ON PREINVEX INTERVAL-VALUED FUNCTIONS AND UNCONSTRAINED INTERVAL-VALUED OPTIMIZATION PROBLEMS

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Abstract. The main objective of this paper is to investigate the generalized convexity of interval-valued functions under the total order relation and apply it to a class of unconstrained interval-valued optimization problems. For this purpose, we present the new definition of preinvex interval-valued functions and obtain its several fascinating characterizations. Then, we introduce the $\leq_{cw}$-semicontinuity and discuss its relationship with preinvex interval-valued functions. As applications related to preinvex interval-valued functions, we study a class of unconstrained interval-valued optimization problems and discuss the existence theorem of its optimal solution.

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1. Introduction

Optimization problems are widespread in engineering field, where uncertain data are often encountered during data collection, and estimating these data with definite numbers for modeling can produce errors that may render the results meaningless in severe cases. How to describe these uncertain data is therefore a very worthy and attractive matter. In recent years, many scholars have worked on the use of random variables, fuzzy numbers and intervals to describe such uncertainty problems. Among them, intervals are used to describe uncertain data in real-world problems, which can validly eliminate errors and be simple and efficient. Interval methods have been extensively applied in aerospace engineering \cite{13}, multi-attribute decision-making \cite{26}, charge transportation problem \cite{17} and other fields.

The convexity of functions has applications in many branches of mathematics. In 1981, Borwein \cite{2} first presented the definition of convex interval-valued functions (IVFs), and then many scholars extended and promoted the convexity of IVFs. For example, invexity \cite{10}, logarithmic convexity \cite{7} and prevexity \cite{25}. It is worth noting that these convex IVFs are defined under the partial order relation, that is, any two intervals may be incomparable. This means that it is impossible to find the largest or smallest interval among them through these orderings, thus unable to solve the maximum-minimum problem. To overcome this deficiency, Hu \cite{5} defined a new order, called the cw-order, using the midpoint and radius of two intervals. This is a total order, and any two interval numbers are comparable under this order. In 2020, Rahman \cite{15} gave a new

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definition of convex IVF using \(cw\)-order, and presented the necessary optimization conditions for the constrained optimization problem of interval-valued objective function. Bhunia [1] discussed the optimality of interval multi-objective optimization problems with the help of \(cw\)-order. Rahman [16] built an interval optimization problem in the production inventory model by using intervals to represent demand rate, productivity and deterioration rate, transformed it into a deterministic optimization problem using the \(cw\)-order, and solved it numerically using different variants of quantum behavior particle swarm optimization techniques.

On the other hand, application the generalized convex IVFs to interval optimization have been studied by many authors. Wu [24] derived the optimization conditions in optimization problem with interval-valued objective function using H-derivative IVFs. However, the H-derivative of IVFs may not necessarily exist, resulting in certain drawbacks. In 2009, Stefanini [18] defined the gH-differentiability of IVFs based on the gH-difference. Chalco-Cano [3] obtained KKT type optimization conditions by using the gH-derivative. Li [9] considered three types of total order relationships on the interval space, and introduced interval-valued convex functions and obtained KKT optimality conditions in an optimization problem with interval-valued objective function. For more investigations on interval optimization problems, interested readers are referred to [21–23,28] and others. We note that there are relatively few literature on interval optimization problems based on \(cw\)-order. In 2009, Stefanini [19] defined the gH-differentiability of IVFs based on the gH-difference. Wu [24] derived the optimization conditions in optimization problem with interval-valued objective function. For certain drawbacks. In 2009, Stefanini [18] defined the gH-differentiability of IVFs based on the gH-difference. Chalco-Cano [3] obtained KKT type optimization conditions by using the gH-derivative. Li [9] considered three types of total order relationships on the interval space, and introduced interval-valued convex functions and obtained KKT optimality conditions in an optimization problem with interval-valued objective function. For more investigations on interval optimization problems, interested readers are referred to [21–23,28] and others. We note that there are relatively few literature on interval optimization problems based on \(cw\)-order. In 2009, Stefanini [19] defined the gH-differentiability of IVFs based on the gH-difference.

The main research framework of this article is as follows. After providing preliminaries in Section 2. Section 3 first defines the preconvex IVF under \(cw\)-order and proves that it is equivalent to a convex IVF on \([0,1]\). Then, a necessary and sufficient condition for it is given under the condition that the endpoint functions are differentiable. Section 4 presents the definition of \(cw\)-semicontinuity and proves its close relationship with preinvex IVFs. As an application, in Section 5, we consider a class of unconstrained interval optimization problems and give existence theorems for optimal solutions through variational inequalities. Finally, Section 6 concludes this article.

2. Preliminaries

We denote by \(\mathbb{R}_I\) the set of all bounded and closed intervals in \(\mathbb{R}\), i.e.,

\[ \mathbb{R}_I = \{ [a, \bar{a}] : a, \bar{a} \in \mathbb{R} \text{ and } a \leq \bar{a} \}. \]

Let \(A = [a, \bar{a}] \in \mathbb{R}_I\), \(A_c = \frac{a + \bar{a}}{2}\) is called the centre of \(A\), \(A_w = \frac{\bar{a} - a}{2}\) is called the radius of \(A\). Then, \(A = [a, \bar{a}]\) can also be presented in centre-radius form as

\[ A = \langle A_c; A_w \rangle = [A_c - A_w, A_c + A_w]. \]

For any \(A = [a, \bar{a}] = \langle A_c; A_w \rangle, B = [b, \bar{b}] = \langle B_c; B_w \rangle \in \mathbb{R}_I\) and every \(\lambda \in \mathbb{R}\), we have

\[ A + B = [a + b, \bar{a} + \bar{b}] = [a + b, \bar{a} + \bar{b}], \]

and

\[ \lambda A = \lambda [a, \bar{a}] = \begin{cases} [\lambda a, \lambda \bar{a}], & \lambda \geq 0, \\ [\lambda \bar{a}, \lambda a], & \lambda < 0. \end{cases} \]

Using centre-radius form, we have

\[ A + B = \langle A_c; A_w \rangle + \langle B_c; B_w \rangle = \langle A_c + B_c; A_w + B_w \rangle, \]

and

\[ \lambda A = \lambda \langle A_c; A_w \rangle = \langle \lambda A_c; |\lambda| A_w \rangle. \]

The gH-difference of two intervals \([a, \bar{a}], [b, \bar{b}] \in \mathbb{R}_I\) is defined by Stefanini [18]:

\[ [a, \bar{a}] \ominus_g [b, \bar{b}] = [\min\{a - b, \bar{a} - \bar{b}\}, \max\{a - b, \bar{a} - \bar{b}\}]. \]
It is well known that \( \lim S \) is the infimum (or supremum) of an IVF ordered on \( S \). If not stated otherwise, Ishibuchi and Tanaka [6] defined an order relation as follows:

\[
A \leq_{cw} B \iff \begin{cases} A_c < B_c, & \text{if } A_c \neq B_c, \\ A_w \geq B_w, & \text{if } A_c = B_c; \end{cases}
\]

Remark 2.2. Ishibuchi and Tanaka [6] defined an order relation \( \leq_{mw} \) between intervals \( A \) and \( B \) as follows:

\[
A \leq_{mw} B \text{ if } A_c \leq B_c \text{ and } A_w \geq B_w.
\]

Obviously, \( \leq_{mw} \) is a partial ordering. Moreover, when \( A \leq_{mw} B \), \( A \leq_{cw} B \) also holds, i.e., \( A \leq_{mw} B \Rightarrow A \leq_{cw} B \). When \( A_c < B_c \) and \( A_w \leq B_w \), \( A \leq_{cw} B \) holds, but \( A \leq_{mw} B \) does not, that is, \( A \leq_{cw} B \not\Rightarrow A \leq_{mw} B \). For example, let \( A = [1, 2], B = [0, 4] \), by calculation we can see that \( A \leq_{cw} B \), but either \( A \leq_{mw} B \) or \( B \leq_{mw} A \). Therefore, ordering \( \leq_{cw} \) is more widely used than ordering \( \leq_{mw} \). The relationship between the ordering \( \leq_{cw} \) and the ordering \( \leq_{mw} \) is shown in Figure 1.

The function \( F: S \to R_I \) is said to be interval-valued function (IVF) on the open subset \( S \) of \( R^n \) if \( F(x) = [F^L(x), F^U(x)] \) such that \( F^L(x) \leq F^U(x) \) for all \( x \in S \), where the endpoint functions \( F^L, F^U: S \to R \) are called lower and upper functions of \( F \), respectively. The length of \( F(x) \) is written as \( \text{len}(F(x)) = F^U(x) - F^L(x) \) for any \( x \in S \). If not stated otherwise, \( S \) is represented as a nonempty open subset of \( R^n \).

Definition 2.3. [11] The IVF \( F: S \to R_I \) is called \( \ell \)-increasing on \( S \) if the function \( x \mapsto \text{len}(F(x)) \) is increasing on \( S \).

It is well known that \( \lim_{x \to x^*} F(x) \) exists if and only if \( \lim_{x \to x^*} F^L(x) \) and \( \lim_{x \to x^*} F^U(x) \) exist, and is given by

\[
\lim_{x \to x^*} F(x) = [\lim_{x \to x^*} F^L(x), \lim_{x \to x^*} F^U(x)].
\]

Moreover, if \( \lim_{x \to x^*} F(x) = F(x^*) \), we say \( F \) is continuous at \( x^* \). If \( F \) is continuous for all \( x \in S \), then \( F \) is continuous on \( S \).

Motivated by [8], we present the following definitions for the infimum (or supremum) of \( K \subseteq R_I \), the infimum (or supremum) of an IVF \( F \) and the lower (upper) limit of an IVF \( F \) under the \( cw \)-order relation, respectively.

Definition 2.4. Let \( K \subseteq R_I \). An interval \( A \) is called a lower bound of \( K \) if \( A \leq_{cw} B \) for any \( B \in K \).

A lower bound \( C \) of \( K \) is said to be the infimum of \( K \) if every lower bound \( A \) of \( K \) have \( A \leq_{cw} C \), denoted as \( C = \text{inf} K \).
Definition 2.5. Let $\mathbb{K} \subseteq \mathbb{R}_x$. An interval $A$ is called a upper bound of $\mathbb{K}$ if $B \preceq cw A$ for any $B \in \mathbb{K}$.

An upper bound $C$ of $\mathbb{K}$ is said to be the supremum of $\mathbb{K}$ if each upper bound $A$ of $\mathbb{K}$ have $C \preceq cw A$, denoted as $C = \sup \mathbb{K}$.

Example 2.6. Let $\mathbb{K} = \{[1, 1 + \frac{1}{k}] : k = 1, 2, \cdots \}$. The set of all lower bounds of $\mathbb{K}$ is
\[
\{[a, b] : a + b < 2 \text{ or } a + b = 2, b - a \geq 0\},
\]
and $\inf \mathbb{K} = [1, 1]$.

The set of all upper bounds of $\mathbb{K}$ is
\[
\{[a, b] : a + b > 3 \text{ or } a + b = 3, b - a \leq 1\},
\]
and $\sup \mathbb{K} = [1, 2]$.

Definition 2.7. Let $F : S \to \mathbb{R}_x$ be an IVF. The infimum of $F$ is denoted as $\inf_{x \in S} F(x)$, which is equivalent to the infimum of the range in values of $F$ over $S$, that is
\[
\inf_{x \in S} F(x) = \inf \{F(x) : x \in S\}.
\]

Similarly, the supremum of $F$ over $S$ is
\[
\sup_{x \in S} F(x) = \sup \{F(x) : x \in S\}.
\]

Example 2.8. Let $F : [-1, 1] \to \mathbb{R}_x$ be defined by
\[
F(x) = \begin{cases}
[1, 2] \sin \frac{1}{x}, & \text{if } x \in [-1, 0) \cup (0, 1], \\
[0, 0], & \text{if } x = 0.
\end{cases}
\]
Then $F\left(-\frac{2}{\pi}\right) = \{-2, -1\}$, that is, $F_c\left(-\frac{2}{\pi}\right) = -\frac{3}{2}$ and $F_w\left(-\frac{2}{\pi}\right) = \frac{1}{2}$. Next, we discuss about $x$ in the following three cases.

(i) When $x \in [-1, 0) \cup (0, 1] \setminus \left\{\frac{1}{k\pi - \frac{\pi}{2}} : k \in \mathbb{Z}\right\}$, where $\mathbb{Z}$ represents the set of all integers, we have
\[
F_c(x) = \frac{3}{2} \sin \frac{1}{x} > -\frac{3}{2} = F_c\left(-\frac{2}{\pi}\right) \text{ for all } x \in [-1, 1] \setminus \left\{\frac{1}{k\pi - \frac{\pi}{2}} : k \in \mathbb{Z}\right\},
\]
which implies that $F\left(-\frac{2}{\pi}\right) \preceq cw F(x)$ for all $x \in [-1, 1] \setminus \left\{\frac{1}{k\pi - \frac{\pi}{2}} : k \in \mathbb{Z}\right\}$.

(ii) When $x \in \left\{\frac{1}{k\pi - \frac{\pi}{2}} : k \in \mathbb{Z}\right\}$, we have
\[
F_c(x) = -\frac{3}{2} = F_c\left(-\frac{2}{\pi}\right) \text{ and } F_w(x) = \frac{1}{2} = F_w\left(-\frac{2}{\pi}\right) \text{ for all } x \in \left\{\frac{1}{k\pi - \frac{\pi}{2}} : k \in \mathbb{Z}\right\},
\]
which implies that $F\left(-\frac{2}{\pi}\right) = F(x)$, that is, $F\left(-\frac{2}{\pi}\right) \preceq cw F(x)$ for all $x \in \left\{\frac{1}{k\pi - \frac{\pi}{2}} : k \in \mathbb{Z}\right\}$.

(iii) When $x = 0$, we have $F_c(0) = 0 > -\frac{3}{2} = F_c\left(-\frac{2}{\pi}\right)$, that is, $F\left(-\frac{2}{\pi}\right) \preceq cw F(0)$.

Combining (i)-(iii), we obtain $F\left(-\frac{2}{\pi}\right) \preceq cw F(x)$ for all $x \in [-1, 1]$. Hence,
\[
\inf_{x \in [-1, 1]} F(x) = [-2, -1].
\]

Similarly, we get
\[
\sup_{x \in [-1, 1]} F(x) = [1, 2].
\]
**Definition 2.9.** Let $F : S \to \mathbb{R}_I$ be an IVF. The lower limit of $F$ at $x^* \in S$, denoted as \( \liminf_{x \to x^*} F(x) \), is defined by

\[
\liminf_{x \to x^*} F(x) = \lim_{\delta \to 0^+} \inf \{ F(x) : x \in N_\delta(x^*) \},
\]

where $N_\delta(x^*) = \{ x : 0 < |x - x^*| < \delta \}$ represents the $\delta$-neighborhood of $x^*$ with some $\delta > 0$, and the upper limit of $F$ at $x^* \in S$, denoted as \( \limsup_{x \to x^*} F(x) \), is defined by

\[
\limsup_{x \to x^*} F(x) = \lim_{\delta \to 0^+} \sup \{ F(x) : x \in N_\delta(x^*) \}.
\]

**Example 2.10.** We consider the same IVF $F$ as in Example 2.8, it follows from Definition 2.9 that

\[
\liminf_{x \to 0} F(x) = \lim_{\delta \to 0^+} \inf \{ [1, 2] \sin \frac{1}{x} : x \in (-\delta, 0) \cup (0, \delta) \} = [-2, -1],
\]

and

\[
\limsup_{x \to 0} F(x) = \lim_{\delta \to 0^+} \sup \{ [1, 2] \sin \frac{1}{x} : x \in (-\delta, 0) \cup (0, \delta) \} = [1, 2].
\]

**Definition 2.11.** [18] The gH-derivative of an IVF $F : S \subseteq \mathbb{R} \to \mathbb{R}_I$ at $x^* \in S$ is defined as

\[
F'(x^*) = \lim_{h \to 0} \frac{1}{h} (F(x^* + h) - F(x^*)).
\] (2.1)

If $F'(x^* ) \in \mathbb{R}_I$ satisfying (2.1) exists, we say that $F$ is gH-differentiable at $x^*$.

**Definition 2.12.** [18] Let $F : S \to \mathbb{R}_I$ be an IVF and $x^* = (x_{i1}^*, x_{i2}^*, \ldots, x_{in}^*)^\top$ be fixed in $S$.

(i) We consider the IVF $P_i(x_i) = F(x_{i1}^*, \ldots, x_{i,i-1}^*, x_i, x_{i,i+1}^*, \ldots, x_{in}^*)$. If $P_i$ is gH-differentiable at $x_{i}^*$, then say $F$ have the $i$th partial gH-derivative at $x^*$ and \( \frac{\partial F(x^*)}{\partial x_i} = P'_i(x_i^*) \).

(ii) We say that $F$ is gH-differentiable at $x^*$ if all the partial gH-derivatives \( \frac{\partial F(x^*)}{\partial x_i} (i = 1, 2, \ldots, n) \) exist on some neighborhood of $x^*$ and are continuous at $x^*$. If $F$ is gH-differentiable at each $x \in S$, the $F$ is said to be gH-differentiable on $S$.

(iii) The gH-gradient of $F$ at $x^*$, denoted as \( \nabla gF(x^*) \), is defined by

\[
\nabla gF(x^*) = \left( \frac{\partial F(x^*)}{\partial x_1}, \frac{\partial F(x^*)}{\partial x_2}, \ldots, \frac{\partial F(x^*)}{\partial x_n} \right)^\top.
\]

For any $n$-dimensional vector $d = (d_1, d_2, \ldots, d_n)^\top \in \mathbb{R}^n$, we define the product of $d$ and $\nabla gF(x^*)$, denoted by $d^\top \nabla gF(x^*)$, as an interval:

\[
d^\top \nabla gF(x^*) = \sum_{i=1}^n d_i \frac{\partial F(x^*)}{\partial x_i}.
\]

**Definition 2.13.** [4] Let $F : S \to \mathbb{R}_I$ be an IVF. Let $x^* \in S$ and $d \in \mathbb{R}^n$. If the limit

\[
\lim_{\lambda \to 0^+} \frac{1}{\lambda} (F(x^* + \lambda d) - F(x^*))
\]

exists, then the limit is said to be gH-directional derivative of $F$ at $x^*$ in the direction $d$, and it is denoted by $F'(x^*; d)$.

**Lemma 2.14.** Let $F : S \to \mathbb{R}_I$ be such that $F(x) = [F(x), \overline{F}(x)]$ for all $x \in S$. If $\underline{F}, \overline{F} : S \to \mathbb{R}$ are differentiable at $x^* \in S$ and $d = (d_1, d_2, \ldots, d_n)^\top \in \mathbb{R}_+^n$ is any fixed direction, then

\[
F'(x^*; d) = d^\top \nabla gF(x^*),
\]

where $d \in \mathbb{R}_+^n$ is denoted as $d_i \geq 0$ for all $i = 1, 2, \ldots, n$. 

Proof. Since $F, \mathcal{F}$ are differentiable at $x^*$, then $F_e$ and $F_w$ are differentiable at $x^*$, and it follows from Theorem 3 in [14] that $F$ is gH-differentiable at $x^*$.

By Theorem 4 in [20] and $d = (d_1, d_2, \ldots, d_n)^T \in \mathbb{R}_+^n$, we have

$$F'(x^*; d) = \left[ \sum_{i=1}^{n} d_i \frac{\partial F_e(x^*)}{\partial x_i} \right] = \frac{\partial F_e(x^*)}{\partial x_i}$$

$$= \sum_{i=1}^{n} d_i \left( \frac{\partial F_e(x^*)}{\partial x_i} ; \frac{\partial F_w(x^*)}{\partial x_i} \right) .$$

By Remark 3 and Proposition 9 in [20], we have

$$\frac{\partial F(x^*)}{\partial x_i} = \left( \frac{\partial F_e(x^*)}{\partial x_i} ; \frac{\partial F_w(x^*)}{\partial x_i} \right) .$$

Combining (2.2) and (2.3), we have,

$$F'(x^*; d) = \sum_{i=1}^{n} d_i \frac{\partial F(x^*)}{\partial x_i} = d^T \nabla_g F(x^*).$$

Next, we will present the relationship between $\nabla_g F(x^*)$, $\nabla F_e(x^*)$ and $\nabla F_w(x^*)$, where “$\nabla$” represents the gradient of real-valued functions.

Remark 2.15. Assuming that all the conditions in Lemma 2.14 hold, it follows from (2.2) and Lemma 2.14 that

$$d^T \nabla_g F(x^*) = F'(x^*; d) = \left[ \sum_{i=1}^{n} d_i \frac{\partial F_e(x^*)}{\partial x_i} \right] = \frac{\partial F_e(x^*)}{\partial x_i}$$

$$= \left. \left( d^T \nabla F_e(x^*) ; d^T \nabla F_w(x^*) \right) \right|$$

$$= \left. \left( d^T \nabla F_e(x^*) - d^T \nabla F_w(x^*) \right| ; d^T \nabla F_e(x^*) + d^T \nabla F_w(x^*) \right) .$$

Thus,

$$(d^T \nabla_g F(x^*))_e = d^T \nabla F_e(x^*)$$

and

$$(d^T \nabla_g F(x^*))_w = |d^T \nabla F_w(x^*)| \geq d^T \nabla F_w(x^*).$$

Since $d \in \mathbb{R}_+^n$, then

$$d^T (\nabla_g F(x^*))_e = d^T \nabla F_e(x^*) \quad \text{and} \quad d^T (\nabla_g F(x^*))_w \geq d^T \nabla F_w(x^*).$$

□

3. Some characterizations of preinvex interval-valued functions

In this section, we will discuss some new characterizations of preinvex IVFs under cw-order.

Definition 3.1. [25] Let $y \in S$. We say that $S$ is invex at $y$ w.r.t. $\eta : S \times S \to \mathbb{R}^n$ if for each $x \in S$ and $\varrho \in [0, 1]$, $y + \varrho \eta(x, y) \in S$. $S$ is said to be an invex set w.r.t. $\eta$ if $S$ is invex at each $y \in S$. 


**Definition 3.2.** The IVF $F : S \rightarrow \mathbb{R}_I$ is said to be preinvex on an invex set $S$ w.r.t. $\eta : S \times S \rightarrow \mathbb{R}$ if for any $x, y \in S$, $\varrho \in [0, 1]$,

$$F(y + \varrho \eta(x, y)) \preceq_{cw} \varrho F(x) + (1 - \varrho)F(y).$$

(3.1)

**Remark 3.3.** When $\eta(x, y) = x - y$, we get the definition of convex IVFs under the $cw$-order from Definition 3.2. Obviously, preinvexity is more extensive than convexity.

**Example 3.4.** Let IVF $F$ be defined by

$$F(x) = \begin{cases} [-2x, -x], & \text{if } x \geq 0, \\ [2x, x], & \text{if } x < 0. \end{cases}$$

For $\bar{x} > 0, \bar{y} < 0$ and $\bar{\varrho} \in (0, 1)$ such that $\bar{\varrho}\bar{x} + (1 - \bar{\varrho})\bar{y} > 0$, then

$$\tilde{F}(\bar{x}) + (1 - \tilde{\varrho})\tilde{F}(\bar{y}) = [-2(\bar{\varrho}\bar{x} + (1 - \bar{\varrho})\bar{y}), -(\bar{\varrho}\bar{x} + (1 - \bar{\varrho})\bar{y})],$$

$$\tilde{\varrho}\tilde{F}(\bar{x}) = [-2\tilde{\varrho}\bar{x}, -\tilde{\varrho}\bar{x}],$$

$$(1 - \tilde{\varrho})F(\bar{y}) = [2(1 - \tilde{\varrho})\bar{y}, (1 - \tilde{\varrho})\bar{y}].$$

Thus,

$$F_{\tilde{\varrho}}(\bar{x} + (1 - \tilde{\varrho})\bar{y}) = -\frac{3}{2}(\bar{x} + (1 - \tilde{\varrho})\bar{y}) > -\frac{3}{2}\bar{x} + \frac{3}{2}(1 - \tilde{\varrho})\bar{y} = \tilde{\varrho}F_{\tilde{\varrho}}(\bar{x}) + (1 - \tilde{\varrho})F_{\tilde{\varrho}}(\bar{y}).$$

That is, $\tilde{F}(\bar{x}) + (1 - \tilde{\varrho})F(\bar{y}) \prec_{cw} F(\bar{x} + (1 - \tilde{\varrho})\bar{y})$. Therefore, $F$ is not a convex IVF.

On the other hand, let

$$\eta(x, y) = \begin{cases} x - y, & \text{if } xy \geq 0, \\ y - x, & \text{if } xy < 0. \end{cases}$$

Then,

$$F(y + \varrho \eta(x, y)) = \begin{cases} [-2(\varrho x + (1 - \varrho)y), -(\varrho x + (1 - \varrho)y)], & \text{if } x \geq 0, \ y \geq 0, \\ [2(\varrho x + (1 - \varrho)y), \varrho x + (1 - \varrho)y], & \text{if } x \leq 0, \ y \leq 0, \\ [-2(\varrho + 1)y + 2\varrho x, -(\varrho + 1)y + \varrho x], & \text{if } x < 0, \ y > 0, \\ [2(\varrho + 1)y - 2\varrho x, (\varrho + 1)y - \varrho x], & \text{if } x > 0, \ y < 0. \end{cases}$$

By calculation, it follows that $F(y + \varrho \eta(x, y)) \preceq_{cw} \varrho F(x) + (1 - \varrho)F(y)$ holds for any $x, y \in \mathbb{R}$. Therefore, $F$ is a preinvex IVF.

In order to verify some results of this paper, we need the following Condition C given by Mohan and Neogy in [12].

**Condition C.** We say that the function $\eta : S \times S \rightarrow \mathbb{R}^n$ that satisfies Condition C if for any $x, y \in S$ and $\varrho \in [0, 1]$:

(C1) $\eta(y, x) + \eta(x, y) = 0$;  
(C2) $\eta(x, y) + \eta(y, x) = (1 - \varrho)\eta(x, y)$.

For example, Let $S = \mathbb{R} \setminus \{0\}$ and $\eta(x, y) = x - y$, then $S$ is an invex set w.r.t. $\eta$ and Condition C is satisfied.

**Theorem 3.5.** Let $S$ be an invex set w.r.t. $\eta : S \times S \rightarrow \mathbb{R}^n$ that satisfies Condition C. Assume that $F : S \rightarrow \mathbb{R}_I$ is an IVF and satisfies $F(y + \eta(x, y)) \preceq_{cw} F(x)$ for all $x, y \in S$. Then, $F$ is preinvex on $S$ if and only if for any $x, y \in S$, $\Psi(\varrho) = F(y + \varrho \eta(x, y))$ is convex on $[0, 1]$. 

\[\square\]
Proof. Suppose that $F$ is preinvex on $S$. For all $x, y \in S$ and any $\varrho, \kappa_1, \kappa_2 \in [0, 1]$.

If $\kappa_1 = \kappa_2$, then
\[
\Psi(\kappa_2 + \varrho(\kappa_1 - \kappa_2)) = \Psi(\kappa_2) = \varrho \Psi(\kappa_1) + (1 - \varrho) \Psi(\kappa_2).
\]
That is, $\Psi(\varrho)$ is convex on $[0, 1]$.

If $\kappa_1 > \kappa_2$, then $\kappa_1 - \kappa_2 > 0$, $\kappa_2 \neq 1$, and
\[
0 < (\kappa_1 - \kappa_2)/(1 - \kappa_2) \leq 1.
\]

From Condition C, for any $x, y \in S$ and every $\varrho \in [0, 1]$,
\[
\eta(x + \varrho \eta(x, y), y) = \eta(x + \varrho \eta(x, y), y + \varrho \eta(x, y) - \varrho \eta(x, y)) \\
= \eta(x + \varrho \eta(x, y), y + \varrho \eta(x, y) + \eta(y, y + \varrho \eta(x, y))) \\
= -\eta(y, y + \varrho \eta(x, y)) = \varrho \eta(x, y).
\]
Once again from Condition C and (3.2), we get
\[
\eta(x + \kappa_1 \eta(x, y), y + \kappa_2 \eta(x, y)) = (\kappa_1 - \kappa_2) \eta(x, y).
\]
From (3.3), we have
\[
\Psi(\kappa_2 + \varrho(\kappa_1 - \kappa_2)) = F(x + (\kappa_2 + \varrho(\kappa_1 - \kappa_2)) \eta(x, y)) \\
= F(x + \kappa_2 \eta(x, y) + \varrho(\kappa_1 - \kappa_2) \eta(x, y)) \\
= F(x + \kappa_2 \eta(x, y) + \eta(y + \kappa_1 \eta(x, y), y + \kappa_2 \eta(x, y))) \\
\leq_{cw} \varrho F(y + \kappa_1 \eta(x, y)) + (1 - \varrho) F(y + \kappa_2 \eta(x, y)) \\
= \varrho \Psi(\kappa_1) + (1 - \varrho) \Psi(\kappa_2).
\]

If $\kappa_1 < \kappa_2$, by a similar way, we have
\[
\Psi(\kappa_2 + \varrho(\kappa_1 - \kappa_2)) \leq_{cw} \varrho \Psi(\kappa_1) + (1 - \varrho) \Psi(\kappa_2).
\]
Hence, by (3.4)–(3.5) we get that $\Psi(\varrho)$ is convex on $[0, 1]$.

Conversely, since $\Psi(\varrho)$ is convex on $[0, 1]$ and $F(y + \eta(x, y)) \leq_{cw} F(x)$, then
\[
F(y + \eta(x, y)) = \Psi(\varrho) = \varrho \cdot 1 + (1 - \varrho) \cdot 0 \\
\leq_{cw} \varrho \Psi(1) + (1 - \varrho) \Psi(0) \\
= \varrho F(y + \eta(x, y)) + (1 - \varrho) F(y) \\
\leq_{cw} \varrho F(x) + (1 - \varrho) F(y).
\]
Thus, $F$ is preinvex on $S$ and the proof is completed.

This shows that a preinvex IVF is equivalent to a convex IVF on $[0, 1]$. It follows that we give a necessary and sufficient condition for preinvex IVFs.

**Theorem 3.6.** Let $S$ be an invex set w.r.t. $\eta : S \times S \to \mathbb{R}_+^n$ that satisfies Condition C. Let $F : S \to \mathbb{R}_+$ be such that $F(x) = [F(x), \overline{F}(x)]$ for all $x \in S$. If $F, \overline{F} : S \to \mathbb{R}$ are differentiable on $S$, then $F$ is preinvex on $S$ if and only if for any $x, y \in S$,
\[
F(y) + \eta(x, y)^\top \nabla y F(y) \leq_{cw} F(x).
\]
Proof. Since $F$ is preinvex on $S$, then for any $x, y \in S$ and $\varrho \in [0, 1]$, 

$$\frac{F(y + \varrho\eta(x, y)) \leq_w \varrho F(x) + (1 - \varrho)F(y).$$

That is, 

$$F_c(y + \varrho\eta(x, y)) < \varrho F_c(x) + (1 - \varrho)F_c(y),$$

or 

$$\begin{cases} F_c(y + \varrho\eta(x, y)) = \varrho F_c(x) + (1 - \varrho)F_c(y), \\
F_w(y + \varrho\eta(x, y)) \geq \varrho F_w(x) + (1 - \varrho)F_w(y). \end{cases}$$

Then, 

$$\frac{1}{\varrho}(F_c(y + \varrho\eta(x, y)) - F_c(y)) < F_c(x) - F_c(y),$$

or 

$$\begin{cases} \frac{1}{\varrho}(F_c(y + \varrho\eta(x, y)) - F_c(y)) = F_c(x) - F_c(y), \\
\frac{1}{\varrho}(F_w(y + \varrho\eta(x, y)) - F_w(y)) \geq F_w(x) - F_w(y). \end{cases}$$

Let $\varrho \to 0^+$, we have 

$$\eta(x, y)^\top \nabla F_c(y) \leq F_c(x) - F_c(y),$$

or 

$$\begin{cases} \eta(x, y)^\top \nabla F_c(y) = F_c(x) - F_c(y), \\
\eta(x, y)^\top \nabla F_w(y) \geq F_w(x) - F_w(y). \end{cases}$$

By Remark 2.15, we have 

$$\eta(x, y)^\top (\nabla_g F(y))_c \leq F_c(x) - F_c(y),$$

or 

$$\begin{cases} \eta(x, y)^\top (\nabla_g F(y))_c = F_c(x) - F_c(y), \\
\eta(x, y)^\top (\nabla_g F(y))_w \geq F_w(x) - F_w(y). \end{cases}$$

Thus, 

$$F(y) + \eta(x, y)^\top \nabla_g F(y) \leq_w F(x).$$

Conversely, for any $x, y \in S$ and $\varrho \in (0, 1)$, let $\tilde{y} = y + \varrho\eta(x, y)$. Note that $\tilde{y} \in S$, then 

$$F(\tilde{y}) + \eta(y, \tilde{y})^\top \nabla_g F(\tilde{y}) \leq_w F(y).$$

That is, 

$$F_c(\tilde{y}) + \eta(y, \tilde{y})^\top (\nabla_g F(\tilde{y}))_c < F_c(y),$$

or 

$$\begin{cases} F_c(\tilde{y}) + \eta(y, \tilde{y})^\top (\nabla_g F(\tilde{y}))_c = F_c(y), \\
F_w(\tilde{y}) + \eta(y, \tilde{y})^\top (\nabla_g F(\tilde{y}))_w \geq F_w(y). \end{cases}$$

By Remark 2.15, we have 

$$F_c(\tilde{y}) + \eta(y, \tilde{y})^\top \nabla F_w(\tilde{y}) < F_c(y),$$

or 

$$\begin{cases} F_c(\tilde{y}) + \eta(y, \tilde{y})^\top \nabla F_c(\tilde{y}) = F_c(y), \\
F_w(\tilde{y}) + \eta(y, \tilde{y})^\top \nabla F_w(\tilde{y}) \geq F_w(y). \end{cases}$$

Similarly, applied to the pair $x, \tilde{y}$ yields 

$$F_c(\tilde{y}) + \eta(x, \tilde{y})^\top \nabla F_c(\tilde{y}) < F_c(x),$$
or
\[
\begin{cases}
F_c(\bar{y}) + \eta(x, \bar{y})^\top \nabla F_c(\bar{y}) = F_c(x), \\
F_w(\bar{y}) + \eta(x, \bar{y})^\top \nabla F_w(\bar{y}) \geq F_w(x).
\end{cases}
\]
Thus,
\[
F_c(\bar{y}) + g\eta(x, \bar{y})^\top \nabla F_c(\bar{y}) + (1 - g)\eta(y, \bar{y})^\top \nabla F_c(\bar{y}) < gF_c(x) + (1 - g)F_c(y),
\]
or
\[
\begin{cases}
F_c(\bar{y}) + g\eta(x, \bar{y})^\top \nabla F_c(\bar{y}) + (1 - g)\eta(y, \bar{y})^\top \nabla F_c(\bar{y}) = gF_c(x) + (1 - g)F_c(y), \\
F_w(\bar{y}) + g\eta(x, \bar{y})^\top \nabla F_w(\bar{y}) + (1 - g)\eta(y, \bar{y})^\top \nabla F_w(\bar{y}) \geq gF_w(x) + (1 - g)F_w(y).
\end{cases}
\]
From Condition C, we have
\[
\eta(y, \bar{y}) = -g\eta(x, y), \quad \eta(x, \bar{y}) = (1 - g)\eta(x, y).
\]
Therefore,
\[
F_c(\bar{y}) < gF_c(x) + (1 - g)F_c(y),
\]
or
\[
\begin{cases}
F_c(\bar{y}) = gF_c(x) + (1 - g)F_c(y), \\
F_w(\bar{y}) \geq gF_w(x) + (1 - g)F_w(y).
\end{cases}
\]
It follows that
\[
F(\bar{y}) \lesssim cw gF(x) + (1 - g)F(y).
\]
Thus, $F$ is preinvex on $S$. \hfill \Box

4. \textit{cw}-Semicontinuity and Preinvex IVFs

We first introduce the \textit{cw}-semicontinuity of IVFs under the \textit{cw}-order.

\textbf{Definition 4.1.} Let $F : S \to \mathbb{R}_I$ be an IVF. We say that $F$ is \textit{cw}-lower semicontinuous (\textit{cw}-lsc) at $x^* \in S$ if
\[
F(x^*) \lesssim cw \liminf_{x \to x^*} F(x).
\]

Further, $F$ is called \textit{cw}-lsc on $S$ if (4.1) holds for every $x^* \in S$.

\textbf{Example 4.2.} Consider the following IVF $F$:
\[
F(x_1, x_2) = \begin{cases}
[1, 2] \sin \frac{1}{x_1} + [e^{x_2}, 2e^{x_2}], & \text{if } x_1 x_2 \neq 0, \\
[-3, 1], & \text{if } x_1 x_2 = 0.
\end{cases}
\]
The lower limit of $F$ at $(0, 0)$ is given by
\[
\liminf_{(x_1, x_2) \to (0, 0)} F(x) = [-1, 1].
\]
Since $F(0, 0) = [-3, 1]$, we have
\[
F(0, 0) = [-3, 1] \lesssim cw [-1, 1] = \liminf_{(x_1, x_2) \to (0, 0)} F(x).
\]
Therefore, $F$ is \textit{cw}-lsc at $(0, 0)$. \hfill \Box

\textbf{Theorem 4.3.} Let $F : S \to \mathbb{R}_I$ be an IVF. Then, $F$ is \textit{cw}-lsc at $x^* \in S$ if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $F(x^*) \lesssim cw F(x) + [\varepsilon, \varepsilon]$ for all $x \in N_{\delta}(x^*)$. 

Proof. Let $F$ be $\preceq_{cw}$-lsc at $x^*$. To the contrary, suppose there exists an $\varepsilon_0 > 0$ such that for all $\delta > 0$, $F(x) + [\varepsilon_0, \varepsilon_0] \preceq_{cw} F(x^*)$ for at least one $x \in N_{\delta}(x^*)$. Then,

$$\inf\{F(x) : x \in N_{\delta}(x^*)\} + [\varepsilon_0, \varepsilon_0] \preceq_{cw} F(x^*)$$

$$\Rightarrow \lim_{\delta \to 0^+} \inf\{F(x) : x \in N_{\delta}(x^*)\} + [\varepsilon_0, \varepsilon_0] \preceq_{cw} F(x^*)$$

$$\Rightarrow \lim_{x \to x^*} \inf F(x) + [\varepsilon_0, \varepsilon_0] \preceq_{cw} F(x^*)$$

$$\Rightarrow \lim_{x \to x^*} \inf F(x) \preceq_{cw} F(x^*),$$

which contradicts that $F$ is $\preceq_{cw}$-lsc at $x^*$. Thus, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $F(x^*) \preceq_{cw} F(x) + [\varepsilon, \varepsilon]$ for all $x \in N_{\delta}(x^*)$.

Conversely, suppose that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $F(x^*) \preceq_{cw} F(x) + [\varepsilon, \varepsilon]$ for any $x \in N_{\delta}(x^*)$. Then,

$$F(x^*) \preceq_{cw} \inf\{F(x) : x \in N_{\delta}(x^*)\} + [\varepsilon, \varepsilon]$$

$$\Rightarrow F(x^*) \preceq_{cw} \lim_{\delta \to 0^+} \inf\{F(x) : x \in N_{\delta}(x^*)\} + [\varepsilon, \varepsilon]$$

$$\Rightarrow F(x^*) \preceq_{cw} \lim_{x \to x^*} \inf F(x) + [\varepsilon, \varepsilon]$$

$$\Rightarrow F(x^*) \preceq_{cw} \lim_{x \to x^*} \inf F(x).$$

Thus, by Definition 4.1, $F$ is $\preceq_{cw}$-lsc at $x^*$.

\[\square\]

**Definition 4.4.** Let $F : S \to \mathbb{R}_I$ be an IVF. We say that $F$ is $\preceq_{cw}$-upper semicontinuous ($\preceq_{cw}$-usc) at $x^* \in S$ if

$$\limsup_{x \to x^*} F(x) \preceq_{cw} F(x^*).$$

Further, $F$ is called $\preceq_{cw}$-usc on $S$ if (4.2) holds for every $x^* \in S$.

**Example 4.5.** Consider the following IVF $F$:

$$F(x_1, x_2) = \begin{cases} [1, 2] \sin \frac{1}{x_1} + [e^{x_2}, 2e^{x_2}], & \text{if } x_1x_2 \neq 0, \\ [3, 4], & \text{if } x_1x_2 = 0. \end{cases}$$

The upper limit of $F$ at $(0, 0)$ is given by

$$\limsup_{(x_1, x_2) \to (0, 0)} F(x) = [2, 4].$$

Since $F(0, 0) = [3, 4]$, we have

$$\limsup_{(x_1, x_2) \to (0, 0)} F(x) = [2, 4] \preceq_{cw} [3, 4] = F(0, 0).$$

Therefore, $F$ is $\preceq_{cw}$-usc at $(0, 0)$.

\[\square\]

**Theorem 4.6.** Let $F : S \to \mathbb{R}_I$ be an IVF. Then, $F$ is $\preceq_{cw}$-usc at $x^* \in S$ if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $F(x) \prec_{cw} F(x^*) + [\varepsilon, \varepsilon]$ for all $x \in N_{\delta}(x^*)$.

**Proof.** Similar to the proof of Theorem 4.3 and is omitted. \[\square\]
Lemma 4.7. Let $S$ be an invex set w.r.t. $\eta : S \times S \to \mathbb{R}^n$ that satisfies Condition C. Let $F : S \to \mathbb{R}$ be an IVF that satisfies

$$F(y + \eta(x, y)) \preceq_{cw} F(x), \forall x, y \in S.$$ 

If there exists a $\kappa \in (0, 1)$ such that

$$F(y + \kappa \eta(x, y)) \preceq_{cw} \kappa F(x) + (1 - \kappa) F(y), \forall x, y \in S,$$

then, the set $K = \{ \varrho \in [0, 1] : F(y + \varrho \eta(x, y)) \preceq_{cw} \varrho F(x) + (1 - \varrho) F(y), \forall x, y \in S \}$ is dense in $[0, 1]$.

Proof. Obviously, $0, 1 \in K$. Suppose that $K$ is not dense in $[0, 1]$, then exists a $\varrho_0 \in (0, 1)$ such that

$$\mathbb{N}_\delta(\varrho_0) \cap K = \emptyset$$

with some $\delta > 0$.

Let

$$\varrho_1 = \inf\{ \varrho \in K : \varrho \geq \varrho_0 \}, \quad (4.3)$$

$$\varrho_2 = \sup\{ \varrho \in K : \varrho \leq \varrho_0 \}. \quad (4.4)$$

Choose $\vartheta_1, \vartheta_2 \in K$ with $\vartheta_1 \geq \varrho_1$ and $\vartheta_2 \leq \varrho_2$ such that

$$\max\{ \kappa, 1 - \kappa \}(\vartheta_1 - \vartheta_2) < \varrho_1 - \varrho_2. \quad (4.5)$$

Then

$$\vartheta_2 \leq \varrho_2 < \varrho_1 \leq \vartheta_1.$$

Next, consider $\tilde{\vartheta} = \kappa \vartheta_1 + (1 - \kappa) \vartheta_2$. From Condition C,

$$y + \vartheta_2 \eta(x, y) + \kappa \eta(y + \vartheta_1 \eta(x, y), y + \vartheta_2 \eta(x, y))$$

$$= y + \vartheta_2 \eta(x, y) + \kappa \eta(y + \vartheta_1 \eta(x, y), y + \vartheta_1 \eta(x, y) - (\vartheta_1 - \vartheta_2) \eta(x, y))$$

$$= y + \tilde{\vartheta} \eta(x, y).$$

Hence, from (3.1) and $\vartheta_1, \vartheta_2 \in K$ we get

$$F(y + \tilde{\vartheta} \eta(x, y))$$

$$= F[y + \vartheta_2 \eta(x, y) + \kappa \eta(y + \vartheta_1 \eta(x, y), y + \vartheta_2 \eta(x, y))]$$

$$\preceq_{cw} \kappa F(y + \vartheta_1 \eta(x, y)) + (1 - \kappa) F(y + \vartheta_2 \eta(x, y))$$

$$\preceq_{cw} \kappa [\vartheta_1 F(x) + (1 - \vartheta_1) F(y)] + (1 - \kappa) [\vartheta_2 F(x) + (1 - \vartheta_2) F(y)]$$

$$= \tilde{\vartheta} F(x) + (1 - \tilde{\vartheta}) F(y).$$

That is, $\tilde{\vartheta} \in K$.

If $\tilde{\vartheta} \geq \varrho_0$, from (4.5) we obtain

$$\tilde{\vartheta} - \vartheta_2 = \kappa (\vartheta_1 - \vartheta_2) < \varrho_1 - \varrho_2,$$

and therefore $\tilde{\vartheta} \leq \varrho_1$. Because $\tilde{\vartheta} \geq \varrho_0$ and $\tilde{\vartheta} \in K$, this contradicts (4.3). Similarly, $\tilde{\vartheta} \leq \varrho_0$ leads to contradicts (4.4). Consequently, $K$ is dense in $[0, 1]$, which completes the proof. $\square$

Remark 4.8. If the IVF $F$ degenerates to a real function, the conclusion in Lemma 4.7 is coupled with Lemma 3.2 of [27].

Based on Lemma 4.7, the following two theorems focus on the relationship between $\preceq_{cw}$-semicontinuity and preinvexity of IVFs.
Theorem 4.9. Let $S$ be an invex set w.r.t. $\eta : S \times S \to \mathbb{R}^n$ that satisfies Condition C and $\|\eta\| < \infty$. Let $F : S \to \mathbb{R}_I$ be an $\preceq_{cw}$-usc IVF that satisfies 

$$F(y + \eta(x, y)) \preceq_{cw} F(x), \ \forall x, y \in S.$$ 

If there exists a $\varrho \in (0, 1)$ such that 

$$F(y + \varrho \eta(x, y)) \preceq_{cw} \varrho F(x) + (1 - \varrho)F(y), \ \forall x, y \in S,$$

Then, $F$ is preinvex on $S$.

Proof. Suppose that $F$ is not a preinvex IVF on $S$, then there exist $x, y \in S$ and $\varrho \in (0, 1)$ such that 

$$\varrho F(x) + (1 - \varrho)F(y) \prec_{cw} F(y + \varrho \eta(x, y)).$$

That is, 

$$\varrho F_c(x) + (1 - \varrho)F_c(y) < F_c(y + \varrho \eta(x, y)), \quad (4.6)$$

or 

$$\begin{cases} 
\varrho F_c(x) + (1 - \varrho)F_c(y) = F_c(y + \varrho \eta(x, y)), \\
\varrho F_w(x) + (1 - \varrho)F_w(y) > F_w(y + \varrho \eta(x, y)). 
\end{cases} \quad (4.7)$$

Now, let 

$$\omega = y + \varrho \eta(x, y),$$

$$K = \{ g \in [0, 1] | F(y + g \eta(x, y)) \preceq_{cw} g F(x) + (1 - g)F(y), \forall x, y \in S \}.$$ 

From Lemma 4.7, $K$ is dense. Thus, there exists a sequence $\{ g_n \} \subseteq K$ such that 

$$g_n \to \varrho \ as \ n \to \infty.$$ 

Let $y_n = y + \frac{\varrho - g_n}{1 - g_n} \eta(x, y)$, we have 

$$\| y_n - y \| = \left\| \frac{\varrho - g_n}{1 - g_n} \eta(x, y) \right\| \to 0 \ as \ n \to \infty.$$ 

Since $S$ is an open set, then $y_n \in S$, and from Condition C, 

$$y_n + g_n \eta(x, y_n) = y + \frac{\varrho - g_n}{1 - g_n} \eta(x, y) + g_n \eta \left( x, y + \frac{\varrho - g_n}{1 - g_n} \eta(x, y) \right)$$

$$= y + \frac{\varrho - g_n}{1 - g_n} \eta(x, y) + g_n \left( 1 - \frac{\varrho - g_n}{1 - g_n} \right) \eta(x, y)$$

$$= y + g_n \eta(x, y) + \left( \frac{\varrho - g_n}{1 - g_n} - \frac{\varrho - g_n}{1 - g_n} g_n \right) \eta(x, y)$$

$$= y + g_n \eta(x, y) = \omega. \quad (4.8)$$

By the $\preceq_{cw}$-usc of $F$, then for any $\varepsilon > 0$, there exists a positive integer $N$ such that 

$$F(y_n) \preceq_{cw} F(y) + \frac{\varepsilon}{1 - g} \ \text{for all} \ n > N.$$ 

From (4.8) and $g_n \in K$, we obtain 

$$F(\omega) = F(y_n + g_n \eta(x, y_n))$$

$$\preceq_{cw} g_n F(x) + (1 - g_n)F(y_n)$$

$$\preceq_{cw} g_n F(x) + (1 - g_n)F(y) + (1 - g_n)\frac{\varepsilon}{1 - g_n}.$$
Due to the arbitrariness of \( \varepsilon \), let \( n \to \infty \), we get
\[
F_c(y + \tilde{\eta}(x, y)) < \tilde{\eta}F_c(x) + (1 - \tilde{\eta})F_c(y)
\]
or
\[
\begin{cases}
F_c(y + \tilde{\eta}(x, y)) = \tilde{\eta}F_c(x) + (1 - \tilde{\eta})F_c(y) + (1 - \tilde{\eta}) \frac{\varepsilon}{1 - \varrho}, \\
F_w(y + \tilde{\eta}(x, y)) \geq \tilde{\eta}F_w(x) + (1 - \tilde{\eta})F_w(y) + (1 - \tilde{\eta}) \frac{\varepsilon}{1 - \varrho}.
\end{cases}
\]
which contradicts the inequality (4.6) or (4.7). Hence, \( F \) is preinvex on \( S \).

**Theorem 4.10.** Let \( S \) be an invex set w.r.t. \( \eta : S \times S \to \mathbb{R}^n \) that satisfies Condition C and \( \| \eta \| < \infty \). Let \( F : S \to \mathbb{R}_I \) be an \( \leq_{cw}\text{-lsc} \) IVF that satisfies
\[
F(y + \eta(x, y)) \leq_{cw} F(x), \; \forall x, y \in S.
\]
If there exists a \( \varrho \in (0, 1) \) such that
\[
F(y + \varrho \eta(x, y)) \leq_{cw} \varrho F(x) + (1 - \varrho) F(y), \; \forall x, y \in S,
\]
Then, \( F \) is preinvex on \( S \).

**Proof.** Similar to the proof of Theorem 4.9 and is omitted. \( \square \)

**Remark 4.11.** If the IVF \( F \) degenerates to a real function, the conclusions in Theorems 4.9–4.10 are coupled with Theorems 3.1–3.2 of [27].

### 5. Unconstrained Interval-Valued Optimization Problems

We consider the following unconstrained interval-valued optimization problem:

\[
\text{(IVOP)} \quad \min_{x \in S} F(x),
\]
where \( F : S \to \mathbb{R}_I \) is an IVF.

**Definition 5.1.** The point \( x^* \in S \) is called a (global optimal) solution to the Problem (IVOP) if for any \( x \in S \), \( F(x^*) \leq_{cw} F(x) \) holds.

**Definition 5.2.** The point \( x^* \in S \) and exists a neighborhood \( N_\delta(x^*) \) of \( x^* \), if for any \( x \neq x^* \in S \cap N_\delta(x^*) \) such that
(i) \( F(x^*) \leq_{cw} F(x) \), then \( x^* \) is called a local optimal solution;
(ii) \( F(x^*) <_{cw} F(x) \), then \( x^* \) is called a strict local optimal solution.

The interval variational-like inequality problem is to find \( x^* \in S \) such that

\[
\text{(IVLI)} \quad [0, 0] \preceq_{cw} \eta(x, x^*)^\top \nabla g F(x), \; \forall x \in S \setminus \{x^*\},
\]
and the interval weak variational-like inequality problem is to find \( x^* \in S \) such that

\[
\text{(IWVLI)} \quad [0, 0] \preceq_{cw} \eta(x, x^*)^\top \nabla g F(x), \; \forall x \in S \setminus \{x^*\}.
\]

**Theorem 5.3** shows the relationship between the (IVLI) and (IVOP).
Theorem 5.3. Let $S$ be an invex set w.r.t. $\eta : S \times S \to \mathbb{R}^n_+$ and $x^* \in S$. Let $F : S \to \mathbb{R}_I$ be such that $F(x) = [F(x), \overline{F}(x)]$ for all $x \in S$. Assume that $F$ is preinvex and $\underline{F}, \overline{F} : S \to \mathbb{R}$ are differentiable on $S$. If $(x^*, \nabla_g F(x^*))$ is a solution of (IVLI), then $x^*$ is a strictly local optimal solution of (IVOP).

Proof. Let $(x^*, \nabla_g F(x^*))$ be a solution of (IVLI). By contradiction, there exists a $\hat{x} \in S \cap N_\delta(x^*)$ such that

$$F(\hat{x}) \not\leq_{cw} F(x^*).$$

Since $F$ is preinvex and $\underline{F}, \overline{F}$ are differentiable on $S$, from Theorem 3.6, we have

$$F(x^*) + \eta(\hat{x}, x^*) \nabla_g F(x^*) \not\leq_{cw} F(\hat{x}).$$

Thus,

$$\eta(\hat{x}, x^*) \nabla_g F(x^*) \not\leq_{cw} [0, 0].$$

This contradicts the fact that $(x^*, \nabla_g F(x^*))$ is a solution of (IVLI). $\square$

In order to get the inverse of the above Theorem 5.3, we will strengthen the conditions to obtain the following results.

Theorem 5.4. Let $S$ be an invex set w.r.t. $\eta : S \times S \to \mathbb{R}^n_+$. Let $F : S \to \mathbb{R}_I$ be such that $F(x) = [F(x), \overline{F}(x)]$ for all $x \in S$ and $\underline{F}, \overline{F} : S \to \mathbb{R}$ are differentiable on $S$. If $x^*$ is a solution of (IVOP), then $(x^*, \nabla_g F(x^*))$ is a strictly local optimal solution of (IVWLI).

Proof. Let $x^*$ be a solution of (IVOP). By contradiction, there exists a $\hat{x} \in S$ such that

$$\eta(\hat{x}, x^*) \nabla_g F(x^*) \not\leq_{cw} [0, 0].$$

Since $x^* + \theta \eta(\hat{x}, x^*) \in S$, it follows from Lemma 2.14 that

$$\lim_{\theta \to 0^+} \frac{1}{\theta} (F(x^* + \theta \eta(\hat{x}, x^*)) \odot_g F(x^*)) = \eta(\hat{x}, x^*) \nabla_g F(x^*).$$

Therefore

$$F(x^* + \theta \eta(\hat{x}, x^*)) \odot_g F(x^*) \nleq_{cw} [0, 0].$$

This contradicts the fact that $x^*$ is a strictly local optimal solution of (IVOP). $\square$

From some results in Section 4, we obtain the equivalent relation between (IVOP) and (IVLI).

Theorem 5.5. Let $S$ be an invex set w.r.t. $\eta : S \times S \to \mathbb{R}^n_+$. Let $F : S \to \mathbb{R}_I$ be such that $F(x) = [F(x), \overline{F}(x)]$ for all $x \in S$ and $\underline{F}, \overline{F} : S \to \mathbb{R}$ are differentiable on $S$. Assume that $F$ is $\leq_{cw}$-usc(lsc) and $\ell$-increasing on $S$ and for any $x, y \in S$ that satisfies

$$F(y + \eta(x, y)) \leq_{cw} F(x).$$

If there exists a $\theta \in (0, 1)$ such that

$$F(y + \theta \eta(x, y)) \leq_{cw} \theta F(x) + (1 - \theta) F(y), \ \forall x, y \in S.$$

Then, $x^*$ is a strict local optimal solution of (IVOP) if and only if $(x^*, \nabla_g F(x^*))$ is a solution of (IVLI).
Proof. Let \((x^*, \nabla_g F(x^*))\) be a solution of (IVLI). By contradiction, then there exists a \(\hat{x} \in S\) such that

\[
F(\hat{x}) \preceq_{cw} F(x^*).
\]

Since \(x^* + \eta \hat{x}, x^*) \in S\), then \(F\) is preinvex by Theorem 4.9, and hence

\[
F(x^* + \eta \hat{x}, x^*)) \preceq_{cw} \theta F(\hat{x}) + (1 - \theta) F(x^*)
\]

\[
eq \theta F(\hat{x}) + F(x^*) \theta \eta \theta F(x^*) \quad \text{(since \(\text{len}(F(x^*)) > \text{len}(\theta F(x^*))\))}
\]

\[
= \theta F(\hat{x}) \ominus \theta F(x^*) + F(x^*) \quad \text{(since \(\text{len}(F(\hat{x})) > \text{len}(F(x^*))\)).}
\]

By \(\text{len}(F(x^* + \eta \hat{x}, x^*)) > \text{len}(F(x^*))\) and \(\text{len}(\theta F(x^*) \ominus \theta F(x^*) + F(x^*)) > \text{len}(F(x^*))\), we get

\[
F(x^* + \eta \hat{x}, x^*)) \ominus \theta F(x^*) \preceq_{cw} (\theta F(x^*) \ominus \theta F(x^*) + F(x^*)) \ominus \theta F(x^*).
\]

It follows that

\[
F(x^* + \eta \hat{x}, x^*)) \ominus \theta F(x^*) \preceq_{cw} \theta F(\hat{x}) \ominus \theta F(x^*),
\]

that is

\[
\frac{1}{\theta} (F(x^* + \eta \hat{x}, x^*)) \ominus \theta F(x^*) \preceq_{cw} F(\hat{x}) \ominus \theta F(x^*).
\]

It follows from Lemma 2.14 that

\[
\lim_{\theta \to 0^+} \frac{1}{\theta} (F(x^* + \eta \hat{x}, x^*)) \ominus \theta F(x^*) = \eta(\hat{x}, x^*)^\top \nabla_g F(x^*).
\]

Therefore, \(\eta(\hat{x}, x^*)^\top \nabla_g F(x^*) \preceq_{cw} [0, 0]\). This contradicts that \((x^*, \nabla_g F(x^*))\) is a solution of (IVLI).

Conversely, let \(x^*\) be a strict local optimal solution of (IVOP). By contradiction, then there exists a \(\hat{x} \in S\) such that

\[
\eta(\hat{x}, x^*)^\top \nabla_g F(x^*) \preceq_{cw} [0, 0].
\]

Let \(\theta \leq \delta/\|\eta(\hat{x}, x^*)\|\), then \(x^* + \eta \hat{x}, x^*) \in S \cap \mathbb{N}_\delta(x^*)\). From Lemma 2.14, it follows that

\[
\lim_{\theta \to 0^+} \frac{1}{\theta} (F(x^* + \eta \hat{x}, x^*)) \ominus \theta F(x^*) = \eta(\hat{x}, x^*)^\top \nabla_g F(x^*).
\]

Since \(F\) is \(\ell\)-increasing, we have

\[
F(x^* + \eta \hat{x}, x^*)) \preceq_{cw} F(x^*).
\]

This contradicts that \(x^*\) is a strict local optimal solution of (IVOP). \(\square\)

Remark 5.6. As can be seen from Remark 2.2, the ordering \(\preceq_{cw}\) is more extensive than the ordering \(\preceq_{m_{cw}}\), then the optimal solution under the ordering \(\preceq_{cw}\) may not necessarily be the optimal solution under the ordering \(\preceq_{m_{cw}}\).

Example 5.7. Let \(S = [-4, 4]\) and consider the following IVOP:

\[
\min_{x \in S} F(x) = [-2x^2 + 2, 4x^2 + 6].
\]

(5.1)

The images of the lower endpoint function \(\underline{F}\) and the upper endpoint function \(\bar{F}\) of \(F\) are shown in Figure 2. By calculation, for any \(x \in [-4, 0) \cup (0, 4]\), we get

\[
F_u(x) = x^2 + 4 > 4 = F_u(0),
\]

and

\[
F_w(x) = 3x^2 + 2 > 2 = F_w(0).
\]

Then for any \(x \in [-4, 0) \cup (0, 4]\), \(F(0) \preceq_{cw} F(x)\) but \(F(0) \preceq_{m_{cw}} F(x)\) does not. That is, \(0\) is an optimal solution for IVOP (5.1) under the ordering \(\preceq_{cw}\), but not the optimal solution for IVOP (5.1) under the ordering \(\preceq_{m_{cw}}\). This result can also be observed from Figure 3.
6. Conclusions

This paper has focused on extending the generalized convexity theory of IVFs and its application to a class of unconstrained interval optimization problems. We first defined the preinvex IVF under $cw$-order and obtained its some new characterizations. Furthermore, we also introduced the concept of $\preccurlyeq_{cw}$-semicontinuity and discussed its relationship with preinvex IVFs. As the applications of preinvex IVFs, we investigated a class of unconstrained interval optimization problems and derived existence theorems for its optimal solutions. In the future, we intend to study these results deeper, perhaps by considering interval dual problems, fuzzy optimization problems and other problems.

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