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ON TOTAL COLORING AND EQUITABLE TOTAL COLORING OF INFINITE SNARK FAMILIES

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Abstract. We show that all members of the SemiBlowup, Blowup and the first Loupekine snark families have equitable total chromatic number equal to 4. These results provide evidence of negative answers for the questions proposed: by (A. Cavicchioli, T.E. Murgolo, B. Ruini and F. Spaggiari, *Acta Appl. Math.* **76** (2003) 56–88.) about the smallest order of a Type 2 snark of girth at least 5; and by (S. Dantas, C.M.H. De Figueiredo, G. Mazzuoccolo, M. Preissmann, V.F. dos Santos and D. Sasaki, *Discret. Appl. Math.* **209** (2016) 84–91.) about the existence of Type 1 cubic graph with girth at least 5 and equitable total chromatic number 5. Moreover, we show new infinite families of snarks obtained by the Kochol superpositions that are Type 1.

Mathematics Subject Classification. 05C15.

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1. Introduction

The graphs $G = (V, E)$ considered in this paper are undirected and simple, where $V = V(G)$ is the set of vertices of G, and $E = E(G)$ is the set of edges of G. A *q-total coloring* of G is an assignment of q colors to the vertices and edges of G, so that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by χ'', is the smallest q for which G has a q-total coloring. The well known *Total Coloring Conjecture (TCC)* [1, 18] states that the total chromatic number of a graph G is at least $\Delta(G) + 1$ (graphs called *Type 1*), and at most $\Delta(G) + 2$ (graphs called *Type 2*), where $\Delta(G)$ is the maximum degree of G. The TCC has been settled for some specific graph families, such as the split graphs, the complete r-partite graphs, and the dually chordal graphs [7], but it remains open for several graph classes for more than fifty years. The total chromatic number of a *cubic* graph (graph where each vertex has degree 3) was proved to be either 4 or 5 by Rosenfeld [14], and independently by Vijayaditya [17].

An *equitable total coloring* is a total coloring such that the difference between the cardinalities of any two color classes is at most one. The *equitable total chromatic number* χ''_e of G is the smallest number for which G admits an equitable total coloring. The *Equitable Total Coloring Conjecture (ETCC)* was posed by Wang [19] in 2002, and states that the equitable total chromatic number of a graph is at most $\Delta(G) + 2$. The ETCC was

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proved for cubic graphs, and it implies that if a cubic graph is Type 2, then both the total chromatic number and the equitable total chromatic number are equal to 5; if a cubic graph is Type 1, then the equitable total chromatic number is either 4 or 5. Dantas *et al.* [5] proved that it is NP-complete to decide whether the equitable total chromatic number is equal to 4 for bipartite cubic graphs.

The search for counterexamples to the *Four Color Conjecture* motivated the study of cubic graphs, and this class is still much studied not only in coloring problems [6]. Based on the poem by Lewis Carroll "The Hunting of the Snark", Gardner [8] introduced the name *Snark* for cyclically 4-edge-connected cubic graph that does not allow a 3-edge-coloring. The importance of these graphs arises also from the fact that several well-known conjectures would have snarks as minimal counterexamples, such as the cycle double cover conjecture and the 5-flow conjecture [6]. The Petersen graph is the smallest and earliest known snark. In [11], Isaacs introduced the dot product, a famous operation used for constructing infinitely many snarks, and defined the *Flower* and the *Blanuša* snark families. In 1976, Isaacs [12] introduced two families of *Loupekine* snarks.

The *girth* of G is the length of the shortest cycle contained in G. In 2003, Cavicchioli *et al.* [4] verified, by using a computer, that all snarks with girth at least 5 and fewer than 30 vertices are Type 1, and asked the following question:

Question 1.1. Does there exist a Type 2 cubic graph with girth at least 5?

In 2011, Campos *et al.* [3] provided (equitable) total colorings with four colors for all Flower [11] and Goldberg [9] snarks, and they all have girth 5. In 2014, Sasaki *et al.* [16] showed that all members of the Blanuša families and part of the Loupekine families are Type 1. In 2015, Brinkmann *et al.* [2] presented the first Type 2 snarks with small girth. About the equitable total coloring of cubic graphs with girth at least 5, Dantas *et al.* [5] in 2016 proposed the question:

Question 1.2. Does there exist a Type 1 cubic graph with girth greater than 4 and equitable total chromatic number 5?

The authors observed that the only known examples of Type 1 cubic graphs with equitable total chromatic number 5 have girth smaller than 5. We highlight that Question 1.2 remains open so far.

In 1996, Kochol [13] introduced the *superposition* construction which provides infinite families of large girth snarks. Furthermore, in 2016, Hägglund [10] defined two other infinite families of snarks: the *Blowup* and the *SemiBlowup* families. In 2016, Sasaki [15] proved that all members of an infinite family of snarks obtained by the superposition are Type 1.

In this paper, we contribute to Dantas *et al.* [5] (Question 1.2) and to Cavicchioli *et al.* [4] (Question 1.1) questions, by providing evidence for a negative answer to their questions. We present equitable 4-total colorings for all members of the SemiBlowup, Blowup, and for all members of the first Loupekine snark families. Furthermore, we determine 4-total colorings for all members of the new infinite families obtained by Kochol superpositions.

2. SemiBlowup snarks

A *semi-graph* is a 3-tuple $G = (V, E, S)$ where V is a finite set of vertices of G, E is a set of edges having two distinct endpoints in V, and S is a multiset of *semiedges* having at most one endpoint in V. A semiedge without endpoints is called an *isolated edge*. A semiedge with endpoint v is denoted by $v\cdot$ and an edge having endpoints v and w is denoted by vw. Given two semiedges $v\cdot$ and $w\cdot$, the *junction* of $v\cdot$ and $w\cdot$ is done by replacing $v\cdot$ and $w\cdot$ by the edge vw. The definitions for simple graphs can be naturally extended to semi-graphs. Indeed, a *graph* $G = (V, E)$ is a semi-graph with an empty set of semiedges. We denote by φ_H the equitable 4-total coloring of the semi-graph (graph) H, and by $|\varphi_H^{-1}(a)|$ the corresponding cardinality of the color class a in φ_H.

The SemiBlowup family of snarks was introduced in 2016 by Hgglund [10], and its construction is as follows. Let S_i be the semi-graph, depicted in Figure 1, where semiedges are represented by lines with only one vertex as endpoint.

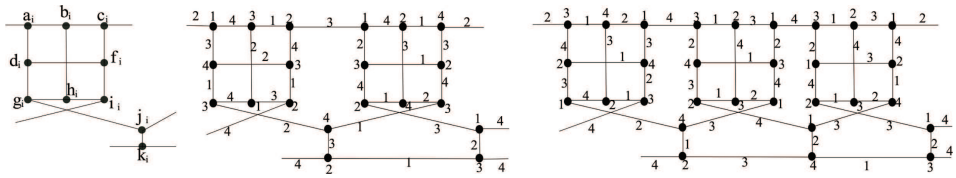


FIGURE 1. The semi-graph S_i, and the semi-graphs B_2 and B_3 with their respective equitable 4-total colorings φ_{B_2} and φ_{B_3}.

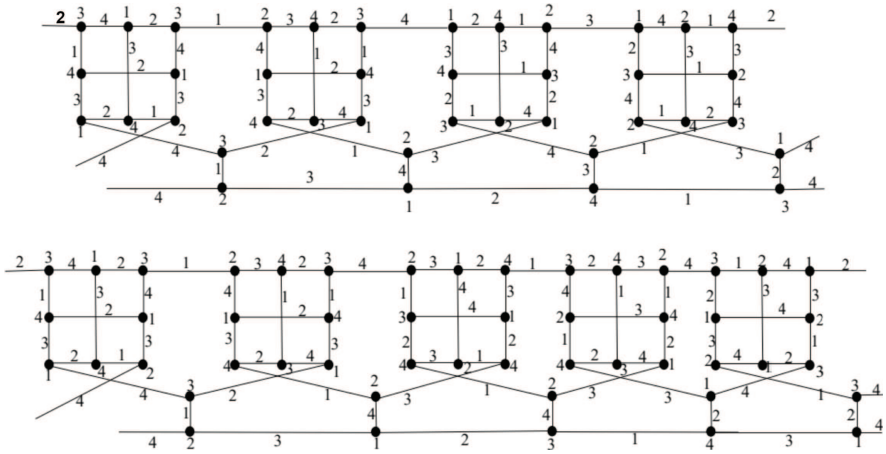


FIGURE 2. The semi-graphs B_4 and B_5 with their respective equitable 4-total colorings φ_{B_4} and φ_{B_5}.

A *t-SemiBlowup* is a snark constructed by connecting $t \geq 5$ copies of semi-graph S_i, for $1 \leq i \leq t$. More precisely, the t-SemiBlowup is constructed through junctions: for $2 \leq i \leq t$, we make the junctions of semiedges $a_i\cdot$ with $c_{i-1}\cdot$, $i_i\cdot$ with $j_{i-1}\cdot$, $k_i\cdot$ with $k_{i-1}\cdot$; and for $i = 1$, $a_1\cdot$ with $c_t\cdot$, $i_1\cdot$ with $j_t\cdot$, and $k_1\cdot$ with $k_t\cdot$. We connect t copies of S_i as a cycle, as is depicted in Figure 5 (left), that presents the 6-SemiBlowup colored by Theorem 2.1.

Now, we observe that, since the cardinality of the elements of S_i is not a multiple of four, the idea of the desired equitable 4-total coloring is to assign four different equitable 4-total colorings to subgraphs that are used in construction process.

Theorem 2.1. *All t-SemiBlowup snarks, $t \geq 5$, have equitable total chromatic number equal to 4.*

Proof. Let G be a t-SemiBlowup composed by t copies of the semi-graph S_i, and let B_2, B_3, B_4 and B_5 be the semi-graphs composed by 2, 3, 4, and 5 copies of the semi-graph S_i, respectively. Their respective equitable 4-total colorings φ_{B_2}, φ_{B_3}, φ_{B_4}, and φ_{B_5} are shown in Figures 1 and 2.

To determine the cardinality of a color class in a semi-graph, we count each semiedge as 0.5, due to the junction that takes place in the construction. Following this rule, we have that $|\varphi_{B_4}^{-1}(1)| = |\varphi_{B_4}^{-1}(2)| = |\varphi_{B_4}^{-1}(3)| = |\varphi_{B_4}^{-1}(4)| = 25$; $|\varphi_{B_5}^{-1}(1)| = 32$, $|\varphi_{B_5}^{-1}(2)| = |\varphi_{B_5}^{-1}(3)| = |\varphi_{B_5}^{-1}(4)| = 31$; $|\varphi_{B_2}^{-1}(1)| = |\varphi_{B_2}^{-1}(4)| = 12$, $|\varphi_{B_2}^{-1}(2)| = |\varphi_{B_2}^{-1}(3)| = 13$; $|\varphi_{B_3}^{-1}(1)| = |\varphi_{B_3}^{-1}(2)| = |\varphi_{B_3}^{-1}(3)| = 19$ and $|\varphi_{B_3}^{-1}(4)| = 18$.

For every t-SemiBlowup with $t \equiv 0 \pmod 4$, we obtain an equitable 4-total coloring by repeating $\frac{t}{4}$ times the coloring φ_{B_4} for $\frac{t}{4}$ copies of the semi-graph B_4. Note that the coloring fits the junctions used in construction. So, $|\varphi_G^{-1}(1)| = |\varphi_G^{-1}(2)| = |\varphi_G^{-1}(3)| = |\varphi_G^{-1}(4)| = \frac{t}{4} \times 25$.

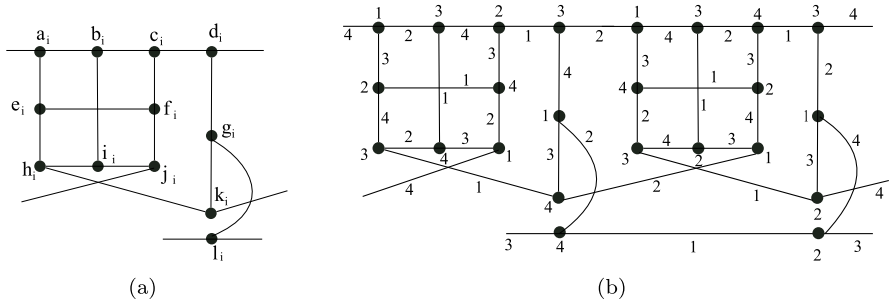


FIGURE 3. The semi-graph S'_i, and the semi-graph B'_2 with its respective 4-total coloring $\varphi_{B'_2}$.

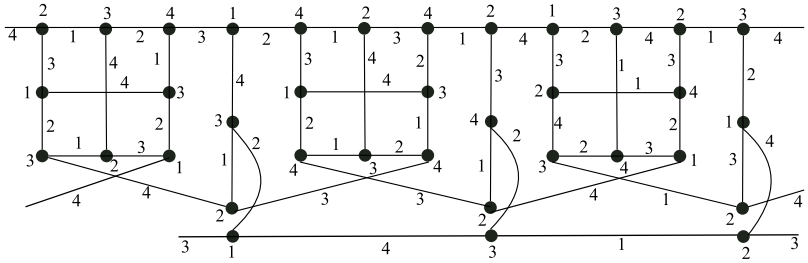


FIGURE 4. The semi-graph B'_3 with its respective 4-total coloring $\varphi_{B'_3}$.

For every t-SemiBlowup with $t \equiv 1 \pmod 4$, we obtain an equitable 4-total coloring by repeating $\frac{t-5}{4}$ times the coloring φ_{B_4} for $\frac{t-5}{4}$ copies of the semi-graph B_4, and we use φ_{B_5} for one copy of B_5. Evidently, if $t = 5$, then the coloring φ_{B_5} is enough. So, $|\varphi_G^{-1}(1)| = \frac{t-5}{4} \times 25 + 32$, and $|\varphi_G^{-1}(2)| = |\varphi_G^{-1}(3)| = |\varphi_G^{-1}(4)| = \frac{t-5}{4} \times 25 + 31$.

For every t-SemiBlowup with $t \equiv 2 \pmod 4$, we obtain an equitable 4-total coloring by repeating $\frac{t-2}{4}$ times the coloring φ_{B_4} for $\frac{t-2}{4}$ copies of the semi-graph B_4, and we use φ_{B_2} for one copy of B_2. So, $|\varphi_G^{-1}(1)| = |\varphi_G^{-1}(4)| = \frac{t-2}{4} \times 25 + 12$, and $|\varphi_G^{-1}(2)| = |\varphi_G^{-1}(3)| = \frac{t-2}{4} \times 25 + 13$.

For every t-SemiBlowup with $t \equiv 3 \pmod 4$, we obtain an equitable 4-total coloring by repeating $\frac{t-3}{4}$ times the coloring φ_{B_4} for $\frac{t-3}{4}$ copies of the semi-graph B_4, and we use φ_{B_3} for one copy of B_3 . So, $|\varphi_G^{-1}(1)| = |\varphi_G^{-1}(2)| = |\varphi_G^{-1}(3)| = \frac{t-3}{4} \times 25 + 19$, and $|\varphi_G^{-1}(4)| = \frac{t-3}{4} \times 25 + 18$.

We observe that, in each case, the equitable 4-total coloring of t-SemiBlowup is constructed by repeatedly using the presented 4-total colorings, and by making the junction of semiedges with the same color such that adjacent vertices have different colors. Hence, the result follows. □

3. Blowup Snarks

The Blowup family of snarks was also introduced in 2016 by Hgglund [10], and its construction is presented as follows. Let S'_i be the semi-graph depicted in Figure 3a with vertex set $\{a_i, b_i, \ldots, l_i\}$ and six semiedges. A *t-Blowup* is a snark constructed by connecting $t \geq 5$ copies of semi-graph S'_i, for $1 \leq i \leq t$.

More precisely, the t-Blowup is constructed through junctions: for $2 \leq i \leq t$, we make the junctions of semiedges $a_i\cdot$ with $d_{i-1}\cdot$, $j_i\cdot$ with $k_{i-1}\cdot$, $l_i\cdot$ with $l_{i-1}\cdot$; and for $i = 1$, $a_1\cdot$ with $d_t\cdot$, $j_1\cdot$ with $k_t\cdot$, and $l_1\cdot$ with $l_t\cdot$. We connect t copies of S'_i similarly to a cycle, as we can see in Figure 5, that presents the 6-Blowup studied in Theorem 3.1.

Theorem 3.1. *All t-Blowup, $t \geq 5$, have equitable total chromatic number equal to 4.*

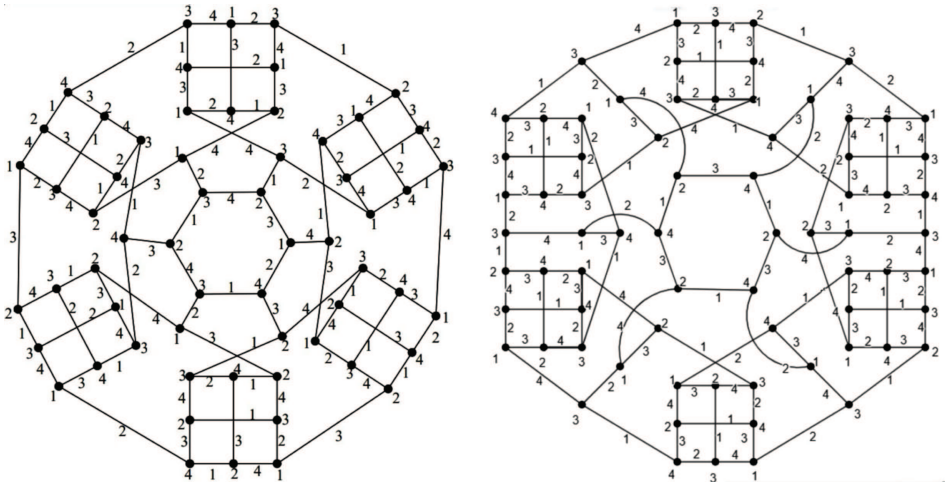


FIGURE 5. Snarks 6-SemiBlowup (left) and 6-Blowup (right) with their respective equitable 4-total colorings.

Proof. Let G' be a t-Blowup composed by t copies of the semi-graph S'_i, and let B'_2, B'_3, and B'_4 be the semi-graphs composed by 2, 3, and 4 copies of the semi-graph S_i, respectively. To construct an equitable 4-total coloring of G', we present the 4-total colorings $\varphi_{B'_2}$ and $\varphi_{B'_3}$ of B'_2 and B'_3 depicted in Figures 3a and 4, respectively. We have that: $|\varphi^{-1}_{B'_2}(1)| = |\varphi^{-1}_{B'_2}(2)| = |\varphi^{-1}_{B'_2}(3)| = |\varphi^{-1}_{B'_2}(4)| = 15$; $|\varphi^{-1}_{B'_3}(1)| = 22$, $|\varphi^{-1}_{B'_3}(2)| = 23$, $|\varphi^{-1}_{B'_3}(3)| = 23$, and $|\varphi^{-1}_{B'_3}(4)| = 22$.

For every t-Blowup with $t \equiv 0 \pmod 2$, we obtain an equitable 4-total coloring by repeating the coloring $\varphi^{-1}_{B'_2}$ for the $\frac{t}{2}$ copies of the semi-graph B'_2. Indeed, $|\varphi^{-1}_{G'}(1)| = |\varphi^{-1}_{G'}(2)| = |\varphi^{-1}_{G'}(3)| = |\varphi^{-1}_{G'}(4)| = \frac{t}{2} \times 15$.

For every t-Blowup with $t \equiv 1 \pmod 2$, we obtain an equitable 4-total coloring by repeating the coloring $\varphi_{B'_2}$ for $\frac{t-3}{2}$ copies of semi-graph B'_2, and $\varphi^{-1}_{B'_3}$ for one copy of semi-graph B'_3. Indeed, $|\varphi^{-1}_{G'}(1)| = |\varphi^{-1}_{G'}(4)| = \frac{t-3}{2} \times 15 + 22$ and $|\varphi^{-1}_{G'}(2)| = |\varphi^{-1}_{G'}(3)| = \frac{t-3}{2} \times 15 + 23$.

Hence, $|\varphi^{-1}_{G'}(1)|$, $|\varphi^{-1}_{G'}(2)|$, $|\varphi^{-1}_{G'}(3)|$, and $|\varphi^{-1}_{G'}(4)|$ differ by at most one, and the obtained 4-total colorings are equitable.

We observe that, in each case, the equitable 4-total coloring of t-Blowup is constructed by repeatedly using the presented 4-total colorings, and by making the junction of semiedges with the same color such that adjacent vertices have different colors. Hence, the result follows. □

4. LOUPEKINE SNARKS

The first snark L_0 of Loupekine family has 22 vertices. We refer to both L_0 and the semi-graph H in Figure 6. The second member of this family, snark L_1 (see Fig. 7), is obtained by deleting the dashed edges w_2v_3, w_6v_4, w_3u_2, w_7u_6, and adding the semiedges $w_2\cdot$, $v_3\cdot$, $w_6\cdot$, $v_4\cdot$, $w_3\cdot$, $u_2\cdot$, $w_7\cdot$, $u_6\cdot$; and then making the junction of the following corresponding semiedges of semi-graph H: $w'_2\cdot$, $v'_3\cdot$, $w'_6\cdot$, $v'_4\cdot$, $w'_3\cdot$, $u'_2\cdot$, $w'_7\cdot$, $u'_6\cdot$.

All subsequent members L_t, with $t \geq 2$, of this family are similarly constructed through junctions of the semiedges of H with the corresponding semiedges of the semi-graph obtained from L_{t-1} by removing the two edges of type wv and the two edges of type wu, and by replacing them with their respective semiedges (a similar process to the construction of L_1 by removing the dashed edges from L_0).

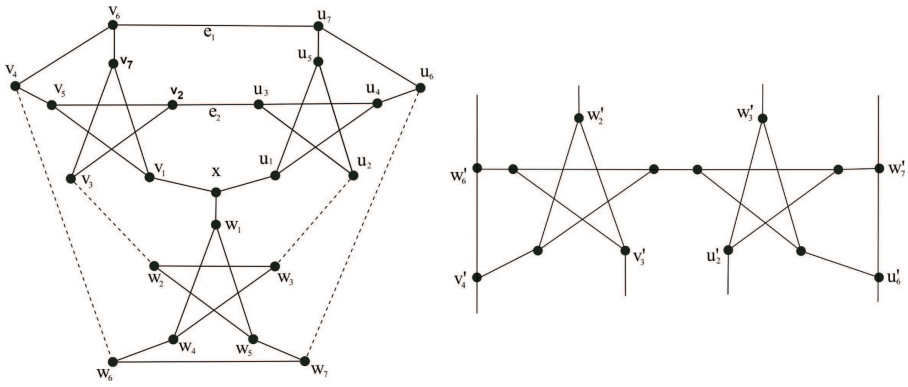


FIGURE 6. The first Loupekine snark L_0 (left) and the semi-graph H (right).

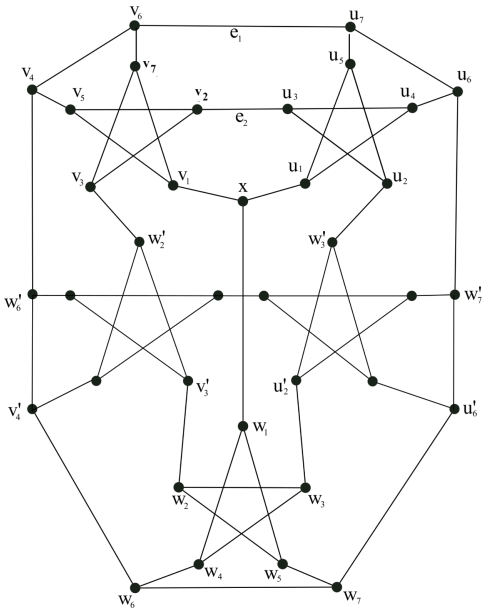


FIGURE 7. The Loupekine snark L_1.

For every snark of this family, let $P = L_0[\{w_1, w_2, \ldots, w_7\}]$ be the cubic semi-graph obtained from the subgraph of L_0 induced by the set of vertices $\{w_1, w_2, \ldots, w_7\} \subseteq V(L_0)$, and adding the semiedges $w_1\cdot$, $w_2\cdot$, $w_3\cdot$, $w_6\cdot$, and $w_7\cdot$.

Recently, equitable 4-total colorings have been determined for all members of four snark families: Flower snark family [3], both Blanuša families [3], and Goldberg family [5].

However, the strategies used on those families are different from the ones presented in this paper. In the Flower and the Goldberg snarks, the semi-graphs used to construct the members of the families do not interfere with the coloring balance. In Blanuša snarks, the balance is maintained by switching two different colorings of the semi-graphs. In the Loupekine's construction, the number of new elements added in each step is not a multiple of 4 and so, in the process of determining the equitable 4-total colorings of these graphs, we add

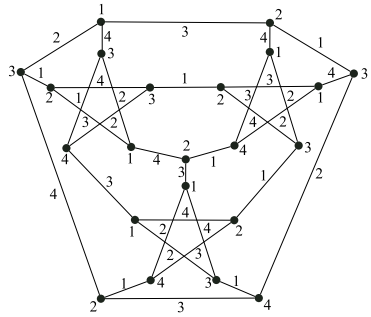


FIGURE 8. The first Loupekine snark L_0 and its equitable 4-total coloring.

different cardinalities of color classes in each step, implying that we cannot use the same total coloring for every new semi-graph added to the previous snark.

Informally, we determine four different 4-total colorings for the semi-graphs H and P, say φ_H^i and φ_P^i, for $0 \leq i \leq 3$. We define $|(\varphi_H^i)^{-1}(a)|$ (resp. $|(\varphi_P^i)^{-1}(a)|$) the cardinality of the color class a in the coloring φ_H^i (resp. φ_P^i). Since the semi-graph H is continuously used in the construction process, we prove that the established 4-total colorings can be linked in sequence indefinitely.

More precisely, when $t \equiv i \pmod 4$, we use φ_H^i to color the semi-graph H that was added to snark L_{t-1} to obtain the snark L_t.

However, this total coloring is not equitable yet. Therefore, we recolor the semigraph $P \subseteq L_t$ with the coloring φ_P^i obtaining an equitable 4-total coloring of snark L_t. In the following, we present equitable 4-total colorings of each Loupekine snark.

Theorem 4.1. *All Loupekine snarks L_t, $t \geq 0$, have equitable total chromatic number equal to 4.*

Proof. The first Loupekine snark admits the equitable 4-total coloring presented in Figure 8.

For L_0 we have $|\varphi_{L_0}^{-1}(1)| = |\varphi_{L_0}^{-1}(2)| = |\varphi_{L_0}^{-1}(3)| = 14$ and $|\varphi_{L_0}^{-1}(4)| = 13$.

For $P = L_0[\{w_1, w_2, \ldots, w_7\}]$, we have that $|\varphi_P^{-1}(1)| = |\varphi_P^{-1}(2)| = |\varphi_P^{-1}(4)| = 4.5$ and $|\varphi_P^{-1}(3)| = 4$.

For snark L_1, since $t \equiv 1 \pmod 4$, we apply the total colorings φ_H^1 and φ_P^1 of Figure 9:

$|(\varphi_H^1)^{-1}(1)| = 9.5$, $|(\varphi_H^1)^{-1}(2)| = 8.5$, $|(\varphi_H^1)^{-1}(3)| = 8$, $|(\varphi_H^1)^{-1}(4)| = 9$, $|(\varphi_P^1)^{-1}(1)| = |(\varphi_P^1)^{-1}(3)| = 4$, $|(\varphi_P^1)^{-1}(2)| = 5$, and $|(\varphi_P^1)^{-1}(4)| = 4.5$.

Since we use new total colorings for P, we cannot just sum the cardinalities of φ_{L_0}, φ_H^1 and φ_P^1. To properly count the colors on the total coloring of L_1, we subtract the respective values of the total coloring φ_P from the sum of φ_{L_0}, and add the values of φ_H^1 and φ_P^1. So, for each color we have:

$$\begin{aligned}
|\varphi_{L_1}^{-1}(1)| &= |\varphi_{L_0}^{-1}(1)| - |\varphi_P^{-1}(1)| + |(\varphi_H^1)^{-1}(1)| + |(\varphi_P^1)^{-1}(1)| \\
&= 14 - 4.5 + 9.5 + 4 = 23 \\
|\varphi_{L_1}^{-1}(2)| &= |\varphi_{L_0}^{-1}(2)| - |\varphi_P^{-1}(1)| + |(\varphi_H^1)^{-1}(2)| + |(\varphi_P^1)^{-1}(2)| \\
&= 14 - 4.5 + 8.5 + 5 = 23 \\
|\varphi_{L_1}^{-1}(3)| &= |\varphi_{L_0}^{-1}(3)| - |\varphi_P^{-1}(1)| + |(\varphi_H^1)^{-1}(3)| + |(\varphi_P^1)^{-1}(3)| \\
&= 14 - 4 + 8 + 4 = 22 \\
|\varphi_{L_1}^{-1}(4)| &= |\varphi_{L_0}^{-1}(4)| - |\varphi_P^{-1}(1)| + |(\varphi_H^1)^{-1}(4)| + |(\varphi_P^1)^{-1}(4)| \\
&= 13 - 4.5 + 9 + 4.5 = 22
\end{aligned}$$

and the obtained 4-total coloring is equitable.

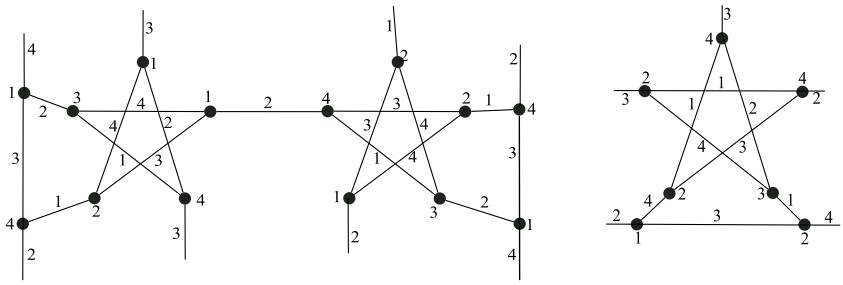


FIGURE 9. The 4-total colorings φ_H^1 and φ_P^1 used in L_t, when $t \equiv 1 \pmod 4$.

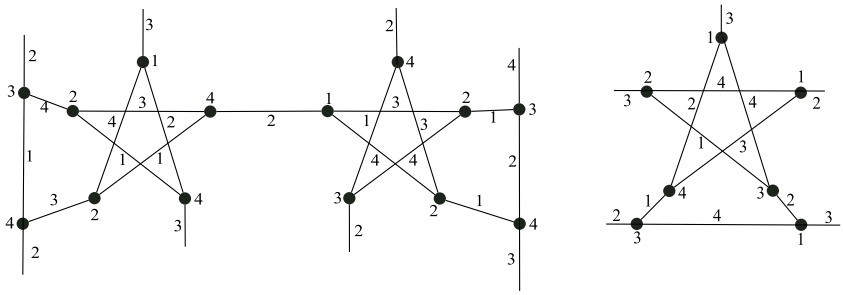


FIGURE 10. The 4-total colorings φ_H^2 and φ_P^2 used in L_t when $t \equiv 2 \pmod 4$.

For the next family members, we repeat the same process. To construct an equitable total coloring of L_2, we color the semi-graph H added to Loupekine L_1 with coloring φ_H^2, and recolor $P \subseteq L_1$ with coloring φ_P^2, as presented in Figure 10. For this pair of semi-graphs we have:

$|(\varphi_H^2)^{-1}(1)| = 8$, $|(\varphi_H^2)^{-1}(2)| = 9$, $|(\varphi_H^2)^{-1}(3)| = 8.5$, $|(\varphi_H^2)^{-1}(4)| = 9.5$, $|(\varphi_P^2)^{-1}(1)| = 5$, $|(\varphi_P^2(2))^{-1}| = |(\varphi_P^2)^{-1}(4)| = 4$ and $|(\varphi_P^2)^{-1}(3)| = 4.5$. Once more, we subtract the respective values of the total coloring φ_P^1 from the sum of φ_{L_1}, and add the values of φ_H^2 and φ_P^2.

So, for L_2 we have:

$$\begin{aligned}
|\varphi_{L_2}^{-1}(1)| &= |\varphi_{L_1}^{-1}(1)| - |(\varphi_P^1)^{-1}(1)| + |(\varphi_H^2)^{-1}(1)| + |(\varphi_P^2)^{-1}(1)| \\
&= 23 - 4 + 8 + 5 = 32 \\
|\varphi_{L_2}^{-1}(2)| &= |\varphi_{L_1}^{-1}(2)| - |(\varphi_P^1)^{-1}(2)| + |(\varphi_H^2)^{-1}(2)| + |(\varphi_P^2)^{-1}(2)| \\
&= 23 - 5 + 9 + 4 = 31 \\
|\varphi_{L_2}^{-1}(3)| &= |\varphi_{L_1}^{-1}(3)| - |(\varphi_P^1)^{-1}(3)| + |(\varphi_H^2)^{-1}(3)| + |(\varphi_P^2)^{-1}(3)| \\
&= 22 - 4 + 8.5 + 4.5 = 31 \\
|\varphi_{L_2}^{-1}(4)| &= |\varphi_{L_1}^{-1}(4)| - |(\varphi_P^1)^{-1}(4)| + |(\varphi_H^2)^{-1}(4)| + |(\varphi_P^2)^{-1}(4)| \\
&= 22 - 4.5 + 9.5 + 4 = 31
\end{aligned}$$

and the obtained 4-total coloring is equitable.

For the equitable 4-total colorings of L_3 and L_4, we repeat the previous steps. The total colorings φ_H^3 and φ_P^3 are presented in Figure 11, and the total colorings φ_H^4 and φ_P^4 are presented in Figure 12.

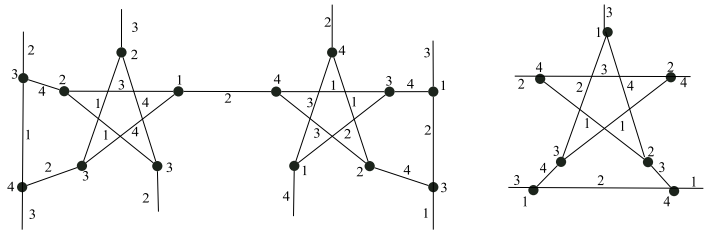


FIGURE 11. The 4-total colorings φ_H^3 and φ_P^3 used in L_t when $t \equiv 3 \pmod 4$.

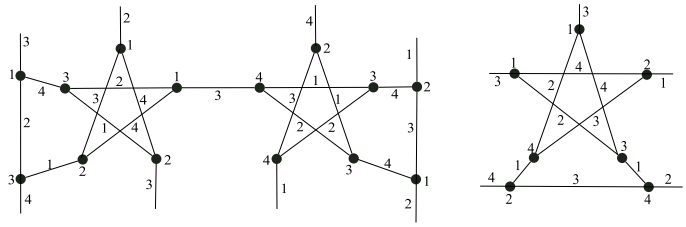


FIGURE 12. The 4-total colorings φ_H^0 and φ_P^0 used in L_t when $t \equiv 0 \pmod 4$.

For L_3, we have $|(\varphi_H^3)^{-1}(1)| = |(\varphi_H^3)^{-1}(2)| = |(\varphi_H^3)^{-1}(4)| = 8.5$, $|(\varphi_H^3)^{-1}(3)| = 9.5$, $|(\varphi_P^3)^{-1}(1)| = |(\varphi_P^3)^{-1}(2)| = |(\varphi_P^3)^{-1}(4)| = 4.5$ and $|(\varphi_P^3)^{-1}(3)| = 4$. The 4-total coloring of L_3 is equitable, since:

$$\begin{aligned}
|\varphi_{L_3}^{-1}(1)| &= |(\varphi_{L_2}^{-1}(1)| - |(\varphi_P^2)^{-1}(1)| + |(\varphi_H^3)^{-1}(1)| + |(\varphi_P^3)^{-1}(1)| \\
&= 32 - 5 + 8.5 + 4.5 = 40 \\
|\varphi_{L_3}^{-1}(2)| &= |\varphi_{L_2}^{-1}(2)| - |(\varphi_P^2)^{-1}(2)| + |(\varphi_H^3)^{-1}(2)| + |(\varphi_P^3)^{-1}(2)| \\
&= 31 - 4 + 8.5 + 4.5 = 40 \\
|\varphi_{L_3}^{-1}(3)| &= |\varphi_{L_2}^{-1}(3)| - |(\varphi_P^2)^{-1}(3)| + |(\varphi_H^3)^{-1}(3)| + |(\varphi_P^3)^{-1}(3)| \\
&= 31 - 4.5 + 9.5 + 4 = 40 \\
|\varphi_{L_3}^{-1}(4)| &= |\varphi_{L_2}^{-1}(4)| - |(\varphi_P^2)^{-1}(4)| + |(\varphi_H^3)^{-1}(4)| + |(\varphi_P^3)^{-1}(4)| \\
&= 31 - 4 + 8.5 + 4.5 = 40.
\end{aligned}$$

For L_4, we have $|(\varphi_H^0)^{-1}(1)| = |(\varphi_H^0)^{-1}(2)| = |(\varphi_H^0)^{-1}(3)| = 9$, $|(\varphi_H^0)^{-1}(4)| = 8$, $|(\varphi_P^0)^{-1}(1)| = |(\varphi_P^0)^{-1}(2)| = |(\varphi_P^0)^{-1}(4)| = 4.5$ and $|(\varphi_P^0)^{-1}(3)| = 4$. The 4-total coloring of L_4 is equitable, since:

$$\begin{aligned}
|\varphi_{L_4}^{-1}(1)| &= |\varphi_{L_3}^{-1}(1)| - |(\varphi_P^3)^{-1}(1)| + |(\varphi_H^0)^{-1}(1)| + |(\varphi_P^0)^{-1}(1)| \\
&= 40 - 4.5 + 9 + 4.5 = 49 \\
|\varphi_{L_4}^{-1}(2)| &= |\varphi_{L_3}^{-1}(2)| - |(\varphi_P^3)^{-1}(2)| + |(\varphi_H^0)^{-1}(2)| + |(\varphi_P^0)^{-1}(2)| \\
&= 40 - 4.5 + 9 + 4.5 = 49 \\
|\varphi_{L_4}^{-1}(3)| &= |\varphi_{L_3}^{-1}(3)| - |(\varphi_P^3)^{-1}(3)| + |(\varphi_H^0)^{-1}(3)| + |(\varphi_P^0)^{-1}(3)| \\
&= 40 - 4 + 9 + 4 = 49 \\
|\varphi_{L_4}^{-1}(4)| &= |\varphi_{L_3}^{-1}(4)| - |(\varphi_P^3)^{-1}(4)| + |(\varphi_H^0)^{-1}(4)| + |(\varphi_P^0)^{-1}(4)| \\
&= 40 - 4.5 + 8 + 4.5 = 48.
\end{aligned}$$

In sum, the number of elements in each color class in the 4-total colorings for the first five Loupekine snarks are:

$$\begin{aligned}
&|\varphi_{L_0}^{-1}(1)| = |\varphi_{L_0}^{-1}(2)| = |\varphi_{L_0}^{-1}(3)| = |\varphi_{L_0}^{-1}(4)| + 1\\
&|\varphi_{L_1}^{-1}(1)| = |\varphi_{L_1}^{-1}(2)| = |\varphi_{L_1}^{-1}(3)| + 1 = |\varphi_{L_1}^{-1}(4)| + 1\\
&|\varphi_{L_2}^{-1}(1)| = |\varphi_{L_2}^{-1}(2)| + 1 = |\varphi_{L_2}^{-1}(3)| + 1 = |\varphi_{L_2}^{-1}(4)| + 1\\
&|\varphi_{L_3}^{-1}(1)| = |\varphi_{L_3}^{-1}(2)| = |\varphi_{L_3}^{-1}(3)| = |\varphi_{L_3}^{-1}(4)|\\
&|\varphi_{L_4}^{-1}(1)| = |\varphi_{L_4}^{-1}(2)| = |\varphi_{L_4}^{-1}(3)| = |\varphi_{L_4}^{-1}(4)| + 1.
\end{aligned}$$

We assume by inductive hypothesis that for $t \geq 0$ the following relations hold:

$$\begin{aligned}
&\text{if } t \equiv 0 \mod 4, \text{ then } |\varphi_{L_0}^{-1}(1)| = |\varphi_{L_0}^{-1}(2)| = |\varphi_{L_0}^{-1}(3)| = |\varphi_{L_0}^{-1}(4)| + 1\\
&\text{if } t \equiv 1 \mod 4, \text{ then } |\varphi_{L_1}^{-1}(1)| = |\varphi_{L_1}^{-1}(2)| = |\varphi_{L_1}^{-1}(3)| + 1 = |\varphi_{L_1}^{-1}(4)| + 1\\
&\text{if } t \equiv 2 \mod 4, \text{ then } |\varphi_{L_2}^{-1}(1)| = |\varphi_{L_2}^{-1}(2)| + 1 = |\varphi_{L_2}^{-1}(3)| + 1 = |\varphi_{L_2}^{-1}(4)| + 1\\
&\text{if } t \equiv 3 \mod 4, \text{ then } |\varphi_{L_3}^{-1}(1)| = |\varphi_{L_3}^{-1}(2)| = |\varphi_{L_3}^{-1}(3)| = |\varphi_{L_3}^{-1}(4)|.
\end{aligned}$$

Thus, φ_{L_t} is an equitable total coloring. In fact, note that if $t-1 \equiv i-1 \pmod 4$, then we have that $\varphi_{L_{t-1}}$ and $\varphi_{L_{i-1}}$ satisfy the same relations as in the inductive hypothesis. Moreover, L_t (resp. L_i) is obtained from L_{t-1} (resp. from L_{i-1}) by recoloring the subgraph $P \subseteq L_{t-1}$ (resp. $P \subseteq L_{i-1}$) with the 4-total coloring φ_P^i, and by coloring the new subgraph H with φ_H^i, where $t \equiv i \pmod 4$ and $0 < i \leq 4$. So, in both cases, to obtain the cardinality of each color, we subtract the respective values of the total coloring φ_P^{i-1} from the sum of $\varphi_{L_{t-1}}$ (resp. $\varphi_{L_{i-1}}$), and add the values of φ_H^i and φ_P^i.

We also observe that, in each case, the equitable 4-total coloring of Loupekine snarks L_t is constructed by making the junction of semiedges with the same color such that adjacent vertices have different colors. Hence, the result follows. □

5. Kochol Superposition

In this section, we present new families of snarks obtained by an application of the Kochol superposition construction [13], and show their 4-total colorings.

Given a cubic semi-graph $M(V, E, S)$, with $S \neq \emptyset$, the set S of semiedges is partitioned into q pairwise disjoint nonempty sets $Q_1, Q_2, \ldots, Q_q$.

Denote by k_i the cardinality of Q_i, for each $i \in \{1, \ldots, q\}$. Following Kochol's notation [13], we call the sets Q_i *connectors*, and denote the semi-graph M by $(k_1, k_2, \ldots, k_q)$-semi-graph M. A *superedge* ξ is a semi-graph with two connectors, and a *supervertex* ϑ is a semi-graph with three connectors. Now, we consider the following semi-graphs depicted in Figure 13:

- $(3,3)$-semi-graph M' (superedge) is obtained by removing two nonadjacent vertices v_1 and v_2 from a snark G, and replacing each edge incident with v_1 or v_2 by semiedges.
- $(1,1)$-semi-graph L' (superedge) is an isolated edge (two semiedges);
- $(1,3,3)$-semi-graph J' (supervertex) consists of two isolated edges and a vertex incident to 3 semi-edges;
- $(1,1,1)$-semi-graph K' (supervertex) consists of a vertex and three semiedges.

Let $G = (V, E)$ be a snark. Replace every edge $e \in E$ by a superedge ξ, and every vertex $v \in V$ by a supervertex ϑ. If $v \in V$ is incident with $e \in E$ then a connector in ϑ_v is linked with a connector in ξ_e through the junction of semiedges. The obtained cubic graph $G(\nu,\vartheta)$ is called *superposition of* G and it is a snark [13].

Next, we show new infinite families of snarks which are Type 1.

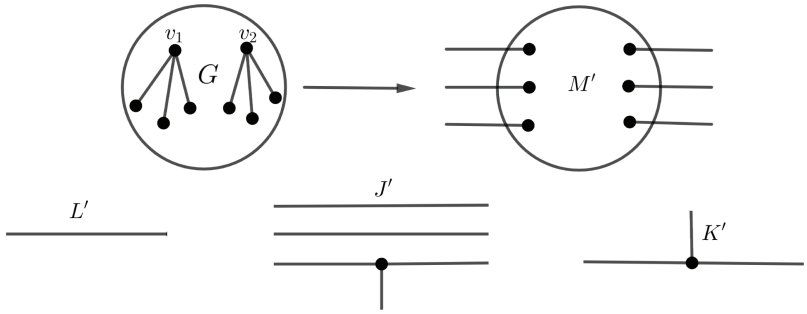
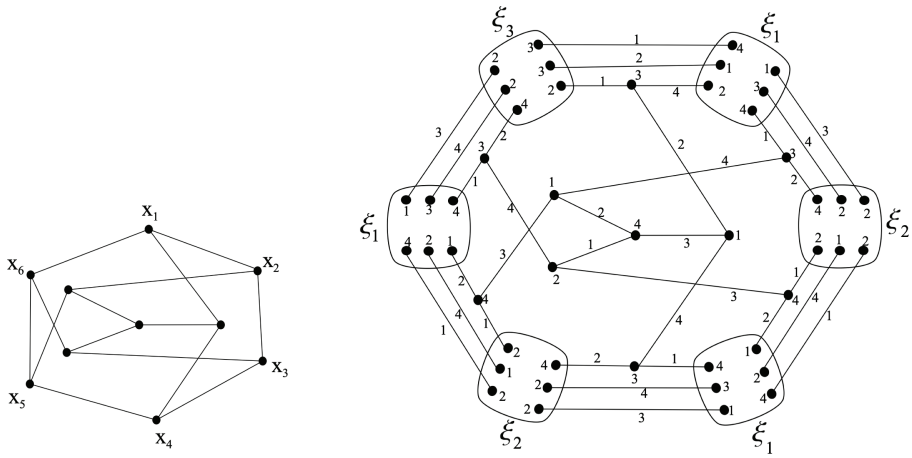


FIGURE 13. Superedge M' obtained from snark G, superedge L', supervertex J', and supervertex K'.



(a) (b)

FIGURE 14. The Petersen graph and a depiction of a 4-total coloring of its superposition with t-SemiBlowup.

5.1. Petersen graph

Let R_t, with $t \geq 6$, be the snark obtained by a superposition of the Petersen graph with a t-SemiBlowup snark as follows.

The superedge ξ_1 is obtained by removing from the t-SemiBlowup snark, $t \geq 6$, two nonadjacent vertices k_4 and c_3, and replacing each edge incident with k_4 and c_3 by the corresponding semiedge. Similarly, superedges ξ_2 (resp. ξ_3) is obtained by removing the nonadjacent vertices k_2 and c_1 (resp. k_2 and g_2), and replacing edges by semiedges accordingly.

Now, consider the Petersen graph depicted in Figure 14. To obtain its superposition, replace:

- the vertex x_i, $1 \leq i \leq 6$, by the supervertex J' depicted in Figure 13;
- edges x_1x_2, x_3x_4 and x_5x_6 by a copy of superedge ξ_1, respectively;
- edges x_2x_3 and x_4x_5 by a copy of superedge ξ_2, respectively; and
- edge x_6x_1 by a copy of superedge ξ_3.
- the remaining edges (resp. vertices) are replaced by superedges L' (resp. supervertices K') which is equivalent to maintain the original edges (resp. vertices) of the Petersen graph.

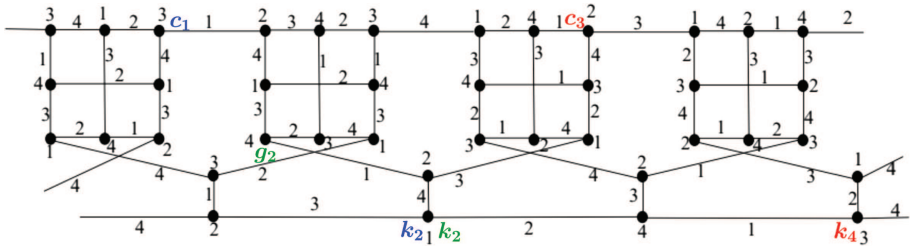


FIGURE 15. A coloring for semi-graph B_4 and the vertices removed in ξ_1 (red), ξ_2 (blue), and ξ_3 (green).

Finally, if $v \in V$ is incident with $e \in E$ then a connector in ϑ_v is linked with a connector in ξ_e through the junction of semiedges.

Theorem 5.1. *All snarks R_t obtained by a superposition of the Petersen graph with a t-SemiBlowup snark, $t \geq 6$, have total chromatic number equal to 4.*

Proof. A 4-total coloring of snarks R_t is constructed as follows. By Theorem 2.1, the 4-total coloring of semi-graph B_4 appears in every t-SemiBlowup, with $t \geq 6$.

So, the coloring of superedge ξ_1 is obtained by removing the two nonadjacent vertices k_4 and c_3 from Figure 15 (vertices depicted in red).

Similarly, the coloring of superedge ξ_2 (resp. ξ_3) is obtained by removing two nonadjacent vertices k_2 and c_1 (resp. k_2 and g_2) (vertices depicted in blue (resp. green)). The color of each semiedge in ξ_i, $1 \leq i \leq 3$, is equal to the color of the respective edge incident with the removed vertices. The coloring of the remaining elements is presented in Figure 14b.

It is easy to verify that this construction produces a 4-total coloring. Indeed, for each t-SemiBlowup, $t \geq 6$, every superedge ξ_1, ξ_2, or ξ_3 have the same coloring depicted in Figure 14b. So, each member of the new family admits a 4-total coloring which is an extension of the total coloring presented in Figure 14b.

Moreover, if $v \in V$ is incident with $e \in E$ then a connector in ϑ_v is linked with a connector in ξ_e through the junction of semiedges with the same color; and adjacent vertices have different colors. Hence, the result follows. □

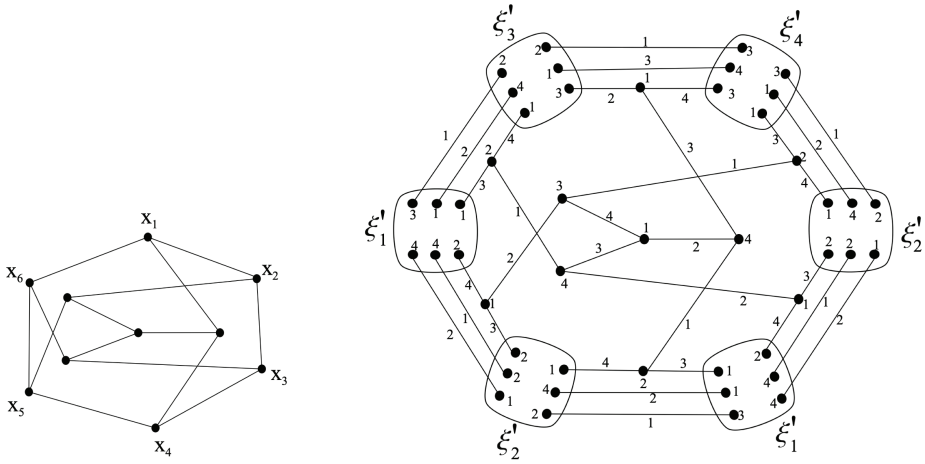
Now, let R'_t, $t \geq 6$, be the Snark obtained by a superposition of the Petersen graph with a t-Blowup snark as follows.

Similarly to the previous superposition, the superedges ξ'_1, ξ'_2, ξ'_3, and ξ'_4 are obtained by removing, from the t-Blowup snark, $t \geq 6$, the nonadjacent vertices $\{h_1, k_3\}$, $\{b_2, f_1\}$, $\{b_2, e_2\}$, and $\{c_1, k_1\}$, respectively. Each edge incident with these vertices is replaced by the corresponding semiedge.

Again, consider the Petersen graph depicted in Figure 16. To obtain its superposition, replace:

- vertex x_i, $1 \leq i \leq 6$, by the supervertex J' depicted in Figure 13;
- edges x_3x_4 and x_5x_6 by a copy of the superedge ξ'_1, respectively;
- edges x_2x_3 and x_4x_5 by a copy of superedge ξ'_2, respectively;
- edge x_6x_1 by a copy of superedge ξ'_3;
- edge x_1x_2 by a copy of superedge ξ'_4;
- the remaining edges (resp. vertices) are replaced by superedges L' (resp. supervertices K') which is equivalent to maintain the original edges (resp. vertices) of the Petersen graph.

Finally, if $v \in V$ is incident with $e \in E$ then a connector in ϑ_v is linked with a connector in ξ_e through the junction of semiedges.



(a) (b)

FIGURE 16. The Petersen graph and a depiction of a 4-total coloring of its superposition with t-Blowup.

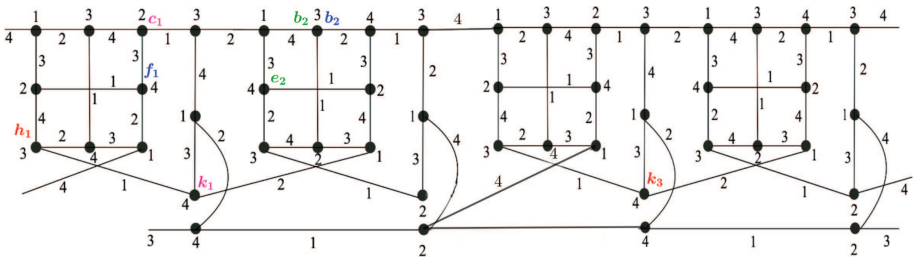


FIGURE 17. A coloring for semi-graph B'_4 and the vertices removed in ξ_i, $1 \leq i \leq 4$, depicted in colors.

Theorem 5.2. *All snarks R'_t obtained by a superposition of the Petersen graph with a t-Blowup snark, $t \geq 6$, have total chromatic number equal to 4.*

Proof. Similarly to the proof of Theorem 5.1, a 4-total coloring of snarks R'_t, $t \geq 6$, is constructed as follows. By Theorem 3.1, the 4-total coloring of semi-graph B'_4 (obtained by using the coloring of B'_2 twice), appears in every t-Blowup, with $t \geq 6$.

So, the colorings of superedges ξ'_1, ξ'_2, ξ'_3, and ξ'_4 are obtained by removing, from Figure 17, two nonadjacent vertices $\{h_1, k_3\}$, $\{b_2, f_1\}$, $\{b_2, e_2\}$, and $\{c_1, k_1\}$, respectively (vertices depicted in colors).

The color of each semiedge in ξ_i, $1 \leq i \leq 4$, is equal to the color of the respective edge incident with the removed vertices. The coloring of the remaining elements is presented in Figure 16b.

It is easy to verify that this construction produces a 4-total coloring. Indeed, for each t-Blowup, $t \geq 6$, every superedge ξ_i, $1 \leq i \leq 4$, has the same coloring depicted in Figure 16b. So, each member of the new family admits a 4-total coloring which is an extension of the total coloring presented in Figure 16b.

Moreover, if $v \in V$ is incident with $e \in E$ then a connector in ϑ_v is linked with a connector in ξ_e through the junction of semiedges with the same color; and adjacent vertices have different colors. Hence, the result follows. $\square$

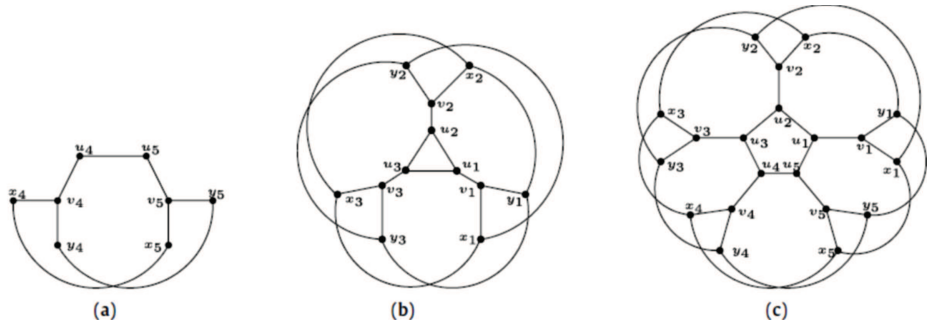


FIGURE 18. Graph F_5 constructed from graphs F_3 and L_5. (a) Link graph L_5. (b) Graph F_3. (C) Graph F_5.

Note that Theorem 2.1 (resp. 3.1) determine the coloring of infinite families of snarks, since each t-SemiBlowup (resp. t-Blowup) with $t \geq 6$ generates a new snark R_t (resp. R'_t) with $60t - 2$ (resp. $66t + 10$) vertices.

5.2. Flower snarks

Now, we present another construction of infinite families of Type 1 graphs obtained from a Kochol superposition of Flower snarks with t-SemiBlowup or t-Blowup snarks.

Let $F_{2i+1}, i \geq 1$ be the members of the family of Flower Snarks. For this family we define the basic block B_k as the claw graph with vertex set $V(B_k) = \{u_k, v_k, x_k, y_k\}$ and edge set $E(B_k) = \{u_k v_k, x_k v_k, y_k v_k\}$. We define the set of the link edges as $E_{kj} = \{u_k u_j, x_k x_j, y_k y_j\}$ and link graph L_k, k odd and $k \geq 5$, as the union of B_{k-1}, B_k, and the graph spanned by $E_{(k-1)k}$, *i.e.*, $V(L_k) = V(B_{k-1}) \cup V(B_k)$ and $E(L_k) = E(B_{k-1}) \cup E(B_k) \cup E_{(k-1)k}$. Figure 18a shows L_5.

The first Flower Snark, F_3, is defined as the union of B_1, B_2, B_3, and the graph spanned by $E_{23} \cup E_{31} \cup \{u_1u_2, x_1y_2, y_1x_2\}$ (see Fig. 18b). For each $i \geq 2$, F_{2i+1} is obtained from graphs F_{2i-1} and L_{2i+1} as follows: $V(F_{2i+1}) = V(F_{2i-1}) \cup V(L_{2i+1})$, and $E(F_{2i+1}) = (E(F_{2i-1}) \backslash E^{out}_{2i-1}) \cup E(L_{2i+1}) \cup E^{in}_{2i+1}$, where $E^{out}_{2i-1} = E_{(2i-1)1}$, and $E^{in}_{2i+1} = E_{(2i-1)(2i)} \cup E_{(2i+1)1}$. Figure 18c shows F_5, constructed from graphs F_3 and L_5.

In [3], the authors construct 4-total coloring for each element of that family taking advantage of the recursive construction of these graphs. This coloring Φ_{2i+1} satisfies that the E^{out}_{2i+1} edges have the same color 1. See Figure 20.

In the following theorems, we consider a specific coloring Φ_{2i+1} obtained by replacing in each step of the construction of F_{2i+1} the edge $u_{2i+1}u_1$, see Figure 20.

Let $R_t(2i+1)$ be the Snark obtained by the superposition of F_{2i+1} with a t-SemiBlowup, $t \geq 6$, $i \geq 1$, as follows.

Similarly to the previous superposition, the superedges ψ_1, ψ_2, ψ_3, ψ_4 and ψ_5 are obtained by removing, from the t-SemiBlowup snark, $t \geq 6$, the nonadjacent vertices $\{h_4, c_3\}$, $\{b_1, f_4\}$, $\{k_3, j_2\}$, $\{k_3, b_3\}$ and $\{d_2, b_1\}$, respectively. Each edge incident with these vertices is replaced by the corresponding semiedge.

Consider the Flower snark F_3 depicted in Figure 18b. To obtain its superposition, replace:

- edge u_1u_2 by a copy of the superedge ψ_1;
- edge u_2u_3 by a copy of the superedge ψ_2, and
- edge u_3u_1 by a copy of the superedge ψ_3.

For the Flower snark F_5, replace:

- edge u_1u_2 by a copy of the superedge ψ_1;
- edge u_2u_3 by a copy of the superedge ψ_2;
- edge u_3u_4 by a copy of the superedge ψ_4;

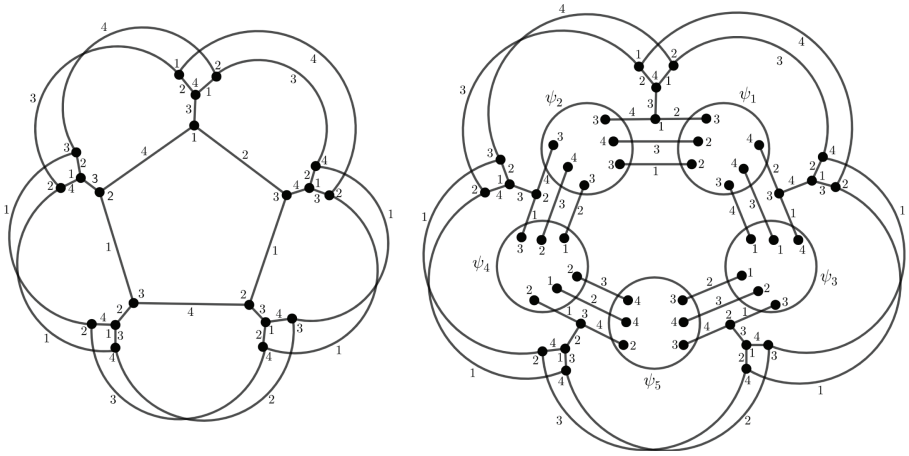


FIGURE 19. A 4-total coloring of F_5, and a 4-total coloring of its superposition with a t-SemiBlowup.

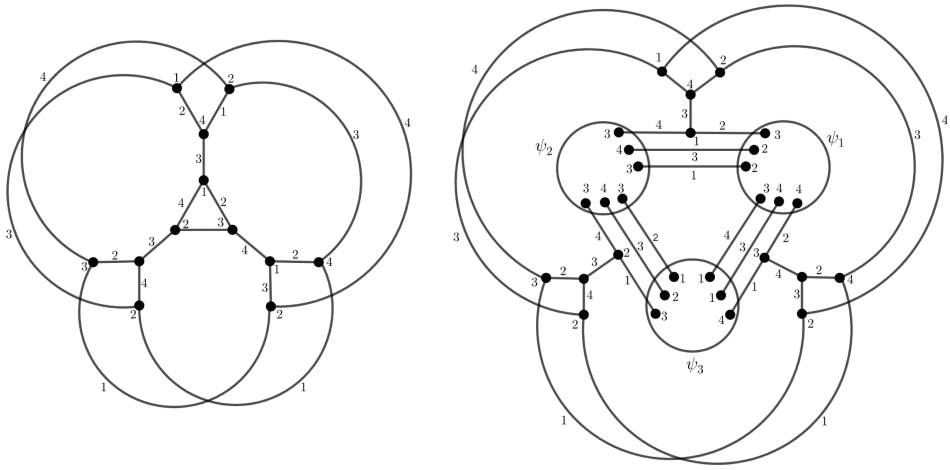


FIGURE 20. A 4-total coloring of F_3, and a 4-total coloring of its superposition with a t-SemiBlowup.

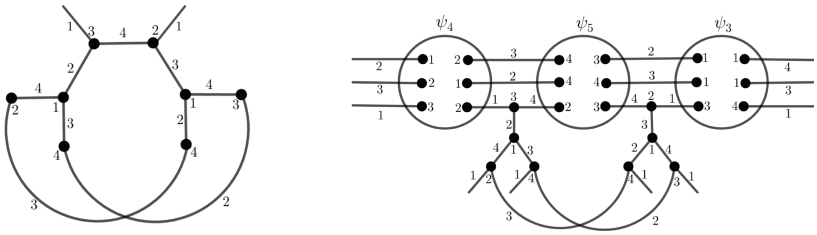


FIGURE 21. A 4-total coloring of Link graph L_{2i+1}, and a 4-total coloring of its superposition with a t-SemiBlowup.

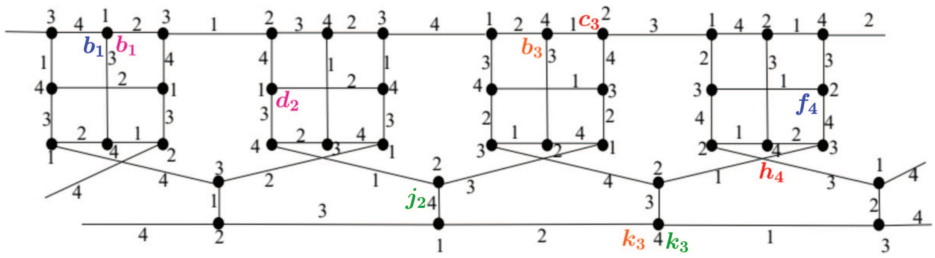


FIGURE 22. A coloring for semi-graph B_4 and the vertices removed in ψ_i, $1 \leq i \leq 5$, depicted in colors.

– edge u_4u_5 by a copy of the superedge ψ_5;
– edge u_5u_1 by a copy of the superedge ψ_3.

For the link graph L_{2i+1}, replace:

– edge $u_{2i}u_{2i+1}$ by a copy of the superedge ψ_5;
– semiedge $u_{2i}\cdot$ by a copy of the superedge ψ_4;
– semiedge $u_{2i+1}\cdot$ by a copy of the superedge ψ_3

Now, for any Flower snark F_{2i+1} and link graph L_{2i+1}, $i \geq 2$, replace:

– the vertex u_j, $1 \leq j \leq 2i+1$ of F_{2i+1} by the supervertex J' depicted in Figure 13.
– the remaining edges (resp. vertices) are replaced by superedges L' (resp. supervertices K') which is equivalent to maintain the original edges (resp. vertices) of F_{2i+1}.

An example of the superposition of the Flower snark F_5 with a t-SemiBlowup snark is depicted in Figure 19 (right).

Theorem 5.3. *All snarks $R_t(2i+1)$ obtained by superposition of a Flower snark graph F_{2i+1}, $i \geq 1$, with a t-SemiBlowup snark, $t \geq 6$, have total chromatic number equal to 4.*

Proof. Analogously to the proof of Theorem 5.2, a 4-total coloring of snarks $R_t(2i+1)$, $t \geq 6$, $i \geq 1$, is constructed as follows. By Theorem 5.1, the 4-total coloring of semi-graph B_4 appears in every t-SemiBlowup, with $t \geq 6$.

So, the colorings of superedges ψ_1, ψ_2, ψ_3, ψ_4 and ψ_5 are obtained by removing, from Figure 22, two nonadjacent vertices $\{h_4, c_3\}$, $\{b_1, f_4\}$, $\{k_3, j_2\}$, $\{k_3, b_3\}$ and $\{d_2, b_1\}$, respectively (vertices depicted in colors).

The color of each semiedge in ψ_i, $1 \leq i \leq 5$, is equal to the color of the respective edge incident with the removed vertices.

The coloring of the remaining elements of the Flower snark F_5 and the link graph L_{2i+1} are presented in Figures 19(right) and 21.

Also, Figure 19(right) presents a depiction of 4-total coloring of $R_t(5)$. Note that the coloring of $R_t(7)$ coincides with the coloring of graph obtained by replacing superedge ψ_3, and edges x_1x_5 and y_1y_5, in $R_t(5)$ by the colored semi-graph L' in Figure 21, and linking the semiedges of L' and the semi-graph (obtained from $R_t(5)$) through the junctions of the following semi-graphs: ψ_4 of L' with ψ_5 of the semi-graph (obtained from $R_t(5)$); and ψ_3 of L' with ψ_1 of the semi-graph (obtained from $R_t(5)$); and the remaining semi-edges $x.$ and $y.$ accordingly.

We assume by inductive hypothesis that there exists a 4-total coloring of $R_t(2i-1)$ obtained from the superposition of a Flower snark F_{2i-1} with a t-SemiBlowup, such that there is only one copy of the colored semi-graph L' in Figure 21 on this graph. We obtain a 4-total coloring of $R_t(2i+1)$ from

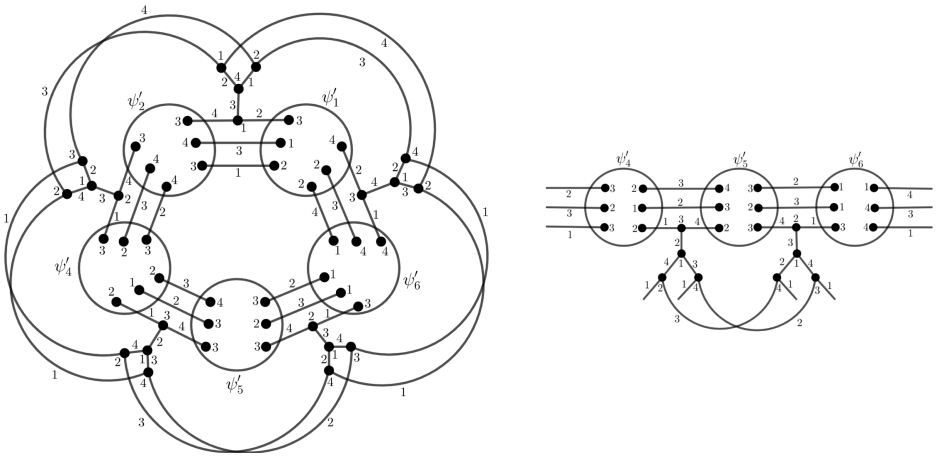


FIGURE 23. A depiction of $R_t(5)$ left and a 4-total coloring of the superposition of L', the Link graph L_{2i+1} with a t-Blowup.

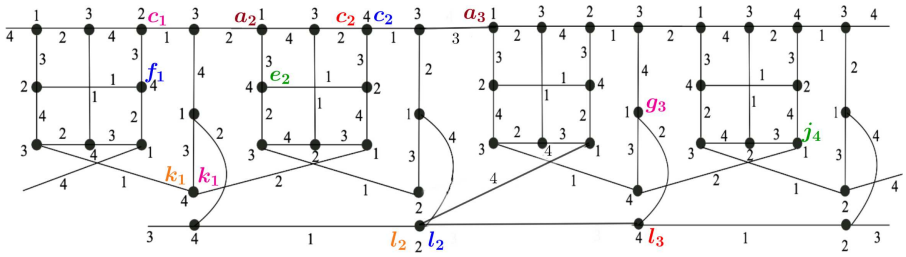


FIGURE 24. A coloring for semi-graph B'_4 and the vertices removed in ψ'_i, $1 \leq i \leq 6$, depicted in colors.

$R_t(2i-1)$ by replacing the superedge of ψ_3 and edges x_1x_{2i-1} and y_1y_{2i-1}, in $R_t(2i-1)$ by the colored semi-graph L' in Figure 21, and linking the semiedges of L' and the semi-graph (obtained from $R_t(2i-1)$) through the junctions of the following semi-graphs: ψ_4 of L' with ψ_5 of the semi-graph (obtained from $R_t(2i-1)$); and ψ_3 of L' with ψ_1 of the semi-graph (obtained from $R_t(2i-1)$); and the remaining semiedges $x.$ and $y.$ accordingly.

It is easy to verify that this construction produces a 4-total coloring. Indeed, for each t-SemiBlowup, $t \geq 6$, every superedge ψ_i, $1 \leq i \leq 5$, has the same coloring depicted in Figure 22. So, each member of the new family admits a 4-total coloring which is an extension of the total coloring presented in Figure 22.

Moreover, if $v \in V(F_{2i+1})$ is incident with $e \in E(F_{2i+1})$ then a connector in ϑ_v is linked with a connector in ξ_e through the junction of semiedges with the same color; and adjacent vertices have different colors. Hence, the result follows. □

Let $R'_t(2i+1)$ be the Snark obtained by the superposition of F_{2i+1} and a t-Blowup, $t \geq 6$, $i \geq 1$, as follows.

Similarly to the previous superposition, the superedges ψ'_1, ψ'_2, ψ'_3, ψ'_4, ψ'_5 and ψ'_6 are obtained by removing, from the t-Blowup snark, $t \geq 6$, the nonadjacent vertices $\{e_2, j_4\}$, $\{g_3, c_1\}$, $\{c_2, l_2\}$, $\{c_2, l_3\}$, $\{a_2,a_3\}$ and $\{k_1, l_2\}$, respectively. Each edge incident with these vertices is replaced by the corresponding semiedge.

Consider the Flower snark F_3 depicted in Figure 18b. To obtain its superposition, replace:

- edge u_1u_2 by a copy of the superedge ψ'_1;
- edge u_2u_3 by a copy of the superedge ψ'_2, and

– edge u_3u_1 by a copy of the superedge ψ'_3.

For the Flower snark F_5, replace:

– edge u_1u_2 by a copy of the superedge ψ'_1;
– edge u_2u_3 by a copy of the superedge ψ'_2;
– edge u_3u_4 by a copy of the superedge ψ'_4;
– edge u_4u_5 by a copy of the superedge ψ'_5;
– edge u_5u_1 by a copy of the superedge ψ'_6.

For the link graph L_{2i+1}, replace:

– edge $u_{2i}u_{2i+1}$ by a copy of the superedge ψ'_5;
– semiedge $u_{2i}\cdot$ by a copy of the superedge ψ'_4;
– semiedge $u_{2i+1}\cdot$ by a copy of the superedge ψ'_6.

Now, for any Flower snark F_{2i+1} and link graph L_{2i+1}, $i \geq 2$, replace:

– the vertex u_j, $1 \leq j \leq 2i+1$ of F_{2i+1} by the supervertex J' depicted in Figure 13.
– the remaining edges (resp. vertices) are replaced by superedges L' (resp. supervertices K') which is equivalent to maintain the original edges (resp. vertices) of F_{2i+1}.

An example of the superposition of the Flower snark F_5 and t-Blowup snark is depicted in Figure 23(left).

Theorem 5.4. *All snarks $R'_t(2i+1)$ obtained by a superposition of a Flower snark graph and a t-Blowup snark with $t \geq 6$ have total chromatic number equal to 4.*

Proof. The proof follows similarly to the proof of Theorem 5.3, by using the coloring in Figure 24, the coloring of $R'(5)$, and the coloring of semi-graph L', Figure 23 in the inductive process. □

Finally, we show a generalization of the previous superpositions where each superege ($(3,3)$-semi-graph M') is obtained from a t_j-SemiBlowup (resp. t_j-Blowup), eventually of different order, $t_j \geq 6$.

Formally, let $R_S(2i+1)$, with $S = \{t_j : t_j \geq 6\}$ be the snark obtained by the superposition of the Petersen graph (resp. Flower snark F_{2i+1}) such that each superedge is constructed from a t_j-SemiBlowup (resp. t_j-Blowup), with $t_j \geq 6$, for $1 \leq j \leq 6$ (resp. $1 \leq j \leq 2i+1$).

Theorem 5.5. *All snarks $R_S(2i+1)$ obtained by superposition of the Petersen graph (resp. Flower snark F_{2i+1}, $i \geq 1$,) $t \geq 6$, have total chromatic number equal to 4.*

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