ON TOTAL COLORING AND EQUITABLE TOTAL COLORING OF INFINITE SNARK FAMILIES

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Abstract. We show that all members of the SemiBlowup, Blowup and the first Loupekine snark families have equitable total chromatic number equal to 4. These results provide evidence of negative answers for the questions proposed: by (A. Cavicchioli, T.E. Murgolo, B. Ruini and F. Spaggiari, Acta Appl. Math. 76 (2003) 56–88.) about the smallest order of a Type 2 snark of girth at least 5; and by (S. Dantas, C.M.H. De Figueiredo, G. Mazzuoccolo, M. Preissmann, V.F. dos Santos and D. Sasaki, Discret. Appl. Math. 209 (2016) 84–91.) about the existence of Type 1 cubic graph with girth at least 5 and equitable total chromatic number 5. Moreover, we show new infinite families of snarks obtained by the Kochol superpositions that are Type 1.

Mathematics Subject Classification. 05C15.

Received January 24, 2023. Accepted August 22, 2023.

1. Introduction

The graphs $G = (V,E)$ considered in this paper are undirected and simple, where $V = V(G)$ is the set of vertices of $G$, and $E = E(G)$ is the set of edges of $G$. A $q$-total coloring of $G$ is an assignment of $q$ colors to the vertices and edges of $G$, so that adjacent or incident elements have different colors. The total chromatic number of $G$, denoted by $\chi''$, is the smallest $q$ for which $G$ has a $q$-total coloring. The well known Total Coloring Conjecture (TCC) [1, 18] states that the total chromatic number of a graph $G$ is at least $\Delta(G) + 1$ (graphs called Type 1), and at most $\Delta(G) + 2$ (graphs called Type 2), where $\Delta(G)$ is the maximum degree of $G$. The TCC has been settled for some specific graph families, such as the split graphs, the complete $r$-partite graphs, and the dually chordal graphs [7], but it remains open for several graph classes for more than fifty years. The total chromatic number of a cubic graph (graph where each vertex has degree 3) was proved to be either 4 or 5 by Rosenfeld [14], and independently by Vijayaditya [17].

An equitable total coloring is a total coloring such that the difference between the cardinalities of any two color classes is at most one. The equitable total chromatic number $\chi''_e$ of $G$ is the smallest number for which $G$ admits an equitable total coloring. The Equitable Total Coloring Conjecture (ETCC) was posed by Wang [19] in 2002, and states that the equitable total chromatic number of a graph is at most $\Delta(G) + 2$. The ETCC was

Keywords. Equitable total coloring, total coloring, Blowup snarks, SemiBlowup snarks, superposition.

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proved for cubic graphs, and it implies that if a cubic graph is Type 2, then both the total chromatic number and the equitable total chromatic number are equal to 5; if a cubic graph is Type 1, then the equitable total chromatic number is either 4 or 5. Dantas et al. [5] proved that it is NP-complete to decide whether the equitable total chromatic number is equal to 4 for bipartite cubic graphs.

The search for counterexamples to the Four Color Conjecture motivated the study of cubic graphs, and this class is still much studied not only in coloring problems [6]. Based on the poem by Lewis Carroll “The Hunting of the Snark”, Gardner [8] introduced the name Snark for cyclically 4-edge-connected cubic graph that does not allow a 3-edge-coloring. The importance of these graphs arises also from the fact that several well-known conjectures would have snarks as minimal counterexamples, such as the cycle double cover conjecture and the 5-flow conjecture [6]. The Petersen graph is the smallest and earliest known snark. In [11], Isaacs introduced the dot product, a famous operation used for constructing infinitely many snarks, and defined the Flower and the Blanuša snark families. In 1976, Isaacs [12] introduced two families of Loupekine snarks.

The girth of $G$ is the length of the shortest cycle contained in $G$. In 2003, Cavicchioli et al. [4] verified, by using a computer, that all snarks with girth at least 5 and fewer than 30 vertices are Type 1, and asked the following question:

**Question 1.1.** Does there exist a Type 2 cubic graph with girth at least 5?

In 2011, Campos et al. [3] provided (equitable) total colorings with four colors for all Flower [11] and Goldberg [9] snarks, and they all have girth 5. In 2014, Sasaki et al. [16] showed that all members of the Blanuša families and part of the Loupekine families are Type 1. In 2015, Brinkmann et al. [2] presented the first Type 2 snarks with small girth. About the equitable total coloring of cubic graphs with girth at least 5, Dantas et al. [5] in 2016 proposed the question:

**Question 1.2.** Does there exist a Type 1 cubic graph with girth greater than 4 and equitable total chromatic number 5?

The authors observed that the only known examples of Type 1 cubic graphs with equitable total chromatic number 5 have girth smaller than 5. We highlight that Question 1.2 remains open so far.


In this paper, we contribute to Dantas et al. [5] (Question 1.2) and to Cavicchioli et al. [4] (Question 1.1) questions, by providing evidence for a negative answer to their questions. We present equitable 4-total colorings for all members of the SemiBlowup, Blowup, and for all members of the first Loupekine snark families. Furthermore, we determine 4-total colorings for all members of the new infinite families obtained by Kochol superpositions.

# 2. SemiBlowup snarks

A *semi-graph* is a 3-tuple $G = (V, E, S)$ where $V$ is a finite set of vertices of $G$, $E$ is a set of edges having two distinct endpoints in $V$, and $S$ is a multiset of *semiedges* having at most one endpoint in $V$. A semiedge without endpoints is called an *isolated edge*. A semiedge with endpoint $v$ is denoted by $v$- and an edge having endpoints $v$ and $w$ is denoted by $vw$. Given two semiedges $v$- and $w$- , the *junction* of $v$- and $w$- is done by replacing $v$- and $w$- by the edge $vw$. The definitions for simple graphs can be naturally extended to semi-graphs. Indeed, a *graph* $G = (V, E)$ is a semi-graph with an empty set of semiedges. We denote by $\varphi_H$ the equitable 4-total coloring of the semi-graph (graph) $H$, and by $|\varphi_H^{-1}(a)|$ the corresponding cardinality of the color class $a$ in $\varphi_H$.

The SemiBlowup family of snarks was introduced in 2016 by Hägglund [10], and its construction is as follows. Let $S_t$ be the semi-graph, depicted in Figure 1, where semiedges are represented by lines with only one vertex as endpoint.
On total coloring and equitable total coloring of infinite snark families

Figure 1. The semi-graph $S_i$, and the semi-graphs $B_2$ and $B_3$ with their respective equitable 4-total colorings $\varphi_{B_2}$ and $\varphi_{B_3}$.

Figure 2. The semi-graphs $B_4$ and $B_5$ with their respective equitable 4-total colorings $\varphi_{B_4}$ and $\varphi_{B_5}$.

A $t$-SemiBlowup is a snark constructed by connecting $t \geq 5$ copies of semi-graph $S_i$, for $1 \leq i \leq t$. More precisely, the $t$-SemiBlowup is constructed through junctions: for $2 \leq i \leq t$, we make the junctions of semiedges $a_i \cdot c_{i-1} \cdot i_i \cdot j_{i-1} \cdot k_{i-1} \cdot i_i \cdot j_{i-1}$ and for $i = 1$, $a_1 \cdot c_1 \cdot i_1 \cdot j_1 \cdot k_1$. We connect $t$ copies of $S_i$ as a cycle, as is depicted in Figure 5 (left), that presents the 6-SemiBlowup colored by Theorem 2.1.

Now, we observe that, since the cardinality of the elements of $S_i$ is not a multiple of four, the idea of the desired equitable 4-total coloring is to assign four different equitable 4-total colorings to subgraphs that are used in construction process.

Theorem 2.1. All $t$-SemiBlowup snarks, $t \geq 5$, have equitable total chromatic number equal to 4.

Proof. Let $G$ be a $t$-SemiBlowup composed by $t$ copies of the semi-graph $S_i$, and let $B_2$, $B_3$, $B_4$ and $B_5$ be the semi-graphs composed by 2, 3, 4, and 5 copies of the semi-graph $S_i$, respectively. Their respective equitable 4-total colorings $\varphi_{B_2}$, $\varphi_{B_3}$, $\varphi_{B_4}$, and $\varphi_{B_5}$ are shown in Figures 1 and 2.

To determine the cardinality of a color class in a semi-graph, we count each semiedge as 0.5, due to the junction that takes place in the construction. Following this rule, we have that $|\varphi_{B_4}(1)| = |\varphi_{B_4}(2)| = |\varphi_{B_4}(3)| = |\varphi_{B_4}(4)| = 25$; $|\varphi_{B_3}(1)| = 32$, $|\varphi_{B_3}(2)| = |\varphi_{B_3}(3)| = |\varphi_{B_3}(4)| = 31$; $|\varphi_{B_2}(1)| = |\varphi_{B_2}(4)| = 12$, $|\varphi_{B_2}(2)| = |\varphi_{B_2}(3)| = 13$; $|\varphi_{B_1}(1)| = |\varphi_{B_1}(2)| = |\varphi_{B_1}(3)| = 19$ and $|\varphi_{B_1}(4)| = 18$.

For every $t$-SemiBlowup with $t \equiv 0 \pmod{4}$, we obtain an equitable 4-total coloring by repeating $\frac{t}{4}$ times the coloring $\varphi_{B_4}$ for $\frac{t}{4}$ copies of the semi-graph $B_4$. Note that the coloring fits the junctions used in construction. So, $|\varphi_{B_4}^{-1}(1)| = |\varphi_{B_4}^{-1}(2)| = |\varphi_{B_4}^{-1}(3)| = |\varphi_{B_4}^{-1}(4)| = \frac{t}{4} \times 25$. 
Figure 3. The semi-graph $S'_i$, and the semi-graph $B'_2$ with its respective 4-total coloring $\varphi_{B'_2}$.

Figure 4. The semi-graph $B'_3$ with its respective 4-total coloring $\varphi_{B'_3}$.

For every $t$-SemiBlowup with $t \equiv 1 \pmod{4}$, we obtain an equitable 4-total coloring by repeating $\frac{t-3}{4}$ times the coloring $\varphi_{B_4}$ for $\frac{t-5}{4}$ copies of the semi-graph $B_4$, and we use $\varphi_{B_5}$ for one copy of $B_5$. Evidently, if $t = 5$, then the coloring $\varphi_{B_5}$ is enough. So, $|\varphi_G^{-1}(1)| = \frac{t-3}{4} \times 25 + 32$, and $|\varphi_G^{-1}(2)| = |\varphi_G^{-1}(3)| = |\varphi_G^{-1}(4)| = \frac{t-5}{4} \times 25 + 31$.

For every $t$-SemiBlowup with $t \equiv 2 \pmod{4}$, we obtain an equitable 4-total coloring by repeating $\frac{t-2}{4}$ times the coloring $\varphi_{B_4}$ for $\frac{t-2}{4}$ copies of the semi-graph $B_4$, and we use $\varphi_{B_2}$ for one copy of $B_2$. So, $|\varphi_G^{-1}(1)| = |\varphi_G^{-1}(4)| = \frac{t-2}{4} \times 25 + 12$, and $|\varphi_G^{-1}(2)| = |\varphi_G^{-1}(3)| = \frac{t-2}{4} \times 25 + 13$.

For every $t$-SemiBlowup with $t \equiv 3 \pmod{4}$, we obtain an equitable 4-total coloring by repeating $\frac{t-3}{4}$ times the coloring $\varphi_{B_4}$ for $\frac{t-3}{4}$ copies of the semi-graph $B_4$, and we use $\varphi_{B_3}$ for one copy of $B_3$. So, $|\varphi_G^{-1}(1)| = |\varphi_G^{-1}(2)| = |\varphi_G^{-1}(3)| = \frac{t-3}{4} \times 25 + 19$, and $|\varphi_G^{-1}(4)| = \frac{t-3}{4} \times 25 + 18$.

We observe that, in each case, the equitable 4-total coloring of $t$-SemiBlowup is constructed by repeatedly using the presented 4-total colorings, and by making the junction of semiedges with the same color such that adjacent vertices have different colors. Hence, the result follows. □

3. Blowup Snarks

The Blowup family of snarks was also introduced in 2016 by Hgglund [10], and its construction is presented as follows. Let $S'_i$ be the semi-graph depicted in Figure 3a with vertex set $\{a_i, b_i, \ldots, l_i\}$ and six semiedges. A $t$-Blowup is a snark constructed by connecting $t \geq 5$ copies of semi-graph $S'_i$, for $1 \leq i \leq t$.

More precisely, the $t$-Blowup is constructed through junctions: for $2 \leq i \leq t$, we make the junctions of semiedges $a_i$ with $d_{i-1}$, $j_i$ with $k_{i-1}$, $l_i$ with $l_{i-1}$; and for $i = 1$, $a_1$ with $d_{t'}$, $j_1$ with $k_{t'}$, and $l_1$ with $l_t$. We connect $t$ copies of $S'_i$ similarly to a cycle, as we can see in Figure 5, that presents the 6-Blowup studied in Theorem 3.1.

Theorem 3.1. All $t$-Blowup, $t \geq 5$, have equitable total chromatic number equal to 4.
Proof. Let $G'$ be a $t$-Blowup composed by $t$ copies of the semi-graph $S'$, and let $B'_2$, $B'_3$, and $B'_4$ be the semi-graphs composed by 2, 3, and 4 copies of the semi-graph $S$, respectively. To construct an equitable 4-total coloring of $G'$, we present the 4-total colorings $\varphi_{B'_2}$ and $\varphi_{B'_3}$ of $B'_2$ and $B'_3$ depicted in Figures 3a and 4, respectively. We have that: $|\varphi_{B'_2}^{-1}(1)| = |\varphi_{B'_2}^{-1}(2)| = |\varphi_{B'_2}^{-1}(3)| = |\varphi_{B'_2}^{-1}(4)| = 15; |\varphi_{B'_3}^{-1}(1)| = 22, |\varphi_{B'_3}^{-1}(2)| = 23, |\varphi_{B'_3}^{-1}(3)| = 23, and $|\varphi_{B'_3}^{-1}(4)| = 22$.

For every $t$-Blowup with $t \equiv 0 \pmod{2}$, we obtain an equitable 4-total coloring by repeating the coloring $\varphi_{B'_2}$ for the $\frac{t}{2}$ copies of the semi-graph $B'_2$. Indeed, $|\varphi_{G'_2}^{-1}(1)| = |\varphi_{G'_2}^{-1}(2)| = |\varphi_{G'_2}^{-1}(3)| = |\varphi_{G'_2}^{-1}(4)| = \frac{t}{2} \times 15$.

For every $t$-Blowup with $t \equiv 1 \pmod{2}$, we obtain an equitable 4-total coloring by repeating the coloring $\varphi_{B'_3}$ for $\frac{t-3}{2}$ copies of semi-graph $B'_2$, and $\varphi_{B'_3}^{-1}$ for one copy of semi-graph $B'_3$. Indeed, $|\varphi_{G'_1}^{-1}(1)| = |\varphi_{G'_1}^{-1}(4)| = \frac{t-3}{2} \times 15 + 22$ and $|\varphi_{G'_1}^{-1}(2)| = |\varphi_{G'_1}^{-1}(3)| = \frac{t-3}{2} \times 15 + 23$.

Hence, $|\varphi_{G'_2}^{-1}(1)|, |\varphi_{G'_2}^{-1}(2)|, |\varphi_{G'_2}^{-1}(3)|$, and $|\varphi_{G'_2}^{-1}(4)|$ differ by at most one, and the obtained 4-total colorings are equitable.

We observe that, in each case, the equitable 4-total coloring of $t$-Blowup is constructed by repeatedly using the presented 4-total colorings, and by making the junction of semiedges with the same color such that adjacent vertices have different colors. Hence, the result follows.

4. Loupekine snarks

The first snark $L_0$ of Loupekine family has 22 vertices. We refer to both $L_0$ and the semi-graph $H$ in Figure 6. The second member of this family, snark $L_1$ (see Fig. 7), is obtained by deleting the dashed edges $w_2v_3$, $w_6v_4$, $w_3u_2$, $w_7u_6$, and adding the semiedges $w_2\cdot v_3\cdot, w_3\cdot v_4\cdot, w_3\cdot u_2\cdot, w_7\cdot u_6\cdot$ and then making the junction of the following corresponding semiedges of semi-graph $H$: $w_2\cdot, v_3\cdot, w'_6\cdot, v'_4\cdot, w'_3\cdot, w'_7\cdot, w'_4\cdot, w'_4\cdot, w'_7\cdot, u'_6\cdot$.

All subsequent members $L_t$, with $t \geq 2$, of this family are similarly constructed through junctions of the semiedges of $H$ with the corresponding semiedges of the semi-graph obtained from $L_{t-1}$ by removing the two edges of type $vw$ and the two edges of type $wu$, and by replacing them with their respective semiedges (a similar process to the construction of $L_1$ by removing the dashed edges from $L_0$).
For every snark of this family, let $P = L_0\{w_1, w_2, \ldots, w_7\}$ be the cubic semi-graph obtained from the subgraph of $L_0$ induced by the set of vertices $\{w_1, w_2, \ldots, w_7\} \subseteq V(L_0)$, and adding the semiedges $w_1\cdot, w_2\cdot, w_3\cdot, w_6\cdot,$ and $w_7\cdot$.

Recently, equitable 4-total colorings have been determined for all members of four snark families: Flower snark family [3], both Blanuša families [3], and Goldberg family [5].

However, the strategies used on those families are different from the ones presented in this paper. In the Flower and the Goldberg snarks, the semi-graphs used to construct the members of the families do not interfere with the coloring balance. In Blanuša snarks, the balance is maintained by switching two different colorings of the semi-graphs. In the Loupekine’s construction, the number of new elements added in each step is not a multiple of 4 and so, in the process of determining the equitable 4-total colorings of these graphs, we add
and the obtained 4-total coloring is equitable.

Informally, we determine four different 4-total colorings for the semi-graphs $H$ and $P$, say $\varphi_H^i$ and $\varphi_P^i$, for $0 \leq i \leq 3$. We define $|\varphi_H^{-1}(a)|$ (resp. $|\varphi_P^{-1}(a)|$) the cardinality of the color class $a$ in the coloring $\varphi_H^i$ (resp. $\varphi_P^i$). Since the semi-graph $H$ is continuously used in the construction process, we prove that the established 4-total colorings can be linked in sequence indefinitely.

More precisely, when $t \equiv i \pmod{4}$, we use $\varphi_H^i$ to color the semi-graph $H$ that was added to snark $L_{i-1}$ to obtain the snark $L_t$.

However, this total coloring is not equitable yet. Therefore, we recolor the semi-graph $P \subseteq L_t$ with the coloring $\varphi_P^i$ obtaining an equitable 4-total coloring of snark $L_t$. In the following, we present equitable 4-total colorings of each Loupekine snark.

**Theorem 4.1.** All Loupekine snarks $L_i$, $t \geq 0$, have equitable total chromatic number equal to 4.

**Proof.** The first Loupekine snark admits the equitable 4-total coloring presented in Figure 8.

For $L_0$ we have $|\varphi_{L_0}^{-1}(1)| = |\varphi_{L_0}^{-1}(2)| = |\varphi_{L_0}^{-1}(3)| = 14$ and $|\varphi_{L_0}^{-1}(4)| = 13$.

For $P = L_0[\{w_1, w_2, \ldots, w_7\}]$, we have that $|\varphi_P^{-1}(1)| = |\varphi_P^{-1}(2)| = |\varphi_P^{-1}(4)| = 4.5$ and $|\varphi_P^{-1}(3)| = 4$.

For snark $L_1$, since $t \equiv 1 \pmod{4}$, we apply the total colorings $\varphi_H^1$ and $\varphi_P^1$ of Figure 9:

$|\varphi_H^{-1}(1)| = 9.5, |\varphi_H^{-1}(2)| = 8.5, |\varphi_H^{-1}(3)| = 8, |\varphi_H^{-1}(4)| = 9, |\varphi_H^{-1}(1)| = |\varphi_H^{-1}(3)| = 4,$

$|\varphi_H^{-1}(2)| = 5, |\varphi_H^{-1}(4)| = 4.5.$

Since we use new total colorings for $P$, we cannot just sum the cardinalities of $\varphi_{L_0}, \varphi_H^1$ and $\varphi_P^1$. To properly count the colors on the total coloring of $L_1$, we subtract the respective values of the total coloring $\varphi_P$ from the sum of $\varphi_{L_0}$, and add the values of $\varphi_H^1$ and $\varphi_P^1$. So, for each color we have:

$|\varphi_{L_1}^{-1}(1)| = |\varphi_{L_0}^{-1}(1)| - |\varphi_P^{-1}(1)| + |\varphi_H^{-1}(1)| + |\varphi_P^{-1}(1)|$

$= 14 - 4.5 + 9.5 + 4 = 23$

$|\varphi_{L_1}^{-1}(2)| = |\varphi_{L_0}^{-1}(2)| - |\varphi_P^{-1}(1)| + |\varphi_H^{-1}(2)| + |\varphi_P^{-1}(2)|$

$= 14 - 4.5 + 8.5 + 5 = 23$

$|\varphi_{L_1}^{-1}(3)| = |\varphi_{L_0}^{-1}(3)| - |\varphi_P^{-1}(1)| + |\varphi_H^{-1}(3)| + |\varphi_P^{-1}(3)|$

$= 14 - 4 + 8 + 4 = 22$

$|\varphi_{L_1}^{-1}(4)| = |\varphi_{L_0}^{-1}(4)| - |\varphi_P^{-1}(1)| + |\varphi_H^{-1}(4)| + |\varphi_P^{-1}(4)|$

$= 13 - 4.5 + 9 + 4.5 = 22$

and the obtained 4-total coloring is equitable.
Figure 9. The 4-total colorings \( \varphi_1^H \) and \( \varphi_1^P \) used in \( L_t \), when \( t \equiv 1 \) (mod 4).

Figure 10. The 4-total colorings \( \varphi_2^H \) and \( \varphi_2^P \) used in \( L_t \) when \( t \equiv 2 \) (mod 4).

For the next family members, we repeat the same process. To construct an equitable total coloring of \( L_2 \), we color the semi-graph \( H \) added to Loupekine \( L_1 \) with coloring \( \varphi_2^H \), and recolor \( P \subseteq L_1 \) with coloring \( \varphi_2^P \), as presented in Figure 10. For this pair of semi-graphs we have:

\[
|\varphi_2^H| = 8, \quad |\varphi_2^P| = 9
\]

For the equitable 4-total colorings of \( L_3 \) and \( L_4 \), we repeat the previous steps. The total colorings \( \varphi_3^H \) and \( \varphi_3^P \) are presented in Figure 11, and the total colorings \( \varphi_4^H \) and \( \varphi_4^P \) are presented in Figure 12.
For $L_3$, we have $|\varphi_H^{-1}(1)| = |\varphi_H^{-1}(2)| = |\varphi_H^{-1}(3)| = 9.5$, $|\varphi_H^{-1}(4)| = 4.5$ and $|\varphi_P^{-1}(1)| = |\varphi_P^{-1}(2)| = |\varphi_P^{-1}(3)| = 9 and $|\varphi_P^{-1}(4)| = 4.5$. The 4-total coloring of $L_3$ is equitable, since:

$$|\varphi_{L_3}^{-1}(1)| = |\varphi_{L_2}^{-1}(1)| - |\varphi_{P}^{-1}(1)| + |\varphi_{H}^{-1}(1)| + |\varphi_{P}^{-1}(1)| = 32 - 5 + 8.5 + 4.5 = 40$$

$$|\varphi_{L_3}^{-1}(2)| = |\varphi_{L_2}^{-1}(2)| - |\varphi_{P}^{-1}(2)| + |\varphi_{H}^{-1}(2)| + |\varphi_{P}^{-1}(2)| = 31 - 4 + 8.5 + 4.5 = 40$$

$$|\varphi_{L_3}^{-1}(3)| = |\varphi_{L_3}^{-1}(3)| - |\varphi_{P}^{-1}(3)| + |\varphi_{H}^{-1}(3)| + |\varphi_{P}^{-1}(3)| = 31 - 4.5 + 9.5 + 4 = 40$$

$$|\varphi_{L_3}^{-1}(4)| = |\varphi_{L_3}^{-1}(4)| - |\varphi_{P}^{-1}(4)| + |\varphi_{H}^{-1}(4)| + |\varphi_{P}^{-1}(4)| = 31 - 4 + 8.5 + 4.5 = 40.$$

For $L_4$, we have $|\varphi_{H}^{-1}(1)| = |\varphi_{H}^{-1}(2)| = |\varphi_{H}^{-1}(3)| = 9$, $|\varphi_{H}^{-1}(4)| = 8.5$, $|\varphi_{P}^{-1}(1)| = |\varphi_{P}^{-1}(2)| = |\varphi_{P}^{-1}(3)| = 9$ and $|\varphi_{P}^{-1}(4)| = 4.5$. The 4-total coloring of $L_4$ is equitable, since:

$$|\varphi_{L_4}^{-1}(1)| = |\varphi_{L_3}^{-1}(1)| - |\varphi_{P}^{-1}(1)| + |\varphi_{H}^{-1}(1)| + |\varphi_{P}^{-1}(1)| = 40 - 4.5 + 9 + 4.5 = 49$$

$$|\varphi_{L_4}^{-1}(2)| = |\varphi_{L_3}^{-1}(2)| - |\varphi_{P}^{-1}(2)| + |\varphi_{H}^{-1}(2)| + |\varphi_{P}^{-1}(2)| = 40 - 4.5 + 9 + 4.5 = 49$$

$$|\varphi_{L_4}^{-1}(3)| = |\varphi_{L_3}^{-1}(3)| - |\varphi_{P}^{-1}(3)| + |\varphi_{H}^{-1}(3)| + |\varphi_{P}^{-1}(3)| = 40 - 4 + 9 + 4 = 49$$

$$|\varphi_{L_4}^{-1}(4)| = |\varphi_{L_3}^{-1}(4)| - |\varphi_{P}^{-1}(4)| + |\varphi_{H}^{-1}(4)| + |\varphi_{P}^{-1}(4)| = 40 - 4.5 + 8 + 4.5 = 48.$$

Figure 11. The 4-total colorings $\varphi_{L_3}$ and $\varphi_{P}$ used in $L_t$ when $t \equiv 3 \pmod{4}$.

Figure 12. The 4-total colorings $\varphi_{L_4}$ and $\varphi_{P}$ used in $L_t$ when $t \equiv 0 \pmod{4}$.
In sum, the number of elements in each color class in the 4-total colorings for the first five Loupekine snarks are:

\[
\begin{align*}
|\varphi_{L_0}^{-1}(1)| &= |\varphi_{L_0}^{-1}(2)| = |\varphi_{L_0}^{-1}(3)| = |\varphi_{L_0}^{-1}(4)| + 1 \\
|\varphi_{L_1}^{-1}(1)| &= |\varphi_{L_1}^{-1}(2)| = |\varphi_{L_1}^{-1}(3)| + 1 = |\varphi_{L_1}^{-1}(4)| + 1 \\
|\varphi_{L_2}^{-1}(1)| &= |\varphi_{L_2}^{-1}(2)| + 1 = |\varphi_{L_2}^{-1}(3)| + 1 = |\varphi_{L_2}^{-1}(4)| + 1 \\
|\varphi_{L_3}^{-1}(1)| &= |\varphi_{L_3}^{-1}(2)| = |\varphi_{L_3}^{-1}(3)| = |\varphi_{L_3}^{-1}(4)| \\
|\varphi_{L_4}^{-1}(1)| &= |\varphi_{L_4}^{-1}(2)| = |\varphi_{L_4}^{-1}(3)| = |\varphi_{L_4}^{-1}(4)| + 1.
\end{align*}
\]

We assume by inductive hypothesis that for \( t \geq 0 \) the following relations hold:

- If \( t \equiv 0 \mod 4 \), then \( |\varphi_{L_0}^{-1}(1)| = |\varphi_{L_0}^{-1}(2)| = |\varphi_{L_0}^{-1}(3)| = |\varphi_{L_0}^{-1}(4)| + 1 \)
- If \( t \equiv 1 \mod 4 \), then \( |\varphi_{L_1}^{-1}(1)| = |\varphi_{L_1}^{-1}(2)| = |\varphi_{L_1}^{-1}(3)| + 1 = |\varphi_{L_1}^{-1}(4)| + 1 \)
- If \( t \equiv 2 \mod 4 \), then \( |\varphi_{L_2}^{-1}(1)| = |\varphi_{L_2}^{-1}(2)| + 1 = |\varphi_{L_2}^{-1}(3)| + 1 = |\varphi_{L_2}^{-1}(4)| + 1 \)
- If \( t \equiv 3 \mod 4 \), then \( |\varphi_{L_3}^{-1}(1)| = |\varphi_{L_3}^{-1}(2)| = |\varphi_{L_3}^{-1}(3)| = |\varphi_{L_3}^{-1}(4)| \)

Thus, \( \varphi_{L_t} \) is an equitable total coloring. In fact, note that if \( t - 1 \equiv i - 1 \mod 4 \), then we have that \( \varphi_{L_{t-1}} \) and \( \varphi_{L_{t-1}} \) satisfy the same relations as in the inductive hypothesis. Moreover, \( L_t \) (resp. \( L_{t-1} \)) is obtained from \( L_{t-1} \) (resp. \( L_{t-1} \)) by recoloring the subgraph \( P \subset L_{t-1} \) (resp. \( P \subset L_{t-1} \)) with the 4-total coloring \( \varphi_p \), and by coloring the new subgraph \( H \) with \( \varphi_H \), where \( t \equiv i \mod 4 \) and \( 0 < i \leq 4 \). So, in both cases, to obtain the cardinality of each color, we subtract the respective values of the total coloring \( \varphi_p^{-1} \) from the sum of \( \varphi_{L_{t-1}} \) (resp. \( \varphi_{L_{t-1}} \)), and add the values of \( \varphi_{H} \) and \( \varphi_{p} \).

We also observe that, in each case, the equitable 4-total coloring of Loupekine snarks \( L_t \) is constructed by making the junction of semiedges with the same color such that adjacent vertices have different colors. Hence, the result follows.

\[ \square \]

5. Kochol Superposition

In this section, we present new families of snarks obtained by an application of the Kochol superposition construction [13], and show their 4-total colorings.

Given a cubic semi-graph \( M(V, E, S) \), with \( S \neq \emptyset \), the set \( S \) of semiedges is partitioned into \( q \) pairwise disjoint nonempty sets \( Q_1, Q_2, \ldots , Q_q \).

Denote by \( k_i \) the cardinality of \( Q_i \), for each \( i \in \{1, \ldots , q\} \). Following Kochol’s notation [13], we call the sets \( Q_i \) connectors, and denote the semi-graph \( M \) by \( (k_1, k_2, \ldots , k_q) \)-semi-graph \( M \). A superedge \( \xi \) is a semi-graph with two connectors, and a supervertex \( \vartheta \) is a semi-graph with three connectors. Now, we consider the following semi-graphs depicted in Figure 13:

- (3, 3)-semi-graph \( M' \) (superedge) is obtained by removing two nonadjacent vertices \( v_1 \) and \( v_2 \) from a snark \( G \), and replacing each edge incident with \( v_1 \) or \( v_2 \) by semiedges.
- (1, 1)-semi-graph \( L' \) (superedge) is an isolated edge (two semiedges);  
- (1, 3, 3)-semi-graph \( J' \) (supervertex) consists of two isolated edges and a vertex incident to 3 semi-edges;  
- (1, 1, 1)-semi-graph \( K' \) (supervertex) consists of a vertex and three semiedges.

Let \( G = (V, E) \) be a snark. Replace every edge \( e \in E \) by a superedge \( \xi \), and every vertex \( v \in V \) by a supervertex \( \vartheta \). If \( v \in V \) is incident with \( e \in E \) then a connector in \( \vartheta_{v} \) is linked with a connector in \( \xi_{e} \) through the junction of semiedges. The obtained cubic graph \( G(v, \vartheta) \) is called superposition of \( G \) and it is a snark [13].

Next, we show new infinite families of snarks which are Type 1.
5.1. Petersen graph

Let $R_t$, with $t \geq 6$, be the snark obtained by a superposition of the Petersen graph with a $t$-SemiBlowup snark as follows.

The superedge $\xi_1$ is obtained by removing from the $t$-SemiBlowup snark, $t \geq 6$, two nonadjacent vertices $k_4$ and $c_3$, and replacing each edge incident with $k_4$ and $c_3$ by the corresponding semiedge. Similarly, superedges $\xi_2$ (resp. $\xi_3$) is obtained by removing the nonadjacent vertices $k_2$ and $c_1$ (resp. $k_2$ and $g_2$), and replacing edges by semiedges accordingly.

Now, consider the Petersen graph depicted in Figure 14. To obtain its superposition, replace:

- the vertex $x_i$, $1 \leq i \leq 6$, by the supervertex $J'$ depicted in Figure 13;
- edges $x_1x_2$, $x_3x_4$ and $x_5x_6$ by a copy of superedge $\xi_1$, respectively;
- edges $x_2x_3$ and $x_4x_5$ by a copy of superedge $\xi_2$, respectively; and
- edge $x_6x_1$ by a copy of superedge $\xi_3$.

- the remaining edges (resp. vertices) are replaced by superedges $L'$ (resp. supervertices $K'$) which is equivalent to maintain the original edges (resp. vertices) of the Petersen graph.
Finally, if \( v \in V \) is incident with \( e \in E \) then a connector in \( \vartheta_v \) is linked with a connector in \( \xi_e \) through the junction of semiedges.

**Theorem 5.1.** All snarks \( R_t \) obtained by a superposition of the Petersen graph with a \( t \)-SemiBlowup snark, \( t \geq 6 \), have total chromatic number equal to 4.

**Proof.** A 4-total coloring of snarks \( R_t \) is constructed as follows. By Theorem 2.1, the 4-total coloring of semi-graph \( B_4 \) appears in every \( t \)-SemiBlowup, with \( t \geq 6 \).

So, the coloring of superedge \( \xi_1 \) is obtained by removing the two nonadjacent vertices \( k_4 \) and \( c_3 \) from Figure 15 (vertices depicted in red).

Similarly, the coloring of superedge \( \xi_2 \) (resp. \( \xi_3 \)) is obtained by removing two nonadjacent vertices \( k_2 \) and \( c_1 \) (resp. \( k_2 \) and \( g_2 \)) (vertices depicted in blue (resp. green)). The color of each semiedge in \( \xi_i \), \( 1 \leq i \leq 3 \), is equal to the color of the respective edge incident with the removed vertices. The coloring of the remaining elements is presented in Figure 14b.

It is easy to verify that this construction produces a 4-total coloring. Indeed, for each \( t \)-SemiBlowup, \( t \geq 6 \), every superedge \( \xi_1 \), \( \xi_2 \), or \( \xi_3 \) have the same coloring depicted in Figure 14b. So, each member of the new family admits a 4-total coloring which is an extension of the total coloring presented in Figure 14b.

Moreover, if \( v \in V \) is incident with \( e \in E \) then a connector in \( \vartheta_v \) is linked with a connector in \( \xi_e \) through the junction of semiedges with the same color; and adjacent vertices have different colors. Hence, the result follows.

Now, let \( R'_t \), \( t \geq 6 \), be the Snark obtained by a superposition of the Petersen graph with a \( t \)-Blowup snark as follows.

Similarly to the previous superposition, the superedges \( \xi'_1 \), \( \xi'_2 \), \( \xi'_3 \), and \( \xi'_4 \) are obtained by removing, from the \( t \)-Blowup snark, \( t \geq 6 \), the nonadjacent vertices \( \{h_1, k_3\}, \{b_2, f_1\}, \{b_2, e_2\}, \) and \( \{c_1, k_1\}, \) respectively. Each edge incident with these vertices is replaced by the corresponding semiedge.

Again, consider the Petersen graph depicted in Figure 16. To obtain its superposition, replace:

- vertex \( x_i \), \( 1 \leq i \leq 6 \), by the supervertex \( J' \) depicted in Figure 13;
- edges \( x_3x_4 \) and \( x_3x_6 \) by a copy of the superedge \( \xi'_1 \), respectively;
- edges \( x_2x_3 \) and \( x_4x_5 \) by a copy of superedge \( \xi'_2 \), respectively;
- edge \( x_6x_1 \) by a copy of superedge \( \xi'_3 \);
- edge \( x_1x_2 \) by a copy of superedge \( \xi'_4 \);
- the remaining edges (resp. vertices) are replaced by superedges \( L' \) (resp. supervertices \( K' \)) which is equivalent to maintain the original edges (resp. vertices) of the Petersen graph.

Finally, if \( v \in V \) is incident with \( e \in E \) then a connector in \( \vartheta_v \) is linked with a connector in \( \xi_e \) through the junction of semiedges.
Figure 16. The Petersen graph and a depiction of a 4-total coloring of its superposition with $t$-Blowup.

Figure 17. A coloring for semi-graph $B'_4$ and the vertices removed in $\xi_i$, $1 \leq i \leq 4$, depicted in colors.

**Theorem 5.2.** All snarks $R'_t$ obtained by a superposition of the Petersen graph with a $t$-Blowup snark, $t \geq 6$, have total chromatic number equal to 4.

**Proof.** Similarly to the proof of Theorem 5.1, a 4-total coloring of snarks $R'_t$, $t \geq 6$, is constructed as follows. By Theorem 3.1, the 4-total coloring of semi-graph $B'_4$ (obtained by using the coloring of $B'_2$ twice), appears in every $t$-Blowup, with $t \geq 6$.

So, the colorings of superedges $\xi'_1$, $\xi'_2$, $\xi'_3$, and $\xi'_4$ are obtained by removing, from Figure 17, two nonadjacent vertices $\{h_1, k_3\}$, $\{b_2, f_1\}$, $\{b_2, e_2\}$, and $\{c_1, k_1\}$, respectively (vertices depicted in colors).

The color of each semiedge in $\xi_i$, $1 \leq i \leq 4$, is equal to the color of the respective edge incident with the removed vertices. The coloring of the remaining elements is presented in Figure 16b.

It is easy to verify that this construction produces a 4-total coloring. Indeed, for each $t$-Blowup, $t \geq 6$, every superedge $\xi_i$, $1 \leq i \leq 4$, has the same coloring depicted in Figure 16b. So, each member of the new family admits a 4-total coloring which is an extension of the total coloring presented in Figure 16b.

Moreover, if $v \in V$ is incident with $e \in E$ then a connector in $\vartheta_v$ is linked with a connector in $\xi_e$ through the junction of semiedges with the same color; and adjacent vertices have different colors. Hence, the result follows. □
Figure 18. Graph $F_5$ constructed from graphs $F_3$ and $L_5$. (a) Link graph $L_5$. (b) Graph $F_3$. (C) Graph $F_5$.

Note that Theorem 2.1 (resp. 3.1) determine the coloring of infinite families of snarks, since each $t$-SemiBlowup (resp. $t$-Blowup) with $t \geq 6$ generates a new snark $R_t$ (resp. $R'_t$) with $60t - 2$ (resp. $66t + 10$) vertices.

5.2. Flower snarks

Now, we present another construction of infinite families of Type 1 graphs obtained from a Kochol superposition of Flower snarks with $t$-SemiBlowup or $t$-Blowup snarks.

Let $F_{2i+1}$, $i \geq 1$ be the members of the family of Flower Snarks. For this family we define the basic block $B_k$ as the claw graph with vertex set $V(B_k) = \{u_k, v_k, x_k, y_k\}$ and edge set $E(B_k) = \{u_kv_k, x_ky_k, y_kv_k\}$. We define the set of the link edges as $E_{kj} = \{u_ku_j, x_kx_j, y_ky_j\}$ and link graph $L_k$, $k$ odd and $k \geq 5$, as the union of $B_k$, $B_k$, and the graph spanned by $E_{(k-1)k}$, i.e., $V(L_k) = V(B_{k-1}) \cup V(B_k)$ and $E(L_k) = E(B_{k-1}) \cup E(B_k) \cup E_{(k-1)k}$. Figure 18a shows $L_5$.

The first Flower Snark, $F_3$, is defined as the union of $B_1$, $B_2$, $B_3$, and the graph spanned by $E_{21} \cup E_{31} \cup \{u_1u_2, x_1y_2, y_1x_2\}$ (see Fig. 18b). For each $i \geq 2$, $F_{2i+1}$ is obtained from graphs $F_{2i-1}$ and $L_{2i+1}$ as follows: $V(F_{2i+1}) = V(F_{2i-1}) \cup V(L_{2i+1})$, and $E(F_{2i+1}) = (E(F_{2i-1}) \setminus E_{2i-1}) \cup E(L_{2i+1}) \cup E_{2i+1}^{in}$, where $E_{out}^{out} = E_{2i-1}^{out}$, and $E_{2i+1}^{in} = E_{(2i-1)(2i)} \cup E_{(2i+1)}^{in}$. Figure 18c shows $F_5$, constructed from graphs $F_3$ and $L_5$.

In [3], the authors construct 4-total coloring for each element of that family taking advantage of the recursive construction of these graphs. This coloring $\Phi_{2i+1}$ satisfies that the $E_{2i+1}^{out}$ edges have the same color 1. See Figure 20.

In the following theorems, we consider a specific coloring $\Phi_{2i+1}$ obtained by replacing in each step of the construction of $F_{2i+1}$ the edge $u_{2i+1}u_1$, see Figure 20.

Let $R_i(2i + 1)$ be the Snark obtained by the superposition of $F_{2i+1}$ with a $t$-SemiBlowup, $t \geq 6$, $i \geq 1$, as follows.

Similarly to the previous superposition, the superedges $\psi_1$, $\psi_2$, $\psi_3$, $\psi_4$ and $\psi_5$ are obtained by removing, from the $t$-SemiBlowup snark, $t \geq 6$, the nonadjacent vertices $\{h_4, c_3\}$, $\{b_1, f_4\}$, $\{k_3, j_2\}$, $\{k_3, b_3\}$ and $\{d_2, b_1\}$, respectively. Each edge incident with these vertices is replaced by the corresponding semiedge.

Consider the Flower snark $F_3$ depicted in Figure 18b. To obtain its superposition, replace:

- edge $u_1u_2$ by a copy of the superedge $\psi_1$;
- edge $u_2u_3$ by a copy of the superedge $\psi_2$, and
- edge $u_3u_1$ by a copy of the superedge $\psi_3$.

For the Flower snark $F_5$, replace:

- edge $u_1u_2$ by a copy of the superedge $\psi_1$;
- edge $u_2u_3$ by a copy of the superedge $\psi_2$;
- edge $u_3u_4$ by a copy of the superedge $\psi_4$;
Figure 19. A 4-total coloring of $F_5$, and a 4-total coloring of its superposition with a $t$-SemiBlowup.

Figure 20. A 4-total coloring of $F_3$, and a 4-total coloring of its superposition with a $t$-SemiBlowup.

Figure 21. A 4-total coloring of Link graph $L_{2i+1}$, and a 4-total coloring of its superposition with a $t$-SemiBlowup.
Figure 22. A coloring for semi-graph $B_4$ and the vertices removed in $\psi_i$, $1 \leq i \leq 5$, depicted in colors.

- edge $u_4u_5$ by a copy of the superedge $\psi_5$;
- edge $u_5u_1$ by a copy of the superedge $\psi_3$.

For the link graph $L_{2i+1}$, replace:

- edge $u_{2i}u_{2i+1}$ by a copy of the superedge $\psi_5$;
- semiedge $u_{2i}$ by a copy of the superedge $\psi_4$;
- semiedge $u_{2i+1}$ by a copy of the superedge $\psi_3$.

Now, for any Flower snark $F_{2i+1}$ and link graph $L_{2i+1}$, $i \geq 2$, replace:

- the vertex $u_j$, $1 \leq j \leq 2i + 1$ of $F_{2i+1}$ by the supervertex $J'$ depicted in Figure 13;
- the remaining edges (resp. vertices) are replaced by superedges $L'$ (resp. supervertices $K'$) which is equivalent to maintain the original edges (resp. vertices) of $F_{2i+1}$.

An example of the superposition of the Flower snark $F_5$ with a $t$-SemiBlowup snark is depicted in Figure 19 (right).

**Theorem 5.3.** All snarks $R_t(2i + 1)$ obtained by superposition of a Flower snark graph $F_{2i+1}$, $i \geq 1$, with a $t$-SemiBlowup snark, $t \geq 6$, have total chromatic number equal to 4.

**Proof.** Analogously to the proof of Theorem 5.2, a 4-total coloring of snarks $R_t(2i + 1)$, $t \geq 6$, $i \geq 1$, is constructed as follows. By Theorem 5.1, the 4-total coloring of semi-graph $B_4$ appears in every $t$-SemiBlowup, with $t \geq 6$.

So, the colorings of superedges $\psi_1$, $\psi_2$, $\psi_3$, $\psi_4$ and $\psi_5$ are obtained by removing, from Figure 22, two nonadjacent vertices $\{h_4, c_3\}$, $\{b_1, f_4\}$, $\{k_3, j_2\}$, $\{k_3, b_1\}$ and $\{d_2, b_1\}$, respectively (vertices depicted in colors).

The color of each semiedge in $\psi_i$, $1 \leq i \leq 5$, is equal to the color of the respective edge incident with the removed vertices.

The coloring of the remaining elements of the Flower snark $F_5$ and the link graph $L_{2i+1}$ are presented in Figures 19(right) and 21.

Also, Figure 19(right) presents a depiction of 4-total coloring of $R_t(5)$. Note that the coloring of $R_t(7)$ coincides with the coloring of graph obtained by replacing superedge $\psi_3$, and edges $x_1x_5$ and $y_1y_5$, in $R_t(5)$ by the colored semi-graph $L'$ in Figure 21, and linking the semiedges of $L'$ and the semi-graph (obtained from $R_t(5)$) through the junctions of the following semi-graphs: $\psi_4$ of $L'$ with $\psi_5$ of the semi-graph (obtained from $R_t(5)$); and $\psi_3$ of $L'$ with $\psi_1$ of the semi-graph (obtained from $R_t(5)$); and the remaining semi-edges $x$. and $y$. accordingly.

We assume by inductive hypothesis that there exists a 4-total coloring of $R_t(2i - 1)$ obtained from the superposition of a Flower snark $F_{2i-1}$ with a $t$-SemiBlowup, such that there is only one copy of the colored semi-graph $L'$ in Figure 21 on this graph. We obtain a 4-total coloring of $R_t(2i + 1)$ from
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Figure 23. A depiction of $R_t(5)$ left and a 4-total coloring of the superposition of $L'$, the Link graph $L_{2i+1}$ with a t-Blowup.

Figure 24. A coloring for semi-graph $B'_4$ and the vertices removed in $\psi_i$, $1 \leq i \leq 6$, depicted in colors.

Let $R_t(2i - 1)$ be the Snark obtained by the superposition of $F_{2i+1}$ and a t-Blowup, $t \geq 6$, $i \geq 1$, as follows. Similarly to the previous superposition, the superedges $\psi_1'$, $\psi_2'$, $\psi_3'$, $\psi_4'$, $\psi_5'$ and $\psi_6'$ are obtained by removing, from the t-Blowup snark, $t \geq 6$, the nonadjacent vertices $\{e_2, j_4\}$, $\{g_3, c_1\}$, $\{c_2, l_2\}$, $\{c_2, l_3\}$, $\{a_2, a_3\}$ and $\{k_1, l_2\}$, respectively. Each edge incident with these vertices is replaced by the corresponding semiedge.

Consider the Flower snark $F_3$ depicted in Figure 18b. To obtain its superposition, replace:

- edge $u_1u_2$ by a copy of the superedge $\psi_1'$;
- edge $u_2u_3$ by a copy of the superedge $\psi_2'$, and
edge $u_3u_1$ by a copy of the superedge $\psi'_3$.

For the Flower snark $F_5$, replace:

- edge $u_1u_2$ by a copy of the superedge $\psi'_1$;
- edge $u_2u_3$ by a copy of the superedge $\psi'_2$;
- edge $u_3u_4$ by a copy of the superedge $\psi'_4$;
- edge $u_4u_5$ by a copy of the superedge $\psi'_5$;
- edge $u_5u_1$ by a copy of the superedge $\psi'_6$.

For the link graph $L_{2i+1}$, replace:

- edge $u_{2i}u_{2i+1}$ by a copy of the superedge $\psi'_5$;
- semiedge $u_{2i} \cdot$ by a copy of the superedge $\psi'_4$;
- semiedge $u_{2i+1} \cdot$ by a copy of the superedge $\psi'_6$.

Now, for any Flower snark $F_{2i+1}$ and link graph $L_{2i+1}$, $i \geq 2$, replace:

- the vertex $u_j$, $1 \leq j \leq 2i + 1$ of $F_{2i+1}$ by the supervertex $J'$ depicted in Figure 13.
- the remaining edges (resp. vertices) are replaced by superedges $L'$ (resp. supervertices $K'$) which is equivalent to maintain the original edges (resp. vertices) of $F_{2i+1}$.

An example of the superposition of the Flower snark $F_5$ and $t$-Blowup snark is depicted in Figure 23(left).

**Theorem 5.4.** All snarks $R'_t(2i+1)$ obtained by a superposition of a Flower snark graph and a $t$-Blowup snark with $t \geq 6$ have total chromatic number equal to 4.

**Proof.** The proof follows similarly to the proof of Theorem 5.3, by using the coloring in Figure 24, the coloring of $R'(5)$, and the coloring of semi-graph $L'$, Figure 23 in the inductive process. □

Finally, we show a generalization of the previous superpositions where each superedge ($3,3$)-semi-graph $M'$) is obtained from a $t_j$-SemiBlowup (resp. $t_j$-Blowup), eventually of different order, $t_j \geq 6$.

Formally, let $R'_S(2i+1)$, with $S = \{t_j : t_j \geq 6\}$ be the snark obtained by the superposition of the Petersen graph (resp. Flower snark $F_{2i+1}$) such that each superedge is constructed from a $t_j$-SemiBlowup (resp. $t_j$-Blowup), with $t_j \geq 6$, for $1 \leq j \leq 6$ (resp. $1 \leq j \leq 2i + 1$).

**Theorem 5.5.** All snarks $R'_S(2i+1)$ obtained by superposition of the Petersen graph (resp. Flower snark $F_{2i+1}$, $i \geq 1$) $t \geq 6$, have total chromatic number equal to 4.

Acknowledgements. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 and CAPES-PrInt project number 88881.310248/2018-01, CNPq and FAPERJ.

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