ON CHARACTERIZATIONS OF SOLUTION SETS OF INTERVAL-VALUED QUASICONVEX PROGRAMMING PROBLEMS

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Abstract. In this article, we study several characterizations of solution sets of $LU$-quasiconvex interval-valued function. Firstly, we provide Gordan’s theorem of the alternative of interval-valued linear system. As a consequence of this theorem, we find the normalized gradient of the interval-valued function is constant over the solution set when its gradient is not zero. Further, we discuss Lagrange multiplier characterizations of solution sets of $LU$-quasiconvex interval-valued function and provide optimality conditions of interval-valued optimization problem under the generalized Mangasarian-Frolovitz constraint qualifications. We provide illustrative examples in the support of our theory.

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1. Introduction

Characterizations of solution sets for different type of nonlinear optimization problems have attracted the attention of many researchers over the years. Mangasarian [18] attained several characterizations of solution sets of smooth convex functions. Burke and Ferris [3] generalized these results for nonsmooth proper convex functions with the help of Fenchel subdifferential. Further, Jeyakumar and Yang [9] extended the results for pseudolinear programming problems.


Keywords. Quasiconvex functions, interval-valued optimization problem, KKT optimality conditions.

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order optimality conditions. Sisarat et al. [27] discussed the characterizations of approximate solutions of convex vector optimization problems.

The real-life optimization problems are associated with uncertain data that occur due to measurement errors. Interval-valued optimization is one of the significant tools to handle uncertainty. Moore [22] gave a detailed explanation of interval analysis. Neumaier [23] provided the development of interval analysis as a tool for computation and computer-favored proofs. Wu [31] obtained the Karush Kuhn Tucker (KKT) optimality conditions for interval-valued optimization problems and derived Wolfe duality and strong duality theorems for interval-valued optimization in [32]. Further, Wu [33] obtained KKT optimality condition in multiobjective interval-valued optimization and proposed Pareto optimal solution for multiobjective optimization problems with interval-valued objective function. Stefanini and Bede introduced the concept of gH-differentiability for interval-valued functions. Chalco-Cano et al. presented the fundamental theorem of Calculus for interval-valued function. Bedregal and Santiago [2] presented some continuity notions for interval functions. They provided the relationship between the three interpretations of intervals (as a set, as an information and as a number) and the topological counterparts. Lai et al. [13] proposed duality results for interval-valued semiinfinite optimization problems with equilibrium constraints using convexificators. Further, Lai et al. [14] introduced stationary conditions and characterizations of solution sets for interval-valued tightened nonlinear problems. Recently, several researches have been done in the field of interval-valued optimization, see for instance [12,15–17,26].

To the best of our knowledge, there are very few articles related to characterizations of solution sets of interval-valued optimization problems. Recently, Treanta [30] established some characterizations of solution sets of interval-valued optimization problems and discussed the relation between LU-optimal solutions of the interval-valued variational control problem and saddle points related to the interval-valued Lagrange functional.

Motivated by Mangasarian [18] and Ivanov [7], we consider interval-valued optimization problem (IVOP) with \( H \)-differentiable \( LU \)-quasiconvex functions and establish several characterizations of solution set of IVOP. Inspired by Gordan [5] and Rohn and Kreslova [25], we consider Gordan’s theorem of alternative for interval linear matrix system of inequalities and with the help of this, we show that normalized gradient of interval-valued \( LU \)-quasiconvex function is non-zero constant when the gradient of the interval-valued function is not equal to zero. Further, we establish the Lagrange multiplier characterizations of solution set with interval-valued \( LU \)-quasiconvex continuously \( H \)-differentiable objective function and inequality constraints with the help of Generalized Mangasarian-Fromovitz constraint qualification (GMFCQ), which is the generalization of a significant constraint qualification named as Mangasarian-Fromovitz constraint qualification (MFCQ) [20]. We construct some solution sets for IVOP and characterize them with the help of KKT optimality conditions for interval-valued objective function given by Wu [33].

The layout of our article is as follows: In Section 2, some basic and essential results and definitions are provided. In Section 3, we characterize the solution sets of interval-valued continuously differentiable \( LU \)-quasiconvex problem. In Section 4, we establish Lagrange multiplier characterizations of solution sets for IVOP and Section 5 is devoted to concluding remarks and future research opportunities.

2. Preliminaries

2.1. Interval analysis

We collect some basic concepts and essential definitions related to interval-valued functions for bounded intervals from Moore [22].

We denote by \( \mathcal{I}(\mathbb{R}) \) the class of all closed intervals in \( \mathbb{R} \). Let \( U = [u^L, u^U] \), where \( u^L \) and \( u^U \) denotes the lower and upper bounds of \( U \), respectively. Let \( U = [u^L, u^U] \) and \( V = [v^L, v^U] \) be in \( \mathcal{I}(\mathbb{R}) \), then, we have

(i) \( U + V = \{ u + v : u \in U, v \in V \} = [u^L + v^L, u^U + v^U] \),
(ii) \( -U = \{-u : u \in U\} = [-u^U, -u^L] \),
(iii) \( U - V = U + (-V) = [u^L - v^U, u^U - v^L] \),
(iv) \( tU = \{ tu : u \in U \} = \begin{cases} [tu^L, tu^U] & \text{if } t \geq 0 \\ [tu^U, tu^L] & \text{for } t < 0 \end{cases} \)

where \( t \) is a real number.

We refer to Moore [22], for further details of interval analysis. We have collected the definition and properties of Hausdorff metric from Wu [33].

Suppose that \( X \subseteq \mathbb{R}^n \) and \( Y \subseteq \mathbb{R}^n \), then the Hausdorff metric between \( X \) and \( Y \) is denoted and defined by

\[
d_H(X, Y) = \max \left\{ \sup_{u \in X} \inf_{v \in Y} \| u - v \|, \sup_{v \in Y} \inf_{u \in X} \| u - v \| \right\},
\]

where \( \| \cdot \| \) is the Euclidean norm.

Let \( U = [u^L, u^U] \) and \( V = [v^L, v^U] \) be two closed intervals, then it is easy to prove that

\[
d_H(U, V) = \max\{|u^L - v^L|, |u^U - v^U|\}.
\]

Let \( \{U_n = [u^L_n, u^U_n]\} \) and \( U \) be closed intervals in \( \mathbb{R} \), then the sequence of closed interval \( \{U_n\} \) converges to \( U \), if for every \( \epsilon > 0 \), there exists a natural number \( N > 0 \) such that, for \( n > N \), we have \( d_H(U_n, U) < \epsilon \). Wu [31] proved that

\[
\lim_{n \to \infty} U_n = U \quad \text{if and only if} \quad \lim_{n \to \infty} u^L_n = u^L \quad \text{and} \quad \lim_{n \to \infty} u^U_n = u^U.
\]

A function \( f : \mathbb{R}^n \to \mathcal{I}(\mathbb{R}) \) is called interval-valued function, this means \( f(u) = f(u_1, \ldots, u_n) \) is a closed interval in \( \mathbb{R} \) for each \( u \in \mathbb{R}^n \). \( f \) can be written as \( f(u) = [f^L(u), f^U(u)] \), where \( f^L \) and \( f^U \) are two real valued functions defined on \( \mathbb{R}^n \) such that \( f^L(u) \leq f^U(u), \forall u \in \mathbb{R}^n \).

Wu [31] discussed limit and continuity of interval-valued functions. Let \( f \) be an interval-valued function defined on \( \mathbb{R}^n \) and \( U = [u^L, u^U] \) be an interval in \( \mathbb{R} \), we say

\[
\lim_{u \to a} f(u) = U, \quad \text{if and only if} \quad \lim_{u \to a} f^L(u) = u^L \quad \text{and} \quad \lim_{u \to a} f^U(u) = u^U.
\]

The interval-valued function \( f \) defined on \( \mathbb{R}^n \) is said to be continuous at \( a \in \mathbb{R}^n \) if

\[
\lim_{u \to a} f(u) = f(a).
\]

**Proposition 2.1.** [33] Suppose \( f \) is an interval-valued function defined on \( \mathbb{R}^n \), then \( f \) is continuous at \( a \in \mathbb{R}^n \) if and only if \( f^L \) and \( f^U \) are continuous at \( a \).

**Definition 2.2.** [33] Suppose \( K \) is an open set in \( \mathbb{R} \). The interval-valued function \( f : K \to \mathcal{I}(\mathbb{R}) \) with \( f(u) = [f^L(u), f^U(u)] \) is called weakly differentiable at \( u^0 \) if the real valued functions \( f^L \) and \( f^U \) are differentiable at \( u^0 \) (in the ordinary sense).

For \( U, V \in \mathcal{I}(\mathbb{R}) \), if there exists a \( W \in \mathcal{I}(\mathbb{R}) \) such that \( U = V + W \), then \( W \) is called the Hukuhara difference of \( U \) and \( V \). Also, \( W \) can be written as \( W = U \ominus V \), considering the Hukuhara difference [1] \( W \) exists, which means that \( u^L - v^L \leq u^U - v^U \) and \( W = [u^L - v^L, u^U - v^U] \).

**Definition 2.3.** [33] Suppose \( K \) is an open set in \( \mathbb{R} \). The interval-valued function \( f : K \to \mathcal{I}(\mathbb{R}) \) is called \( H \)–differentiable at \( u^0 \) if there exists a closed interval \( U(u^0) \in \mathcal{I}(\mathbb{R}) \) such that the limits

\[
\lim_{h \to 0^+} \frac{f(u^0 + h) \ominus f(u^0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(u^0) \ominus f(u^0 - h)}{h}
\]

both exist and equal to \( U(u^0) \), which is called the \( H \)– derivative of \( f \) at \( u^0 \).
2.2. Solution concepts

Suppose \( U = [u^L, u^V] \) and \( V = [v^L, v^V] \) are two closed intervals in \( \mathbb{R} \). We write \( U \preceq_{LU} V \) if and only if \( u^L \leq v^L \) and \( u^V \leq v^V \).

Consider multiobjective programming problem with multiple interval-valued objective functions

\[
(IVOP1) \quad \min f(u) = (f_1(u), \ldots, f_p(u))
\]

subject to \( u = (u_1, \ldots, u_n) \in K \subseteq \mathbb{R}^n \),

where each \( f_k(u) = [f^L_k(u), f^U_k(u)] \) is an interval-valued function for \( k = 1, \ldots, p \).

We write \( U \prec_{LU} V \) if and only if \( U \preceq_{LU} V \) and \( U \neq V \). We say \( U = (U_1, \ldots, U_p) \) is an interval-valued vector if each component \( U_k = [u^L_k, u^U_k] \) is closed interval for \( k = 1, \ldots, p \). Suppose \( U = (U_1, \ldots, U_p) \) and \( V = (V_1, \ldots, V_p) \) are two interval-valued vectors. We write \( U \preceq_{LU} V \) if and only if \( U_k \preceq_{LU} V_k \) for all \( k = 1, \ldots, p \), and \( U \prec_{LU} V \) if and only if \( U_k \preceq_{LU} V_k \) for at least one \( q \). Suppose \( u^* \) is a feasible solution of \((IVOP1)\), then \( f(u^*) \) is an interval-valued vector. The concepts of Pareto optimal (efficient) solution is given below.

**Definition 2.4.** [31] Suppose \( u^0 \) is a feasible solution to the problem \((IVOP1)\).

(i) \( u^0 \) is said to be an efficient solution to the problem \((IVOP1)\) if there exists no \( \bar{u} \) such that \( f(\bar{u}) \prec_{LU} f(u^0) \).

(ii) \( u^0 \) is said to be a strong efficient solution to the problem \((IVOP1)\) if there exists no \( \bar{u} \) such that \( f(\bar{u}) \preceq_{LU} f(u^0) \).

(iii) \( u^0 \) is said to be a weak efficient solution to the problem \((IVOP1)\) if there exists no \( \bar{u} \) such that \( f_k(\bar{u}) \prec_{LU} f_k(u^0) \) \( \forall k = 1, \ldots, p \).

**Definition 2.5.** [31] Suppose \( u^0 \) is feasible solution of the problem \((IVOP1)\). \( u^0 \) is said to be a local weak efficient solution of the problem \((IVOP1)\), if there exists a neighborhood \( N \) of \( u^0 \) such that for all \( \bar{u} \in K \cap N \), then the following cannot be satisfied for any \( k = 1, \ldots, p \)

\[
f_k(\bar{u}) \prec_{LU} f_k(u^0).
\]

2.3. Interval-valued convex functions and generalized convex functions

Wu [31] introduced the concept of convexity for interval-valued functions.

**Definition 2.6.** [31] Suppose \( f \) is an interval-valued function defined on a convex set \( X \subseteq \mathbb{R}^n \). Then \( f \) is said to be \( LU \)-convex at \( \bar{u} \) if

\[
f(\lambda \bar{u} + (1 - \lambda)u) \preceq_{LU} \lambda f(\bar{u}) + (1 - \lambda)f(u) \quad \forall \lambda \in (0, 1) \text{ and } \forall u \in X.
\]

**Proposition 2.7.** [31] Suppose \( f(u) = [f^L(u), f^U(u)] \) is an interval-valued function defined on \( X \subseteq \mathbb{R}^n \) then \( f \) is called \( LU \)-convex at \( \bar{u} \) if and only if \( f^L \) and \( f^U \) are convex at \( \bar{u} \).

We consider a continuous and \( H \)-differentiable interval-valued function \( f(u) = [f^L(u), f^U(u)] \) on an open convex set \( X \). We define \( LU \)-quasiconvex and \( LU \)-pseudoconvex function motivated by Wu [31] and Zhang et al. [34].

**Definition 2.8.** Let \( f : X \rightarrow \mathcal{I}(\mathbb{R}) \) be a continuous and \( H \)-differentiable interval-valued function on an open convex set \( X \subseteq \mathbb{R}^n \). \( f \) is said to be a \( LU \)-quasiconvex function if

\[
f^L(u) \leq f^L(v) \quad \text{and} \quad f^U(u) \leq f^U(v)
\]

\[
\implies (\nabla f^L(v) + \nabla f^U(v), u - v) \leq 0 \quad \forall \ u, v, \in X,
\]

where \( \nabla \) denotes the gradient operator.
Lemma 3.1. Suppose a solution \( v \). Then either each matrix system (1) has a solution \( u \). We consider a family of systems of linear inequalities written, in short, as system (1).

3. Characterizations of solution sets of interval-valued optimization problems

We introduce Gordan’s theorem of the alternative of interval-valued linear system motivated by [5] and [25] to obtain the characterizations of solution sets.

3.1. Gordan’s theorem of the alternative of interval-valued linear system

Consider an interval linear system of inequalities

\[
\begin{align*}
a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n &> 0 \\
a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n &> 0 \\
&\vdots \\
a_{r1}u_1 + a_{r2}u_2 + \cdots + a_{rn}u_n &> 0.
\end{align*}
\]

In short, \( A_x x > 0_x \), (1)

where \( A_x = \{ A : A^L \leq A \leq A^U \} \) (componentwise inequalities) is an \( r \times n \) interval matrix and \( 0_x \) is an interval \( r \)-vector, \( a_{ij} (i = 1, 2, \ldots, r, j = 1, 2, \ldots, n) \) and \( u_j (j = 1, 2, \ldots, n) \) are intervals.

\( A u > 0 \forall A \in A_x, 0 \in 0_x \). (2)

Then either each matrix system (1) has a solution \( u \) or the system,

\[
A_x^T v = 0, \ v \geq 0, \ v \neq 0,
\]

has a solution \( v \) but never both.

Lemma 3.1. Suppose \( X \subseteq \mathbb{R}^n \) is an open and convex set, \( K \subseteq X \) is a convex subset and \( u, v \in K, d \in \mathbb{R}^n \). Let the function \( f : X \rightarrow \mathcal{I}(\mathbb{R}) \) be \( LU \)-quasiconvex and \( \nabla f^L(u) + \nabla f^U(u) \neq 0, \ \nabla f^L(v) + \nabla f^U(v) \neq 0 \), then

\[
\langle \nabla f^L(u) + \nabla f^U(u), d \rangle < 0 \implies \langle \nabla f^L(v) + \nabla f^U(v), d \rangle \leq 0.
\]
Proof. On contrary, suppose that there exists $d \in \mathbb{R}^n$ with
\[
\langle \nabla f^L(u) + \nabla f^U(u), d \rangle < 0 \quad \text{and} \quad \langle \nabla f^L(v) + \nabla f^U(v), -d \rangle < 0.
\]
It follows that there exists $\lambda > 0$ such that
\[
f^L(p) < f^L(u) = f^L(v) \quad \text{and} \quad f^U(p) < f^U(u) = f^U(v),
\]
\[
f^L(q) < f^L(v) = f^L(u) \quad \text{and} \quad f^U(q) < f^U(v) = f^U(u),
\]
where $p = u + \lambda d \in X$, $q = v - \lambda d \in X$. Suppose $w = \frac{p + q}{2}$, hence $w = \frac{u + v}{2}$.

Since $f$ is $LU$-quasiconvex, we have from Lemma 2.12, $K$ is convex and $f^L(w) = f^L(u) = f^L(v)$ and $f^U(w) = f^U(u) = f^U(v)$.

As $f$ is $LU$-quasiconvex, so from (3) and (4),
\[
f^L(w) \leq \max\{f^L(p), f^L(q)\} < f^L(u) = f^L(w)
\]
\[
f^U(w) \leq \max\{f^U(p), f^U(q)\} < f^U(u) = f^U(w),
\]
which is contradictory. This completes the proof. \hfill \Box

**Lemma 3.2.** Let $u, v \in \mathcal{I}(\mathbb{R}^n); u \neq 0, v \neq 0$. Suppose
\[
\langle u, d \rangle < 0, \quad d \in \mathbb{R}^n \implies \langle v, d \rangle \leq 0,
\]
where $u = [u^L, u^U]$ and $v = [v^L, v^U]$. Then there exists $a > 0$ such that $v = au$.

*Proof.* The equivalent system of (7) is to claim that the system
\[
\langle v, d \rangle > 0, \quad \langle -u, d \rangle > 0
\]
has not a solution $d$. Then, it follows from Gordan’s theorem of the alternative of interval-valued linear system, there exist real numbers $a_1$ and $a_2$ such that
\[
va_1 = 0 \quad \text{and} \quad ua_2 = 0 \quad \text{or} \quad va_1 - ua_2 = 0, \quad a_1 \geq 0, a_2 \geq 0, (a_1, a_2) \neq (0, 0).
\]
Without loss of generality, let $a_1 > 0$ and $a = \frac{a_2}{a_1}$ then $a > 0$ and satisfies $v = au$. \hfill \Box

**Lemma 3.3.** Suppose $X \subseteq \mathbb{R}^n$ is an open and convex set, $K \subseteq X$ is a convex subset. Let the function $f : X \rightarrow \mathcal{I}(\mathbb{R})$ be $LU$-quasiconvex and $f^L(u)$ and $f^U(u)$ be continuously differentiable functions, then the interval-valued normalized gradient is constant over the nonempty set $\{u \in K : \nabla f^L(u) + \nabla f^U(u) \neq 0\}$.

*Proof.* Suppose $u$ and $v$ are two distinct points of the set $K$ with $\nabla f^L(u) + \nabla f^U(u) \neq 0$ and $\nabla f^L(v) + \nabla f^U(v) \neq 0$, then from Lemmas 3.1 and 3.2, there exists $a > 0$ with condition
\[
\nabla f^L(v) + \nabla f^U(v) = a[\nabla f^L(u) + \nabla f^U(u)].
\]
Then we get
\[
\frac{\nabla f^L(v) + \nabla f^U(v)}{\|\nabla f^L(v) + \nabla f^U(v)\|} = a \frac{\nabla f^L(u) + \nabla f^U(u)}{\|\nabla f^L(u) + \nabla f^U(u)\|}
\]
\[
\Rightarrow \quad \frac{\nabla f^L(v) + \nabla f^U(v)}{\|\nabla f^L(v) + \nabla f^U(v)\|} = \frac{\nabla f^L(u) + \nabla f^U(u)}{\|\nabla f^L(u) + \nabla f^U(u)\|}.
\]
Then, we can get the claim immediately. \hfill \Box
Lemma 3.4. Suppose $X \subseteq \mathbb{R}^n$ is an open and convex set, $K \subseteq X$ is a convex subset. Let the function $f : X \to T(\mathbb{R})$ be LU-quasiconvex and $f^L(u)$ and $f^U(u)$ be continuously differentiable functions then either of the following conditions holds but not both

1. $\nabla f^L(u) + \nabla f^U(u) \neq 0$ for all $u \in K$ and the normalized gradient
   $\frac{\|\nabla f^L(u) + \nabla f^U(u)\|}{\nabla f^L(u) + \nabla f^U(u)}$ is constant over the set $K$;
2. $\nabla f^L(u) + \nabla f^U(u) = 0$ for all $u \in K$.

Proof. We consider the two possible cases:

Case (1): Let $u \in K$, $\nabla f^L(u) + \nabla f^U(u) \neq 0$. We prove for another arbitrary point $v \in K$, $\nabla f^L(v) + \nabla f^U(v) \neq 0$. On contrary suppose that $\nabla f^L(v) + \nabla f^U(v) = 0$.

Consider the sets

$$P := \{\lambda \in [0, 1] : \nabla f^L(u + \lambda(v - u)) + \nabla f^U(u + \lambda(v - u)) = 0\},$$

$$Q := \{\lambda \in [0, 1] : \nabla f^L(u + \lambda(v - u)) + \nabla f^U(u + \lambda(v - u)) \neq 0\}.$$

Since $K$ is convex, so $u + \lambda(v - u) \in K$, let $u + \lambda(v - u) = w$.

Consider the open sets (intervals) in the interval $[0, 1]$ which is of the type $(u, v)$ such that $0 < u < v < 1$, $(u, 1]$ such that $0 < u < 1$, $[0, v)$ such that $0 < v < 1$ and their unions.

We claim that $P$ is closed. Consider a sequence $\{\lambda_n\}$, where $\lambda_n \in P$ and $\lambda_n \to \lambda_0$.

Since $f^L$ and $f^U$ are continuously differentiable, then we have

$$\nabla f^L(u + \lambda_0(v - u)) + \nabla f^U(u + \lambda_0(v - u)) = \lim_{n \to \infty} \nabla f^L(u + \lambda_n(v - u)) + \nabla f^U(u + \lambda_n(v - u)) = 0.$$

Hence, $\lambda_0 \in P$ and $P$ is a closed set.

Now, we claim that $Q$ is closed set. Consider a sequence $\{\lambda_n\}$ where $\lambda_n \in Q$ and $\lambda_n \to \lambda_0$.

From Lemma 3.3, there exists a vector $a \neq 0$ such that

$$\frac{\nabla f^L(w) + \nabla f^U(w)}{\|\nabla f^L(w) + \nabla f^U(w)\|} = a \ \forall w \in K.$$

As the normalized gradient is constant, therefore

$$\lim_{n \to \infty} \frac{\nabla f^L(u + \lambda_n(v - u)) + \nabla f^U(u + \lambda_n(v - u))}{\|\nabla f^L(u + \lambda_n(v - u)) + \nabla f^U(u + \lambda_n(v - u))\|} = a \neq 0.$$

Since $f^L$ and $f^U$ are continuously differentiable, then we have

$$\frac{\nabla f^L(u + \lambda_0(v - u)) + \nabla f^U(u + \lambda_0(v - u))}{\|\nabla f^L(u + \lambda_0(v - u)) + \nabla f^U(u + \lambda_0(v - u))\|} = \lim_{n \to \infty} \frac{\nabla f^L(u + \lambda_n(v - u)) + \nabla f^U(u + \lambda_n(v - u))}{\|\nabla f^L(u + \lambda_n(v - u)) + \nabla f^U(u + \lambda_n(v - u))\|} = a.$$

Therefore, $\|\nabla f^L(u + \lambda_0(v - u)) + \nabla f^U(u + \lambda_0(v - u))\| \neq 0$, this implies $\lambda_0 \in Q$ and $Q$ is closed.

Since $P \cup Q = [0, 1]$, then either $P = [0, 1]$, $Q = \emptyset$ or $Q = [0, 1]$, $P = \emptyset$, but both cases are not possible. This contradicts the assumption $\nabla f^L(v) + \nabla f^U(v) = 0$. Therefore, $\nabla f^L(v) + \nabla f^U(v) \neq 0 \ \forall v \in K$. Then by Lemma 3.3, the normalized gradient is non-zero constant over $K$.

Case (2): Let $\nabla f^L(u) + \nabla f^U(u) = 0$. From the proof of Case (1), there is impossibility of the assumption $\nabla f^L(u) + \nabla f^U(u) \neq 0$ for arbitrary $v \in K$. Therefore $\nabla f^L(u) + \nabla f^U(u) = 0, \ \forall v \in K$. This completes the proof. □
Example 3.5. Consider an interval-valued $LU$-quasiconvex function $f : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ defined by $f(u) = [u^3 - 1, 2(1 - u^2)]$ then $f^L(u) = u^3 - 1$ and $f^U(u) = 2(1 - u^2)$.
\[ \nabla f^L(u) + \nabla f^U(u) = -2x \] and $||\nabla f^L(u) + \nabla f^U(u)|| = 2x$.

Then the normalized gradient $\frac{\nabla f^L(u) + \nabla f^U(u)}{||\nabla f^L(u) + \nabla f^U(u)||} = -1$ for any $v \in \mathbb{R}$ where $\nabla f^L(v) + \nabla f^U(v) \neq 0$.

Suppose $\bar{u} \in \mathcal{K}$. We denote the two sets by $\hat{K}_1$ and $\hat{K}_2$ and define by
\[
\hat{K}_1 := \{u \in K : \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} = 0, \frac{\nabla f^L(u) + \nabla f^U(u)}{||\nabla f^L(u) + \nabla f^U(u)||} = \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), \nabla f^L(u) + \nabla f^U(u) \neq 0\};
\]
\[
\hat{K}_2 := \{u \in K : \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \leq 0, \frac{\nabla f^L(u) + \nabla f^U(u)}{||\nabla f^L(u) + \nabla f^U(u)||} = \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), \nabla f^L(u) + \nabla f^U(u) \neq 0\}.
\]

Theorem 3.6. Let $K \subseteq X$ be a convex subset of open and convex set $X \subseteq \mathbb{R}^n$. Let the interval-valued function $f : X \to \mathcal{I}(\mathbb{R})$ be $LU$-quasiconvex and $f^L$ and $f^U$ be continuously differentiable. Suppose $\bar{u} \in \mathcal{K}$ is a solution of (IVOP1) and $\nabla f^L(u) + \nabla f^U(u) \neq 0$, then $\bar{K} = \hat{K}_1 = \hat{K}_2$.

Proof. First we prove that $\mathcal{K} \subseteq \hat{K}_1$. Let $u \in \mathcal{K}$, since $\mathcal{K}$ is convex, $\bar{u} \in \mathcal{K}$, and we have $\bar{u} + \lambda(u - \bar{u}) \in \mathcal{K}$ for all $\lambda \in [0, 1]$, hence
\[
f^L(\bar{u} + \lambda(u - \bar{u})) = f^L(\bar{u}) \quad \forall \lambda \in [0, 1]
\]
and
\[
f^L(\bar{u} + \lambda(u - \bar{u})) = f^L(\bar{u}) \quad \forall \lambda \in [0, 1].
\]
From (9) and (10), we get
\[
\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle = 0.
\]
Then from Lemma 3.3, $\nabla f^L(u) + \nabla f^U(u) \neq 0$, and $u \in \hat{K}_1$. Hence $\mathcal{K} \subseteq \hat{K}_1$.

It is obvious to show that $\hat{K}_1 \subseteq \hat{K}_2$.

Now, we prove that $\hat{K}_2 \subseteq \mathcal{K}$. Suppose $u \in \hat{K}_2$. We show that $u \in \mathcal{K}$. On contrary suppose that $u \notin \mathcal{K}$, then $f^L(\bar{u}) < f^L(u)$ and $f^U(\bar{u}) < f^U(u)$.

Since $\mathcal{K}$ is convex, we have $\bar{u} + \lambda(u - \bar{u}) \in \mathcal{K}$ for all $\lambda \in [0, 1]$. As $K \subseteq \mathcal{K}$, we get that
\[
f^L(\bar{u} + \lambda(u - \bar{u})) \geq f^L(\bar{u})
\]
and
\[
f^L(\bar{u} + \lambda(u - \bar{u})) \geq f^L(\bar{u}).
\]

Hence, $\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle \geq 0$ and from the assumption $u \in \hat{K}_2$, it follows that
\[
\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle = 0.
\]

Since $f$ is continuous and $f^L(\bar{u}) < f^L(u)$ and $f^U(\bar{u}) < f^U(u)$, there exists a number $t > 0$ such that
\[
f^L(\bar{u} + t(\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}))) < f^L(u)
\]
and
\[
f^L(\bar{u} + t(\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}))) < f^L(u).
\]

Then, from Definition 2.8,
\[
\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} + t(\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})) \rangle \leq 0.
\]

Then, from $\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle = 0$, we have $||\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})||^2 \leq 0$, which contradicts the assumption $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \neq 0$. This completes the theorem. \qed
Example 3.7. Consider an interval-valued optimization problem (IVOP1) \( \min f(x) \), where \( f : \mathbb{R}^2 \to \mathcal{I}(\mathbb{R}) \) defined by

\[
f(u_1, u_2) = \left[-u_2 - 1, \frac{2(u_2 + 1)}{u_1}\right]
\]

then

\[
\nabla f^L(u) = \left(\frac{u_2 + 1}{u_1^4}, -\frac{1}{u_1}\right) \quad \text{and} \quad \nabla f^U(u) = \left(\frac{u_2 + 1}{u_1^4}, -\frac{1}{u_1}\right).
\]

Consider a set \( K = \{u = (u_1, u_2) : 1 \leq u_1 \leq 3, 0 \leq u_2 < u_1\} \). Then \( f \) is \( L_U \)-quasiconvex on the set \( X = \{u = (u_1, u_2) : u_1 > 0, -\infty < u_2 < \infty\} \).

Since \( f^L(u) \) and \( f^U(u) \) are quasiconvex on the set \( X \) and it satisfies \( f^L(u) \leq f^L(v) \) and \( f^U(u) \leq f^U(v) \) \( \implies \langle \nabla f^L(v) + \nabla f^U(v), u - v \rangle \leq 0 \ \forall \ u, v \in X \).

The solution set is

\[
K = \{u = (u_1, u_2) : 1 \leq u_1 \leq 3, u_2 = -1\}.
\]

Suppose \( \bar{u} = (1, -1) \) is the given solution. Now

\[
\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle = u_2 + 1,
\]

\[
\frac{\nabla f^L(u) + \nabla f^U(u)}{\|\nabla f^L(u) + \nabla f^U(u)\|} = \frac{\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})}{\|\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})\|} = (0, 1) \quad \text{provided} \ u_2 = -1.
\]

Hence, \( \bar{K}_1 = \{u = (u_1, u_2) : 1 \leq u_1 \leq 3, u_2 = -1\} \), which is same as \( K \).

Theorem 3.8. Let \( K \subseteq X \) be a convex subset of an open and convex set \( X \subseteq \mathbb{R}^n \). Let the interval-valued function \( f : X \to \mathcal{I}(\mathbb{R}) \) be \( L_U \)-quasiconvex and \( f^L \) and \( f^U \) be continuously differentiable. Suppose \( \bar{u} \in K \) is a solution of (IVOP1) and \( \nabla f^L(u) + \nabla f^U(u) \neq 0 \), then \( \overline{K} \subseteq \bar{K} \), where \( \bar{K} = \{u \in K : \nabla f^L(u) + \nabla f^U(u) = 0\} \).

In addition to that if \( f \) is \( L_U \)-pseudoconvex on \( K \), then \( \overline{K} = \bar{K} \).

Proof. From the second alternative of Lemma 3.3, it can be easily seen that \( \overline{K} \subseteq \bar{K} \). Hence, \( \nabla f^L(u) + \nabla f^U(u) = 0 \ \forall u \in \overline{K} \).

Now, we suppose that \( f \) is \( L_U \)-pseudoconvex function on \( K \). We show that \( \overline{K} = \bar{K} \). Since every \( L_U \)-pseudoconvex function is \( L_U \)-quasiconvex, so we have to prove only \( K \subseteq \bar{K} \).

On contrary, suppose that \( \exists u \in \bar{K} \) but \( u \notin \overline{K} \). Hence, \( f^L(\bar{u}) < f^L(u) \) and \( f^U(\bar{u}) < f^U(u) \).

By the assumption of \( L_U \)-pseudoconvexity,

\[
\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle < 0.
\]

Which contradicts the assumption \( \nabla f^L(u) + \nabla f^U(u) = 0 \). \( \square \)
Example 3.9. Consider an interval-valued $LU$-quasiconvex function $f : \mathbb{R} \rightarrow \mathcal{I}(\mathbb{R})$ defined by $f(u) = [u^3, 2u^3]$ then $f^L(u) = u^3$ and $f^U(u) = 2u^3$. Then $\nabla f^L(u) + \nabla f^U(u) = 9u^2$.

Consider $u \in \overline{K}$, $u = 0$ is a known solution and $\langle \nabla f^U(\bar{u}), u - \bar{u} \rangle = 0$, hence $\overline{K} \subseteq \tilde{K}$.

We construct the following sets to discuss the characterizations of solution sets.

$$K_1 := \{ u \in K : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle = 0, \nabla f^L(u) + \nabla f^U(u) \neq 0 \},$$

$$K_2 := \{ u \in K : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle \geq 0, \nabla f^L(u) + \nabla f^U(u) \neq 0 \},$$

$$K_3 := \{ u \in K : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle = \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle, \nabla f^L(u) + \nabla f^U(u) \neq 0 \},$$

$$K_4 := \{ u \in K : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle \geq \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle, \nabla f^L(u) + \nabla f^U(u) \neq 0 \},$$

$$K_5 := \{ u \in K : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle = \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle = 0, \nabla f^L(u) + \nabla f^U(u) \neq 0 \}.$$

Theorem 3.10. Let $K \subseteq X$ be a convex subset of open and convex set $X \subseteq \mathbb{R}^n$. Let the interval-valued function $f : X \rightarrow \mathcal{I}(\mathbb{R})$ be $LU$-quasiconvex and $f^L$ and $f^U$ be continuously differentiable. Suppose $\bar{u} \in \overline{K}$ is a solution of (IVOP1) such that $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) = 0$, then $\overline{K} = K_1 = K_2 = K_3 = K_4 = K_5$.

Proof. It can be easily seen that $K_5 \subseteq K_3 \subseteq K_4$ and $K_5 \subseteq K_1 \subseteq K_2$.

First we prove that $\overline{K} \subseteq K_5$. Let $u \in \overline{K}$, then $f^L(u) = f^L(\bar{u})$ and $f^U(u) = f^U(\bar{u})$. Since $f$ is $LU$-quasiconvex, the solution set $\overline{K}$ is convex and

$$f^L[\bar{u} + \lambda(u - \bar{u})] = f^L(\bar{u}) \text{ and } f^U[\bar{u} + \lambda(u - \bar{u})] = f^U(\bar{u}) \forall \lambda \in [0, 1].$$

Hence, $\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle = 0$. Interchanging the role of $\bar{u}$ and $u$, and by similar arguments, we can easily show that $\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle = 0$. From Lemma 3.4, we get $\nabla f^L(u) + \nabla f^U(u) \neq 0$. Therefore, $u \in K_5$.

Next, we prove that $K_4 \subseteq K_2$. Suppose $u \in K_4$, hence

$$\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle \geq \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle. \tag{11}$$

Since $K$ is convex, $u, \bar{u} \in K$, $\bar{u} \in \overline{K}$, then

$$f^L[\bar{u} + \lambda(u - \bar{u})] \geq f^L(\bar{u}) \text{ and } f^U[\bar{u} + \lambda(u - \bar{u})] \geq f^U(\bar{u}) \forall \lambda \in [0, 1].$$

Hence, $\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle \geq 0$.

This implies from (11) that $x \in K_2$.

Now, we show that $K_2 \subseteq \overline{K}$. Suppose $u \in K_2$, we have to show that $u \in \overline{K}$. On contrary, suppose that $x \notin \overline{K}$, then $f^L(u) < f^L(\bar{u})$ and $f^U(u) < f^U(\bar{u})$. From quasiconvexity of $f$, it follows that $\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle \leq 0$. Since $u \in K_2$, we have $\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle = 0$. Then, it follows from the continuity of $f$ and $f^L(\bar{u}) < f^L(u)$.
and \( f^U(\bar{u}) < f^U(u) \), there exists \( t > 0 \) with \( f^L[\bar{u} + t(\nabla f^L(u) + \nabla f^U(u))] < f^L(u) \) and \( f^U[\bar{u} + t(\nabla f^L(u) + \nabla f^U(u))] < f^U(u) \).

Then, from quasiconvexity of \( f \),
\[
\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} + t(\nabla f^L(u) + \nabla f^U(u)) - u \rangle \leq 0.
\]
From \( \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle = 0 \), we have \( \nabla f^L(u) + \nabla f^U(u) = 0 \), this contradicts the assumption \( u \in K_2 \).

This completes the theorem. \( \square \)

4. **Lagrange multiplier characterizations of the solution sets of interval-valued optimization problems**

\[
(IVOP2) \quad \min f(u) \quad \text{subject to } g_i(u) \leq 0, \ i = 1, 2, \cdots, m \quad x \geq 0,
\]

where \( f : X \subseteq \mathbb{R}^n \rightarrow \mathcal{I}(\mathbb{R}) \) is an interval-valued function and \( g_i : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, \ i = 1, 2, \cdots, m \) are real valued functions. Let \( M \) be a convex subset of \( X \), here \( M \) be not necessarily open.

Let \( I(u) := \{ i \in \{ 1, 2, \cdots, m \} : g_i(u) = 0 \} \) be index set of the active constraints. Suppose \( K := \{ u \in M : g_i(u) \leq 0, \ i = 1, 2, \cdots, m \} \) is the feasible set.

Let \( C \subseteq \mathbb{R}^n \) be a cone. Then the negative polar cone of \( C \) is defined as:
\[
C^- := \{ c \in \mathbb{R}^n : \langle c, u \rangle \leq 0 \ \forall \ u \in C \},
\]
which is also called normal cone and denoted as \( N_M(u) \).

**Definition 4.1.** [20] MFCQ is said to be satisfy at point \( \bar{u} \in K \) if \( \nabla g_i(\bar{u}), \ i \in I(\bar{u}) \) are linearly independent and there exists a point \( d \in \mathbb{R}^n \) such that
\[
\langle \nabla g_i(\bar{u}), d \rangle = 0, \ i \in I = \{ 1, 2, \cdots, l \}, \ 0 \leq l \leq m, \ \langle \nabla g_i(\bar{u}), d \rangle < 0, \ i \in I(\bar{u}).
\]

**Definition 4.2.** [7] Generalized Mangasarian-Fromovitz constraint qualification (GMFCQ) is said to be satisfy at \( \bar{u} \) if and only if there is a direction \( d \in (N_M(\bar{u}))^- \) such that \( \langle \nabla g_i(\bar{u}), d \rangle < 0, \ i \in I(\bar{u}) \).

Motivated by Wu [32] and Ivanov [7], we establish the Karush Kuhn Tucker (KKT) optimality conditions for interval-valued functions with the help of GMFCQ.

4.1. **KKT optimality conditions for interval-valued optimization problem**

Suppose \( \bar{u} \) is a nondominated solution of (IVOP2) and \( f \) and \( g_i, \ i \in I(\bar{u}) \) are differentiable at \( \bar{u} \) on convex set \( M \). Let the GMFCQ holds at \( \bar{u} \), then there exist Lagrange multipliers \( \lambda^L > 0, \lambda^U > 0 \) and \( 0 \leq \mu_i \in \mathbb{R}, \ i = 1, 2, \cdots, m \) such that
\[
\left\langle \lambda^L \nabla f^L(\bar{u}) + \lambda^U \nabla f^U(\bar{u}) + \sum_{i \in I(\bar{u})} \mu_i \nabla g_i(\bar{u}), u - \bar{u} \right\rangle \geq 0 \ \forall \ u \in K,
\]
\[
\mu_i g_i(\bar{u}) = 0 \ \forall \ i = 1, 2, \cdots, m.
\]

Let us denote the following index set by \( I(\bar{u}, \mu) \) and defined as:
\[
I(\bar{u}, \mu) := \{ i \in \{ 1, 2, \cdots, m \} : g_i(\bar{u}) = 0, \ \mu_i > 0 \},
\]
and a set by
\[
M_1(\mu) := \{ u \in M : g_i(u) = 0 \ \forall i \in I(\bar{u}, \mu), \ g_i(u) \leq 0 \ \forall i \in \{ 1, 2, \cdots, m \} \ \setminus I(\bar{u}, \mu) \}.\]
Lemma 4.3. Suppose the interval-valued function $f$ is differentiable and $LU$-quasiconvex and $g$ is differentiable and quasiconvex. Let $\bar{u} \in \overline{K}$ be solution and $M$ be the convex set. Let the KKT optimality conditions and GMFCQ be satisfied at $\bar{u}$ with multipliers $\lambda^L$, $\lambda^U$, $\mu_i, (i = 1, 2, \cdots, m)$. Then, $\overline{K} \subseteq M_1(\mu)$ and the function (Lagrangian)

$$L = f(\cdot) + \sum_{i \in I(\bar{u})} \mu_i g_i(\cdot)$$

is constant over $\overline{K}$, where $f(u) = \lambda^L f^L_k + \lambda^U f^U_k$.

Proof. First we show that $\overline{K} \subseteq M_1(\mu)$. Suppose $u$ is an arbitrary point of $\overline{K}$ then, for the claim it is enough to prove that $g_i(u) = 0 \forall i \in I(\bar{u}, \mu)$. On contrary, suppose that there exists $j \in I(\bar{u}, \mu)$ such that $g_j(u) < 0$.

As $g_j(u) = 0$, we have $g_j(u) < g_j(\bar{u})$ and $\exists \tau > 0$ such that

$$g_j(u + \tau \nabla g_j(\bar{u})) < g_j(\bar{u}).$$

Then from the quasiconvexity of $g$, we have

$$\langle \nabla g_j(\bar{u}), u + \tau \nabla g_j(\bar{u}) - \bar{u} \rangle \leq 0. \quad (12)$$

Since $f$ is $LU$-quasiconvex so from Definition 2.8, it follows that

$$f^L(u) = f^L(\bar{u}), \ f^U(u) = f^U(\bar{u}) \quad (13)$$

$$\implies \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle \leq 0. \quad (14)$$

As $g_i$ are quasiconvex and $g_i(u) \leq 0 = g_i(\bar{u}) \forall i \in I(\bar{u})$, then by quasiconvexity of $g_i$, we have

$$\langle \nabla g_i(\bar{u}), u - \bar{u} \rangle \leq 0 \ i \in I(\bar{u}).$$

Also by KKT optimality conditions, we have

$$\langle \lambda^L \nabla f^L(\bar{u}) + \lambda^U \nabla f^U(\bar{u}), u - \bar{u} \rangle = 0, \quad (15)$$

$$\langle \mu_i \nabla g_i(\bar{u}), u - \bar{u} \rangle = 0 \ i \in I(\bar{u}). \quad (16)$$

From (15), we have $\langle \nabla g_j(\bar{u}), u - \bar{u} \rangle = 0$.

Then from (12), we have $\|\nabla g_j(\bar{u})\|^2 \leq 0$.

Since GMFCQ is satisfied at $\bar{u}$, then $\nabla g_j(\bar{u}) \neq 0$. This is a contradiction to assumption $\nabla g_j(\bar{u}) = 0$. Therefore, $u \in M_1(\mu)$.

It immediately follows from (13),

$$L(u) = L(\bar{u}).$$

This completes the proof. \hfill \square

We consider the following sets:

$$\hat{K}_1(\mu) := \{u \in M_1(\mu) : \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle = 0, \quad \| \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \| \| \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \| \| \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \| \| \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \| \}.$$
ON CHARACTERIZATIONS OF SOLUTION SETS

Theorem 4.4. Suppose the interval-valued function $f$ is continuously differentiable and $LU$-quasiconvex and $g_i$, $i \in I(\bar{u})$ is differentiable and quasiconvex, $g_i$, $i \notin I(\bar{u})$ is continuous at $\bar{u}$. Let $K$ be convex set, $\bar{u} \in \overline{K}$, $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \neq 0$, GMFCQ be satisfied and the Lagrange multipliers be known. Then,

$$\overline{K} = \hat{K}'_1(\mu) = \hat{K}'_2(\mu).$$

Proof. It can be easily seen that $\hat{K}'_1(\mu) = M_1(\mu) \cap \hat{K}_1$ and $\hat{K}'_2(\mu) = M_1(\mu) \cap \hat{K}_2$. Then, from Lemma 4.3 and from relation $\overline{K} = \hat{K}_1 = \hat{K}_2$, the claim is a part of Theorem 3.6. □

Remark 4.5. We assume that $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \neq 0$ in Theorem 4.4, if we assume $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) = 0$, then from Theorem 3.8, $\nabla f^L(u) + \nabla f^U(u) = 0 \ \forall u \in \overline{K}$.

Now, we construct the following sets to study the Lagrange multiplier characterization:

$$K'_1(\mu) := \{ u \in K_1(\mu) : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle = 0, \nabla f^L(u) + \nabla f^U(u) \neq 0 \},$$

$$K'_2(\mu) := \{ u \in K_1(\mu) : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle \geq 0, \nabla f^L(u) + \nabla f^U(u) \neq 0 \},$$

$$K'_3(\mu) := \{ u \in K_1(\mu) : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle \geq 0, \nabla f^L(u) + \nabla f^U(u) \neq 0 \},$$

$$K'_4(\mu) := \{ u \in K_1(\mu) : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u \rangle = 0, \nabla f^L(u) + \nabla f^U(u) \neq 0 \}.$$

Theorem 4.6. Suppose the interval-valued function $f$ is continuously differentiable and $LU$-convex set and $g_i$, $i \in I(\bar{u})$ is differentiable and quasiconvex, $g_i$, $i \notin I(\bar{u})$ is continuous at $\bar{u}$. Let $K$ be convex set, $\bar{u} \in \overline{K}$, $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \neq 0$, GMFCQ be satisfied and the Lagrange multipliers be known. Then,

$$\overline{K} = K'_1(\mu) = K'_2(\mu) = K'_3(\mu) = K'_4(\mu) = K'_5(\mu).$$

Proof. It can be seen that $K'_i = K_i(\mu) \cap K_i, i = 1, 2, 3, 4, 5$. Then, from Lemma 4.3 and relation $\overline{K} = K_1 = K_2 = K_3 = K_4 = K_5$, the claim follows immediately. □

Now, we assume that $K$ is an open set. Then, the KKT optimality conditions reduce to:

$$\lambda^L \nabla f^L(\bar{u}) + \lambda^U \nabla f^U(\bar{u}) + \sum_{i \in I(\bar{u})} \mu_i \nabla g_i(\bar{u}) = 0, \ \mu_i g_i(\bar{u}) = 0 \ \forall i = 1, 2, \cdots, m. \quad (17)$$

In this case, GMFCQ reduces to MFCQ, which is stated and defined as:

$$\exists \ d \in \mathbb{R}^n \text{ such that } \langle g_i(\bar{u}), d \rangle < 0 \ \forall i \in I(\bar{u}).$$

Consider the sets:

$$\hat{K}'_0(\mu) := \{ u \in K_1(\mu) : \langle \nabla g_i(\bar{u}), u - \bar{u} \rangle = 0, \ i \in I(\bar{u}), \mu \},$$

$$\frac{\nabla f^L(u) + \nabla f^U(u)}{\| \nabla f^L(u) + \nabla f^U(u) \|} = \frac{\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})}{\| \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \|}, \ \nabla f^L(u) + \nabla f^U(u) \neq 0;$$
\[ K_2''(\mu) := \{ u \in K_1(\mu) : \langle \nabla g_i(\bar{u}), u - \bar{u} \rangle \geq 0, \; i \in I(\bar{u}, \mu), \]
\[ \frac{\nabla f^L(u) + \nabla f^U(u)}{\| \nabla f^L(u) + \nabla f^U(u) \|} = \frac{\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})}{\| \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \|} , \nabla f^L(u) + \nabla f^U(u) \neq 0 \}. \]

**Theorem 4.7.** Suppose the interval-valued function \( f \) is continuously differentiable and \( LU \)-quasiconvex and \( g_i, i \in I(\bar{u}) \) is differentiable and quasiconvex, \( g_i, i \notin I(\bar{u}) \) is continuous at \( \bar{u} \). Let \( K \) be an open and convex set, \( \overline{u} \in K \), \( \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \neq 0 \), \( MFCQ \) be satisfied and the Lagrange multipliers be known. Then,
\[ K = \hat{K}'(\mu) = \hat{K}''(\mu). \]

**Proof.** It can be seen that \( \hat{K}'(\mu) \subseteq \hat{K}''(\mu) \). Next, we show that \( \hat{K}''(\mu) \subseteq K \). Suppose \( \bar{u} \in \hat{K}''(\mu) \) is an arbitrary point then from relation (17), we get \( (\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}) \leq 0 \), Hence, \( u \in \hat{K}'(\mu) \). Then, from Theorem 4.3, it follows that \( \hat{K}'(\mu) = K \) and \( u \in K \).

Now, we show that \( K \subseteq \hat{K}''(\mu) \). Suppose \( u \in K \) is an arbitrary point. Since, \( f \) is \( LU \)-quasiconvex and \( f^L(u) = f^L(\bar{u}), f^U(u) = f^U(\bar{u}) \), then it follows that \( (\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}) \leq 0 \). Also, \( g_i \) are quasiconvex then,
\[ g_i(u) \leq 0 = g_i(\bar{u}) \implies (\nabla g_i(\bar{u}), u - \bar{u}) \geq 0 \forall \; i \in I(\bar{u}). \]

From KKT conditions (17), \( (\nabla g_i(\bar{u}), u - \bar{u}) = 0 \forall \; i \in I(\bar{u}, \mu) \). It follows from Theorem 4.4, \( \hat{K}'(\mu) = K \) and \( u \in \hat{K}'(\mu) \). We conclude from here, \( u \in \hat{K}''(\mu) \). This completes the proof. \( \Box \)

5. **Conclusions and future remarks**

Motivated by Zhang et al. [34] and Wu [31], we define \( LU \)-quasiconvex and \( LU \)-pseudoconvex functions and show that the lower level sets are convex in case of \( LU \)-quasiconvex function. In this article, we obtain the characterizations of solution sets of interval-valued optimization problem with the \( H \)-differentiable \( LU \)-quasiconvex objective function. We construct Gordon’s theorem of alternative for interval linear matrix system of inequalities and prove that the normalized gradient of interval-valued function is non-zero constant over the solution set when the interval-valued gradient is not zero at a point. Further, we study Lagrange multiplier characterizations of solutions set of interval-valued optimization problem in form of \( LU \)-quasiconvex objective function and quasiconvex inequality constraint function. We also derive KKT optimality conditions with the help of generalized Mangasarian-Fromovitz constraint qualifications (GMFCQ). In the future, we can study the second order characterizations of solution sets of interval-valued function motivated by Ivanov [8]. We can obtain the sequential characterizations of approximate solutions in interval-valued convex optimization problems motivated by Sisarat et al. [27].

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**References**


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