ON THE CONFORMABILITY OF REGULAR LINE GRAPHS

Luerbio Faria, Mauro Nigro* and Diana Sasaki

Abstract. Let $G = (V, E)$ be a graph and the deficiency of $G$ be $\text{def}(G) = \sum_{v \in V(G)} (\Delta(G) - d_G(v))$, where $d_G(v)$ is the degree of a vertex $v$ in $G$. A vertex coloring $\phi : V(G) \rightarrow \{1, 2, \ldots, \Delta(G) + 1\}$ is called conformable if the number of color classes (including empty color classes) of parity different from that of $|V(G)|$ is at most $\text{def}(G)$. A general characterization for conformable graphs is unknown. Conformability plays a key role in the total chromatic number theory. It is known that if $G$ is Type 1, then $G$ is conformable. In this paper, we prove that if $G$ is $k$-regular and Class 1, then $L(G)$ is conformable. As an application of this statement we establish that the line graph of complete graph $L(K_n)$ is conformable, which is a positive evidence towards the Vignesh et al.'s conjecture that $L(K_n)$ is Type 1.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph. A $k$-vertex coloring of $G$ is an assignment of $k$ colors to the vertices of $G$ so that adjacent vertices have different colors. A $k$-edge coloring of $G$ is an assignment of $k$ colors to the edges of $G$ so that adjacent edges have different colors. The chromatic index of $G$, denoted by $\chi'(G)$, is the smallest $k$ for which $G$ has a $k$-edge coloring. Vizing's theorem states that the chromatic index $\chi'(G)$ is at least $\Delta(G)$ and at most $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the graph $G$ [13]. Graphs with $\chi'(G) = \Delta(G)$ are called Class 1, and graphs with $\chi'(G) = \Delta(G) + 1$ are called Class 2. Similarly, a $k$-total coloring of $G$ is an assignment of $k$ colors to the vertices and edges of $G$ so that adjacent or incident elements have different colors. The total chromatic number of $G$, denoted by $\chi''(G)$, is the smallest $k$ for which $G$ has a $k$-total coloring. Clearly, $\chi''(G) \geq \Delta(G) + 1$ and the Total Coloring Conjecture (TCC) states that the total chromatic number of any graph is at most $\Delta(G) + 2$ [2,13]. Graphs with $\chi''(G) = \Delta(G) + 1$ are called Type 1, and graphs with $\chi''(G) = \Delta(G) + 2$ are called Type 2.

In 1971, Rosenfeld [10] proved that the TCC holds for cubic graphs. In 1989, Sánchez-Arroyo [11] proved that determining the total chromatic number of an arbitrary graph is a NP-hard problem. The problem remains NP-hard even when $G$ is cubic and bipartite.

Keywords. Vertex coloring, total coloring, conformable coloring.

Rio de Janeiro State University, Rio de Janeiro, Brazil.

*Corresponding author: mauro.nigro@pos.ime.uerj.br; mauronigro94@gmail.com

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A Type 1 graph has a nice structural property due to Chetwynd and Hilton [5]. Let the deficiency of $G$ be $\text{def}(G) = \sum_{v \in V(G)} (\Delta(G) - d_G(v))$, where $d_G(v)$ is the degree of vertex $v$ in $G$. A vertex coloring $\varphi : V(G) \to \{1, 2, \ldots, \Delta(G) + 1\}$ is called conformable if the number of color classes (including empty color classes) of parity different from that of $|V(G)|$ is at most $\text{def}(G)$. Note that if $G$ is a regular graph, then $\varphi$ is called conformable if each color class has the same parity as $|V(G)|$. A graph is said to be conformable if it has a conformable vertex coloring; otherwise, it is said to be non-conformable.

An important connection between Type 1 graphs and conformability was established in Theorem 1.1 and an example is presented in Figure 1.

**Theorem 1.1** (Chetwynd and Hilton [5]). If $G$ is Type 1, then $G$ is conformable. □

Equivalently, if $G$ is non-conformable, then $G$ is not Type 1. A classical example of non-conformable graphs are complete graphs with even order $K_{2n}$ (Fig. 1b). However, there are conformable graphs which are Type 2: For instance, the complete bipartite graphs $K_{n,n}$, for even $n > 1$, and the Mbius ladders $M_{2n}$ (Fig. 1a), for $n > 3$, are conformable and Type 2 [5].

Constructing conformable vertex colorings or establishing their non-existence can be used to tackle total coloring problem. Indeed, every non-conformable graph is not Type 1 and a suitable conformable coloring might be extended to a $(\Delta(G) + 1)$-total coloring. Several recent studies have been conducted on the problem of conformability.

For $n \geq 2$, a star graph $S_n$ is the complete bipartite graph $K_{1,n-1}$. A graph is a power of cycle, denoted by $C_n^k$, if $V(C_n^k) = \{v_0, \ldots, v_{n-1}\}$ and $E(C_n^k) = \bigcup_{i=1}^{k} E'$, where $E' = \{v_jv_{(j+i) \mod n} \mid j \in \{0, \ldots, n-1\}\}$. For $k \geq \lfloor n/2 \rfloor$, $C_n^k$ is, up to multiple edges, isomorphic to the complete graph $K_n$. In 2006, Campos and de Mello [4] proved that for positive integers $n$ and $k$, if $n < 3(k + 1)$ and $n$ is odd, then a power of cycle graph $C_n^k$ is non-conformable, they determined the total chromatic number of $C_n^k$ and conjectured that these non-conformable power of cycle graphs are the unique Type 2 graphs of this class. In 2022, Zorzi et al. [14] proved that if $C_n^k$ is non-conformable, then $n < 3(k + 1)$ and $n$ is odd, determining the conformability of power of cycle graphs.

In 2018, Vignesh et al. [12] conjectured that all line graphs of complete graphs $L(K_n)$ are Type 1. In 2021, Mohan et al. [8] verified the TCC to the set of quasi-line graphs, which is a generalization of line graphs, and presented some infinite families of Type 1 graphs. In 2022, Jayaraman et al. [6] determined the total chromatic number for certain line graphs. In this paper, we establish a relationship between the chromatic index of $G$ and the conformability of $L(G)$, proving that if $G$ is $k$-regular Class 1, then $L(G)$ is conformable. We apply this statement to prove that $L(K_n)$ is conformable, supporting the Vignesh et al.’s [12] conjecture. In addition, we
ON THE CONFORMABILITY OF REGULAR LINE GRAPHS

Figure 2. The case when edge $a_i$ of color $i$ is adjacent to the two edges $a_j^1$ and $a_j^2$ of color $j$.

propose a question about the existence of $L(G)$ non-conformable of a $k$-regular $G$ and investigate this question, by presenting non-conformable graphs which are not line graphs.

2. Line graphs of $k$-regular Class 1 graphs are conformable

In this section, we prove that if $G$ is $k$-regular and Class 1, then $L(G)$ is conformable and that $L(K_n)$ is conformable. Lemma 2.1 relates the regularity of both $G$ and $L(G)$.

Lemma 2.1. If $G$ is $k$-regular, then $L(G)$ is $(2k - 2)$-regular.

The following result is well-known.

Lemma 2.2 (Niessen [9]). If $G = (V, E)$ is $k$-regular and Class 1, then $n = |V|$ is even.

Lemma 2.3. If $G = (V, E)$ is $k$-regular and Class 1 with $|V| \geq 6$, then every $k$-edge coloring $\varphi$ has each color class $\mathcal{C}_i$, $i \in \{1, \ldots, k\}$, of size at least 3.

Proof. Let $G = (V, E)$ be a $k$-regular and Class 1 graph and $\varphi$ a $k$-edge coloring of $G$. Note that each color class of $\varphi$ is a perfect matching of $G$ and these $k$ perfect matchings comprise a partition of $E(G)$. Since $|V| \geq 6$, we have that each perfect matching must have at least 3 edges.

Lemma 2.4. Let $G$ be a graph, with a $k$-edge coloring $\varphi$, where $\mathcal{C}_i$ are color classes of $\varphi$, with $i \in \{1, \ldots, k\}$ and $|\mathcal{C}_i| \geq 3$, then for every pair $i, j \in \{1, \ldots, k\}$ with $i \neq j$, there is a pair of edges $a_i \in \mathcal{C}_i$ and $a_j \in \mathcal{C}_j$ such that $\{a_i, a_j\}$ is a matching.

Proof. Let $i, j \in \{1, \ldots, k\}$ with $i \neq j$. Take $a_i = x_iy_i \in \mathcal{C}_i$. Let $a_j^1, a_j^2, a_j^3 \in \mathcal{C}_j$. Since $\varphi$ is an edge coloring, we claim that at most 2 of those edges share a same endpoint with $a_i$ (Fig. 2). For instance, considering $a_j^1$ and $a_j^2$ adjacent to $a_i$, we have that $\{a_i, a_j^3\}$ is a matching.

Theorem 2.5. Let $G$ be a $k$-regular graph. If $G$ is Class 1, then $L(G)$ is conformable.

Proof. Let $G$ be a $k$-regular graph. If $k = 1$, then $G = K_2$ and $L(G)$ is the trivial graph which is conformable. Assume that $k \geq 2$. Since $G$ is Class 1, we have that $G$ has a $k$-edge coloring $\varphi$ with color classes $\mathcal{C}_i$, $1 \leq i \leq k$. Note that since every vertex has degree $k$ and there are $k$ color classes, then each color class is a perfect matching, $|V(G)| = 2q$ and $|\mathcal{C}_i| = q$, for $q \in \mathbb{N}$. By Lemma 2.1, $L(G)$ is $(2k - 2)$-regular. We construct a $(2k - 1)$-vertex coloring $\phi$ to $L(G)$ using the $k$-edge coloring $\varphi$ of $G$. Note that $l = |V(L(G))| = \frac{|V(G)|k}{2} = qk$ and consider the two cases:
Corollary 2.6. Let $L$ be a $k$-regular graph. If $L(G)$ is non-conformable, then $G$ is Class 2.

Theorem 2.7 (Baranyai [1]). The complete graph $K_n$ is Class 1 if, and only if, $n$ is even.

Theorem 2.8. The graphs $L(K_n)$ are conformable.

Proof. First consider $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$. By definition of line graphs, each edge of $K_n$ is identified with a vertex in $L(K_n)$. Remark that $L(K_n)$ is a $(2n-4)$-regular graph (Lem. 2.1). In each case we provide a conformable vertex coloring for $L(G)$ with $2n-3$ color classes, each one of them with the same parity as $|V(L(K_n))|$.

We split the proof into two cases according to the parity of $n$:

(1) $n$ even – From Theorems 2.5 and 2.7, $L(K_n)$ is conformable.

(2) $n$ odd – Given $n = 2k-1$, where $k$ is a positive integer. Hence, the number of vertices of $L(K_n)$ is $\frac{n(n-1)}{2} = (2k-1)(k-1)$. Next, we consider the two cases, depending on the parity of $k$.

(a) $k$ odd – In this case the number of vertices of $L(K_n)$ is even. We define the $(2n-3)$-vertex coloring of $L(K_n)$ where each color class is even. We set for each $p \in \{1, \ldots, n\}$, the color class of $L(K_n)$, $C_p = \{v_{(p-1)-q} \mod n v_{(p-1)+q} \mod n \mid 1 \leq q \leq \frac{n-1}{2}\}$ (Fig. 3) and for each $p \in \{n+1, \ldots, 2n-3\}$, $C_p = \emptyset$. Note that $C_p$ is a maximum matching of $K_n$, and so it is a maximal independent set of $L(K_n)$. Moreover, $\bigcup_{p=1}^{n} C_p = E(K_n) = V(L(K_n))$. Consequently this vertex coloring is conformable, since for each $p \in \{1, \ldots, n\}$, $|C_p| = \frac{n-1}{2} = k-1$ is even and for each $p \in \{n+1, \ldots, 2n-3\}$, we have $n-3$ empty color classes.

(b) $k$ even – In this case the number of vertices $|V(L(K_n))|$ is $(2k-1)(k-1)$ of $L(K_n)$ is odd. Our conformable coloring consists in $n-2$ maximal matchings of $K_n$ of size $\frac{n-1}{2}$ and $n-1$ matchings of $K_n$ of size 1. Observe that $K_n$ has $\frac{n(n-1)}{2} = (n-1)(n-2) + (n-1)$ edges. We define a conformable coloring to $L(K_n)$, taking for each $p \in \{1, \ldots, n-2\}$, $C_p = \{v_{(p-1)-q} \mod n v_{(p-1)+q} \mod n \mid 1 \leq q \leq \frac{n-1}{2}\}$ (Fig. 4). Since we have $\frac{(n-2)(n-1)}{2}$ colored edges, we have $n-1$ edges which will be colored each one
Figure 3. An example of conformable coloring to $L(K_5)$, where the edges of $K_5$ are colored. For each copy of $K_5$, we have the color class $\mathcal{C}_p$ with $p \in \{1, \ldots, n\}$ and we have $\mathcal{C}_p = \emptyset$, when $p \in \{n+1, \ldots, 2n-3\}$.

Figure 4. An example of conformable coloring to $L(K_7)$, where the edges of $K_7$ are colored. In the first line, for each copy of $K_7$ we present the color class $\mathcal{C}_p$ with $p \in \{1, \ldots, n-2\}$; in the second and third lines we have the $n-1$ unitary color classes.

with a different color. Let $A = \{a_1, \ldots, a_{n-1}\}$ be the set of elements which are not colored with colors $p \in \{1, \ldots, n-2\}$. We define for each $p \in \{n-1, \ldots, 2n-3\}, \mathcal{C}_p = \{a_p-(n-2)\}$. For each $p \in \{1, \ldots, n-2\}$, we have that $|\mathcal{C}_p| = \frac{n-1}{2} = k - 1$ is odd; for each $p \in \{n-1, \ldots, 2n-3\}$, we have that $|\mathcal{C}_p| = 1$, and we use $n - 2 + n - 1 - 2n - 3 = \Delta(L(K_n)) + 1$ color classes. We conclude that this vertex coloring is conformable.

While there exist conformable graphs that are of Type 2, the proof of Theorem 2.8 represents a promising approach towards verifying the conjecture of Vignesh et al. [12]. As an example, the conformable coloring provided in the proof can be extended to a total coloring for the line graph of $K_5$ (Fig. 5).

3. Discussion about the existence of $L(G)$ non-conformable, of $k$-regular graph $G$

We present a connection between the chromatic index of a $k$-regular graph $G$ and the conformability of its line graph $L(G)$ by proving that if $G$ is a $k$-regular and Class 1 graph, then $L(G)$ is conformable. Therefore,
we can conclude that if the line graph \( L(G) \) is non-conformable, then \( G \) is Class 2, where \( G \) is \( k \)-regular. As an application of this result we prove that every line graph of a complete graph \( L(K_n) \) is conformable which is a piece of evidence to the Vignesh et al.’s conjecture [12] which states that every \( L(K_n) \) is Type 1. König [7] proved that bipartite graphs are Class 1. As a consequence of this result together with Theorem 2.5 we have the following corollary.

**Corollary 3.1.** If \( G \) is \( k \)-regular and bipartite, then \( L(G) \) is conformable.

Note that the \( k \)-regularity condition is necessary in Theorem 2.5. For instance, the star graph with an odd order \( S_{2n+1} \) is non-regular and Class 1, while \( L(S_{2n+1}) = K_{2n} \) is non-conformable. For \( k = 2 \), there is a \( k \)-regular graph \( G = C_5 \), such that the line graph \( L(G) = C_5 \) is non-conformable.

These facts motivate us to pose the following question.

**Question 3.2.** Is there a \( k \)-regular graph \( G \), \( k \geq 3 \), such that the line graph \( L(G) \) is non-conformable?

Beineke [3] proved that there are nine minimal graphs that are not line graphs, such that any graph that is not a line graph has some of these nine graphs as an induced subgraph.

**Theorem 3.3 (Beineke [3]).** Let \( H \) be a graph. There exists a graph \( G \) such that \( H = L(G) \) is the line graph of \( G \) if and only if \( H \) contains no induced subgraph from the following set presented in Figure 6.

Those graphs in Figure 6 are forbidden induced subgraphs for line graphs. In an attempt to answer Question 3.2, knowing that there are some known non-conformable power of cycle \( C_n \), we wonder if there is a \( k \)-regular graph \( G \), such that some non-conformable \( C_n^k = L(G) \). Answering, in this case, positively Question 3.2. Unfortunately, we prove that there is no such a graph \( G \) at all, since we use the Beineke’s characterization to prove that every non-conformable power of cycle graph \( C_n^k \) is not a line graph.

We remark that Zorzi et al. [14] characterized the non-conformable power of cycle graphs.

**Theorem 3.4 (Campos and de Mello [4] and Zorzi et al. [14]).** Let \( C_n^k \) be a power of cycle which is neither a cycle nor a complete graph. The power of cycle \( C_n^k \) is non-conformable if, and only if, \( n \) is odd and \( n < 3(k + 1) \).

**Theorem 3.5.** Let \( C_n^k \) be a power of cycle which is neither a cycle nor a complete graph. If \( C_n^k \) is non-conformable, then \( C_n^k \) is not a line graph.
Suppose that $C_n$ from Theorem 3.4.

Since $n$ is odd and $n < 3(k + 1)$. As $C_n^k$ is not the complete graph, $n > 2k + 1$. Since $n$ is odd, $n \geq 2k + 3$.

(1) If $k$ is even, then $3(k + 1)$ is odd. Since $n < 3(k + 1) = 3k + 3$, $n \leq 3k + 1$ and we conclude that $2k + 3 \leq n \leq 3k + 1$. Then, $n$ belongs to the range $2k + 3 = 2k + (2 + 1) \leq n \leq 2k + (2 \cdot \frac{k}{2}) + 1 = 3k + 1$, or $n = 2k + 2j + 1$ with $j \in \{1, \ldots, \frac{k}{2}\}$. Let $C_n^{k} = C_{2k+2j+1}^{k}$ be the power of cycle with $j \in \{1, \ldots, \frac{k}{2}\}$. We provide the forbidden induced subgraph $H[S]$ of $H = C_n^{k}$ induced by the set $S = \{v_0, v_1, v_k, v_{k+2j}, v_{2k+2j}\}$ corresponding to a Beineke graph in Figure 6b. From the definition of power of cycle:

- $N(v_0) = \{v_{k+2j+1}, \ldots, v_{2k+2j}\}$ and so:
  - vertex $v_0$ is adjacent to $v_1$, $v_k$, and $v_{k+2j}$ in $H[S]$;
  - vertex $v_0$ is not adjacent to $v_{k+2j}$ in $H$.
- $N(v_1) = \{v_{k+2j+2}, \ldots, v_{2k+2j}, v_0\}$ and so:
  - vertex $v_1$ is adjacent to $v_0$, $v_k$, and $v_{k+2j}$ in $H[S]$;
  - vertex $v_1$ is not adjacent to $v_{k+2j}$ in $H$.
- $N(v_0) = \{v_0, \ldots, v_{k-1}\}$ and so:
  - vertex $v_k$ is adjacent to $v_0$, $v_1$, and $v_{k+2j}$ in $H[S]$;
  - vertex $v_k$ is not adjacent to $v_{2k+2j}$ in $H$.
- $N(v_{k+2j}) = \{v_{k+2j+1}, \ldots, v_{2k+2j}\}$ and so:
  - vertex $v_{k+2j}$ is adjacent to $v_k$ and $v_{2k+2j}$ in $H[S]$;
  - vertex $v_{k+2j}$ is not adjacent to $v_0$ and $v_1$ in $H$.

Hence, $H[S]$ is isomorphic to a Beineke graph depicted in Figure 6b and so there is a forbidden induced subgraph of $C_n^{k}$ with 5 vertices. From Beineke’s characterization [3], $C_n^{k}$ is not a line graph. For the convenience of the reader we offer in Figure 7 one example with $k = 2$ and $n = 2 \cdot 2 + 2j + 1$ with $j \in \{1\}$; and two examples with $k = 4$ and $n = 2 \cdot 4 + 2j + 1$ with $j \in \{1, 2\}$.

**Figure 6.** The nine minimal graphs that are not line graphs.
If \( k \) is odd, then \( 3(k+1) \) is even. Since \( n < 3(k+1) = 3k + 3 \), \( n \leq 3k + 2 \) and \( 2k + 3 \leq n \leq 3k + 2 \). Then, \( n = 2k + 2j + 1 \) with \( j \in \{1, \ldots, \frac{k-1}{2}, \frac{k+1}{2}\} \).

(a) Suppose that \( j \in \{1, \ldots, \frac{k-1}{2}\} \). Consider the power of cycle \( C_{2k+2j+1}^k \). We provide \( H[S] \) the forbidden induced subgraph for line graphs, subgraph of \( C_n^k \) induced by the set \( S = \{v_0, v_1, v_k, v_{k+2j}, v_{2k+2j}\} \) corresponding to a Beineke graph in Figure 6b. From the definition of power of cycle:

- \( N(v_0) = \{v_{k+2j+1}, \ldots, v_{2k+2j}\} \cup \{v_1, v_2, \ldots, v_k\} \) and so:
  - vertex \( v_0 \) is adjacent to \( v_1, v_k \) and \( v_{2k+2j} \) in \( H[S] \);
  - vertex \( v_0 \) is not adjacent to \( v_{k+2j} \) in \( H \).
- \( N(v_1) = \{v_{k+2j+2}, \ldots, v_{2k+2j}, v_0\} \cup \{v_2, v_3, \ldots, v_k, v_{k+1}\} \) and so:
  - vertex \( v_1 \) is adjacent to \( v_0, v_k \) and \( v_{2k+2j} \) in \( H[S] \);
  - vertex \( v_1 \) is not adjacent to \( v_{k+2j} \) in \( H \).
- \( N(v_k) = \{v_0, \ldots, v_{k-1}\} \cup \{v_{k+1}, \ldots, v_{2k}\} \) and so:
  - vertex \( v_k \) is adjacent to \( v_0, v_1 \) and \( v_{k+2j} \) in \( H[S] \);
  - vertex \( v_k \) is not adjacent to \( v_{2k+2j} \) in \( H \).
- \( N(v_{k+2j}) = \{v_2j, \ldots, v_{2k+2j-1}\} \cup \{v_{k+2j+1}, \ldots, v_{2k+2j}\} \) and so:
  - vertex \( v_{k+2j} \) is adjacent to \( v_k \) and \( v_{2k+2j} \) in \( H[S] \);
  - vertex \( v_{k+2j} \) is not adjacent to \( v_0 \) and \( v_1 \) in \( H \).

Hence, \( H[S] \) is isomorphic to a Beineke graph depicted in Figure 6b and so there is a forbidden induced subgraph of \( C_n^k \) with 5 vertices. From the Beineke’s characterization [3], \( C_n^k \) is not a line graph. For the convenience of the reader we offer in Figure 8 an example with \( k = 3 \) and \( n = 2 \cdot 3 + 2j + 1 \) with \( j \in \{1\} \).

(b) Suppose that \( j = \frac{k+1}{2} \). Consider the power of cycle \( C_{3k+2}^k \). We provide \( H[S] \) the forbidden induced subgraph for line graphs, subgraph of \( C_n^k \) induced by the set \( S = \{v_0, v_1, v_k, v_{2k+1}, v_{3k+1}\} \) corresponding to a Beineke graph in Figure 6h. From the definition of power of cycle:

- \( N(v_0) = \{v_{2k+2}, \ldots, v_{3k+1}\} \cup \{v_1, v_2, \ldots, v_k\} \) and so:
  - vertex \( v_0 \) is adjacent to \( v_1, v_k \) and \( v_{3k+1} \) and in \( H[S] \);
  - vertex \( v_0 \) is not adjacent to \( v_{2k} \) and \( v_{2k+1} \) in \( H \).
- \( N(v_1) = \{v_{2k+3}, \ldots, v_{3k+1}, v_0\} \cup \{v_2, v_3, \ldots, v_k, v_{k+1}\} \) and so:
  - vertex \( v_1 \) is adjacent to \( v_0, v_k \) and \( v_{3k+1} \) in \( H[S] \);
  - vertex \( v_1 \) is not adjacent to \( v_{2k} \) and \( v_{2k+1} \) in \( H \).
- \( N(v_k) = \{v_k, v_{k+1}, \ldots, v_{2k-1}\} \cup \{v_{2k+1}, \ldots, v_{3k}\} \) and so:
  - vertex \( v_k \) is adjacent to \( v_k \) and \( v_{2k+1} \) in \( H[S] \);
  - vertex \( v_{2k} \) is not adjacent to \( v_0, v_1 \) and \( v_{3k+1} \) in \( H \).
- \( N(v_{2k+1}) = \{v_{k+1}, v_{k+2}, \ldots, v_{2k}\} \cup \{v_{2k+2}, \ldots, v_{3k+1}\} \) and so:
ON THE CONFORMABILITY OF REGULAR LINE GRAPHS

Figure 8. The power of cycle graph $H = C_n^k$ with $k$ odd, $j \in \{1, \ldots, \frac{k-1}{2}, \frac{k+1}{2}\}$, $n = 2k+2j+1$ and the corresponding $H[S]$ isomorphic to a forbidden induced subgraph of Beineke (edges in red). In (8a), $H = C_9^3$, $S = \{v_0, v_1, v_6, v_k, v_{2k+2j}\}$ and a Beineke graph is depicted in Figure 6b. In (8b), $H = C_{11}^3$, $S = \{v_0, v_1, v_k, v_{2k}, v_{2k+1}, v_{3k+1}\}$ and a Beineke graph is depicted in Figure 6h.

- vertex $v_{2k+1}$ is adjacent to $v_{2k}$ and $v_{3k+1}$ in $H[S]$;
- vertex $v_{2k+1}$ is not adjacent to $v_0$, $v_k$ and $v_1$ in $H$.

Hence, there is a forbidden induced subgraph of $C_n^k$ with 6 vertices and from Beineke’s characterization [3], $C_n^k$ is not a line graph. For the convenience of the reader we offer in Figure 8 an example with $k = 3$ and $n = 2 \cdot 3 + 2j + 1$ with $j = \frac{3+1}{2} = 2$.

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References


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