseteq L^N$ such that $p(\gamma) = 1$, then the probabilistic graph game (N, v, p) reduces to a graph game (N, v, γ). Obviously, a graph game (N, v, γ) is a special kind of probabilistic graph game. Thus, we also write a graph game (N, v, γ) in the form (N, v, p^γ), where $p^\gamma(\gamma') = 1$ if $\gamma' = \gamma$ and $p^\gamma(\gamma') = 0$ otherwise.

Gómez *et al.* [11] proposed the probabilistic Myerson value $My(N, v, p)$, which is defined as the Shapley value of game v^p. Similar to Meessen [19], Ghintran *et al.* [9] defined the probabilistic link game (p, r^p) as $r^v_p(p|_\xi) = v^{p|_\xi}(N)$ for any $\xi \subseteq L^N$. Moreover, the authors defined the probabilistic position value:

$$\pi_i(N, v, p) = \sum_{p|_l\in L_i(p)} \frac{1}{2} Sh_{p|_l}(p, r^v_p), \qquad i \in N,$$

where $L_i(p)$ is the set of probabilistic links $\{p|_l | l \in L_i(L_p)\}$.

3. Probabilistic HARSANYI power solutions

This section defines the PHPSs by extending the Harsanyi power solutions for graph games [30] to the setting of probabilistic graph structure. Before defining the PHPSs, we first review the definition of Harsanyi power solutions.

For any $\gamma \subseteq L^N$, a graph power measure is a function σ^γ that assigns a nonnegative weighted vector $\sigma^\gamma(S,\gamma) \in \mathbb{R}_+^{|S|}$ to every graph (S,γ), where $S \subseteq N$. A graph power measure is *symmetric* if $R_{(S,\gamma)}(i)\backslash\{j\} = R_{(S,\gamma)}(j)\backslash\{i\}$ for (S,γ) such that $i,j \in S$, then $\sigma_i^\gamma(S,\gamma) = \sigma_j^\gamma(S,\gamma)$. A graph power measure is said to be positive for any (S,γ), when the power of any player $i \in S$ is positive if and only if i is non-isolated in graph (S,γ) (and if node i is isolated, its power is equal to 0). According to van den Brink *et al.* [30], we introduce the following two well-known graph power measures.

(1) The degree measure is defined by assigning the degree vector $d^\gamma(S,\gamma)$ to each graph (S,γ), $S \subseteq N$.

(2) If $\sigma_i^\gamma(S,\gamma) = \begin{cases} 1, & i \in D(S,\gamma) \\ 0, & \text{otherwise} \end{cases}$ for any $S \subseteq N$, then we call the graph power measure equal power measure, denoted by κ^γ.

There are also other graph power measures. For example, the positional power measure [15][1], the β^γ-measure [29][2], and the H^γ-measure [16][3]. These graph power measures are all positive and symmetric. For a given graph power measure σ^γ, van den Brink *et al.* [30] defined the Harsanyi power solution for any graph game $(N,v,p^\gamma) \in CG(N)$ as:

$$\psi_i^{\sigma^\gamma}(N,v,p^\gamma) = \sum_{S \subseteq N, i \in S} \frac{\sigma_i^\gamma(S,\gamma)}{\sum_{j \in S} \sigma_j^\gamma(S,\gamma)} \Delta_{v^\gamma}(S), \qquad i \in N.$$

Next, we define the probabilistic power measure for probabilistic graph based on the graph power measure.

Definition 3.1. A probabilistic power measure is a function σ^p that assigns to any probabilistic graph (S,p) a nonnegative vector $\sigma^p(S,p) \in \mathbb{R}_+^{|S|}$ and satisfies $\sigma_i^p(S,p) = \sum_{\gamma \subseteq L^N} p(\gamma)\sigma_i^\gamma(S,\gamma)$ for any $i \in S$, where σ^γ is a graph power measure for any $\gamma \subseteq L^N$, and $S \subseteq N$.

Definition 3.1 means that the probabilistic power measure is the expectation of the graph power measures on all graphs $\gamma \subseteq L^N$, which takes into account the power measures of the player on all possible deterministic graphs. Thus, it reflects the player's power more accurately. If the graph power measure σ^γ is taken to be d^γ, κ^γ, δ^γ, β^γ, and H^γ, then the corresponding probabilistic power measure σ^p is denoted as d^p, κ^p, δ^p, β^p, and H^p, respectively.

A probabilistic power measure is symmetric if for any graph (S,γ), the graph power measure σ^γ is symmetric. Furthermore, it is positive if for any (S,p), the power of node i is positive if and only if i is non-isolated in (S,p) (and if node i is isolated, its power is equal to 0). One can easily check that if σ^γ is positive for any $\gamma \subseteq L^N$, then σ^p is positive. This paper only considers symmetric and positive probabilistic power measure.

Now we use an example to illustrate the definition of probabilistic power measure.

Example 3.1. Consider the probabilistic graph (N,p) on $N = \{1,2,3\}$, where $p(\gamma_1) = 0.6$, $p(\gamma_2) = 0.4$ such that $\gamma_1 = \{12,23\}$ and $\gamma_2 = \{13\}$. If we take the degree measure, then $\sigma_1^{\gamma_1}(N,\gamma_1) = 1$, $\sigma_2^{\gamma_1}(N,\gamma_1) = 2$, $\sigma_3^{\gamma_1}(N,\gamma_1) = 1$, $\sigma_1^{\gamma_2}(N,\gamma_2) = 1$, $\sigma_2^{\gamma_2}(N,\gamma_2) = 0$, and $\sigma_3^{\gamma_2}(N,\gamma_2) = 1$. By Definition 3.1, we get $\sigma_1^p(N,p) = 1$, $\sigma_2^p(N,p) = 1.2$, and $\sigma_3^p(N,p) = 1$, respectively. The power of player 2 is overstated in graph (N,γ_1) due to

[1] The positional power measure: $\delta_i^\gamma(S,\gamma) = \left|R_{(S,\gamma)}(i)\right| + \frac{1}{|S|}\sum_{j \in R_{(S,\gamma)}(i)} \delta_j^\gamma(S,\gamma)$, $i \in S$, $S \subseteq N$.

[2] The β^γ-measure: $\beta_i^\gamma(S,\gamma) = \sum_{j \in R_{(S,\gamma)}(i)} \frac{1}{\left|R_{(S,\gamma)}(j)\right|}$, $i \in S$, $S \subseteq N$.

[3] The H^γ-measure: $H_i^\gamma(S,\gamma) = \frac{1+\sum_{h \in N\backslash S, ih \in \gamma}\left|C^{ih}\right|}{|S|+\sum_{j \in S}\sum_{h \in N\backslash S, jh \in \gamma}\left|C^{jh}\right|}$, $i \in S$, $S \subseteq N$, where C^{ih} and C^{jh} as listed in equation (2.1).

ignoring the development potential of link 13. In contrast, it is understated in graph (N, γ_2) due to ignoring the development potential of links 12 and 23. Obviously, the probabilistic power measure combines these two extremes, which is more in line with the actual situation. Moreover, the power of player 2 is larger than that of players 1 and 3 because he holds a key position in γ_1, and γ_1 has a higher possibility of happening than γ_2.

Given a positive probabilistic power measure σ^p, we give the definition of PHPS for PGGs on player set N as follows:

Definition 3.2. Let $(N, v, p) \in PG_{cf}(N)$, and σ^p be a positive probabilistic power measure. The PHPS for (N, v, p) is defined as:

$$\varphi_i^{\sigma^p}(N, v, p) = \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p) \Delta_{v^p}(S), \qquad i \in N \tag{3.1}$$

where $w^S(\sigma^p) = (w^S(\sigma^p))_{S \subseteq N}$ and $w_i^S(\sigma^p) = \frac{\sigma_i^p(S,p)}{\sum_{j \in S} \sigma_j^p(S,p)}$ for any $i \in S$ when $\sum_{j \in S} \sigma_j^p(S, p) \neq 0$, and $w_i^S(\sigma^p) = \frac{1}{|S|}$ for any $i \in S$ when $\sum_{j \in S} \sigma_j^p(S, p) = 0$.

By the definition of the Harsanyi solution for game $(N, v) \in G(N)$, equation (3.1) can also be written as $\varphi^{\sigma^p}(N, v, p) = h^{w(\sigma^p)}(N, v^p)$. Therefore, the PHPS assigns to each (N, v, p) the Harsanyi solution of the corresponding restricted game v^p with the sharing system $w(\sigma^p)$. The probabilistic power measure σ^p determines the sharing system $w(\sigma^p)$, which ensures that the distribution of any dividend $\Delta_{v^p}(S)$ in the restricted game v^p is proportionate to the power of the players in the probabilistic subgraph $(S, p|_{L^S})$. Observe that the shares do not matter when all powers are zero, because it only happens when all players of S are isolated in $(S, p|_{L^S})$. Further, the dividend of S in v^p equals to zero.

Remark 3.1. Let $(N, v, p) \in PG_{cf}(N)$, and σ^p be a positive probabilistic power measure. If there is $\gamma \subseteq L^N$ such that $p(\gamma) = 1$, the PHPS $\varphi^{\sigma^p}(N, v, p)$ is in line with the Harsanyi power solution $\psi^{\sigma}(N, v, p^{\gamma})$ for the graph game (N, v, p^{γ}). Then, following the work of van den Brink *et al.* [30], we have $\varphi^{\kappa^p}(N, v, p) = My(N, v, p^{\gamma})$ under $p(\gamma) = 1$, and $\varphi^{\kappa^p}(N, v, p) = Sh(N, v)$ under $p(L^N) = 1$.

Remark 3.2. Let $(N, v, p) \in PG^{cf}(N)$. According to van den Brink *et al.* [30], we have $\varphi^{d^p}(N, v, p) = \pi(N, v, p^{\gamma})$ if $\gamma \subseteq L^N$ such that $p(\gamma) = 1$. Moreover, following the work of Herings *et al.* [16], if we take the H^p-measure, then $\varphi^{H^p}(N, v, p)$ coincides with the average tree value [16].

The next theorem relates the PHPSs for PGGs to those of all possible deterministic graphs.

Theorem 3.1. *Let $(N, v, p) \in PG(N)$. If $\sigma^p(S, p) = \sigma^{\gamma}(S, \gamma)$ for any $S \subseteq N$ and any $\gamma \subseteq L^N$, then $\varphi^{\sigma^p}(N, v, p) = \sum_{\gamma \subseteq L^N} p(\gamma) \psi^{\sigma^{\gamma}}(N, v, p^{\gamma})$.*

Proof. Since $\sigma^p(S, p) = \sigma^{\gamma}(S, \gamma)$ for any $S \subseteq N$ and any $\gamma \subseteq L^N$, then $\frac{\sigma_i^p(S,p)}{\sum_{j \in S} \sigma_j^p(S,p)} = \frac{\sigma_i^{\gamma}(S,\gamma)}{\sum_{j \in S} \sigma_j^{\gamma}(S,\gamma)}$ for any $S \subseteq N$ and any $\gamma \subseteq L^N$. From equation (3.1), we have

$$\begin{aligned}
\varphi_i^{\sigma^p}(N, v, p) &= \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p) \Delta_{v^p}(S) \\
&= \sum_{S \subseteq N, i \in S} \frac{\sigma_i^p(S, p)}{\sum_{j \in S} \sigma_j^p(S, p)} \sum_{T \subseteq S} (-1)^{s-t} v^p(T) \\
&= \sum_{S \subseteq N, i \in S} \frac{\sigma_i^p(S, p)}{\sum_{j \in S} \sigma_j^p(S, p)} \sum_{T \subseteq S} (-1)^{s-t} \sum_{\gamma \subseteq L^N} p(\gamma) v^{\gamma}(T) \\
&= \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{S \subseteq N, i \in S} \frac{\sigma_i^p(S, p)}{\sum_{j \in S} \sigma_j^p(S, p)} \sum_{T \subseteq S} (-1)^{s-t} v^{\gamma}(T)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{S \subseteq N, i \in S} \frac{\sigma_i^{\gamma}(S,\gamma)}{\sum_{j \in S} \sigma_j^{\gamma}(S,\gamma)} \Delta_{v^{\gamma}}(T) \\
&= \sum_{\gamma \subseteq L^N} p(\gamma) \psi^{\sigma^{\gamma}}(N, v, \gamma).
\end{aligned}
$$

□

Theorem 3.1 shows that PHPS is the expectation of Harsanyi power solution on all graphs $\gamma \subseteq L^N$ if $\sigma^p(S,p) = \sigma^{\gamma}(S,\gamma)$ for any $S \subseteq N$ and $\gamma \subseteq L^N$. Remark 3.1 and the definition of probabilistic Myerson value [11] derive $\varphi^{\kappa^p}(N,v,p) = My(N,v,p)$ for any $(N,v,p) \in PG(N)$ under the condition of $\sigma^p(S,p) = \sigma^{\gamma}(S,\gamma)$ for any $S \subseteq N$ and any $\gamma \subseteq L^N$. Moreover, Remark 3.2 and the definition of probabilistic position value [10] show $\varphi^{d^p}(N,v,p) = \pi(N,v,p)$ for any $(N,v,p) \in PG_{cf}(N)$ under the condition of $\sigma^p(S,p) = \sigma^{\gamma}(S,\gamma)$ for any $S \subseteq N$ and any $\gamma \subseteq L^N$.

4. Axiomatic system of the PHPSs on $PG(N)$

To show the rationality of the PHPSs, this section provides an axiomatic system on PGGs by generalizing the axioms of the Harsanyi power solutions for graph games given by van den Brink *et al.* [30].

First, we need the following notions for PGGs. A link $l \in L^N$ is superfluous in $(N,v,p) \in PG(N)$ if its removal does not affect v^p, *i.e.*, $v^{p|_{\xi}}(S) = v^{p|_{\xi \cup l}}(S)$ for all $S \subseteq N$ and $\xi \subseteq L^N$. (N,v,p) is point anonymous if there is a function $g^p : (1,2,\ldots,|D(N,p)|) \to \mathbb{R}$ such that $v^p(S) = g^p(|S|)$ for all $S \subseteq N$, where $D(N,p)$ is $\{i \in N$ such that $j \in N$ and $\gamma \in \mathcal{S}_p$ with $\{i,j\} \in \gamma\}$. (N,v,p) is point unanimous if $v^p = v^p(N)u^{D(N,p)}$, *i.e.*, it is point anonymous for $g^p(|D(N,p)|) = v^p(N)$ and $g^p(k) = 0$, $k < |D(N,p)|$. (N,v,p) is link anonymous if there is a function $g^L : (1,2,\ldots,|L_p|) \to \mathbb{R}$ satisfying $r^p(p|_{\xi}) = g^L(|L_{p|_{\xi}}|)$ for all $\xi \subseteq L_p$. (N,v,p) is link unanimous if $r^p = v^p(N)u^p$, *i.e.*, it is link anonymous with $g^p(|L_p|) = v^p(N)$ and $g^p(k) = 0$ for $k < |L_p|$.

Based on the above notions, we state the following five axioms for a solution f on PGGs.

Probabilistic component efficiency (PCE). A solution $f : PG(N) \to \mathbb{R}^n$ satisfies PCE if, for any $(N,v,p) \in PG(N)$ and any $C^p \in N/p$, $\sum_{i \in C^p} f_i(N,v,p) = v^p(C^p)$ holds.

Probabilistic additivity (PA). A solution $f : PG(N) \to \mathbb{R}^n$ is probabilistic additive if

$$f(N,v,p) + f(N,\omega,p) = f(N, v+\omega, p)$$

for any $(N,v,p) \in PG(N)$ and any $(N,\varpi,p) \in PG(N)$.

Probabilistic σ^p-communication ability property (σ^p-PCAP). Let $(N,v,p) \in PG(N)$. If (N,v,p) is point unanimous, then there is $\alpha \in \mathbb{R}$ such that $f(N,v,p) = \alpha\sigma^p(N,p)$.

Probabilistic inessential link property (PILP). Let $(N,v,p) \in PG(N)$. If $T \subseteq N$ is connected and $l \in L_p$ contains a player, not in T, then $f(N,v,p) = f(N,v,p|_{-l})$, where $T \neq \varnothing$, $v = cu^T$, and $c \in \mathbb{R}$.

Probabilistic connectedness (PC). A solution $f : PG(N) \to \mathbb{R}^n$ satisfies PC if $v^p = \varpi^p$ for any $(N,v,p) \in PG(N)$ and any $(N,\varpi,p) \in PG(N)$, then $f(N,v,p) = f(N,\varpi,p)$.

The PILP states that when v is a unanimity game on a connected nonempty coalition T, the solution does not depend on links containing at least one player not in T. Such links are called inessential in that unanimity probabilistic graph game. The PC property states that the solution only depends on the worth of the connected coalitions.

If there is $\gamma \subseteq L^N$ such that $p(\gamma) = 1$, the above axioms degenerates to *component efficiency*, *additivity*, σ^{γ}-*communication ability property*, *the inessential link property* and *connectedness* for the graph game (N,v,p^{γ}), respectively, which can characterize the Harsanyi power solutions [30]. However, for PGGs, the probabilistic version of the above axioms can't characterize the PHPSs. Thus, we further establish the following axiom.

Probabilistic decomposable property (PDP). A solution $f : PG(N) \to \mathbb{R}^n$ meets PDP if, for any $(N,v,p) \in PG(N)$, $f(N,v,p) = \sum_{\gamma \subseteq L^N} p(\gamma) f(N,v,p^\gamma)$.

PDP states that player i's payoff in the probabilistic graph game $(N,v,p) \in PG(N)$ is the expectation of his payoffs on all graphs if the solution employed in the probabilistic graph game $(N,v,p) \in PG(N)$ and the graph game $(N,v,p^\gamma) \in CG(N)$ with $\gamma \subseteq L^N$ is identical.

The first five axioms are the counterpart for probabilistic graph games of the axioms characterizing Harsanyi power solution associated with a given positive power measure in the deterministic case (cf. van den Brink *et al.* [30], Prop. 4.4). However, these five axioms are not sufficient for the analogue characterization in the probabilistic case. For example, let $f^1(N,v,p) = \sum_{S \subseteq N, i \in S} \sum_{\gamma \subseteq L^N} p(\gamma) \frac{\sigma_i^\gamma(S,\gamma)}{\sum_{j \in S} \sigma_j^\gamma(S,\gamma)} \Delta_{v^\gamma}(S)$, where σ^γ is the graph power measure of graph (S,γ) and $\gamma \subseteq L^N$. Then, $f^1(N,v,p)$ satisfies PCE, PA, PILP, σ^p-PCAP and PC, but $f^1(N,v,p) \neq \varphi^{\sigma^p}(N,v,p)$. Thus, some extra axiom is needed to characterize the PHPSs for PGGs. We show that PDP is one of the additional axioms that, together with the other axioms, is sufficient by Theorem 4.1.

Theorem 4.1. *Let $(N,v,p) \in PG_{cf}(N)$, and σ^p be a positive probabilistic power measure. The probabilistic Harsanyi power solution $\varphi^{\sigma^p}(N,v,p)$ is uniquely determined by PCE, PA, σ^p-PCAP, PILP, PC, and PDP.*

Proof. **Existence. PCE.** Let $(N,v,p) \in PG(N)$ and $C^p \in N/p$, then

$$\begin{aligned}\sum_{i \in C} \varphi_i^{\sigma^p}(N,v,p) &= \sum_{i \in C} \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p) \Delta_{v^p}(S) = \sum_{i \in C} \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p) \sum_{\gamma \subseteq L^N} p(\gamma) \Delta_{v^\gamma}(S) \\ &= \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{i \in C} \sum_{S \subseteq N, i \in S} w_i^S \Delta_{v^\gamma}(S) = \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{i \in C} \sum_{S \subseteq C, i \in S} w_i^S \Delta_{v^\gamma}(S) \\ &= \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{S \subseteq C} \Delta_{v^\gamma}(S).\end{aligned}$$

As $C^p \in N/p$, for each $\gamma \in \mathcal{S}_p$, C^p is a union of (deterministic) connected components of N in γ, *i.e.*, $C^p = \bigcup_{k=1}^{r(C)} T_{k,\gamma}(C)$, where $T_{k,\gamma}(C) \in N/\gamma$ for any $k = 1, 2, \ldots, r(C)$. Hence, we get

$$\sum_{\gamma \subseteq L^N} p(\gamma) \sum_{S \subseteq C^p} \Delta_{v^\gamma}(S) = \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{k=1}^{r(C)} \sum_{S \subseteq T_{k,\gamma}(C)} \Delta_{v^\gamma}(S).$$

Since, for any $\gamma \in \mathcal{S}_p$, $\sum_{S \subseteq T_{k,\gamma}(C)} \Delta_{v^\gamma}(S) = v(T_{k,\gamma}(C))$, then

$$\sum_{\gamma \subseteq L^N} p(\gamma) \sum_{S \subseteq C^p} \Delta_{v^\gamma}(S) = \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{k=1}^{r(C)} \sum_{S \subseteq T_{k,\gamma}(C)} \Delta_{v^\gamma}(S) = \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{k=1}^{r(C)} v(T_{k,\gamma}(C)) = \sum_{\gamma \subseteq L^N} p(\gamma) v^\gamma(C).$$

Therefore, from the definition of v^p, we get

$$\sum_{i \in C^p} \varphi_i^{\sigma^p}(N,v,p) = \sum_{\gamma \subseteq L^N} p(\gamma) v^\gamma(C) = v^p(C^p).$$

Thus, $\varphi^{\sigma^p}(N,v,p)$ satisfies PCE.

PA. For any $v, \varpi \in G(N)$,

$$\begin{aligned}\varphi^{\sigma^p}(N,v,p) + \varphi^{\sigma^p}(N,\varpi,p) &= \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p) \Delta_{v^p}(S) + \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p) \Delta_{\varpi^p}(S) \\ &= \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p) \Delta_{v^\gamma}(S) + \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p) \Delta_{\varpi^\gamma}(S)\end{aligned}$$

$$= \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p)(\Delta_{v^\gamma}(S) + \Delta_{\varpi^\gamma}(S))$$
$$= \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p)(\Delta_{(v+\varpi)^\gamma}(S)) = \varphi^{\sigma^p}(N, v+\varpi, p),$$

which gets PA.

σ^p-PCAP. If $(N,v,p) \in PG(N)$ is point unanimous, then $v^p = v^p(N)u^{D(N,P)}$. $\varphi^\sigma(N,v,p)$ is obtained by distributing the unique nonzero dividend $\Delta_{v^p}(D(N,p))$ among players in $D(N,\, p)$ for the σ^p-measure. Hence, σ^p-PCAP holds.

PILP. For convenience, we denote $p(T) = p|_{L^T}$ for any $p \in P(N)$. To show that $\varphi^{\sigma^p}(N,v,p)$ satisfies PILP, we consider $v = cu^T$ for some nonempty connected T, $c \in \mathbb{R}$ and $l \not\subset T$. Then, $\Delta_{v^p}(T) = \Delta_{v^{p|_{-l}}}(T) = c$ and $\Delta_{v^p}(S) = \Delta_{v^{p|_{-l}}}(S) = 0$ for all $S \neq T$. Since $(T, p(T)) = (T, p|_{-l}(T))$ for each $l \not\subset T$ and $\sigma_i^p(T, p(T)) = \sigma_i^p(T, p|_{-l}(T))$ for any $i \in T$, we have

$$\varphi^{\sigma^p}(N,v,p) = \varphi^{\sigma^p}(N,v,p|_{-l}),$$

where $v = cu^T$ and $l \not\subset T$.

PC. Since $\Delta_{v^p}(S) = \Delta_{\varpi^p}(S)$ for any $S \subseteq N$ whenever $v^p = \varpi^p$, PC holds obviously.

PDP. From equation (3.1), for any $(N,v,p) \in PG(N)$,

$$\varphi_i^{\sigma^p}(N,v,p) = \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p)\Delta_{v^p}(S) = \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p) \sum_{\gamma \subseteq L^N} p(\gamma)\Delta_{v^\gamma}(S)$$
$$= \sum_{\gamma \subseteq L^N} p(\gamma) \sum_{S \subseteq N, i \in S} w_i^S(\sigma^p)\Delta_{v^\gamma}(S) = \sum_{\gamma \subseteq L^N} p(\gamma)\varphi_i^{\sigma^p}(N,v,p^\gamma).$$

Hence, it satisfies PDP.

Uniqueness. We first consider $v = cu^T$ for some $c \in \mathbb{R}$ and $T \subseteq N$. We distinguish two cases:

Case 1. T is connected for (N,p). (i) If T is not connected by γ for any $\gamma \in \mathcal{S}_p$, then $(u^T)^p = \sum_{\gamma \subseteq L^N} p(\gamma)(u^T)^\gamma = \sum_{\gamma \subseteq L^N} p(\gamma)u^T = u^T$. From PILP, $f(N,u^T,p) = f(N,u^T,p|_{L^T})$. Since $(N,u^T,p|_{L^T})$ is point unanimous, we have $f(N,u^T,p) = f(N,u^T,p|_{L^T}) = \alpha\sigma^p(N,p|_{L^T})$ by σ^p-PCAP for a certain $\alpha \in \mathbb{R}$. Then α is determined by PCE; (ii) If there is $\gamma \in \mathcal{S}_p$ such that T is not connected by γ, let $\mathcal{S}_p'$ denote the set of graphs that T is not connected. Then,

$$(u^T)^p = \sum_{\gamma \in \mathcal{S}_p'} p(\gamma)(u^T)^\gamma + \sum_{\gamma \in \mathcal{S}_p \backslash \mathcal{S}_p'} p(\gamma)u^T. \tag{4.1}$$

Let $\mathcal{T}$ be a set of connected subsets of N for γ containing T. There is δ_S, $S \in \mathcal{T}$, such that $(u^T)^\gamma = \sum_{S \in \mathcal{T}} \delta^S u^S$ [12]. Hence, equation (4.1) can be written as:

$$(u^T)^p = \sum_{\gamma \in \mathcal{S}_p'} p(\gamma) \sum_{S \in \mathcal{T}} \delta^S u^S + \sum_{\gamma \in \mathcal{S}_p \backslash \mathcal{S}_p'} p(\gamma)u^T = \sum_{\gamma \in \mathcal{S}_p'} p(\gamma) \sum_{S \in \mathcal{T}} \delta^S (u^S)^\gamma + \sum_{\gamma \in \mathcal{S}_p \backslash \mathcal{S}_p'} p(\gamma)(u^T)^\gamma.$$

For any $\gamma \in \mathcal{S}_p'$, there is a probability graph p' that satisfies $p'(\gamma) = 1$ and $p'(\varsigma) = 0$ for any $\varsigma \subseteq L^N$ with $\varsigma \neq \gamma$. Thus,

$$\sum_{\gamma \in \mathcal{S}_p'} p(\gamma) \sum_{S \in \mathcal{T}} (\delta^S u^S)^\gamma = \sum_{\gamma \in \mathcal{S}_p'} p(\gamma) \sum_{S \in \mathcal{T}} \left(\delta^S u^S\right)^{p'}.$$

Since any $S \in \mathcal{T}$ is connected in probability graph p' for any $\gamma \in \mathcal{S}_p'$, from (i) we know that $f(N, \delta^S u^S, p') = f(N, \delta^S u^S, p'|_{L^S})$ is uniquely determined for every $S \in \mathcal{T}$. By PA, $f(N,u^T,p^\gamma)$ is uniquely determined for any $\gamma \in \mathcal{S}_p'$. Similarly, we can show the uniqueness of $f(N,u^T,p^\gamma)$ for any $\gamma \in \mathcal{S}_p/\mathcal{S}_p'$. Thus $f(N,u^T,p)$ is uniquely determined by PDP.

Case 2. If T is not connected for (N,p), then T is also not connected in graph γ for any $\gamma \in S_p$. Similar to (ii) in case 1, one can easily check that $f(N,u^T,p)$ is uniquely determined.

By PA, $f(N,v,p) = \sum_{T\subseteq N} f(N,\Delta_{v^p}(T)u^T,p)$ is uniquely determined. □

Next, we demonstrate that the axioms in Theorem 4.1 are logically independent. Since $f^1(N,v,p)$ satisfies all axioms in Theorem 4.1 except for PDP, then we just need to consider the axioms of PCE, PA, PILP, σ^p-PCAP and PC.

Example 4.1. Let $f^2(N,v,p)$ be defined as $f^2(N,v,p) = 0$ for all $i \in N$. Then, $f^2(N,v,p)$ satisfies PA, PILP, σ^p-PCAP (take $\alpha = 0$), PC and PDP, but $f^2(N,v,p)$ does not meet PCE.

Example 4.2. Let $f^3(N,v,p) = \sum_{S\subseteq N, i\in S} \frac{\sigma_i^p(N,p)}{\sum_{j\in S}\sigma_j^p(N,p)}\Delta_{v^p}(S)$, *i.e.*, the dividend of coalition S in v^p is divided proportionally to the σ^p-measure of players in S in the probabilistic graph (N,p) (rather than their powers on the probabilistic subgraph $(S,v,p|_{L^S})$). Then, $f^3(N,v,p)$ satisfies all axioms Theorem 4.1 except for PILP.

Example 4.3. For any probabilistic power measure $\bar\sigma^p = \alpha\sigma^p$ for any $\alpha > 0$, define $f^4(N,v,p)$ as the probabilistic Harsanyi power solution with $\bar\sigma^p$. Then, $f^4(N,v,p)$ satisfies all axioms Theorem 4.1 except for σ^p-PCAP.

Example 4.4. For any $(N,v,p) \in PG(N)$, let $p|_\eta \subseteq p$ be the set of probabilistic subgraph games such that $L_{p|_\eta} = \{l \in L_p|$ there is an $S \subseteq N$ with $l \subseteq S$ and $\Delta_v(S) = 0\}$. Let $f^5(N,v,p)$ be defined as $f^5(N,v,p) = \sum_{S\subseteq N, i\in S} \frac{\sigma_i^p(N,p|_\eta)}{\sum_{j\in S}\sigma_j^p(N,p|_\eta)}\Delta_{v^{p|_\eta}}(S)$. Then, $f^5(N,v,p)$ meets all axioms Theorem 4.1 except for PA.

Example 4.5. Let $f^6(N,v,p) = \frac{\sigma_i^p(N,p|_\eta)}{\sum_{j\in B_i}\sigma_j^p(N,p|_\eta)}v(B_i)$, where B_i is the component in $(N,p|_\eta)$ that contains player i. Here, $f^6(N,v,p)$ satisfies all axioms Theorem 4.1 except for PC.

5. Axiomatic systems of the PHPSs on $\mathrm{PG}_{cf}(N)$

In this section, we provide two axiomatic systems to characterize the PHPSs on the subclass of CFPGGs.

For this purpose, we first define the probabilistic superfluous link property on the subclass of CFPGGs as follows.

Probabilistic superfluous link property (PSLP). A solution $f : PG_{cf}(N) \to \mathbb{R}^n$ satisfies PSLP, if for any $(N,v,p) \in PG_{cf}(N)$ and each link $l \in L^p$ who is superfluous in (N,v,p) it holds that: $f(N,v,p|_\xi) = f(N,v,p|_{\xi\cup l})$ for all $\xi \subseteq L_N$.

If there is $\gamma \subseteq L^N$ such that $p(\gamma) = 1$, then PSLP degenerates to the superfluous link property for graph game (N,v,p^γ). The superfluous link property for graph games means that the payoffs of players will not change if the possibility for $l \in \gamma$ disappears, other things being the same (*i.e.*, the graph changes to another one in which the link $l \in \gamma$ is removed). In this new context, the natural generalization of PSLP states that, when changing from a probabilistic graph to the probabilistic subgraph obtained by removing the link l, there is no effect on the players' payoffs.

Now we characterize the PHPSs of game $(N,v,p) \in PG_{cf}(N)$ by replacing the PILP and PC with PSLP.

Theorem 5.1. *Let $(N,v,p) \in PG_{cf}(N)$, and σ^p be a positive probabilistic power measure. The probabilistic Harsanyi power solution $\varphi^{\sigma^p}(N,v,p)$ is uniquely determined by PCE, PA, σ^p-PCAP, PDP, and PSLP.*

Proof. **Existence.** From Theorem 4.1 we know that $\varphi^{\sigma^p}(N,v,p)$ satisfies the first four properties on CFPGGs.

Now we prove that it satisfies PSLP.

PSLP. Recall that link $l \in L^N$ is superfluous if $v^{p|_\xi}(N) = v^{p|_{\xi \cup l}}(N)$ for all $\xi \subseteq L^N$. Since it is assumed that any game is zero-normalized, it follows straightforwardly from the definition of the restricted game that this condition holds if and only if $v^p(N) = v^{p|_{-l}}(N)$. When l is superfluous,

$$\Delta_{v^p}(S) = \Delta_{v^{p|_{-l}}}(S), \qquad S \subseteq N,$$

i.e., $\sum_{\gamma \subseteq L^N} p(\gamma)\Delta_{v^\gamma}(S) = \sum_{l \notin \gamma \subseteq L^N} p|_{-l}(\gamma)\Delta_{v^\gamma}(S)$. When $l \not\subset S$, then $(\gamma \backslash l)(S) = \gamma(S)$ for any $\gamma \subseteq L^N$ and $\sigma_i^\gamma(S, \gamma|_S) = \sigma_i^\gamma(S, (\gamma \backslash l)|_S)$ for any $i \in S$, implying that the share of $i \in S$ in $v^\gamma(S)$ equals the share of i in $v^{\gamma_{-l}}(S)$. Thus, $\varphi^{\sigma^p}(N, v, p) = \varphi^{\sigma^p}(N, v, p|_{-l})$. If $l \subseteq S$, we get $\Delta_{v^{p|_{-l}}}(S) = 0$ since (N, p) is cycle-free. So, $\Delta_{v^p}(S) = \Delta_{v^{p|_{-l}}}(S) = 0$ and the shares don't matter. Therefore,

$$\varphi^{\sigma^p}(N, v, p) = \varphi^{\sigma^p}(N, v, p|_{-l}),$$

which gets PSLP on CFPGGs.

Uniqueness. Since the proof of uniqueness is similar to that of Theorem 3.2 in [4], then we will give a brief review here. We first consider the probabilistic graph game (N, cu^T, p) for some $c \in R$, and $T \subseteq N$, in which p is cycle-free. We distinguish two cases:

Case 1. If (N, p) doesn't contain any component S with $T \subseteq S$, then there is no component S in (N, γ) such that $T \subseteq S$ for any $\gamma \in \mathcal{S}_p$. This implies that all links are superfluous for any graph game (N, cu^T, p^γ). Then by equation (3.1), all links are superfluous for (N, cu^T, p). From PSLP, $f(N, cu^T, p) = f(N, cu^T, p^\varnothing)$, where $p^\varnothing = \varnothing$ or $L_p = \varnothing$. PCE uniquely determines $f(N, cu^T, p^\varnothing)$.

Case 2. If (N, p) contains a component S with $T \subseteq S$, then there are also two cases. (i) There is no component T' for any graph $\gamma \in \mathcal{S}_p$ such that $T \subseteq T'$. This implies that all links are superfluous for any graph game (N, cu^T, p^γ). According to case 1, $f(N, cu^T, p)$ is uniquely determined. (ii) There is a component T' for some graph $\gamma \in \mathcal{S}_p$ such that $T \subseteq T'$. Denote $\mathcal{S}'_p$ as the set of graphs for which there is a component T' such that $T \subseteq T'$. Thus, $(u^T)^p = \sum_{\gamma \in \mathcal{S}'_p} p(\gamma)(u^T)^\gamma$. By PDP $f(N, u^T, p) = \sum_{\gamma \in \mathcal{S}'_P} p(\gamma) f(N, u^T, p^\gamma)$. Thus, it just needs to show the uniqueness of $f(N, u^T, p^\gamma)$ for any $\gamma \in \mathcal{S}'_p$. Following the work of van den Brink *et al.* [30], for any graph $\gamma \in \mathcal{S}'_p$,

$$(u^T)^\gamma(N) = (u^T)^{\gamma|_{H(T)}}(N),$$

where $H(T) = \{h \in N |$ there are $i, j \in T$ such that h belongs to the path between i and $j\}$, which implies that $(u^T)^p = (u^T)^{p|_{L^N|_{H(T)}}}$. For any $\gamma \in \mathcal{S}'_p$, there is a probability graph p' satisfying $p'(\gamma) = 1$, and $p'(\varsigma) = 0$ for any $\varsigma \subseteq L^N$ with $\varsigma \neq \gamma$. Thus, $(u^T)^\gamma(N) = (u^T)^{\gamma|_{H(T)}}(N) = (u^T)^{p'}(N)$. From PSLP, it has $f(N, cu^T, p^\gamma) = f(N, cu^T, p') = f(N, cu^T, p'|_{L^N|_{H(T)}})$. Since $(N, cu^T, p'|_{L^N|_{H(T)}})$ is link unanimous, σ^p-PCAP and PCE uniquely determine $f(N, cu^T, p'|_{L^N|_{H(T)}})$. Then by PA, f is unique for any $v \in PG_{cf}(N)$.

□

We now demonstrate that the axioms in Theorem 5.1 are logical independent. On CFPGGs, each of the solutions $f^1(N, v, p)$, $f^2(N, v, p)$, $f^3(N, v, p)$ and $f^6(N, v, p)$ meets the five axioms of Theorem 5.1, but PCE, PSLP, σ^p-PCAP, and PDP, respectively. It remains to consider the axiom of PA.

Example 5.1. For any $(N, v, p) \in PG_{cf}(N)$, let $p|_\xi \subseteq p$ be the set of probabilistic subgraph games that all the links in $L_{p|_\xi}$ are not superfluous in (N, v, p). Let $f^7(N, v, p) = \sum_{S \subseteq N, i \in S} \frac{\sigma_i^p(N, p|_\xi)}{\sum_{j \in S} \sigma_j^p(N, p|_\xi)} \Delta_{v^{p|_\xi}}(S)$, *i.e.*, the dividends in the restricted game on the probabilistic subgraph $(N, p|_\xi)$ are distributed proportionally to the σ^p-measure on $(N, v, p|_\xi)$. Then $f^7(N, v, p)$ satisfies PCE, PSLP, and σ^p-PCAP but PA.

Next, we generalize the σ-measure property for graph games to PGGs.

Probabilistic $\boldsymbol{\sigma^p}$-measure property ($\boldsymbol{\sigma^p}$-PMP). Let $(N, v, p) \in PG(N)$. If (N, v, p) is link unanimous, then there is $\alpha \in \mathbb{R}$ such that $f(N, v, p) = \alpha\sigma^p(N, p)$.

If there is $\gamma \subseteq L^N$ such that $p(\gamma) = 1$, the axiom of σ^p-PMP degenerates to σ^γ*-measure property* for graph game (N, v, p^γ).

Following the work of van den Brink [30], the relationship between σ^p-PMP and σ^p-PCAP can be obtained directly.

Theorem 5.2. *(i) Let $(N, v, p) \in PG(N)$. If (N, v, p) is link unanimous, then (N, v, p) is also point unanimous; (ii) Let $(N, v, p) \in PG_{cf}(N)$. Then (N, v, p) is link unanimous if and only if (N, v, p) is point unanimous.*

Proof. Since the proof is similar to that of Lemma 3.5 in van den Brink *et al.* [30], we omit it here. □

Part (i) of Theorem 5.2 means that for PGGs the σ^p-PCAP implies the σ^p-PMP. Part (ii) of Theorem 5.2 implies that on CFPGGs the σ^p-PCAP is equivalent to the σ^p-PMP. Then from Theorems 4.1 and 5.2 (i), we obtain the following corollary.

Corollary 5.1. *For a positive probabilistic power measure σ^p. The probabilistic Harsanyi power solution $\varphi^{\sigma^p}(N, v, p)$ satisfies σ^p-PMP on PGGs.*

Moreover, based on Theorems 5.1 and 5.2 (ii), one can easily get the following corollary.

Corollary 5.2. *Let $(N, v, p) \in PG_{cf}(N)$, and σ^p be a positive probabilistic power measure. The probabilistic Harsanyi power solution $\varphi^{\sigma^p}(N, v, p)$ is uniquely determined by PCE, PA, PSLP, σ^p-PMP, and PDP.*

Because the σ^p-PCAP is equivalent with the σ^p-PMP on CFPGGs, the five solutions $f^1(N, v, p)$, $f^2(N, v, p)$, $f^3(N, v, p)$, $f^6(N, v, p)$ and $f^7(N, v, p)$ can also demonstrate the logical independence of the axioms in Corollary 5.2.

Since $\varphi^{\sigma^p}(N, v, p)$ coincides with the average tree value when $\sigma^\gamma = H^\gamma$ and $p(\gamma) = 1$, one can immediately obtain two new axiomatizations of the average tree value for the subclass of cycle-free graph games on N. Similarly, one can also obtain two new axiomatic systems of the Myerson value and position value, which have been shown in van den Brink *et al.* [30].

6. Concluding remarks

This paper studied a new type of solution for PGGs, called the PHPSs. The PHPSs is defined by two steps. First, we proposed a particular power measure based on the graph power measure, called probabilistic power measure, to measure the power of every player in PGGs; Second, we defined the so-called PHPS by dividing the Harsanyi dividend among players in the corresponding coalition in proportion to the probabilistic power measure. To show the rationality of the PHPSs, we offered an axiomatic system on PGGs by employing PCE, PA, σ^p-PCAP, PILP, PC, and PDP. Replacing PILP and PC by PSLP, we gave two axiomatic systems of the PHPSs on CFPGGs. One uses σ^p-PCAP, and the other employs σ^p-PMP.

However, this paper only considered the case where all players fully participated in the coalition. In some practical situations, players may not fully participate in a game, but to a certain degree, to reduce the risk or increase the payoffs. Therefore, the PHPSs on fuzzy PGGs can be considered in future work. In addition, we can extend the new results to games with probabilistic graphs and coalition structures in a similar way as references [23, 26]. Moreover, theoretical results can also be applied to other fields, such as shipping and irrigation.

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