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STRONGLY GEODESIC PREINVEXITY AND STRONGLY INVARIANT η-MONOTONICITY ON RIEMANNIAN MANIFOLDS AND ITS APPLICATION

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Abstract. This paper introduces the concepts of strongly geodesic preinvexity, strongly η-invexity of order m, and strongly invariant η-monotonicity of order m on Riemannian manifolds. Additionally, it discusses an important characterization of these functions under a condition, known as **Condition C** (The **Condition C** is defined in Remark 1 of this article), defined by Barani and Pouryayevali [*J. Math. Anal. Appl.* **328** (2007) 767–779]. The paper provides various non-trivial examples to support these definitions. Furthermore, it presents a significant characterization of strict η-minimizers (or η-minimizers) of order m for multi-objective optimization problems and a solution to the vector variational-like inequality problem.

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1. Introduction

Monotonicity and the variational inequality problem play crucial roles in finding solutions for practical problems in various fields such as optimization theory, science, and engineering. Generalized monotonicity is a useful tool for analyzing and obtaining solutions for variational inclusion and complementarity problems. Convexity is closely linked to monotonicity, as the monotonicity of the gradient function corresponds to convexity of the real-valued function. Recent developments in convexity and monotonicity have been made by various authors, see [6, 7, 19–22]. While manifolds differ from linear spaces, they have been studied in relation to convexity. Rapcsak [23] and Udriste [25] extended convexity techniques from linear spaces to Riemannian manifolds, introducing the concept of geodesic convexity.

Barani [3] introduced various types of invariant monotone vector fields on Riemannian manifolds. It has been demonstrated that many results and properties established in Euclidean space also hold true for invariant monotone vector fields on Riemannian manifolds. The field of optimization has seen significant advancements on Riemannian manifolds as well, as evidenced by works such as those by Yang [27] and Garzon [8]. Pini [18] defined the concept of invex functions on Riemannian manifolds and discussed several associated properties. Generalized invexity, which is closely related to generalized invariant monotonicity, has been developed within convex analysis and extensively studied by Lang [24]. Yang *et al.* [27] introduced the notion of invariant monotonicity as

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a generalization of monotonicity. The existence of solutions to the vector variational-like inequality problem was established under the framework of generalized monotonicity [24]. Nemeth [16] presented the concept of a monotone vector field on Riemannian manifolds, which serves as an important generalization of monotone operators. Noor [17] discussed the notion of α-invexity and α-monotonicity. Iqbal *et al.* [14] extended these concepts to Riemannian manifolds, introducing the notions of strong α-invexity and invariant α-monotonicity.

Based on the research works mentioned in [1,3,9,10,12,14,17], we propose the concepts of strongly geodesic preinvexity of order m, strongly η-invexity of order m, strongly quasi η-invexity of order m, and strongly pseudo η-invexity of order m on Riemannian manifolds. These concepts generalize the notions of strongly geodesic convex functions of order m defined in [11], as well as strongly α-invexity and invariant α-monotonicity introduced by Iqbal *et al.* in [14]. The article proceeds as follows: Section 2 provides essential definitions and facts, focusing on understanding the fundamental concepts of Riemannian geometry, which are crucial for this article. We present nontrivial examples that support these definitions and establish various interesting properties and results. In Section 3, we introduce an important characterization of strongly geodesic preinvexity of order m and strongly η-invexity of order m.

In Section 4, we define generalized invariant η-monotonicity of order m on Riemannian manifolds, specifically focusing on strongly invariant η-monotonicity and strongly invariant pseudo η-monotonicity. We establish a relationship between strongly η-invexity (or strongly pseudo η-invexity) and strongly invariant η-monotonicity (or strongly invariant pseudo η-monotonicity). This relationship demonstrates the close connection between strongly η-invexity (or strongly pseudo η-invexity) and strongly invariant η-monotonicity (or strongly invariant pseudo η-monotonicity). Moving on to Section 5, we introduce the concept of η-minimizers of order m on Riemannian manifolds. We demonstrate the application of these concepts in solving multi-objective optimization problems for strongly η-invex functions of order m. Additionally, we establish a relationship between strict η-minimizers of order m for multi-objective optimization problems and solutions to the vector variational-like inequality problem.

2. Preliminaries

In this section, we provide a brief review of essential definitions and fundamental results concerning Riemannian manifolds, which will be utilized throughout the paper. For a comprehensive introduction to differential geometry, we refer the readers to [24]. In our discussion, we consider M as a C^∞ smooth manifold, modeled on a Hilbert space H that can be either finite-dimensional or infinite-dimensional. At each point $p \in M$, the tangent space T_pM is isomorphic to H, and we equip it with a Riemannian metric denoted by $\langle \cdot, \cdot \rangle_p$. Thus, we have a smooth assignment of $\langle \cdot, \cdot \rangle_p$ to each T_pM. Usually, we write

$$g_p(X_1, X_2) = \langle X_1, X_2 \rangle_p \qquad \forall X_1, X_2 \in T_pM.$$

Therefore, M is a Riemannian manifold. The length of tangent vector corresponding to a norm induced by the inner product $\langle \cdot, \cdot \rangle_p$ is denoted by $\| \cdot \|_p$.

The length of piece wise C^1 curve $r : [a, b] \to M$ is defined by

$$L(r) = \int_a^b \|r'(s)\|_{r(s)}\, \mathrm{d}s.$$

For $x_1, x_2 \in M,\ x_1 \neq x_2$, we define the distance between x_1 and x_2 as follows:

$$d(x_1, x_2) = \inf\{L(r) : r \text{ is piecewise } C^1 \text{ curve joining } x_1 \text{ to } x_2\}.$$

Subsequently, the distance function d gives rise to the original structure of open sets on the smooth manifold M. The collection of all vector fields on M is represented as $\chi(M)$. By means of the metric, we establish a correspondence where each function h is associated with its gradient, denoted as $h \to \operatorname{grad} h \in \chi(M)$. *i.e.*,

$$\langle \mathrm{d}h, X \rangle_p = \mathrm{d}h(X), \qquad \forall X \in \chi(M).$$

On every Riemannian manifold, there exists a unique covariant derivative known as the Levi-Civita connection, denoted by $\nabla_X Y$, for any vector fields $X, Y \in \chi(M)$. Recall that a geodesic is a smooth path r in the manifold M that connects two points x_1 and x_2, denoted as $r : [a, b] \to M$, such that the length of the curve C, denoted as $L(r)$, is equal to the distance $d(x_1, x_2)$, where $x_1 \neq x_2$. A geodesic is called a minimal geodesic if it satisfies the equation $\nabla_{\frac{\mathrm{d}r(t)}{\mathrm{d}t}} \frac{\mathrm{d}r(t)}{\mathrm{d}t} = 0$. The existence theorem for ordinary differential equations states that for every vector v in the tangent space TM, there exists an open interval $J(v)$ containing 0 and exactly one geodesic $r_v : J(v) \to M$ with $\frac{\mathrm{d}r(0)}{\mathrm{d}t} = v$. This implies that there exists an open neighborhood $\widetilde{T}M$ of the manifold M such that for all $v \in \widetilde{T}M$, the geodesic $r_v(t)$ is defined for $|t| < 2$.

Using the parallel translation operator for vectors along a smooth curve, given a smooth curve $r : I \to M$ and a vector $v_0 \in T_{r(t_0)}M$ for any $t_0 \in I$, there exists a unique parallel vector field $V(t)$ along $r(t)$ such that $V(t_0) = v_0$. The Hilbert space $(T_p M, \|\cdot\|_p)$ is a linear isometric identification with its dual space $(T_p^* M, \|\cdot\|_p)$. The linear isometric map between the tangent spaces $T_{r(t_0)}M$ and $T_{r(t)}M$, for all $t \in I$, is defined by $v_0 \to V(t)$ and denoted as $P^t_{t_0,r}$, which is called the parallel translation from $T_{r(t_0)}M$ to $T_{r(t)}M$ along $r(t)$. If h is a differentiable map from the manifold M into the manifold N, there exists a linear map $\mathrm{d}h_p : T_p M \to T_{h(p)}N$, where $\mathrm{d}h_p$ represents the differential of h at p.

A finite-dimensional Riemannian manifold is called complete if its geodesics are defined for all values of t. Hopf-Rinow's theorem guarantees that a Riemannian manifold M is complete if every pair of points in M can be connected by a (not necessarily unique) minimal geodesic segment.

Consider a Riemannian manifold M, and let $\eta : M \times M \to TM$ be a map such that for every $u, v \in M$, $\eta(u, v) \in TM$, where TM is the tangent bundle defined as $TM = \bigcup_{p\in M} T_p M$. Some of the basic definitions are as follows:

Definition 1 ([18])**.** Let $r_{u,v} : [0, 1] \to M$ be a curve on Riemannian manifold M such that $r_{u,v}(0) = v$ and $r_{u,v}(1) = u$. Then $r_{u,v}$ is said to possess the property (P) with respect to (w.r.t.) $v, u \in M$, if

$$r'_{u,v}(t)(s - t) = \eta(r_{u,v}(s), r_{u,v}(t)), \qquad \forall s, t \in [0, 1].$$

Definition 2 ([18])**.** Let M be a Riemannian manifold. A map $\eta : M \times M \to TM$ is called integrable if $\forall u, v \in M,\ \ \exists$ at least one curve $r_{u,v}$ which possesses the property (P) w.r.t. $v, u \in M$.

Remark 1 ([2])**.** Let M be a Riemannian manifold. A map $\eta : M \times M \to TM$ is called integrable, if

$$r'_{u,v}(0) = \eta(r_{u,v}(1), r_{u,v}(0)) = \eta(u, v).$$

In the case, if $r_{u,v}(s)$ is a geodesic, then the map satisfies **Condition C**, which is defined by Barani *et al.* [2] as follows:

$$\mathbf{C_1} \quad \eta(v, r_{u,v}(s)) = -s r_{u,v}(0) = -s P^s_{0,r_{u,v}}\left[r'_{u,v}(0)\right] = -s P^s_{0,r_{u,v}}[\eta(u, v)]$$

$$\mathbf{C_2} \quad \eta(r_{u,v}(1), r_{u,v}(s)) = (1 - s) r_{u,v}(0) = (1 - s) P^s_{0,r_{u,v}}\left[r'_{u,v}(0)\right] = (1 - s) P^s_{0,r_{u,v}}[\eta(u, v)].$$

Conditions $\mathbf{C_1}$ and $\mathbf{C_2}$ together comprise **Condition C**.

The strongly geodesic convex of order m defined by Akhlad *et al.* [11].

Definition 3 ([11])**.** A function $h : S \to R$ is called strongly geodesic convex of order m on geodesic convex set S, if $\exists\ \delta > 0$ such that

$$h(r_{u,v}(s)) \leq sh(u) + (1 - s)h(v) - \delta s(1 - s)\|r'{}_{u,v}(s)\|^m,$$

for every $u, v \in S$, $s \in [0, 1]$ and $r_{u,v}(s)$ is a geodesic.

The concept of geodesic invex sets was defined by Barani *et al.* [2], which is given as follow:

Definition 4 ([2])**.** Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a map such that for each $u, v \in M,\ \eta(u,v) \in T_vM$. A non empty set $S \subseteq M$ is called geodesic invex w.r.t. η if $\exists$ exactly one geodesic $r_{u,v} : [0,1] \to M$ such that

$$r_{u,v}(0) = v, \quad r'_{u,v}(0) = \eta(u,v), \quad r_{u,v}(s) \in S, \qquad \forall s \in [0,1],\ u, v \in S.$$

Definition 5 ([2])**.** A function $h : S \to R$ is called geodesic η-preinvex on geodesic invex set S, if $\forall u, v \in S, s \in [0,1]$, we have

$$h(r_{u,v}(s)) \le sh(u) + (1-s)h(v),$$

where $r_{u,v}(s)$ is the unique geodesic defined in Definition 4.

Definition 6 ([3])**.** Let M be a Riemannian manifold. A differentiable function $h : M \to R$ is called strongly η-invex of order 2 w.r.t. η, if $\exists\ \ \delta > 0$ such that

$$h(u) \ge h(v) + \mathrm{d}h_v(\eta(u,v)) + \delta \|\eta(u,v)\|_v^2, \qquad \forall u, v \in M.$$

3. Strongly geodesic preinvexity on Riemannian manifolds

In this section, we introduce strongly geodesic preinvexity on Riemannian manifold M and strongly η-invexity of functions of order m on Riemannian manifold M.

Definition 7. Let m be a positive integer. Let $S \subseteq M$ be a geodesic invex subset of Riemannian manifold M. A function $h : S \to R$ is called strongly geodesic preinvex of order m w.r.t. η, if $\exists \delta > 0$, such that $\forall u, v \in S$, $s \in [0,1]$, we have

$$h(r_{u,v}(s)) \le sh(u) + (1-s)h(v) - \delta s(1-s)\|\eta(u,v)\|_v^m,$$

where $r_{u,v}(s)$ is the unique geodesic defined in Definition 4.

Remark 2. If $\eta(u,v) = u - v,\ \forall u, v \in S$, then it reduces to strongly geodesic convexity of order m, see [11].

Remark 3. For $\delta = 0$, Definition 7 reduces to geodesic preinvex defined in [2].

Theorem 1. *If $h_1, h_2, \ldots, h_k$, are strongly geodesic preinvex functions of order m on geodesic invex set S, then $h = \sum_{j=1}^k a_j h_j$ and $H = \max_{1\le j\le k} h_j$ are also strongly geodesic preinvex functions of order m, where $a_j > 0,\ 1 \le j \le k$.*

Proof. Since $h_j, 1 \le j \le k$, are strongly geodesic preinvex functions of order m, for every $u, v \in S,\ s \in [0,1]$, we have

$$h_j(r_{u,v}(s)) \le sh_j(u) + (1-s)h_j(v) - \delta s(1-s)\|\eta(u,v)\|_v^m,\ 1 \le j \le k.$$

Now taking $\sum_{j=1}^k a_j$ on both sides, we get

$$\sum_{j=1}^k a_j h_j(r_{u,v}(s)) \le s\sum_{j=1}^k a_j h_j(u) + (1-s)\sum_{j=1}^k a_j h_j(v) - \sum_{j=1}^k a_j \delta s(1-s)\|\eta(u,v)\|_v^m,$$
$$h(r_{u,v}(s)) \le sh(u) + (1-s)h(v) - \delta' s(1-s)\|\eta(u,v)\|_v^m,$$

choose $\delta' = \sum_{j=1}^k a_j \delta > 0$. Hence, the function h is strongly geodesic preinvex of order m.

For other part, since $h_j, 1 \le j \le k$, are strongly geodesic preinvex functions of order m, we have

$$h_j(r_{u,v}(s)) \le sh_j(u) + (1-s)h_j(v) - \delta s(1-s)\|\eta(u,v)\|_v^m, \qquad \forall 1 \le j \le k.$$

Now taking $\max_{1\le j\le k} h_j$, we get

$$\max_{1\le j\le k} h_j(r_{u,v}(s)) \le s \max_{1\le j\le k} h_j(u) + (1-s) \max_{1\le j\le k} h_j(v) - \delta s(1-s)\|\eta(u,v)\|_v^m.$$
$$H(r_{u,v}(s)) \le sH(u) + (1-s)H(v) - \delta s(1-s)\|\eta(u,v)\|_v^m.$$

□

Theorem 2. *Let M be a complete Riemannian manifold, $S \subseteq M$ be a geodesic invex set w.r.t. η and $F\colon S\times S \to R$ be a continuous strongly geodesic preinvex function of order m w.r.t. (η,η), i.e., F is strongly geodesic preinvex function of order m to each variable. Then, the function $\Psi\colon S \to R$ defined by*

$$\Psi(u) = \inf_{v\in S} F(u,v),$$

is strongly geodesic preinvex function of order m w.r.t. η.

Proof. Let $u_0, u_1 \in S$ and $\epsilon > 0$ is given. Since S is geodesic invex set w.r.t. η, then there exist a geodesic $r_{u_0,u_1}\colon [0,1] \to M$ such that

$$r_{u_0,u_1}(0) = u_1,\ r'_{u_0,u_1}(0) = \eta(u_0,u_1),\ r_{u_0,u_1}(s) \in S, \qquad \forall s \in [0,1].$$

By definition of infimum, $\exists\ v_0, v_1 \in S$ such that

$$F(u_1,v_1) < \Psi(u_1) + \epsilon, F(u_0,v_0) < \Psi(u_0) + \epsilon.$$

By geodesic invexity of S w.r.t. η, $\exists$ a geodesic $t_{v_0,v_1}\colon [0,1] \to M$ such that

$$t_{v_0,v_1}(0) = v_1,\ t'_{v_0,v_1}(0) = \eta(v_0,v_1),\ t_{v_0,v_1}(s) \in S, \qquad \forall s \in [0,1].$$

Clearly, the curve $\alpha_{(u_0,v_0),(u_1,v_1)} = (r_{u_0,u_1}, t_{u_0,u_1})\colon [0,1] \to M\times M$ is geodesic in $S\times S$ with $\alpha_{(u_0,v_0),(u_1,v_1)}(0) = (u_1,v_1)$ such that for every $s\in[0,1]$, we have

$$\alpha_{(u_0,v_0),(u_1,v_1)}(s) = (r_{u_0,u_1}(s), t_{v_0,v_1}(s)) \in S\times S$$

and

$$\begin{aligned}\alpha'_{(u_0,u_1),(v_0,v_1)}(0) &= \left(r'_{u_0,u_1}(0), t'_{v_0,v_1}(0)\right)\\ &= (\eta(u_0,u_1), \eta(v_0,v_1))\\ &= \eta_0((u_0,v_0),(u_1,v_1)),\end{aligned}$$

where the map $\eta_0\colon (M\times M)\times(M\times M) \to TM\times TM$. By definition of infimum and the strongly geodesic preinvexity of order m of F, we have

$$\begin{aligned}\Psi(r_{u_0,u_1}(s)) &= \inf_{v\in S} F(r_{u_0,u_1}(s), v)\\ &\le F(r_{u_0,u_1}(s), t_{v_0,v_1}(s))\\ &\le sF(u_0,v_0) + (1-s)F(u_1,v_1) - \delta s(1-s)\|\eta_0((u_0,v_0),(u_1,v_1))\|^m_{(u_1,v_1)}\\ &= sF(u_0,v_0) + (1-s)F(u_1,v_1) - \delta s(1-s)\|(\eta(u_0,u_1),\eta(v_0,v_1))\|^m_{(u_1,v_1)}\\ &= sF(u_0,v_0) + (1-s)F(u_1,v_1) - \delta s(1-s)\left\{\|\eta(u_0,u_1)\|^m_{u_1} + \|\eta(v_0,v_1)\|^m_{v_1}\right\}\\ &\le s(\Psi(u_0)+\epsilon) + (1-s)(\Psi(u_1)+\epsilon) - \delta s(1-s)\|\eta(u_0,u_1)\|^m_{u_1}\\ &= (s\Psi(u_0) + (1-s)\Psi(u_1)) + \epsilon - \delta s(1-s)\|\eta(u_0,u_1)\|^m_{u_1}\\ &\le s\Psi(u_0) + (1-s)\Psi(u_1) - \delta s(1-s)\|\eta(u_0,u_1)\|^m_{u_1}.\end{aligned}$$

Therefore, the function $\Psi(u) = \inf_{v\in S} F(u,v)$ is strongly geodesic preinvex of order m w.r.t. η. □

Now, we define strongly invex function of order m w.r.t. η.

Definition 8. Let m be a positive integer. Let M be a Riemannian manifold. A differentiable function $h : M \to R$ is called strongly η-invex of order m w.r.t. η, if $\exists\ \ \delta > 0$ such that

$$h(u) \geq h(v) + \mathrm{d}h_v^T(\eta(u,v)) + \delta \|\eta(u,v)\|_v^m, \qquad \forall u, v \in M,$$

where $\mathrm{d}h_v^T$ is the transpose of the differential $\mathrm{d}h_v$ of the map h at v.

Remark 4. For $m = 2$, then Definition 8 reduce to strongly η-invex of order 2 w.r.t. η defined by [3].

In the following example, we show the existence of strongly η-invex function of order m.

Example 1. Let $h : M \to R$ be a differentiable function on Riemannian manifold M such that $\forall\ v \in M$, $\mathrm{d}h_v \neq 0$. A map $\eta : M \times M \to TM$ is defined by

$$\eta(u,v) = \frac{h(u) - h(v) - \delta \|\eta(u,v)\|_v^m}{\|\mathrm{d}h_v\|_v^2} \mathrm{d}h_v, \qquad \forall u, v \in M,\ \delta > 0.$$

Since M is Riemannian manifold, then for every $u, v \in M$, we get

$$\begin{aligned}
\langle \mathrm{d}h_v, \eta(u,v) \rangle_v &= \left\langle \mathrm{d}h_v, \frac{h(u) - h(v) - \delta \|\eta(u,v)\|_v^m}{\|\mathrm{d}h_v\|_v^2} \mathrm{d}h_v \right\rangle_v \\
&= h(u) - h(v) - \delta \|\eta(u,v)\|_v^m \frac{\langle \mathrm{d}h_v, \mathrm{d}h_v \rangle_v}{\|\mathrm{d}h_v\|^2}, \\
&= h(u) - h(v) - \delta \|\eta(u,v)\|_v^m. \\
h(u) &= h(v) + \langle \mathrm{d}h_v, \eta(u,v) \rangle_v + \delta \|\eta(u,v)\|_v^m.
\end{aligned}$$

Hence, h is strongly η-invex function of order m.

Example 2. Let $M = \{(u_1, u_2) \in R^2 : u_1, u_2 > 0\}$ be a Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle = \langle Q(u)x, y \rangle$ for any $x, y \in T_u(M)$, where $Q(u) = (g_{ij}(u))$ defines a 2×2 matrix given by $g_{ij}(u) = \frac{\delta_{ij}}{u_i u_j}$. The geodesic curve $r : \mathbb{R} \to M$ satisfying $r(0) = u = (u_1, u_2) \in M$ and $r'(0) = x = (x_1, x_2) \in T_u(M) = \mathbb{R}^2$ is given by

$$r(t) = \left(u_1 e^{\frac{x_1}{u_1} t},\ u_2 e^{\frac{x_2}{u_2} t}\right).$$

For any $u = (u_1, u_2) \in M$ and any $x = (x_1, x_2) \in T_u(M)$, the map $\exp_u : T_u(M) \to M$ is given by

$$\exp_u(x) = r(1) = \left(u_1 e^{\frac{x_1}{u_1}},\ u_2 e^{\frac{x_2}{u_2}}\right).$$

Define a function $h : M \to R$, for any $u = (u_1, u_2) \in M$, by

$$h(u_1, u_2) = u_1 + u_2^2.$$

Then, h is strongly η-invex of any order w.r.t η (see Def. 8), where $\eta : M \times M \to TM$ is a map defined, for any $u = (u_1, u_2), v = (v_1, v_2) \in M$, by

$$\begin{aligned}
&\eta((u_1, u_2), (v_1, v_2)) = (-1 - v_1, -v_2), \\
\text{and} \quad &\mathrm{d}h_v^T(\eta(u,v)) = \frac{\mathrm{d}}{\mathrm{d}t} h(\exp_v(t \cdot \eta(u,v)))|_{t=0} = -\left(1 + v_1 + 2v_2^2\right).
\end{aligned}$$

However, h is not strongly geodesic preinvex function of any order. For this, let $u = \left(\frac{1}{4}, \frac{1}{4}\right)$, $v = \left(\frac{1}{9}, \frac{1}{9}\right)$ and $s = \frac{1}{10}$, then for any $\delta > 0$ and any $m \geq 1$, it is easy to see the following holds:

$$h(r_{u,v}(s)) \nleq sh(u) + (1-s)h(v) - \delta s(1-s)\|\eta(u,v)\|_v^m,$$

where the geodesic segment joining $u = (u_1, u_2)$ and $v = (v_1, v_2)$ is given by

$$r_{(u,v)}(s) = \left(u_1\left(\frac{v_1}{u_1}\right)^s, u_2\left(\frac{v_2}{u_2}\right)^s\right).$$

In the following theorem, we show that differentiable strongly geodesic preinvex function is strongly η-invex.

Theorem 3. *Let $S \subseteq M$ be an open geodesic invex set and $h : S \to R$ be continuously differentiable function. If h is strongly geodesic preinvex of order m w.r.t. η, then h is strongly η-invex of order m w.r.t. η.*

Proof. Assume h is strongly geodesic preinvex of order m w.r.t. η, for every $u, v \in S$, then there exists exactly one geodesic $r_{u,v} : [0,1] \to M$ such that

$$r_{u,v}(0) = v,\ r'_{u,v}(0) = \eta(u,v),\ r_{u,v}(s) \in S, \qquad \forall s \in [0,1],$$

and

$$h(r_{u,v}(s)) \leq sh(u) + (1-s)h(v) - \delta s(1-s)\|\eta(u,v)\|_v^m,$$
$$h(r_{u,v}(s)) - h(v) \leq s\{h(u) - h(v)\} - \delta s(1-s)\|\eta(u,v)\|_v^m.$$

Since h is differentiable, dividing by s on both side and taking $s \to 0$, we get

$$\mathrm{d}h^T_{r_{u,v}(0)}(r'_{u,v}(0)) \leq h(u) - h(v) - \delta\|\eta(u,v)\|_v^m$$

or

$$h(u) \geq h(v) + \mathrm{d}h_v^T(\eta(u,v)) + \delta\|\eta(u,v)\|_v^m.$$

□

Remark 5. The converse of Theorem 3 is not true in general as is evident from Example 2. However, its converse holds when η satisfies **Condition C** as follows next.

Theorem 4. *Let $S \subseteq M$ be an open geodesic invex set w.r.t. η. Assume $h : S \to R$ is continuously differentiable function. The function h is strongly η-invex of order m w.r.t. η and η satisfies* **Condition C** *if and only if h is strongly geodesic preinvex of order m w.r.t. η.*

Proof. By Theorem 3, then the function h is strongly η-invex of order m w.r.t. η.

Conversely, suppose h is strongly η-invex of order m w.r.t. η on open geodesic invex set S w.r.t. η *i.e.*, for every $u, v \in S$ $\exists$ a exactly one curve $r_{u,v} : (0,1) \to M$ such that

$$r_{u,v}(0) = v,\ r'_{u,v}(0) = \eta(u,v),\ r_{u,v}(s) \in S, \qquad \forall s \in (0,1).$$

Fixed $s \in (0,1)$ and setting $\bar{v} = r_{u,v}(s)$. Then, we have

$$h(u) \geq h(\bar{v}) + \mathrm{d}h_{\bar{v}}^T(\eta(u,\bar{v})) + \delta\|\eta(u,\bar{v})\|^m \tag{3.1}$$
$$h(v) \geq h(\bar{v}) + \mathrm{d}h_{\bar{v}}^T(\eta(v,\bar{v})) + \delta\|\eta(v,\bar{v})\|^m. \tag{3.2}$$

By multiplying s in (3.1) and $(1-s)$ in (3.2) respectively, adding and applying

$$\mathrm{d}h_{\bar{v}}^T(s\eta(u,\bar{v}) + (1-s)\eta(v,\bar{v})) = \mathrm{d}h_{\bar{v}}\Big(s(1-s)P^s_{0,r_{u,v}}[\eta(u,v)] + (1-s)(-s)P^s_{0,r_{u,v}}[\eta(u,v)]\Big)$$

$$= \mathrm{d}h_{\bar{v}}(0)$$
$$= 0,$$

we have

$$h(\bar{v}) + \delta s\left\|(1-s)P^s_{0,r_{u,v}}[\eta(u,v)]\right\|_v^m + \delta(1-s)\left\|(-s)P^s_{0,r_{u,v}}[\eta(u,v)]\right\|_v^m \leq sh(u) + (1-s)h(v)$$
$$h(\bar{v}) + \delta s(1-s)^m\left\|P^s_{0,r_{u,v}}\eta(u,v)\right\|_v^m + \delta(1-s)s^m\left\|P^s_{0,r_{u,v}}\eta(u,v)\right\|_v^m \leq sh(u) + (1-s)h(v)$$
$$h(\bar{v}) + \delta s(1-s)\big[(1-s)^{m-1} + s^{m-1}\big]\left\|P^s_{0,r_{u,v}}\eta(u,v)\right\|_v^m \leq sh(u) + (1-s)h(v). \tag{3.3}$$

Since $P^s_{0,r_{u,v}}\eta(u,v) = \eta(u,v)$, then (3.3) become

$$h(\bar{v}) + \delta s(1-s)\big[(1-s)^{m-1} + s^{m-1}\big]\|\eta(u,v)\|_v^m \leq sh(u) + (1-s)h(v). \tag{3.4}$$

Case i. $0 < m \leq 2$, then $(1-s) + s = 1 \leq (1-s)^{m-1} + s^{m-1}$.
Case ii. $m > 2$, then the real valued function $\phi(s) = s^{m-1}$ is convex on $(0,1)$, thus we have

$$\left(\frac{1}{2}\right)^{m-2} \leq (1-s)^{m-1} + s^{m-1}.$$

It follows that (3.4), $\exists\ \delta' > 0$, which is independent from u, v, s such that

$$h(r_{u,v}(s)) \leq sh(u) + (1-s)h(v) - \delta' s(1-s)\|\eta(u,v)\|_v^m, \qquad \forall u,v \in S,\ s \in (0,1).$$

Hence, the function h is strongly geodesic preinvex function of order m.

□

Now, we generalize strongly η-invex function of order m as follows:

Definition 9. Let m be a positive integer. Let $h : M \to R$ be a differentiable function on Riemannian manifold M. Then, the function h is called:

(i) Strongly pseudo η-invex type 1 of order m, if $\exists\ \ \delta > 0$ such that

$$\mathrm{d}h_v^T(\eta(u,v)) \geq 0 \implies h(u) \geq h(v) + \delta\|\eta(u,v)\|_v^m, \qquad \forall u,v \in M,$$

or equivalently,

$$h(u) < h(v) + \delta\|\eta(u,v)\|_v^m \implies \mathrm{d}h_v^T\eta(u,v) < 0.$$

(ii) Strongly pseudo η-invex type 2 of order m, if $\exists\ \ \delta > 0$ such that

$$\mathrm{d}h_v^T(\eta(u,v)) + \delta\|\eta(u,v)\|_v^m \geq 0 \implies h(u) \geq h(v), \qquad \forall u,v \in M.$$

In the following example, we show the existence of strongly pseudo η-invex function of order m.

Example 3 (Manifold (or Cone) of symmetric positive definite matrices)**.** The collection, S^n_{++}, of $n \times n$ symmetric positive definite matrices with real entries forms a Hadamard manifold with Riemannian metric:

$$g_A(X,Y) = Tr\big(A^{-1}XA^{-1}Y\big), \qquad \forall A \in S^n_{++}, \quad X,Y \in T_P\big(S^n_{++}\big).$$

The minimal geodesic joining $A, B \in S^n_{++}$ is given by

$$\gamma(t) = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}, \qquad \forall t \in [0,1].$$

For more details, one can refer to [20].

Define a function $h : S^n_{++} \to R$, for any $A \in S^n_{++}$, by

$$h(A) = -(\ln(\det(A)))^2.$$

Then, h is strongly pseudo η-invex type 1 of any order w.r.t η, where $\eta : S^n_{++} \times S^n_{++} \to TM$ is a map defined, for any $A, B \in S^n_{++}$, by

$$\eta(A, B) = B^{\frac{1}{2}}(\ln B)B^{\frac{1}{2}},$$

where ln is the matrix logarithmic function,

$$\text{and} \quad \mathrm{d}h^T_B(\eta(A, B)) = \frac{\mathrm{d}}{\mathrm{d}t} h(\exp_B(t \cdot \eta(A, B)))|_{t=0} = -2(\ln(\det(B)))^2,$$

with the exponential map $\exp_B : T_B(S^n_{++}) \to M$ given by

$$\exp_B(W) = B^{\frac{1}{2}} \exp\Big(B^{\frac{-1}{2}} W B^{\frac{-1}{2}}\Big) B^{\frac{1}{2}}, \quad W \in T_B\big(S^n_{++}\big),$$

where exp is the matrix exponential function.

However, h is not strongly η-invex function of any order. For this, let $n = 2,\ A = 2I_2\ , B = I_2$, then it is easy to see the following

$$h(A) \ngeq h(B) + \mathrm{d}h^T_B(\eta(A, B)) + \delta\|\eta(A, B)\|^m_B,$$

where I_2 is a 2×2 Identity matrix.

Definition 10. Let m be a positive integer. Let $h : M \to R$ be a differentiable function on Riemannian manifold M.Then, the function h is called:

(i) Strongly quasi η-invex type 1 order of m, if $\exists\ \ \delta > 0$ such that

$$h(u) \leq h(v) \implies \mathrm{d}h^T_v(\eta(u, v)) + \delta\|\eta(u, v)\|^m_v \leq 0, \qquad \forall\ u, v \in M.$$

(ii) Strongly quasi η-invex type 2 order of m, if $\exists \delta > 0$ such that

$$h(u) \leq h(v) + \delta\|\eta(u, v)\|^m_v \implies \mathrm{d}h^T_v \eta(u, v) \leq 0, \qquad \forall u, v \in M.$$

In the following example, we show the existence of strongly quasi η-invex function of order m.

Example 4. Let $M = \big\{(u_1, u_2) \in R^2 : u_1, u_2 > 0\big\}$ be a Riemannian manifold with Riemannian metric as defined in Example 2.

Define a function $h : M \to R$, for any $u = (u_1, u_2) \in M$, by

$$h(u_1, u_2) = u_1^3 + \ln(u_2).$$

Then, h is strongly quasi η-invex type 1 of any order w.r.t η, where $\eta : M \times M \to TM$ is a map defined, for any $u = (u_1, u_2), v = (v_1, v_2) \in M$, by

$$\eta((u_1, u_2), (v_1, v_2)) = \big(-v_1^2, -v_2^2\big),$$

$$\text{and} \quad \mathrm{d}h^T_v(\eta(u, v)) = \frac{\mathrm{d}}{\mathrm{d}t} h(\exp_v(t \cdot \eta(u, v)))|_{t=0} = -3v_1^4 - v_2 \leq 0.$$

However, h is not strongly η-invex function of any order. For this, let $u = \big(1, \frac{1}{e^5}\big)\ , v = (1, 1)$, then it is easy to see the following

$$h(u) \ngeq h(v) + \mathrm{d}h^T_v(\eta(u, v)) + \delta\|\eta(u, v)\|^m_v.$$

4. Strongly invariant η-monotone on Riemannian manifold

The monotonicity of vector field on Riemannian manifold defined by Nemeth [16] as follows:

Definition 11 ([16]). Let X be a vector field on Riemannian manifold M. Then, X is called monotone on M if $\forall u, v \in M$, we get

$$\left\langle r'_{u,v}(0), P^0_{1,r_{u,v}}[X(u) - X(v)]\right\rangle_v \geq 0,$$

where $r_{u,v}$ is a geodesic joining u and v.

Barani *et al.* [3] generalized it and defined invariant monotonicity on Riemannian manifold M. Later Iqbal *et al.* [11] extended the notion of invariant monotonicity to strongly invariant α-monotonicity on Riemannian manifold M. Motivated by Iqbal *et al.* [11], we extend it as follows:

Definition 12. Let m be a positive integer. Let M be a Riemannian manifold. A vector field X on M is called:

(i) strongly invariant η-monotone of order m if $\exists\ \delta > 0$ such that

$$\langle X(v), \eta(u,v)\rangle_v + \langle X(u), \eta(v,u)\rangle_u \leq -\delta\{\|\eta(u,v)\|^m_v + \|\eta(v,u)\|^m_u\}.$$

For $m = 2$, it reduces to strongly invariant monotone defined by Barani *et al.* [3].

(ii) strongly invariant pseudo η-monotone of order m if $\exists\ \delta > 0$ such that

$$\langle X(u), \eta(v,u)\rangle_u \geq 0 \implies \langle X(v), \eta(u,v)\rangle_v \leq -\delta\|\eta(u,v)\|^m_v.$$

For $m = 2$, it reduces to strongly invariant pseudo monotone defined by Barani *et al.* [3].

In the support of Definition 12, we give the following example.

Example 5. Let $h : M \to R$ be a differentiable function on Riemannian manifold M such that $\forall v \in M$, $\mathrm{d}h_v \neq 0$. A map $\eta : M \times M \to TM$ is defined by

$$\eta(u,v) = -\frac{\|\eta(u,v)\|^m_v}{\|\mathrm{d}h_v\|^2_v}\mathrm{d}h_v, \qquad \forall u, v \in M.$$

Given M is Riemannian manifold, then for every $u, v \in M$, we have

$$\begin{aligned}\langle \mathrm{d}h_v, \eta(u,v)\rangle_v &= \left\langle \mathrm{d}h_v, \frac{-\|\eta(u,v)\|^m_v}{\|\mathrm{d}h_v\|^2_v}\mathrm{d}h_v\right\rangle_v\\ &= -\|\eta(u,v)\|^m_v\frac{\langle \mathrm{d}h_v, \mathrm{d}h_v\rangle_v}{\|\mathrm{d}h_v\|^2},\\ &= -\|\eta(u,v)\|^m_v,\\ \langle \mathrm{d}h_v, \eta(u,v)\rangle_v &= -\|\eta(u,v)\|^m_v. \end{aligned} \tag{4.1}$$

Similarly, we have

$$\langle \mathrm{d}h_u, \eta(v,u)\rangle_u = -\|\eta(v,u)\|^m_u. \tag{4.2}$$

By adding (4.1) and (4.2), we get

$$\langle \mathrm{d}h_v, \eta(u,v)\rangle_v + \langle \mathrm{d}h_u, \eta(v,u)\rangle_u = -\{\|\eta(u,v)\|^m_v + \|\eta(v,u)\|^m_u\}.$$

Hence, $\mathrm{d}h_v$ is strongly invariant η-monotone vector field of order m.

In the next theorems, we discuss a relationship between strongly η-invex of order m and strongly invariant η-monotone of order m.

Theorem 5. *Let $h : M \to R$ be a differentiable function on M. Suppose h is strongly η-invex of order m. Then,* dh *is strongly invariant η-monotone of order m.*

Proof. Let h be a strongly η-invex of order m on M. Then,

$$h(u) - h(v) \geq \mathrm{d}h_v(\eta(u,v)) + \delta\|\eta(u,v)\|_v^m, \qquad \forall u, v \in M, \tag{4.3}$$
$$h(v) - h(u) \geq \mathrm{d}h_u(\eta(v,u)) + \delta\|\eta(v,u)\|_u^m, \qquad \forall u, v \in M. \tag{4.4}$$

Adding (4.3) and (4.4), we get

$$\mathrm{d}h_v(\eta(u,v)) + \mathrm{d}h_u(\eta(v,u)) + \delta\{\|\eta(u,v)\|_v^m + \|\eta(v,u)\|_u^m\} \leq 0,$$

or

$$\mathrm{d}h_v(\eta(u,v)) + \mathrm{d}h_u(\eta(v,u)) \leq -\delta\{\|\eta(u,v)\|_v^m + \|\eta(v,u)\|_u^m\}, \qquad \forall u, v \in M.$$

Thus, dh is strongly invariant η-monotone of order m. □

Theorem 6. *Let $h : M \to R$ be a differentiable function on geodesically complete Riemannian manifold M. If a map $\eta : M \times M \to TM$ is integrable and* dh *is strongly invariant η-monotone of order m. Then, h is strongly η-invex of order m.*

Proof. Since M is a geodesically complete Riemannian manifold, $\forall u, v \in M$, then there exists a geodesic $r_{u,v} : [0,1] \to M$ such that $r_{u,v}(0) = v, r_{u,v}(1) = u$. Let $w = r_{u,v}(\frac{1}{2})$, then by the mean value theorem, $\exists\ s_1, s_2 \in (0,1)$ such that $0 < s_2 < \frac{1}{2} < s_1 < 1$, we get

$$h(u) - h(w) = \frac{1}{2}\mathrm{d}h_x\big(r'_{u,v}(s_1)\big), \tag{4.5}$$
$$h(w) - h(v) = \frac{1}{2}\mathrm{d}h_y\big(r'_{u,v}(s_2)\big), \tag{4.6}$$

where $x = r_{u,v}(s_1), y = r_{u,v}(s_2)$. Since d$h$ is strongly invariant η-monotone of order m, then we get

$$\mathrm{d}h_x^T(\eta(v,x)) + \mathrm{d}h_v^T(\eta(x,v)) \leq -\delta\{\|\eta(v,x)\|_x^m + \|\eta(x,v)\|_v^m\}. \tag{4.7}$$

Since η is integrable, we get

$$\eta(v, r_{u,v}(s_1)) = -s_1 P_{0,r_{u,v}}^{s_1}[\eta(u,v)] \tag{4.8}$$

and

$$\eta(v, r_{u,v}(s_1)) = s_1\eta(u,v). \tag{4.9}$$

Using (4.8) and (4.9) in (4.7), we get

$$\mathrm{d}h_x^T\Big(-s_1 P_{0,r_{u,v}}^{s_1}[\eta(u,v)]\Big) + \mathrm{d}h_v^T(s_1\eta(u,v)) \leq -\delta\Big\{\Big\|-s_1 P_{0,r_{u,v}}^{s_1}[\eta(u,v)]\Big\|_x^m + \|s_1\eta(u,v)\|_v^m\Big\}$$

or

$$-\mathrm{d}h_x^T\Big(P_{0,r_{u,v}}^{s_1}\eta(u,v)\Big) + \mathrm{d}h_v^T(\eta(u,v)) \leq -2s_1^{m-1}\delta\|\eta(u,v)\|_v^m.$$

From **Condition C**, $P_{0,r_{u,v}}^{s_1}[\eta(u,v)] = r'_{u,v}(s_1)$, then we have

$$\frac{1}{2}\mathrm{d}h_x^T\big(r'_{u,v}(s_1)\big) \geq \frac{1}{2}\mathrm{d}h_v^T(\eta(u,v)) + s_1^{m-1}\delta\|\eta(u,v)\|_v^m, \tag{4.10}$$

similarly, we have

$$\frac{1}{2}\mathrm{d}h_y^T\big(r'_{u,v}(s_2)\big) \geq \frac{1}{2}\mathrm{d}h_v^T(\eta(u,v)) + s_2^{m-1}\delta\|\eta(u,v)\|_v^m. \tag{4.11}$$

Hence, (4.5) and (4.6) become

$$h(u) - h(w) \geq \frac{1}{2}\mathrm{d}h_v^T(\eta(u,v)) + s_1^{m-1}\delta\|\eta(u,v)\|_v^m, \tag{4.12}$$

$$h(w) - h(v) \geq \frac{1}{2}\mathrm{d}h_v^T(\eta(u,v)) + s_2^{m-1}\delta\|\eta(u,v)\|_v^m. \tag{4.13}$$

Adding (4.12) and (4.12), we get

$$\begin{aligned} h(u) - h(v) &\geq \mathrm{d}h_v^T(\eta(u,v)) + \delta\big\{s_1^{m-1} + s_2^{m-1}\big\}\|\eta(u,v)\|_v^m \\ &= \mathrm{d}h_v^T(\eta(u,v)) + \delta'\|\eta(u,v)\|_v^m, \end{aligned}$$

where $\delta' = \delta\big(s_1^{m-1} + s_2^{m-1}\big) > 0$. □

Theorem 7. *Let M be a geodesically complete Riemannian manifold and $h : M \to R$ be a differentiable function. If a map $\eta : M \times M \to TM$ is integrable and $\mathrm{d}h$ is strongly invariant pseudo η-monotone of order m. Then, h is strongly pseudo η-invex of order m.*

Proof. Since M is geodesically complete Riemannian manifold, then $\forall u, v \in M$, $\exists$ a geodesic $r_{u,v} : [0,1] \to M$ such that $r_{u,v}(0) = v,\ r_{u,v}(1) = u$. Let $w = r_{u,v}(\frac{1}{2})$, then from the mean value theorem, $\exists s_1, s_2 \in (0,1)$ such that $0 < s_2 < \frac{1}{2} < s_1 < 1$, we get

$$h(u) - h(w) = \frac{1}{2}\mathrm{d}h_x^T\big(r'_{u,v}(s_1)\big), \tag{4.14}$$

$$h(w) - h(v) = \frac{1}{2}\mathrm{d}h_y^T\big(r'_{u,v}(s_2)\big), \tag{4.15}$$

where $x = r_{u,v}(s_1), y = r_{u,v}(s_2)$. Using the property (P), (4.14) can be written as

$$h(u) - h(w) = -\frac{1}{2s_1}\mathrm{d}h_x^T(\eta(v,x)) \tag{4.16}$$

and

$$h(w) - h(v) = -\frac{1}{2s_2}\mathrm{d}h_y^T(\eta(v,y)). \tag{4.17}$$

Assume that

$$\mathrm{d}h_v^T(\eta(u,v)) \geq 0, \tag{4.18}$$

using property (P) in (4.18), we get

$$0 \leq \mathrm{d}h_v^T(\eta(u,v)) = \frac{1}{s_1}\mathrm{d}h_v^T(\eta(v,x)) = \frac{1}{s_2}\mathrm{d}h_v^T(\eta(v,y)). \tag{4.19}$$

Since $\mathrm{d}h$ is strongly invariant pseudo η-monotone of order m, we have

$$\begin{aligned} \mathrm{d}h_x^T(\eta(v,x)) &\leq -\delta\|\eta(v,x)\|_x^m \\ &= -\delta\Big\|-s_1 P_{0,r_{u,v}}^{s_1}[\eta(u,v)]\Big\|_x^m \\ &= -\delta s_1^m\Big\|P_{0,r_{u,v}}^{s_1}[\eta(u,v)]\Big\|_x^m \\ &= -\delta s_1^m\|\eta(u,v)\|_v^m, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} \mathrm{d}h_y^T(\eta(v,y)) &\leq -\delta\|\eta(v,y)\|_y^m \\ &= -\delta\left\|-s_2 P_{0,r_{u,v}}^{s_2}[\eta(u,v)]\right\|_y^m \\ &= -\delta s_2^m \left\|P_{0,r_{u,v}}^{s_2}[\eta(u,v)]\right\|_y^m \\ &= -\delta s_2^m \|\eta(u,v)\|_v^m. \end{aligned} \tag{4.21}$$

Using (4.20) and (4.21) in (4.16) and (4.17) respectively, we get

$$h(u) - h(w) \geq \frac{\delta s_1^{m-1}}{2}\|\eta(u,v)\|_v^m \tag{4.22}$$

where $\frac{\delta s_1^{m-1}}{2} > 0$,

$$h(w) - h(v) \geq \frac{\delta s_2^{m-1}}{2}\|\eta(u,v)\|_v^m, \tag{4.23}$$

where $\frac{\delta s_2^{m-1}}{2} > 0$. Adding (4.22) and (4.23), we get

$$h(u) - h(v) \geq \delta'\|\eta(u,v)\|_v^m$$

or

$$h(u) \geq h(v) + \delta'\|\eta(u,v)\|_v^m,$$

where $\delta' = \frac{\delta\left(s_1^{m-1}+s_2^{m-1}\right)}{2} > 0$. □

5. Vector variational-like inequality problem on Riemannian manifold

In this section, we consider the multi-objective optimization problem as an application for strongly η-invex functions of order m, known as strongly η-invex multi-objective optimization problem, which generalizes the results obtained by Iqbal *et al.* [11].

Suppose $H = (h_1, h_2, \ldots, h_k)$, where $h_i : M \to 2^{TM}$ are set valued vector fields on M. The vector variational-like inequality problem is to find a solution $u^* \in M$, and $X \in H(u^*)$ such that

$$\langle X, \eta(u,u^*)\rangle \nless 0, \qquad \forall u \in M,$$

where

$$\langle X, \eta(u,u^*)\rangle = \langle X_1, \eta(u,u^*)\rangle, \langle X_2, \eta(u,u^*)\rangle, \ldots, \langle X_k, \eta(u,u^*)\rangle.$$

The multi-objective optimization problem is to find a strict η-minimizers of order m for:

$$\text{Min } H(u) = (h_1(u), h_2(u), \ldots, h_k(u)), u \in M.$$

Motivated by Iqbal *et al.* [11] and Bhatia *et al.* [5], we define a local strict η-minimizers of order m with respect to a nonlinear function on Riemannian manifold for multi-objective optimization problem.

Definition 13. A point $u^* \in M$ is a local strict minimizers if $\exists\, \epsilon > 0$ such that $h(u) \nless h(u^*)\ \forall\, u \in B(u^*, \epsilon) \cap M$, *i.e.*, there exist no $u \in B(u^*, \epsilon) \cap M$, such that $h(u) < h(u^*)$.

Definition 14. Let m be a positive integer. A point $u^* \in M$ is a local strict η-minimizers of order m, if $\exists\ \epsilon > 0$ and $\delta > 0$ such that

$$h(u) \nless h(u^*) + \delta \|\eta(u, u^*)\|^m, \qquad \forall u \in B(u^*, \epsilon) \cap M.$$

The local strict η-minimizers change to the strict η-minimizers if an open ball $B(u^*, \epsilon)$ is replaced by Riemannian manifold M.

Definition 15. Let m be a positive integer. A point $u^* \in M$ is a strict η-minimizers of order m, if $\exists \delta > 0$ such that

$$h(u) \nless h(u^*) + \delta \|\eta(u, u^*)\|^m, \qquad \forall u \in M.$$

In next Theorem, we show an important characterization of a solution of the vector variational like-inequality problem and a strict η-minimizers of order m for multi-objective optimization problem.

Theorem 8. *Let $h_i, 1 \leq i \leq k$, be a strongly η-invex functions of order m on M. Then, $u^* \in M$ is a solution of the vector variational like-inequality problem $\iff$ u^* is a strict η-minimizers of order m for multi-objective optimization problem.*

Proof. Assume u^* is a solution of the vector variational like-inequality problem but u^* is not a strict η-minimizers of order m for multi-objective optimization problem. Then, for all $\delta > 0$, $\exists$ some $\bar{u} \in M$, such that

$$h(\bar{u}) < h(u^*) + \delta \|\eta(\bar{u}, u^*)\|^m,$$

i.e.,

$$h_i(\bar{u}) < h_i(u^*) + \delta \|\eta(\bar{u}, u^*)\|^m. \tag{5.1}$$

Since $h_i,\ 1 \leq i \leq k$, are strongly η-invex functions of order m, then equation (5.1) yields

$$\langle X_i, \eta(\bar{u}, u^*)\rangle < 0, \qquad \forall X_i \in T_i(u^*) = \nabla h_i(u^*),\ 1 \leq i \leq k,$$

i.e.,

$$\langle X, \eta(\bar{u}, u^*)\rangle < 0, \qquad \forall X \in T(u^*), \bar{u} \in M,$$

which contradicts that u^* is a solution to the vector variational like-inequality problem. Hence, u^* is a strict η-minimizers of order m for multi-objective optimization problem.

Conversely, assume u^* is a strict η- minimizers of order m for multi-objective optimization problem, but u^* is not a solution to the vector variational like-inequality problem. Then, there exists $\bar{u} \in M$ such that

$$\langle X_i, \eta(\bar{u}, u^*)\rangle < 0, \qquad \forall X_i \in T_i(u^*) = \nabla h_i(u^*),\ 1 \leq i \leq k.$$

By Definition 8, we get

$$h_i(\bar{u}) < h_i(u^*) + \delta \|\eta(\bar{u}, u^*)\|^m,\ 1 \leq i \leq k,$$

i.e.,

$$h(\bar{u}) < h(u^*) + \delta \|\eta(\bar{u}, u^*)\|^m,$$

which contradicts u^* is a strict η-minimizers of order m. Thus, u^* is a solution to the vector variational like-inequality problem. □

6. Conclusion

This paper introduces the concepts of strongly geodesic preinvexity of order m with respect to η on geodesic invex sets, strongly η-invex functions of order m, and strongly invariant η-monotonicity of order m on Riemannian manifolds. The definitions presented in this paper are supported by non-trivial examples, illustrating their applicability. Furthermore, various interesting properties related to these concepts are discussed. An important application of strongly η-invex functions of order m is demonstrated in the context of multi-objective optimization problems. The paper also establishes the characterization of strict η-minimizers of order m for multi-objective optimization problems and their connection to solutions of the vector variational-like inequality problem. The results presented in this paper generalize and extend the existing knowledge and findings from different authors in the field.

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