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ON CHARACTERIZATIONS OF SOLUTION SETS OF INTERVAL-VALUED QUASICONVEX PROGRAMMING PROBLEMS

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Abstract. In this article, we study several characterizations of solution sets of LU-quasiconvex interval-valued function. Firstly, we provide Gordan's theorem of the alternative of interval-valued linear system. As a consequence of this theorem, we find the normalized gradient of the interval-valued function is constant over the solution set when its gradient is not zero. Further, we discuss Lagrange multiplier characterizations of solution sets of LU-quasiconvex interval-valued function and provide optimality conditions of interval-valued optimization problem under the generalized Mangasarian-Fromovitz constraint qualifications. We provide illustrative examples in the support of our theory.

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1. Introduction

Characterizations of solution sets for different type of nonlinear optimization problems have attracted the attention of many researchers over the years. Mangasarian [18] attained several characterizations of solution sets of smooth convex functions. Burke and Ferris [3] generalized these results for nonsmooth proper convex functions with the help of Fenchel subdifferential. Further, Jeyakumar and Yang [9] extended the results for pseudolinear programming problems.

In 2003, Penot [24] provided the characterizations of sets of solutions to quasiconvex programming problems using Greenberg-Pierskalla subdifferential. Jeyakumar *et al.* [10] gave Lagrange multiplier characterizations of solutions set over cone constrained convex programming problems. Further, Jeyakumar *et al.* [11] extended the results over convex vector optimization problems.

Mishra *et al.* [21] obtained the Lagrange multiplier characterizations of solution sets of nonsmooth pseudolinear optimization problems. Ivanov [6] provided some necessary and sufficient conditions for semistrictly quasiconvexity and pseudoconvexity in connection with Clarke-Rockafellar subdifferential. Suzuki and Kuroiwa [29] characterized the solution sets for quasiconvex programming problem with the help of invariance property of Greenberg-Pierskalla subdifferential. Recently, Ivanov [7] studied the characterization of solution sets of differentiable quasiconvex programming problems. Further, Ivanov [8] extended the characterizations for second

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order optimality conditions. Sisarat *et al.* [27] discussed the characterizations of approximate solutions of convex vector optimization problems.

The real-life optimization problems are associated with uncertain data that occur due to measurement errors. Interval-valued optimization is one of the significant tools to handle uncertainty. Moore [22] gave a detailed explanation of interval analysis. Neumaier [23] provided the development of interval analysis as a tool for computation and computer-favored proofs. Wu [31] obtained the Karush Kuhn Tucker (KKT) optimality conditions for interval-valued optimization problems and derived Wolfe duality and strong duality theorems for interval-valued optimization in [32]. Further, Wu [33] obtained KKT optimality condition in multiobjective interval-valued optimization and proposed Pareto optimal solution for multiobjective optimization problems with interval-valued objective function. Stefanini and Bede introduced the concept of gH-differentiability for interval-valued functions. Chalco-Cano *et al.* discussed the calculus for interval-valued function. Chalco-Cano *et al.* presented the fundamental theorem of Calculus for interval-valued function. Bedregal and Santiago [2] presented some continuity notions for interval functions. They provided the relationship between the three interpretations of intervals (as a set, as an information and as a number) and the topological counterparts. Lai *et al.* [13] proposed duality results for interval-valued semiinfinite optimization problems with equilibrium constraints using convexificators. Further, Lai *et al.* [14] introduced stationary conditions and characterizations of solution sets for interval-valued tightened nonlinear problems. Recently, several researches have been done in the field of interval-valued optimization, see for instance [12, 15–17, 26].

To the best of our knowledge, there are very few articles related to characterizations of solution sets of interval-valued optimization problems. Recently, Treanta [30] established some characterizations of solution sets of interval-valued optimization problems and discussed the relation between LU-optimal solutions of the interval-valued variational control problem and saddle points related to the interval-valued Lagrange functional.

Motivated by Mangasarian [18] and Ivanov [7], we consider interval-valued optimization problem (IVOP) with H-differentiable LU-quasiconvex functions and establish several characterizations of solution set of IVOP. Inspired by Gordan [5] and Rohn and Kreslova [25], we consider Gordan's theorem of alternative for interval linear matrix system of inequalities and with the help of this, we show that normalized gradient of interval-valued LU-quasiconvex function is non-zero constant when the gradient of the interval-valued function is not equal to zero. Further, we establish the Lagrange multiplier characterizations of solution set with interval-valued LU-quasiconvex continuously H-differentiable objective function and inequality constraints with the help of Generalized Mangasarian-Fromovitz constraint qualification (GMFCQ), which is the generalization of a significant constraint qualification named as Mangasarian-Fromovitz constraint qualification (MFCQ) [20]. We construct some solution sets for IVOP and characterize them with the help of KKT optimality conditions for interval-valued objective function given by Wu [33].

The layout of our article is as follows: In Section 2, some basic and essential results and definitions are provided. In Section 3, we characterize the solution sets of interval-valued continuously differentiable LU-quasiconvex problem. In Section 4, we establish Lagrange multiplier characterizations of solution sets for IVOP and Section 5 is devoted to concluding remarks and future research opportunities.

2. Preliminaries

2.1. Interval analysis

We collect some basic concepts and essential definitions related to interval-valued functions for bounded intervals from Moore [22].
We denote by $\mathcal{I}(\mathbb{R})$ the class of all closed intervals in $\mathbb{R}$. Let $U = [u^L, u^U]$, where u^L and u^U denotes the lower and upper bounds of U, respectively. Let $U = [u^L, u^U]$ and $V = [v^L, v^U]$ be in $\mathcal{I}(\mathbb{R})$, then, we have

(i) $U + V = \{u + v : u \in U,\ v \in V\} = [u^L + v^L, u^U + v^U]$,
(ii) $-U = \{-u : u \in U\} = [-u^U, -u^L]$,
(iii) $U - V = U + (-V) = [u^L - v^U, u^U - v^L]$,

(iv) $tU = \{tu : u \in U\} = \begin{cases} [tu^L, tu^U] & \text{if } t \geq 0 \\ [tu^U, tu^L] & \text{for } t < 0 \end{cases}$
where t is a real number.

We refer to Moore [22], for further details on interval analysis. We have collected the definition and properties of Hausdorff metric from Wu [33].

Suppose that $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$, then the Hausdorff metric between X and Y is denoted and defined by

$$d_H(X,Y) = \max\Big\{ \sup_{u\in X} \inf_{v\in Y} \|u-v\|, \sup_{v\in Y} \inf_{u\in X} \|u-v\| \Big\},$$

where $\|.\|$ is the Euclidean norm.
Let $U = [u^L, u^U]$ and $V = [v^L, v^U]$ be two closed intervals, then it is easy to prove that

$$d_H(U,V) = \max\{|u^L - v^L|, |u^U - v^U|\}.$$

Let $\{U_n = [u_n^L, u_n^U]\}$ and U be closed intervals in $\mathbb{R}$, then the sequence of closed interval $\{U_n\}$ converges to U, if for every $\epsilon > 0$, there exists a natural number $N > 0$ such that, for $n > N$, we have $d_H(U_n, U) < \epsilon$. Wu [31] proved that

$$\lim_{n\to\infty} U_n = U \ \text{ if and only if } \lim_{n\to\infty} u_n^L = u^L \text{ and } \lim_{n\to\infty} u_n^U = u^U.$$

A function $f : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is called interval-valued function, this means $f(u) = f(u_1, \cdots, u_n)$ is a closed interval in $\mathbb{R}$ for each $u \in \mathbb{R}^n$. f can be written as $f(u) = [f^L(u), f^U(u)]$, where f^L and f^U are two real valued functions defined on $\mathbb{R}^n$ such that $f^L(u) \leq f^U(u), \ \forall u \in \mathbb{R}^n$.
Wu [31] discussed limit and continuity of interval-valued functions. Let f be an interval-valued function defined on $\mathbb{R}^n$ and $U = [u^L, u^U]$ be an interval in $\mathbb{R}$, we say

$$\lim_{u\to a} f(u) = U, \text{ if and only if } \lim_{u\to a} f^L(u) = u^L \text{ and}$$

$$\lim_{u\to a} f^U(u) = u^U.$$

The interval-valued function f defined on $\mathbb{R}^n$ is said to be continuous at $a \in \mathbb{R}^n$ if

$$\lim_{u\to a} f(u) = f(a).$$

Proposition 2.1. *[33] Suppose f is an interval-valued function defined on $\mathbb{R}^n$, then f is continuous at $a \in \mathbb{R}^n$ if and only if f^L and f^U are continuous at a.*

Definition 2.2. [33] Suppose K is an open set in $\mathbb{R}$. The interval-valued function $f : K \to \mathcal{I}(\mathbb{R})$ with $f(u) = [f^L(u), f^U(u)]$ is called weakly differentiable at u^0 if the real valued functions f^L and f^U are differentiable at u^0 (in the ordinary sense).

For $U, V \in \mathcal{I}(\mathbb{R})$, if there exists a $W \in \mathcal{I}(\mathbb{R})$ such that $U = V + W$, then W is called the Hukuhara difference of U and V. Also, W can be written as $W = U \ominus V$, considering the Hukuhara difference [1] W exists, which means that $u^L - v^L \leq u^U - v^U$ and $W = [u^L - v^L, u^U - v^U]$.

Definition 2.3. [33] Suppose K is an open set in $\mathbb{R}$. The interval-valued function $f : K \to \mathcal{I}(\mathbb{R})$ is called $H-$differentiable at u^0 if there exists a closed interval $U(u^0) \in \mathcal{I}(\mathbb{R})$ such that the limits

$$\lim_{h\to 0_+} \frac{f(u^0 + h) \ominus f(u^0)}{h} \text{ and}$$

$$\lim_{h\to 0_+} \frac{f(u^0) \ominus f(u^0 - h)}{h}$$

both exist and equal to $U(u^0)$, which is called the $H-$ derivative of f at u^0.

2.2. Solution concepts

Suppose $U = [u^L, u^U]$ and $V = [v^L, v^U]$ are two closed intervals in $\mathbb{R}$. We write $U \preceq_{LU} V$ if and only if $u^L \leq v^L$ and $u^U \leq v^U$.
Consider multiobjective programming problem with multiple interval-valued objective functions

$$(IVOP1) \qquad \min f(u) = (f_1(u), \cdots, f_p(u))$$
$$\text{subject to } u = (u_1, \cdots, u_n) \in K \subseteq \mathbb{R}^n,$$

where each
$f_k(u) = [f_k^L(u), f_k^U(u)]$ is an interval-valued function for $k = 1, \cdots, p$.
We write $U \prec_{LU} V$ if and only if $U \preceq_{LU} V$ and $U \neq V$. We say $U = (U_1, \cdots, U_p)$ is an interval-valued vector if each component $U_k = [u_k^L, u_k^U]$ is closed interval for $k = 1, \cdots, p$. Suppose $U = (U_1, \cdots, U_p)$ and $V = (V_1, \cdots, V_p)$ are two interval-valued vectors. We write $U \preceq_{LU} V$ if and only if $U_k \preceq_{LU} V_k \ \forall k = 1, \cdots, p$, and $U \prec_{LU} V$ if and only if $U_k \preceq_{LU} V_k$, $\forall k = 1, \cdots, p$ and $U_q \prec_{LU} V_q$ for at least one q. Suppose u^* is a feasible solution of $(IVOP1)$, then $f(u^*)$ is an interval-valued vector. The concepts of Pareto optimal (efficient) solution is given below.

Definition 2.4. [31] Suppose u^0 is a feasible solution to the problem $(IVOP1)$.

(i) u^0 is said to be an efficient solution to the problem $(IVOP1)$ if there exists no $\bar{u}$ such that $f(\bar{u}) \prec_{LU} f(u^0)$.
(ii) u^0 is said to be a strong efficient solution to the problem $(IVOP1)$ if there exists no $\bar{u}$ such that $f(\bar{u}) \preceq_{LU} f(u^0)$.
(iii) u^0 is said to be a weak efficient solution to the problem $(IVOP1)$ if there exists no $\bar{u}$ such that $f_k(\bar{u}) \prec_{LU} f_k(u^0) \ \forall k = 1, \cdots, p$.

Definition 2.5. [31] Suppose u^0 is feasible solution of the problem $(IVOP1)$. u^0 is said to be a local weak efficient solution of the problem $(IVOP1)$, if there exists a neighborhood N of u^0 such that for all $\bar{u} \in K \cap N$, then the following cannot be satisfied for any $k = 1, \cdots, p$

$$f_k(\bar{u}) \prec_{LU} f_k(u^0).$$

2.3. Interval-valued convex functions and generalized convex functions

Wu [31] introduced the concept of convexity for interval-valued functions.

Definition 2.6. [31] Suppose f is an interval-valued function defined on a convex set $X \subseteq \mathbb{R}^n$. Then f is said to be LU-convex at $\bar{u}$ if

$$f(\lambda\bar{u} + (1-\lambda)u) \preceq_{LU} \lambda f(\bar{u}) + (1-\lambda) f(u) \ \forall \ \lambda \in (0,1) \text{ and } \forall \ u \in X.$$

Proposition 2.7. *[31] Suppose $f(u) = [f^L(u), f^U(u)]$ is an interval-valued function defined on $X \subseteq \mathbb{R}^n$ then f is called LU-convex at $\bar{u}$ if and only if f^L and f^U are convex at $\bar{u}$.*

We consider a continuous and H-differentiable interval-valued function $f(u) = [f^L(u), f^U(u)]$ on an open convex set X. We define LU-quasiconvex and LU-pseudoconvex function motivated by Wu [31] and Zhang *et al.* [34].

Definition 2.8. Let $f : X \to \mathcal{I}(\mathbb{R})$ be a continuous and H-differentiable interval-valued function on an open convex set $X \subseteq \mathbb{R}^n$. f is said to be a LU-quasiconvex function if

$$f^L(u) \leq f^L(v) \text{ and } f^U(u) \leq f^U(v)$$
$$\implies \langle \nabla f^L(v) + \nabla f^U(v), u - v \rangle \ \leq 0 \ \forall \ u, v, \in X,$$

where ∇ denotes the gradient operator.

Definition 2.9. Let $f : X \to \mathcal{I}(\mathbb{R})$ be a continuous and H-differentiable interval-valued function on an open convex set $X \subseteq \mathbb{R}^n$. f is said to be a LU-pseudoconvex function if

$$\langle \nabla f^L(v) + \nabla f^U(v), u - v \rangle \geq 0 \implies f^L(u) \geq f^L(v) \text{ and } f^U(u) \geq f^U(v) \; \forall \; u, v, \in X.$$

Remark 2.10. Wu [31] proposed the concept of pseudoconvexity for an interval-valued function f at $\bar{u}$ if the real valued functions f^L and f^U are pseudoconvex at $\bar{u}$.

Remark 2.11. Every LU-pseudoconvex function is also LU-quasiconvex function.

Lemma 2.12. *[19] A real valued function $f : X \to \mathbb{R}$ is quasiconvex on a convex set $X \subseteq \mathbb{R}^n$ if and only if its lower level sets $f_{\leq \alpha} := \{u \in X : f(u) \leq \alpha\}$ are convex for all $\alpha \in \mathbb{R}$.*

Remark 2.13. Let $f(u) = [f^L(u), f^U(u)] \in \mathcal{I}(\mathbb{R})$ be an interval-valued function. Then the real valued functions $f^L(u)$ and $f^U(u)$ are quasiconvex if and only if their respective lower level sets $f^L_{\leq \alpha}(u)$ and $f^U_{\leq \alpha}(u)$ are convex sets.

3. Characterizations of solution sets of interval-valued optimization problems

We introduce Gordan's theorem of the alternative of interval-valued linear system motivated by [5] and [25] to obtain the characterizations of solution sets.

3.1. Gordan's theorem of the alternative of interval-valued linear system

Consider an interval linear system of inequalities

$$\begin{aligned} &a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n > 0 \\ &a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n > 0 \\ &\vdots \\ &a_{r1}u_1 + a_{r2}u_2 + \cdots + a_{rn}u_n > 0. \end{aligned}$$

$$\text{In short,} \qquad A_{\mathcal{I}}x \succ 0_{\mathcal{I}}, \tag{1}$$

where $A_{\mathcal{I}} = \{A : A^L \leq A \leq A^U\}$ (componentwise inequalities) is an $r \times n$ interval matrix and $0_{\mathcal{I}}$ is an interval $r-$vector, a_{ij} $(i = 1, 2, \cdots, r, j = 1, 2, \cdots, n)$ and u_j $(j = 1, 2, \cdots, n)$ are intervals.
We consider a family of systems of linear inequalities written, in short, as system (1).

$$Au \succ 0 \; \forall \; A \in A_{\mathcal{I}}, \; 0 \in 0_{\mathcal{I}}. \tag{2}$$

Then either each matrix system (1) has a solution u or the system,

$$A^T_{\mathcal{I}}v = 0, \; v \geq 0, \; v \neq 0,$$

has a solution v but never both.

Lemma 3.1. *Suppose $X \subseteq \mathbb{R}^n$ is an open and convex set, $K \subseteq X$ is a convex subset and $u, v \in \overline{K}$, $d \in \mathbb{R}^n$. Let the function $f : X \to \mathcal{I}(\mathbb{R})$ be LU- quasiconvex and $\nabla f^L(u) + \nabla f^U(u) \neq 0$, $\nabla f^L(v) + \nabla f^U(v) \neq 0$, then*

$$\langle \nabla f^L(u) + \nabla f^U(u), d \rangle < 0 \implies \langle \nabla f^L(v) + \nabla f^U(v), d \rangle \leq 0.$$

Proof. On contrary, suppose that there exists $d \in \mathbb{R}^n$ with

$$\langle \nabla f^L(u) + \nabla f^U(u), d\rangle < 0 \text{ and } \langle \nabla f^L(v) + \nabla f^U(v), -d\rangle < 0.$$

It follows that there exists $\lambda > 0$ such that

$$f^L(p) < f^L(u) = f^L(v) \text{ and } f^U(p) < f^U(u) = f^U(v), \tag{3}$$
$$f^L(q) < f^L(v) = f^L(u) \text{ and } f^U(q) < f^U(v) = f^U(u), \tag{4}$$

where $p = u + \lambda d \in X,\ q = v - \lambda d \in X$.
Suppose $w = \frac{p+q}{2}$, hence $w = \frac{u+v}{2}$.
Since f is LU-quasiconvex, we have from Lemma 2.12, $\overline{K}$ is convex and $f^L(w) = f^L(u) = f^L(v)$ and $f^U(w) = f^U(u) = f^U(v)$.
As f is LU-quasiconvex, so from (3) and (4),

$$f^L(w) \le \max\{f^L(p), f^L(q)\} < f^L(u) = f^L(w) \tag{5}$$
$$f^U(w) \le \max\{f^U(p), f^U(q)\} < f^U(u) = f^U(w), \tag{6}$$

which is contradictory. This completes the proof. □

Lemma 3.2. *Let $u, v \in \mathcal{I}(\mathbb{R}^n); u \neq 0, v \neq 0$. Suppose*

$$\langle u, d\rangle < 0,\ d \in \mathbb{R}^n \implies \langle v, d\rangle \le 0, \tag{7}$$

where $u = [u^L, u^U]$ and $v = [v^L, v^U]$. Then there exists $a > 0$ such that $v = au$.

Proof. The equivalent system of (7) is to claim that the system

$$\langle v, d\rangle > 0, \langle -u, d\rangle > 0 \tag{8}$$

has not a solution d. Then, it follows from Gordan's theorem of the alternative of interval-valued linear system, there exist real numbers a_1 and a_2 such that
$va_1 = 0$ and $ua_2 = 0$ or $va_1 - ua_2 = 0,\ a_1 \ge 0, a_2 \ge 0, (a_1, a_2) \neq (0, 0)$.
Without loss of generality, let $a_1 > 0$ and $a = \frac{a_2}{a_1}$ then $a > 0$ and satisfies $v = au$. □

Lemma 3.3. *Suppose $X \subseteq \mathbb{R}^n$ is an open and convex set, $K \subseteq X$ is a convex subset. Let the function $f : X \to \mathcal{I}(\mathbb{R})$ be LU- quasiconvex and $f^L(u)$ and $f^U(u)$ be continuously differentiable functions, then the interval-valued normalized gradient is constant over the nonempty set $\{u \in \overline{K} : \nabla f^L(u) + \nabla f^U(u) \neq 0\}$.*

Proof. Suppose u and v are two distinct points of the set $\overline{K}$ with $\nabla f^L(u)+\nabla f^U(u) \neq 0$ and $\nabla f^L(v)+\nabla f^U(v) \neq 0$, then from Lemmas 3.1 and 3.2, there exists $a > 0$ with condition

$$\nabla f^L(v) + \nabla f^U(v) = a[\nabla f^L(u) + \nabla f^U(u)].$$

Then we get

$$\frac{\nabla f^L(v) + \nabla f^U(v)}{\|\nabla f^L(v) + \nabla f^U(v)\|} = a\frac{\nabla f^L(u) + \nabla f^U(u)}{\|a[\nabla f^L(u) + \nabla f^U(u)]\|}$$
$$\implies \frac{\nabla f^L(v) + \nabla f^U(v)}{\|\nabla f^L(v) + \nabla f^U(v)\|} = \frac{\nabla f^L(u) + \nabla f^U(u)}{\|\nabla f^L(u) + \nabla f^U(u)\|}.$$

Then, we can get the claim immediately. □

Lemma 3.4. *Suppose $X \subseteq \mathbb{R}^n$ is an open and convex set, $K \subseteq X$ is a convex subset. Let the function $f : X \to \mathcal{I}(\mathbb{R})$ be LU- quasiconvex and $f^L(u)$ and $f^U(u)$ be continuously differentiable functions then either of the following conditions holds but not both*

1. *$\nabla f^L(u) + \nabla f^U(u) \neq 0$ for all $u \in \overline{K}$ and the normalized gradient $\dfrac{\nabla f^L(u) + \nabla f^U(u)}{\|\nabla f^L(u) + \nabla f^U(u)\|}$ is constant over the set $\overline{K}$;*
2. *$\nabla f^L(u) + \nabla f^U(u) = 0$ for all $u \in \overline{K}$.*

Proof. We consider the two possible cases:
Case (1): Let $u \in \overline{K}$, $\nabla f^L(u)+\nabla f^U(u) \neq 0$. We prove for another arbitrary point $v \in \overline{K}$, $\nabla f^L(v)+\nabla f^U(v) \neq 0$. On contrary suppose that $\nabla f^L(v) + \nabla f^U(v) = 0$.
Consider the sets

$$P := \{\lambda \in [0,1] : \nabla f^L(u + \lambda(v-u)) + \nabla f^U(u + \lambda(v-u)) = 0\},$$
$$Q := \{\lambda \in [0,1] : \nabla f^L(u + \lambda(v-u)) + \nabla f^U(u + \lambda(v-u)) \neq 0\}.$$

Since $\overline{K}$ is convex, so $u + \lambda(v-u) \in \overline{K}$, let $u + \lambda(v-u) = w$.
Consider the open sets (intervals) in the interval $[0,1]$ which is of the type (u,v) such that $0 < u < v < 1$, $(u,1]$ such that $0 < u < 1$, $[0,v)$ such that $0 < v < 1$ and their unions.
We claim that P is closed. Consider a sequence $\{\lambda_n\}$, where $\lambda_n \in P$ and $\lambda_n \to \lambda_0$.
Since f^L and f^U are continuously differential, then we have

$$\begin{aligned}\nabla f^L(u + \lambda_0(v-u)) + \nabla f^U(u + \lambda_0(v-u) &= \lim_{n\to\infty} \nabla f^L(u + \lambda_n(v-u)) \\ +\nabla f^U(u + \lambda_n(v-u) &= \lim_{n\to\infty}(0+0) = 0.\end{aligned}$$

Hence, $\lambda_0 \in P$ and P is a closed set.
Now, we claim that Q is closed set. Consider a sequence $\{\lambda_n\}$ where $\lambda_n \in Q$ and $\lambda_n \to \lambda_0$.
From Lemma 3.3, there exists a vector $a \neq 0$ such that

$$\frac{\nabla f^L(w) + \nabla f^U(w)}{\|\nabla f^L(w) + \nabla f^U(w)\|} = a \ \forall w \in \overline{K}.$$

As the normalized gradient is constant, therefore

$$\lim_{n\to\infty} \frac{\nabla f^L(u + \lambda_n(v-u)) + \nabla f^U(u + \lambda_n(v-u))}{\|\nabla f^L(u + \lambda_n(v-u)) + \nabla f^U(u + \lambda_n(v-u))\|} = a \neq 0.$$

Since f^L and f^U are continuously differentiable, then we have

$$\begin{aligned}&\frac{\nabla f^L(u + \lambda_0(v-u)) + \nabla f^U(u + \lambda_0(v-u))}{\|\nabla f^L(u + \lambda_0(v-u)) + \nabla f^U(u + \lambda_0(v-u))\|} \\ &= \lim_{n\to\infty} \frac{\nabla f^L(u + \lambda_n(v-u)) + \nabla f^U(u + \lambda_n(v-u))}{\|\nabla f^L(u + \lambda_n(v-u)) + \nabla f^U(u + \lambda_n(v-u))\|} = a.\end{aligned}$$

Therefore, $\|\nabla f^L(u + \lambda_0(v-u)) + \nabla f^U(u + \lambda_0(v-u))\| \neq 0$, this implies $\lambda_0 \in Q$ and Q is closed.
Since $P \cup Q = [0,1]$, then either $P = [0,1]$, $Q = \emptyset$ or $Q = [0,1]$, $P = \emptyset$, but both cases are not possible. This contradicts the assumption $\nabla f^L(v) + \nabla f^U(v) = 0$. Therefore, $\nabla f^L(v) + \nabla f^U(v) \neq 0 \ \forall v \in \overline{K}$. Then by Lemma 3.3, the normalized gradient is non-zero constant over $\overline{K}$.
Case (2): Let $\nabla f^L(u) + \nabla f^U(u) = 0$. From the proof of Case (1), there is impossibility of the assumption $\nabla f^L(v) + \nabla f^U(v) \neq 0$ for arbitrary $v \in \overline{K}$. Therefore $\nabla f^L(v) + \nabla f^U(v) = 0$, $\forall v \in \overline{K}$. This completes the proof. □

Example 3.5. Consider an interval-valued LU-quasiconvex function $f:\mathbb{R}\to\mathcal{I}(\mathbb{R})$ defined by $f(u)=[u^2-1,2(1-u^2)]$ then $f^L(u)=u^2-1$ and $f^U(u)=2(1-u^2)$.
$\nabla f^L(u)+\nabla f^U(u)=-2x$ and $\|\nabla f^L(u)+\nabla f^U(u)\|=2x$.
Then the normalized gradient $\frac{\nabla f^L(u)+\nabla f^U(u)}{\|\nabla f^L(u)+\nabla f^U(u)\|}=-1=\frac{\nabla f^L(v)+\nabla f^U(v)}{\|\nabla f^L(v)+\nabla f^U(v)\|}$ for any $v\in\mathbb{R}$ where $\nabla f^L(v)+\nabla f^U(v)\neq 0$.

Suppose $\bar{u}\in\overline{K}$. We denote the two sets by $\hat{K}_1$ and $\hat{K}_2$ and define by

$$\begin{aligned}\hat{K}_1:=&\{u\in K:\langle\nabla f^L(\bar{u})+\nabla f^U(\bar{u}),u-\bar{u}\rangle=0,\\ &\frac{\nabla f^L(u)+\nabla f^U(u)}{\|\nabla f^L(u)+\nabla f^U(u)\|}=\frac{\nabla f^L(\bar{u})+\nabla f^U(\bar{u})}{\|\nabla f^L(\bar{u})+\nabla f^U(\bar{u})\|},\nabla f^L(u)+\nabla f^U(u)\neq 0\};\end{aligned}$$

$$\begin{aligned}\hat{K}_2:=&\{u\in K:\langle\nabla f^L(\bar{u})+\nabla f^U(\bar{u}),u-\bar{u}\rangle\leq 0,\\ &\frac{\nabla f^L(u)+\nabla f^U(u)}{\|\nabla f^L(u)+\nabla f^U(u)\|}=\frac{\nabla f^L(\bar{u})+\nabla f^U(\bar{u})}{\|\nabla f^L(\bar{u})+\nabla f^U(\bar{u})\|},\nabla f^L(u)+\nabla f^U(u)\neq 0\}.\end{aligned}$$

Theorem 3.6. *Let $K\subseteq X$ be a convex subset of open and convex set $X\subseteq\mathbb{R}^n$. Let the interval-valued function $f:X\to\mathcal{I}(\mathbb{R})$ be LU-quasiconvex and f^L and f^U be continuously differentiable. Suppose $\bar{u}\in\overline{K}$ is a solution of $(IVOP1)$ and $\nabla f^L(u)+\nabla f^U(u)\neq 0$, then $\overline{K}=\hat{K}_1=\hat{K}_2$.*

Proof. First we prove that $\overline{K}\subseteq\hat{K}_1$. Let $u\in\overline{K}$, since $\overline{K}$ is convex, $\bar{u}\in\overline{K}$, and we have $\bar{u}+\lambda(u-\bar{u})\in\overline{K}$ for all $\lambda\in[0,1]$, hence

$$f^L(\bar{u}+\lambda(u-\bar{u}))=f^L(\bar{u})\ \forall\lambda\in[0,1] \tag{9}$$

$$\text{and } f^L(\bar{u}+\lambda(u-\bar{u}))=f^L(\bar{u})\ \forall\lambda\in[0,1]. \tag{10}$$

From (9) and (10), we get

$$\langle\nabla f^L(\bar{u})+\nabla f^U(\bar{u}),u-\bar{u}\rangle=0.$$

Then from Lemma 3.3, $\nabla f^L(u)+\nabla f^U(u)\neq 0$, and $u\in\hat{K}_1$. Hence $\overline{K}\subseteq\hat{K}_1$.
It is obvious to show that $\hat{K}_1\subseteq\hat{K}_2$.
Now, we prove that $\hat{K}_2\subseteq\overline{K}$. Suppose $u\in\hat{K}_2$. We show that $u\in\overline{K}$. On contrary suppose that $u\notin\overline{K}$, then $f^L(\bar{u})<f^L(u)$ and $f^U(\bar{u})<f^U(u)$.
Since K is convex, we have $\bar{u}+\lambda(u-\bar{u})\in K\ \forall\lambda\in[0,1]$. As $K\subset\overline{K}$, we get that

$$f^L(\bar{u}+\lambda(u-\bar{u}))\geq f^L(\bar{u})$$
$$\text{and } f^L(\bar{u}+\lambda(u-\bar{u}))\geq f^L(\bar{u}).$$

Hence, $\langle\nabla f^L(\bar{u})+\nabla f^U(\bar{u}),u-\bar{u}\rangle\geq 0$ and from the assumption $u\in\hat{K}_2$, it follows that

$$\langle\nabla f^L(\bar{u})+\nabla f^U(\bar{u}),u-\bar{u}\rangle=0.$$

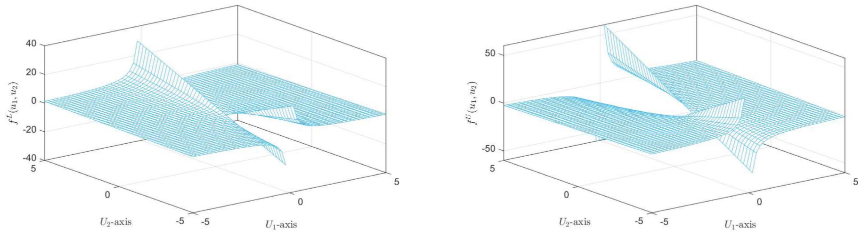
Since f is continuous and $f^L(\bar{u})<f^L(u)$ and $f^U(\bar{u})<f^U(u)$, there exists a number $t>0$ such that

$$f^L[\bar{u}+t(\nabla f^L(\bar{u})+\nabla f^U(\bar{u}))]<f^L(u)$$
$$\text{and } f^L[\bar{u}+t(\nabla f^L(\bar{u})+\nabla f^U(\bar{u}))]<f^L(u).$$

Then, from Definition 2.8,

$$\langle\nabla f^L(u)+\nabla f^U(u),\bar{u}+t(\nabla f^L(\bar{u})+\nabla f^U(\bar{u}))\rangle\leq 0.$$

Then, from $\langle\nabla f^L(\bar{u})+\nabla f^U(\bar{u}),u-\bar{u}\rangle=0$, we have $\|\nabla f^L(\bar{u})+\nabla f^U(\bar{u})\|^2\leq 0$, which contradicts the assumption $\nabla f^L(\bar{u})+\nabla f^U(\bar{u})\neq 0$. This completes the theorem. □



(a) (b)

FIGURE 1. The lower and upper bound functions. (a) $f^L(u)$. (b) $f^U(u)$.

Example 3.7. Consider an interval-valued optimization problem (IVOP1) $\min f(x)$, where $f : \mathbb{R}^2 \to \mathcal{I}(\mathbb{R})$ defined by

$$f(u_1, u_2) = \left[\frac{-u_2 - 1}{u_1}, \frac{2(u_2+1)}{u_1}\right]$$

then
$\nabla f^L(u) = \left(\frac{u_2+1}{u_1^2}, \frac{-1}{u_1}\right)$ and $\nabla f^L(u) = \left(\frac{u_2+1}{u_1^2}, \frac{-1}{u_1}\right)$.
Consider a set $K = \{u = (u_1, u_2) : 1 \leq u_1 \leq 3,\ 0 \leq u_2 \leq u_1\}$. Then f is LU-quasiconvex on the set

$$X = \{u = (u_1, u_2) : u_1 > 0,\ -\infty < u_2 < \infty\}.$$

Since $f^L(u)$ and $f^U(u)$ are quasiconvex on the set X and it satisfies $f^L(u) \leq f^L(v)$ and $f^U(u) \leq f^U(v) \implies \langle \nabla f^L(v) + \nabla f^U(v), u - v\rangle \leq 0\ \forall\ u, v \in X$.
The solution set is

$$\overline{K} = \{u = (u_1, u_2) : 1 \leq u_1 \leq 3,\ u_2 = -1\}.$$

Suppose $\bar{u} = (1, -1)$ is the given solution. Now

$$\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle = u_2 + 1,$$

$$\frac{\nabla f^L(u) + \nabla f^U(u)}{\|\nabla f^L(u) + \nabla f^U(u)\|} = \frac{\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})}{\|\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})\|} = (0, 1) \text{ provided } u_2 = -1.$$

Hence, $\hat{K}_1 = \{u = (u_1, u_2) : 1 \leq u_1 \leq 3,\ u_2 = -1\}$, which is same as $\overline{K}$.

Theorem 3.8. *Let $K \subseteq X$ be a convex subset of an open and convex set $X \subseteq \mathbb{R}^n$. Let the interval-valued function $f : X \to \mathcal{I}(\mathbb{R})$ be LU-quasiconvex and f^L and f^U be continuously differentiable. Suppose $\bar{u} \in \overline{K}$ is a solution of (IVOP1) and $\nabla f^L(u) + \nabla f^U(u) \neq 0$, then, $\overline{K} \subseteq \tilde{K}$, where $\tilde{K} = \{u \in K : \nabla f^L(u) + \nabla f^U(u) = 0\}$. In addition to that if f is LU-pseudoconvex on K, then $\overline{K} = \tilde{K}$.*

Proof. From the second alternative of Lemma 3.3, it can be easily seen that $\overline{K} \subseteq \tilde{K}$. Hence, $\nabla f^L(u) + \nabla f^U(u) = 0\ \forall u \in \overline{K}$.
Now, we suppose that f is LU-pseudoconvex function on K. We show that $\overline{K} = \tilde{K}$. Since every LU-pseudoconvex function is LU-quasiconvex, so we have to prove only $\tilde{K} \subseteq \overline{K}$.
On contrary, suppose that $\exists\ u \in \tilde{K}$ but $u \notin \overline{K}$. Hence, $f^L(\bar{u}) < f^L(u)$ and $f^U(\bar{u}) < f^U(u)$.
By the assumption of LU-pseudoconvexity,

$$\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle < 0.$$

Which contradicts the assumption $\nabla f^L(u) + \nabla f^U(u) = 0$. □

Example 3.9. Consider an interval-valued LU-quasiconvex function $f : \mathbb{R} \to \mathcal{I}(\mathbb{R})$ defined by $f(u) = [u^3, 2u^3]$ then $f^L(u) = u^3$ and $f^U(u) = 2u^3$.
$\nabla f^L(u) + \nabla f^U(u) = 9u^2$.
Consider $u \in \overline{K},\ u = 0$ is a known solution and $\langle \nabla f^U(\bar{u}), u - \bar{u}\rangle = 0$, hence $\overline{K} \subseteq \tilde{K}$.

We construct the following sets to discuss the characterizations of solution sets.

$$K_1 := \{u \in K : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle = 0, \nabla f^L(u) + \nabla f^U(u) \neq 0\},$$

$$K_2 := \{u \in K : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle \geq 0, \nabla f^L(u) + \nabla f^U(u) \neq 0\},$$

$$\begin{aligned} K_3 := \{u \in K : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle =& \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle, \\ & \nabla f^L(u) + \nabla f^U(u) \neq 0\}, \end{aligned}$$

$$\begin{aligned} K_4 := \{u \in K : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle \geq& \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle, \\ & \nabla f^L(u) + \nabla f^U(u) \neq 0\}, \end{aligned}$$

$$\begin{aligned} K_5 := \{u \in K : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle =& \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle = 0, \\ & \nabla f^L(u) + \nabla f^U(u) \neq 0\}. \end{aligned}$$

Theorem 3.10. *Let $K \subseteq X$ be a convex subset of open and convex set $X \subseteq \mathbb{R}^n$. Let the interval-valued function $f : X \to \mathcal{I}(\mathbb{R})$ be LU-quasiconvex and f^L and f^U be continuously differentiable. Suppose $\bar{u} \in \overline{K}$ is a solution of $(IVOP1)$ such that $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \neq 0$, then $\overline{K} = K_1 = K_2 = K_3 = K_4 = K_5$.*

Proof. It can be easily seen that $K_5 \subseteq K_3 \subseteq K_4$ and $K_5 \subseteq K_1 \subseteq K_2$.
First we prove that $\overline{K} \subseteq K_5$. Let $u \in \overline{K}$, then $f^L(u) = f^L(\bar{u})$ and $f^U(u) = f^U(\bar{u})$. Since f is LU-quasiconvex, the solution set $\overline{K}$ is convex and

$$f^L[\bar{u} + \lambda(u - \bar{u})] = f^L(\bar{u}) \text{ and } f^U[\bar{u} + \lambda(u - \bar{u})] = f^U(\bar{u})\ \forall \lambda \in [0, 1].$$

Hence, $\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle = 0$. Interchanging the role of $\bar{u}$ and u, and by similar arguments, we can easily show that $\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle = 0$. From Lemma 3.4, we get $\nabla f^L(u) + \nabla f^U(u) \neq 0$. Therefore, $u \in K_5$.
Next, we prove that $K_4 \subseteq K_2$. Suppose $u \in K_4$, hence

$$\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle \geq \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle. \tag{11}$$

Since K is convex, $u, \bar{u} \in K,\ \bar{u} \in \overline{K}$, then

$$f^L[\bar{u} + \lambda(u - \bar{u})] \geq f^L(\bar{u}) \text{ and } f^U[\bar{u} + \lambda(u - \bar{u})] \geq f^U(\bar{u})\ \forall \lambda \in [0, 1].$$

Hence,

$$\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle \geq 0.$$

This implies from (11) that $x \in K_2$.
Now, we show that $K_2 \subseteq \overline{K}$. Suppose $u \in K_2$, we have to show that $u \in \overline{K}$. On contrary, suppose that $x \notin \overline{K}$, then $f^L(\bar{u}) < f^L(u)$ and $f^U(\bar{u}) < f^U(u)$. From quasiconvexity of f, it follows that $\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle \leq 0$. Since $u \in K_2$, we have $\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle = 0$. Then, it follows from the continuity of f and $f^L(\bar{u}) < f^L(u)$

and $f^U(\bar{u}) < f^U(u)$, there exists $t > 0$ with $f^L[\bar{u} + t(\nabla f^L(u) + \nabla f^U(u))] < f^L(u)$ and $f^U[\bar{u} + t(\nabla f^L(u) + \nabla f^U(u))] < f^U(u)$.
Then, from quasiconvexity of f,

$$\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} + t(\nabla f^L(u) + \nabla f^U(u)) - u\rangle \leq 0.$$

From $\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle = 0$, we have $\nabla f^L(u) + \nabla f^U(u) = 0$, this contradicts the assumption $u \in K_2$. This completes the theorem. □

4. Lagrange multiplier characterizations of the solution sets of interval-valued optimization problems

$$\begin{aligned} (IVOP2) \qquad & \min f(u) \\ \text{subject to } & g_i(u) \leq 0,\ i = 1, 2, \cdots, m \\ & x \geq 0, \end{aligned}$$

where $f : X \subseteq \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is an interval-valued function and $g_i : X \subseteq \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \cdots, m$ are real valued functions. Let M be a convex subset of X, here M be not necessarily open.
Let $I(u) := \{i \in \{i, 2, \cdots, m\} : g_i(u) = 0\}$ be index set of the active constraints. Suppose $K := \{u \in M : g_i(u) \leq 0,\ i = 1, 2, \cdots, m\}$ is the feasible set.
Let $C \subseteq \mathbb{R}^n$ be a cone. Then the negative polar cone of C is defined as:

$$C^- := \{c \in \mathbb{R}^n : \langle c, u\rangle \leq 0\ \forall\ u \in C\},$$

which is also called normal cone and denoted as $N_M(u)$.

Definition 4.1. [20] MFCQ is said to be satisfy at point $\bar{u} \in K$ if $\nabla g_i(\bar{u}),\ i \in I(\bar{u})$ are linearly independent and there exists a point $d \in \mathbb{R}^n$ such that

$$\langle \nabla g_i(\bar{u}), d\rangle = 0,\ i \in I = \{1, 2, \cdots, l\},\ 0 \leq l \leq m,\ \langle \nabla g_i(\bar{u}), d\rangle < 0,\ i \in I(\bar{u}).$$

Definition 4.2. [7] Generalized Mangasarian-Fromovitz constraint qualification (GMFCQ) is said to be satisfy at $\bar{u}$ if and only if there is a direction $d \in (N_M(\bar{u}))^-$ such that $\langle \nabla g_i(\bar{u}), d\rangle < 0,\ i \in I(\bar{u})$.

Motivated by Wu [32] and Ivanov [7], we establish the Karush Kuhn Tucker (KKT) optimality conditions for interval-valued functions with the help of GMFCQ.

4.1. KKT optimality conditions for interval-valued optimization problem

Suppose $\bar{u}$ is a nondominated solution of (IVOP2) and f and $g_i,\ i \in I(\bar{u})$ are differentiable at $\bar{u}$ on convex set M. Let the GMFCQ holds at $\bar{u}$, then there exist Lagrange multipliers $\lambda^L > 0,\ \lambda^U > 0$ and $0 \leq \mu_i \in \mathbb{R},\ i = 1, 2, \cdots, m$ such that

$$\left\langle \lambda^L \nabla f^L(\bar{u}) + \lambda^U \nabla f^U(\bar{u}) + \sum_{i \in I(\bar{u})} \mu_i \nabla g_i(\bar{u}), u - \bar{u} \right\rangle \geq 0\ \forall\ u \in K,$$
$$\mu_i g_i(\bar{u}) = 0\ \forall\ i = 1, 2, \cdots, m.$$

Let us denote the following index set by $I(\bar{u}, \mu)$ and defined as:

$$I(\bar{u}, \mu) := \{i \in \{i, 2, \cdots, m\} : g_i(\bar{u}) = 0,\ \mu_i > 0\},$$

and a set by

$$M_1(\mu) := \{u \in M : g_i(u) = 0\ \forall i \in I(\bar{u}, \mu),\ g_i(u) \leq 0\ \forall i \in \{i, 2, \cdots, m\} \setminus I(\bar{u}, \mu)\}.$$

Lemma 4.3. *Suppose the interval-valued function f is differentiable and LU-quasiconvex and g is differentiable and quasiconvex. Let $\bar{u} \in \overline{K}$ be solution and M be the convex set. Let the KKT optimality conditions and GMFCQ be satisfied at $\bar{u}$ with multipliers $\lambda^L,\ \lambda^U,\ \mu_i, (i = 1, 2, \cdots, m)$. Then, $\overline{K} \subseteq M_1(\mu)$ and the function(Lagrangian)*

$$L = f(\cdot) + \sum_{i \in I(\bar{u})} \mu_i g_i(\cdot)$$

is constant over $\overline{K}$, where $f(u) = \lambda^L f_k^L + \lambda^U f_k^U$.

Proof. First we show that $\overline{K} \subseteq M_1(\mu)$. Suppose u is an arbitrary point of $\overline{K}$ then, for the claim it is enough to prove that $g_i(u) = 0\ \forall\ i \in I(\bar{u}, \mu)$. On contrary, suppose that there exists $j \in I(\bar{u}, \mu)$ such that $g_j(u) < 0$.
As $g_j(u) = 0$, we have $g_j(u) < g_j(\bar{u})$ and $\exists\ \tau > 0$ such that

$$g_j(u + \tau \nabla g_j(\bar{u})) < g_j(\bar{u}).$$

Then from the quasiconvexity of g, we have

$$\langle \nabla g_j(\bar{u}), u + \tau \nabla g_j(\bar{u}) - \bar{u} \rangle \leq 0. \tag{12}$$

Since f is LU-quasiconvex so from Definition 2.8, it follows that

$$f^L(u) = f^L(\bar{u}),\ \ f^U(u) = f^U(\bar{u}) \tag{13}$$

$$\implies \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle \leq 0. \tag{14}$$

As g_i are quasiconvex and $g_i(u) \leq 0 = g_i(\bar{u})\ \forall\ i \in I(\bar{u})$, then by quasiconvexity of g_i, we have

$$\langle \nabla g_i(\bar{u}), u - \bar{u} \rangle \leq 0\ i \in I(\bar{u}).$$

Also by KKT optimality conditions, we have

$$\langle \lambda^L \nabla f^L(\bar{u}) + \lambda^U \nabla f^U(\bar{u}), u - \bar{u} \rangle = 0, \tag{15}$$

$$\langle \mu_i \nabla g_i(\bar{u}), u - \bar{u} \rangle = 0\ i \in I(\bar{u}). \tag{16}$$

From (15), we have $\langle \nabla g_j(\bar{u}), u - \bar{u} \rangle = 0$.
Then from (12), we have $\|\nabla g_j(\bar{u})\|^2 \leq 0$.
Since GMFCQ is satisfied at $\bar{u}$, then $\nabla g_j(\bar{u}) \neq 0$. This is a contradiction to assumption $\nabla g_j(\bar{u}) = 0$. Therefore, $u \in M_1(\mu)$.
It immediately follows from (13),

$$L(u) = L(\bar{u}).$$

This completes the proof. □

We consider the following sets:

$$\hat{K}_1'(\mu) := \{u \in M_1(\mu) : \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle = 0,$$
$$\frac{\nabla f^L(u) + \nabla f^U(u)}{\|\nabla f^L(u) + \nabla f^U(u)\|} = \frac{\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})}{\|\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})\|}, \nabla f^L(u) + \nabla f^U(u) \neq 0\};$$

$$\hat{K}_2'(\mu) := \{u \in M_1(\mu) : \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u} \rangle \leq 0,$$
$$\frac{\nabla f^L(u) + \nabla f^U(u)}{\|\nabla f^L(u) + \nabla f^U(u)\|} = \frac{\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})}{\|\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})\|}, \nabla f^L(u) + \nabla f^U(u) \neq 0\}.$$

Theorem 4.4. *Suppose the interval-valued function f is continuously differentiable and LU-quasiconvex and g_i, $i \in I(\bar{u})$ is differentiable and quasiconvex, g_i, $i \notin I(\bar{u})$ is continuous at $\bar{u}$. Let K be convex set, $\bar{u} \in \overline{K}$, $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \neq 0$, GMFCQ be satisfied and the Lagrange multipliers be known. Then,*

$$\overline{K} = \hat{K}_1'(\mu) = \hat{K}_2'(\mu).$$

Proof. It can be easily seen that $\hat{K}_1'(\mu) = M_1(\mu) \cap \hat{K}_1$ and $\hat{K}_2'(\mu) = M_1(\mu) \cap \hat{K}_2$. Then, from Lemma 4.3 and from relation $\overline{K} = \hat{K}_1 = \hat{K}_2$, the claim is a part of Theorem 3.6. □

Remark 4.5. We assume that $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \neq 0$ in Theorem 4.4, if we assume $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) = 0$, then from Theorem 3.8, $\nabla f^L(u) + \nabla f^U(u) = 0 \ \forall u \in \overline{K}$.

Now, we construct the following sets to study the Lagrange multiplier characterization:

$$K_1'(\mu) := \{u \in K_1(\mu) : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle = 0, \nabla f^L(u) + \nabla f^U(u) \neq 0\},$$

$$K_2'(\mu) := \{u \in K_1(\mu) : \langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle \geq 0, \nabla f^L(u) + \nabla f^U(u) \neq 0\},$$

$$\begin{aligned} K_3'(\mu) : = \{u \in K_1(\mu) : &\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle \\ &= \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle, \nabla f^L(u) + \nabla f^U(u) \neq 0\}, \end{aligned}$$

$$\begin{aligned} K_4'(\mu) : = \{u \in K_1(\mu) : &\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle \\ &\geq \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle, \nabla f^L(u) + \nabla f^U(u) \neq 0\}, \end{aligned}$$

$$\begin{aligned} K_5'(\mu) : = \{u \in K_1(\mu) : &\langle \nabla f^L(u) + \nabla f^U(u), \bar{u} - u\rangle \\ &= \langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle = 0, \nabla f^L(u) + \nabla f^U(u) \neq 0\}. \end{aligned}$$

Theorem 4.6. *Suppose the interval-valued function f is continuously differentiable and LU-quasiconvex and g_i, $i \in I(\bar{u})$ is differentiable and quasiconvex, g_i, $i \notin I(\bar{u})$ is continuous at $\bar{u}$. Let K be convex set, $\bar{u} \in \overline{K}$, $\nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \neq 0$, GMFCQ be satisfied and the Lagrange multipliers be known. Then,*

$$\overline{K} = K_1'(\mu) = K_2'(\mu) = K_3'(\mu) = K_4'(\mu) = K_5'(\mu).$$

Proof. It can be seen that $K_i' = K_1(\mu) \cap K_i, i = 1, 2, 3, 4, 5$. Then, from Lemma 4.3 and relation $\overline{K} = K_1 = K_2 = K_3 = K_4 = K_5$, the claim follows immediately. □

Now, we assume that K is an open set. Then, the KKT optimality conditions reduce to:

$$\lambda^L \nabla f^L(\bar{u}) + \lambda^U \nabla f^U(\bar{u}) + \sum_{i \in I(\bar{u})} \mu_i \nabla g_i(\bar{u}) = 0, \ \mu_i g_i(\bar{u}) = 0 \ \forall \ i = 1, 2, \cdots, m. \tag{17}$$

In this case, GMFCQ reduces to MFCQ, which is stated and defined as:

$$\exists \ d \in \mathbb{R}^n \text{ such that } \langle g_i(\bar{u}), d\rangle < 0 \ \forall i \in I(\bar{u}).$$

Consider the sets:

$$\begin{aligned} &\hat{K}_1''(\mu) := \{u \in K_1(\mu) : \langle \nabla g_i(\bar{u}), u - \bar{u}\rangle = 0, \ i \in I(\bar{u}, \mu), \\ &\frac{\nabla f^L(u) + \nabla f^U(u)}{\|\nabla f^L(u) + \nabla f^U(u)\|} = \frac{\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})}{\|\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})\|}, \nabla f^L(u) + \nabla f^U(u) \neq 0\}; \end{aligned}$$

$$\hat{K}_2''(\mu) := \{u \in K_1(\mu) : \langle \nabla g_i(\bar{u}), u - \bar{u}\rangle \geq 0,\ i \in I(\bar{u}, \mu),$$
$$\frac{\nabla f^L(u) + \nabla f^U(u)}{\|\nabla f^L(u) + \nabla f^U(u)\|} = \frac{\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})}{\|\nabla f^L(\bar{u}) + \nabla f^U(\bar{u})\|}, \nabla f^L(u) + \nabla f^U(u) \neq 0\}.$$

Theorem 4.7. *Suppose the interval-valued function f is continuously differentiable and LU-quasiconvex and $g_i,\ i \in I(\bar{u})$ is differentiable and quasiconvex, $g_i,\ i \notin I(\bar{u})$ is continuous at $\bar{u}$. Let K be open and convex set, $\bar{u} \in \overline{K},\ \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}) \neq 0$, MFCQ be satisfied and the Lagrange multipliers be known. Then,*

$$\overline{K} = \hat{K}_1''(\mu) = \hat{K}_2''(\mu).$$

Proof. It can be seen that $\hat{K}_1''(\mu) \subseteq \hat{K}_2''(\mu)$. Next, we show that $\hat{K}_2''(\mu) \subseteq \overline{K}$. Suppose $\bar{u} \in \hat{K}_2''(\mu)$ is an arbitrary point then from relation (17), we get $\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle \leq 0$, Hence, $u \in \hat{K}_2'(\mu)$. Then, from Theorem 4.3, it follows that $\hat{K}_2'(\mu) = \overline{K}$ and $u \in \overline{K}$.
Now, we show that $\overline{K} \subseteq \hat{K}_1''(\mu)$. Suppose $u \in \overline{K}$ is an arbitrary point. Since, f is LU-quasiconvex and $f^L(u) = f^L(\bar{u}),\ f^U(u) = f^U(\bar{u})$, then it follows that $\langle \nabla f^L(\bar{u}) + \nabla f^U(\bar{u}), u - \bar{u}\rangle \leq 0$. Also, g_i are quasiconvex then,

$$g_i(u) \leq 0 = g_i(\bar{u}) \implies \langle \nabla g_i(\bar{u}), u - \bar{u}\rangle \geq 0\ \forall\ i \in I(\bar{u}).$$

From KKT conditions (17), $\langle \nabla g_i(\bar{u}), u - \bar{u}\rangle = 0\ \forall\ i \in I(\bar{u}, \mu)$. It follows from Theorem 4.4, $\hat{K}_1'(\mu) = \overline{K}$ and $u \in \hat{K}_1'(\mu)$. We conclude from here, $u \in \hat{K}_1''(\mu)$. This completes the proof. □

5. Conclusions and future remarks

Motivated by Zhang *et al.* [34] and Wu [31], we define *LU*-quasiconvex and *LU*-pseudoconvex functions and show that the lower level sets are convex in case of *LU*-quasiconvex function. In this article, we obtain the characterizations of solution sets of interval-valued optimization problem with the *H*-differentiable *LU*-quasiconvex objective function. We construct Gordan's theorem of alternative for interval linear matrix system of inequalities and prove that the normalized gradient of interval-valued function is non-zero constant over the solution set when the interval-valued gradient is not zero at a point. Further, we study Lagrange multiplier characterizations of solutions set of interval-valued optimization problem in form of *LU*-quasiconvex objective function and quasiconvex inequality constraint function. We also derive KKT optimality conditions with the help of generalized Mangasarian-Fromovitz constraint qualifications (GMFCQ). In the future, we can study the second order characterizations of solution sets of interval-valued function motivated by Ivanov [8]. We can obtain the sequential characterizations of approximate solutions in interval-valued convex optimization problems motivated by Sisarat *et al.* [27].

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