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ON THE CONFORMABILITY OF REGULAR LINE GRAPHS

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Abstract. Let $G = (V, E)$ be a graph and the *deficiency of* G be $def(G) = \sum_{v \in V(G)}(\Delta(G) - d_G(v))$, where $d_G(v)$ is the degree of a vertex v in G. A vertex coloring $\varphi : V(G) \to \{1, 2, \ldots, \Delta(G) + 1\}$ is called *conformable* if the number of color classes (including empty color classes) of parity different from that of $|V(G)|$ is at most $def(G)$. A general characterization for conformable graphs is unknown. Conformability plays a key role in the total chromatic number theory. It is known that if G is *Type 1*, then G is conformable. In this paper, we prove that if G is k-regular and *Class 1*, then $L(G)$ is conformable. As an application of this statement we establish that the line graph of complete graph $L(K_n)$ is conformable, which is a positive evidence towards the Vignesh *et al.*'s conjecture that $L(K_n)$ is *Type 1*.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph. A *k-vertex coloring of* G is an assignment of k colors to the vertices of G so that adjacent vertices have different colors. A *k-edge coloring of* G is an assignment of k colors to the edges of G so that adjacent edges have different colors. The *chromatic index of* G, denoted by $\chi'(G)$, is the smallest k for which G has a k-edge coloring. Vizing's theorem states that the chromatic index $\chi'(G)$ is at least $\Delta(G)$ and at most $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the graph G [13]. Graphs with $\chi'(G) = \Delta(G)$ are called *Class 1*, and graphs with $\chi'(G) = \Delta(G) + 1$ are called *Class 2*. Similarly, a *k-total coloring of* G is an assignment of k colors to the vertices and edges of G so that adjacent or incident elements have different colors. The *total chromatic number of* G, denoted by $\chi''(G)$, is the smallest k for which G has a k-total coloring. Clearly, $\chi''(G) \geq \Delta(G) + 1$ and the Total Coloring Conjecture (TCC) states that the total chromatic number of any graph is at most $\Delta(G) + 2$ [2, 13]. Graphs with $\chi''(G) = \Delta(G) + 1$ are called *Type 1*, and graphs with $\chi''(G) = \Delta(G) + 2$ are called *Type 2*.

In 1971, Rosenfeld [10] proved that the TCC holds for cubic graphs. In 1989, Sánchez-Arroyo [11] proved that determining the total chromatic number of an arbitrary graph is a NP-hard problem. The problem remains NP-hard even when G is cubic and bipartite.

Keywords. Vertex coloring, total coloring, conformable coloring.

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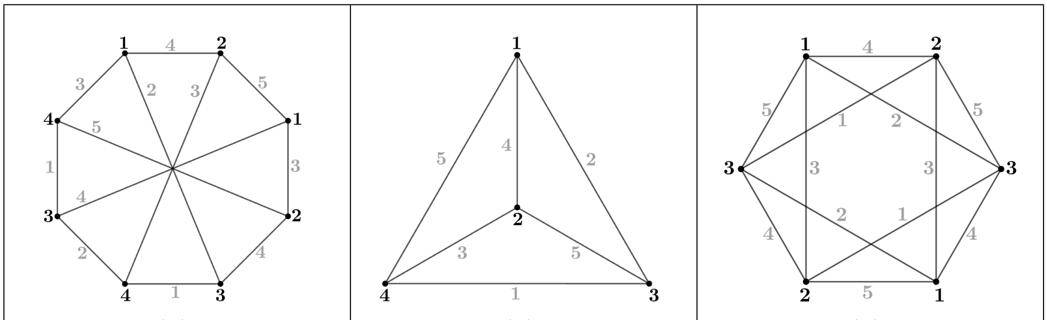

FIGURE 1. In (1a), the Mbius ladder graph M_8 with 8 vertices which is conformable, but is *Type 2*. In (1b), the complete graph $L(S_5) = K_4$ which is non-conformable and consequently is *Type 2*. In (1c), the power of cycle $L(K_4) = C_6^2$ with 6 vertices which is *Type 1* and consequently conformable.

A *Type 1* graph has a nice structural property due to Chetwynd and Hilton [5]. Let the *deficiency of* G be $def(G) = \sum_{v \in V(G)} (\Delta(G) - d_G(v))$, where $d_G(v)$ is the degree of vertex v in G. A vertex coloring $\varphi : V(G) \to \{1, 2, \ldots, \Delta(G)+1\}$ is called *conformable* if the number of color classes (including empty color classes) of parity different from that of $|V(G)|$ is at most $def(G)$. Note that if G is a regular graph, then φ is called conformable if each color class has the same parity as $|V(G)|$. A graph is said to be *conformable* if it has a conformable vertex coloring; otherwise, it is said to be *non-conformable*.

An important connection between *Type 1* graphs and conformability was established in Theorem 1.1 and an example is presented in Figure 1.

Theorem 1.1 (Chetwynd and Hilton [5])**.** *If G is Type 1, then G is conformable.* □

Equivalently, if G is non-conformable, then G is not *Type 1*. A classical example of non-conformable graphs are complete graphs with even order K_{2n} (Fig. 1b). However, there are conformable graphs which are *Type 2*: For instance, the complete bipartite graphs $K_{n,n}$, for even $n > 1$, and the Mbius ladders M_{2n} (Fig. 1a), for $n > 3$, are conformable and *Type 2* [5].

Constructing conformable vertex colorings or establishing their non-existence can be used to tackle total coloring problem. Indeed, every non-conformable graph is not *Type 1* and a suitable conformable coloring might be extended to a $(\Delta(G) + 1)$-total coloring. Several recent studies have been conducted on the problem of conformability.

For $n \geq 2$, a *star graph* S_n is the complete bipartite graph $K_{1,n-1}$. A graph is a *power of cycle*, denoted by C_n^k, if $V(C_n^k) = \{v_0, \ldots, v_{n-1}\}$ and $E(C_n^k) = \bigcup_{i=1}^{n} E^i$, where $E^i = \{v_j v_{(j+i) \bmod n} \mid j \in \{0, \ldots n-1\}\}$. For $k \geq \lfloor n/2 \rfloor$, C_n^k is, up to multiple edges, isomorphic to the complete graph K_n. In 2006, Campos and de Mello [4] proved that for positive integers n and k, if $n < 3(k+1)$ and n is odd, then a power of cycle graph C_n^k is non-conformable, they determined the total chromatic number of C_n^2 and conjectured that these non-conformable power of cycle graphs are the unique *Type 2* graphs of this class. In 2022, Zorzi *et al.* [14] proved that if C_n^k is non-conformable, then $n < 3(k+1)$ and n is odd, determining the conformability of power of cycle graphs.

In 2018, Vignesh *et al.* [12] conjectured that all line graphs of complete graphs $L(K_n)$ are *Type 1*. In 2021, Mohan *et al.* [8] verified the TCC to the set of quasi-line graphs, which is a generalization of line graphs, and presented some infinite families of *Type 1* graphs. In 2022, Jayaraman *et al.* [6] determined the total chromatic number for certain line graphs. In this paper, we establish a relationship between the chromatic index of G and the conformability of $L(G)$, proving that if G is k-regular *Class 1*, then $L(G)$ is conformable. We apply this statement to prove that $L(K_n)$ is conformable, supporting the Vignesh *et al.*'s [12] conjecture. In addition, we

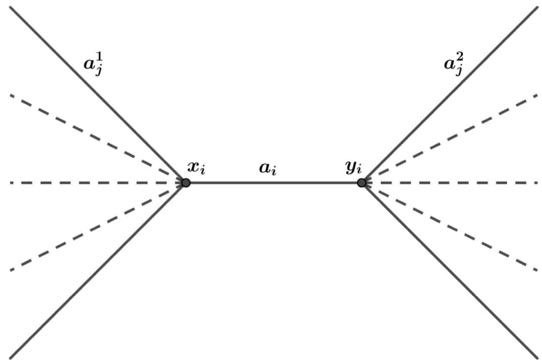

FIGURE 2. The case when edge a_i of color i is adjacent to the two edges a_j^1 and a_j^2 of color j.

propose a question about the existence of $L(G)$ non-conformable of a k-regular G and investigate this question, by presenting non-conformable graphs which are not line graphs.

2. Line graphs of k-regular Class 1 graphs are conformable

In this section, we prove that if G is k-regular and *Class 1*, then $L(G)$ is conformable and that $L(K_n)$ is conformable. Lemma 2.1 relates the regularity of both G and $L(G)$.

Lemma 2.1. *If G is k-regular, then $L(G)$ is $(2k-2)$-regular.* □

The following result is well-known.

Lemma 2.2 (Niessen [9])**.** *If $G = (V, E)$ is k-regular and Class 1, then $n = |V|$ is even.* □

Lemma 2.3. *If $G = (V, E)$ is k-regular and Class 1 with $|V| \geq 6$, then every k-edge coloring φ has each color class $\mathcal{C}_i$, $i \in \{1, \dots, k\}$, of size at least 3.*

Proof. Let $G = (V, E)$ be a k-regular and *Class 1* graph and φ a k-edge coloring of G. Note that each color class of φ is a perfect matching of G and these k perfect matchings comprise a partition of $E(G)$. Since $|V| \geq 6$, we have that each perfect matching must have at least 3 edges. □

Lemma 2.4. *Let G be a graph, with a k-edge coloring φ, where $\mathcal{C}_i$ are color classes of φ, with $i \in \{1, \dots, k\}$ and $|\mathcal{C}_i| \geq 3$, then for every pair $i, j \in \{1, \dots, k\}$ with $i \neq j$, there is a pair of edges $a_i \in \mathcal{C}_i$ and $a_j \in \mathcal{C}_j$ such that $\{a_i, a_j\}$ is a matching.*

Proof. Let $i, j \in \{1, \dots, k\}$ with $i \neq j$. Take $a_i = x_i y_i \in \mathcal{C}_i$. Let a_j^1, a_j^2, $a_j^3 \in \mathcal{C}_j$. Since φ is an edge coloring, we claim that at most 2 of those edges share a same endpoint with a_i (Fig. 2). For instance, considering a_j^1 and a_j^2 adjacent to a_i, we have that $\{a_i, a_j^3\}$ is a matching. □

Theorem 2.5. *Let G be a k-regular graph. If G is Class 1, then $L(G)$ is conformable.*

Proof. Let G be a k-regular graph. If $k = 1$, then $G = K_2$ and $L(G)$ is the trivial graph which is conformable. Assume that $k \geq 2$. Since G is *Class 1*, we have that G has a k-edge coloring φ with color classes $\mathcal{C}_i$, $1 \leq i \leq k$. Note that since every vertex has degree k and there are k color classes, then each color class is a perfect matching, $|V(G)| = 2q$ and $|\mathcal{C}_i| = q$, for $q \in \mathbb{N}$. By Lemma 2.1, $L(G)$ is $(2k-2)$-regular. We construct a $(2k-1)$-vertex coloring ϕ to $L(G)$ using the k-edge coloring φ of G. Note that $l = |V(L(G))| = \frac{|V(G)| \cdot k}{2} = qk$ and consider the two cases:

(1) If k is even, then l is even and we consider the following two cases:
   (a) If q is even, then we define a $(2k-1)$-vertex coloring ϕ to $L(G)$ with color classes $\mathcal{D}_i = \mathcal{C}_i$ where $i \in \{1, \ldots, k\}$; and $\mathcal{D}_i = \emptyset$ where $k+1 \leq i \leq 2k-1$. Since each color class of ϕ is even and l is even, we conclude that ϕ is conformable and $L(G)$ is conformable.
   (b) If q is odd, then we define a $(2k-1)$-vertex coloring ϕ to $L(G)$ where each color class is even. Our approach consists in taking for each $i \in \{1, \ldots, k\}$ one vertex a_i of $\mathcal{C}_i$ and assign the color j to the vertices $a_{2j-2k-1}$ and a_{2j-2k}, where $j \in \{k+1, \ldots, \frac{3k}{2}\}$. By Lemma 2.4, we have that for each $j \in \{k+1, \ldots, \frac{3k}{2}\}$ the matching of G is $\{a_{2j-2k-1}, a_{2j-2k}\}$. Therefore, we give a procedure to yield a conformable coloring to $L(G)$, taking as the color classes of ϕ the sets: $\mathcal{D}_i = \mathcal{C}_i \setminus \{a_i\}$ where $a_i \in \mathcal{C}_i$ and $i \in \{1, \ldots, k\}$; $\mathcal{D}_i = \{a_{2i-2k-1}, a_{2i-2k}\}$ where $i \in \{k+1, \ldots, \frac{3k}{2}\}$; and $\mathcal{D}_i = \emptyset$ where $i \in \{\frac{3k}{2}+1, \ldots, 2k-1\}$. Since ϕ has k, $\frac{k}{2}$, and $\frac{k}{2}-1$ color classes of sizes $q-1$, 2, and 0, respectively, it follows that ϕ is conformable and thus $L(G)$ is conformable.

(2) If k is odd, then we consider the following two cases:
   (a) If q is even, then l is even. We consider a $(2k-1)$-vertex coloring ϕ to $L(G)$ similarly to case (1a) and since l and q are even, we conclude that G is conformable.
   (b) If q is odd, then $l = k \cdot q$ is odd. We define a $(2k-1)$-vertex coloring ϕ to $L(G)$ where each color class is odd. Our approach consists in taking for each $i \in \{1, \ldots, \frac{k-1}{2}\}$ two vertices a_i and b_i of $\mathcal{C}_i$ and assign, respectively, the colors $k+2i-1$ and $k+2i$. We give a procedure to yield a conformable coloring to $L(G)$, taking as the color classes of ϕ the sets: $\mathcal{D}_i = \mathcal{C}_i \setminus \{a_i, b_i\}$ where $a_i, b_i \in \mathcal{C}_i$ and $i \in \{1, \ldots, \frac{k-1}{2}\}$; $\mathcal{D}_i = \mathcal{C}_i$ where $i \in \{\frac{k+1}{2}, \ldots, k\}$ and; $\{a_i\}$, $\{b_i\}$ where $i \in \{1, \ldots, \frac{k-1}{2}\}$. Since ϕ has $\frac{k-1}{2}$, $\frac{k+1}{2}$ and $k-1$ color classes of sizes $q-2$, q and 1, respectively, we conclude that each color class of ϕ is odd and G is conformable.

$\square$

Corollary 2.6. *Let G be a k-regular graph. If $L(G)$ is non-conformable, then G is Class 2.* $\square$

Theorem 2.7 (Baranyai [1])**.** *The complete graph K_n is Class 1 if, and only if, n is even.* $\square$

Theorem 2.8. *The graphs $L(K_n)$ are conformable.*

Proof. First consider $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$. By definition of line graphs, each edge of K_n is identified with a vertex in $L(K_n)$. Remark that $L(K_n)$ is a $(2n-4)$-regular graph (Lem. 2.1). In each case we provide a conformable vertex coloring for $L(G)$ with $2n-3$ color classes, each one of them with same parity as $|V(L(K_n))|$.

We split the proof into two cases according to the parity of n:

(1) n even – From Theorems 2.5 and 2.7, $L(K_n)$ is conformable.

(2) n odd – Given $n = 2k-1$, where k is a positive integer. Hence, the number of vertices of $L(K_n)$ is $\frac{n(n-1)}{2} = (2k-1)(k-1)$. Next, we consider the two cases, depending on the parity of k.
   (a) k odd – In this case the number of vertices of $L(K_n)$ is even. We define the $(2n-3)$-vertex coloring of $L(K_n)$ where each color class is even. We set for each $p \in \{1, \ldots, n\}$, the color class of $L(K_n)$, $\mathcal{C}_p = \left\{v_{(p-1-q) \bmod n} v_{(p-1+q) \bmod n} \mid 1 \leq q \leq \frac{n-1}{2}\right\}$ (Fig. 3) and for each $p \in \{n+1, \ldots, 2n-3\}$, $\mathcal{C}_p = \emptyset$. Note that $\mathcal{C}_p$ is a maximum matching of K_n, and so it is a maximal independent set of $L(K_n)$. Moreover, $\bigcup_{p=1}^{n} \mathcal{C}_p = E(K_n) = V(L(K_n))$. Consequently this vertex coloring is conformable, since for each $p \in \{1, \ldots n\}$, $|\mathcal{C}_p| = \frac{n-1}{2} = k-1$ is even and for each $p \in \{n+1, \ldots, 2n-3\}$, we have $n-3$ empty color classes.
   (b) k is even – In this case the number of vertices $|V(L(K_n))| = (2k-1)(k-1)$ of $L(K_n)$ is odd. Our conformable coloring consists in $n-2$ maximal matchings of K_n of size $\frac{n-1}{2}$ and $n-1$ matchings of K_n of size 1. Observe that K_n has $\frac{n(n-1)}{2} = \frac{(n-1)(n-2)}{2} + (n-1)$ edges. We define a conformable coloring to $L(K_n)$, taking for each $p \in \{1, \ldots, n-2\}$, $\mathcal{C}_p = \left\{v_{(p-1-q) \bmod n} v_{(p-1+q) \bmod n} \mid 1 \leq q \leq \frac{n-1}{2}\right\}$ (Fig. 4). Since we have $\frac{(n-2)(n-1)}{2}$ colored edges, we have $n-1$ edges which will be colored each one

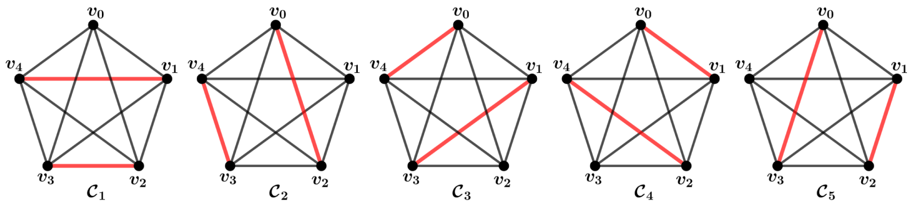

FIGURE 3. An example of conformable coloring to $L(K_5)$, where the edges of K_5 are colored. For each copy of K_5, we have the color class $\mathcal{C}_p$ with $p \in \{1, \ldots, n\}$ and we have $\mathcal{C}_p = \emptyset$, when $p \in \{n+1, \ldots, 2n-3\}$.

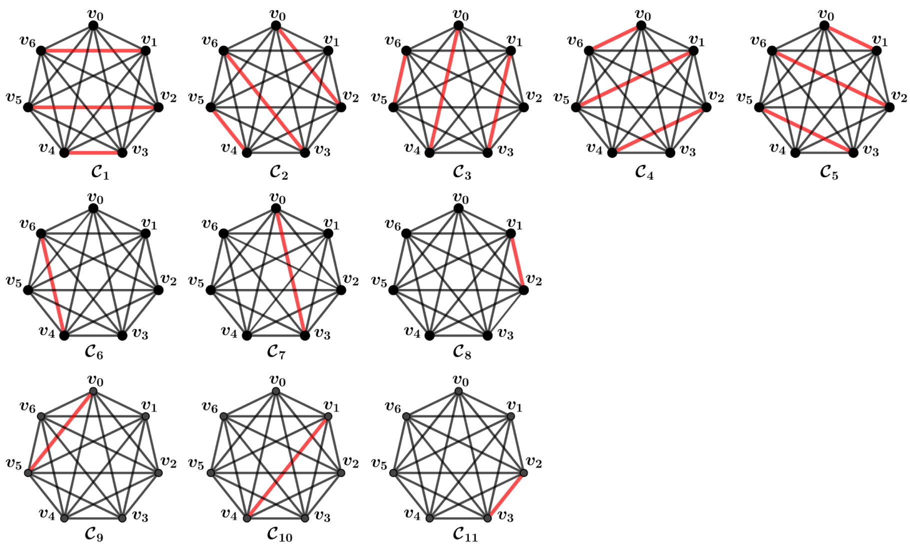

FIGURE 4. An example of conformable coloring to $L(K_7)$, where the edges of K_7 are colored. In the first line, for each copy of K_7 we present the color class $\mathcal{C}_p$ with $p \in \{1, \ldots, n-2\}$; in the second and third lines we have the $n-1$ unitary color classes.

with a different color. Let $A = \{a_1, \ldots, a_{n-1}\}$ be the set of elements which are not colored with colors $p \in \{1, \ldots, n-2\}$. We define for each $p \in \{n-1, \ldots, 2n-3\}$, $\mathcal{C}_p = \{a_{p-(n-2)}\}$. For each $p \in \{1, \ldots, n-2\}$, we have that $|\mathcal{C}_p| = \frac{n-1}{2} = k-1$ is odd; for each $p \in \{n-1, \ldots, 2n-3\}$, we have that $|\mathcal{C}_p| = 1$, and we use $n-2+n-1 = 2n-3 = \Delta(L(K_n))+1$ color classes. We conclude that this vertex coloring is conformable.

□

While there exist conformable graphs that are of *Type 2*, the proof of Theorem 2.8 represents a promising approach towards verifying the conjecture of Vignesh *et al.* [12]. As an example, the conformable coloring provided in the proof can be extended to a total coloring for the line graph of K_5 (Fig. 5).

3. Discussion about the existence of $L(G)$ non-conformable, of k-regular graph G

We present a connection between the chromatic index of a k-regular graph G and the conformability of its line graph $L(G)$ by proving that if G is a k-regular and *Class 1* graph, then $L(G)$ is conformable. Therefore,

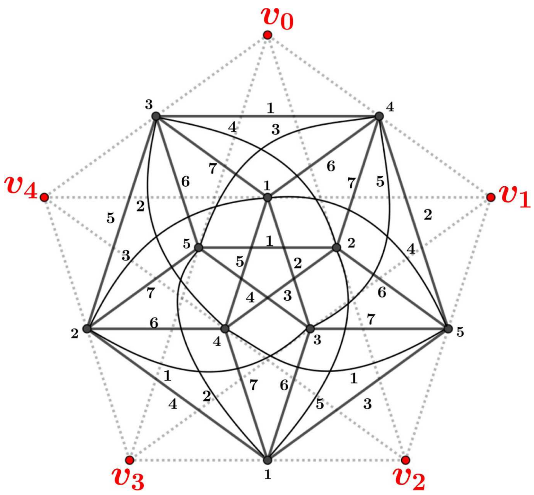

FIGURE 5. A $(\Delta(L(K_5)) + 1)$-total coloring to $L(K_5)$.

we can conclude that if the line graph $L(G)$ is non-conformable, then G is *Class 2*, where G is k-regular. As an application of this result we prove that every line graph of a complete graph $L(K_n)$ is conformable which is a piece of evidence to the Vignesh *et al.*'s conjecture [12] which states that every $L(K_n)$ is *Type 1*. König [7] proved that bipartite graphs are *Class 1*. As a consequence of this result together with Theorem 2.5 we have the following corollary.

Corollary 3.1. *If G is k-regular and bipartite, then $L(G)$ is conformable.* □

Note that the k-regularity condition is necessary in Theorem 2.5. For instance, the star graph with an odd order S_{2n+1} is non-regular and *Class 1*, while $L(S_{2n+1}) = K_{2n}$ is non-conformable. For $k = 2$, there is a k-regular graph $G = C_5$, such that the line graph $L(G) = C_5$ is non-conformable.

These facts motivate us to pose the following question.

Question 3.2. Is there a k-regular graph G, $k \geq 3$, such that the line graph $L(G)$ is non-conformable?

Beineke [3] proved that there are nine minimal graphs that are not line graphs, such that any graph that is not a line graph has some of these nine graphs as an induced subgraph.

Theorem 3.3 (Beineke [3])**.** *Let H be a graph. There exists a graph G such that $H = L(G)$ is the line graph of G if and only if H contains no induced subgraph from the following set presented in Figure 6.* □

Those graphs in Figure 6 are forbidden induced subgraphs for line graphs. In an attempt to answer Question 3.2, knowing that there are some known non-conformable power of cycle C_n^k, we wonder if there is a k-regular graph G, such that some non-conformable $C_n^k = L(G)$. Answering, in this case, positively Question 3.2. Unfortunately, we prove that there is no such a graph G at all, since we use the Beineke's characterization to prove that every non-conformable power of cycle graph C_n^k is not a line graph.

We remark that Zorzi *et al.* [14] characterized the non-conformable power of cycle graphs.

Theorem 3.4 (Campos and de Mello [4] and Zorzi *et al.* [14])**.** *Let C_n^k be a power of cycle which is neither a cycle nor a complete graph. The power of cycle C_n^k is non-conformable if, and only if, n is odd and $n < 3(k + 1)$.* □

Theorem 3.5. *Let C_n^k be a power of cycle which is neither a cycle nor a complete graph. If C_n^k is non-conformable, then C_n^k is not a line graph.*

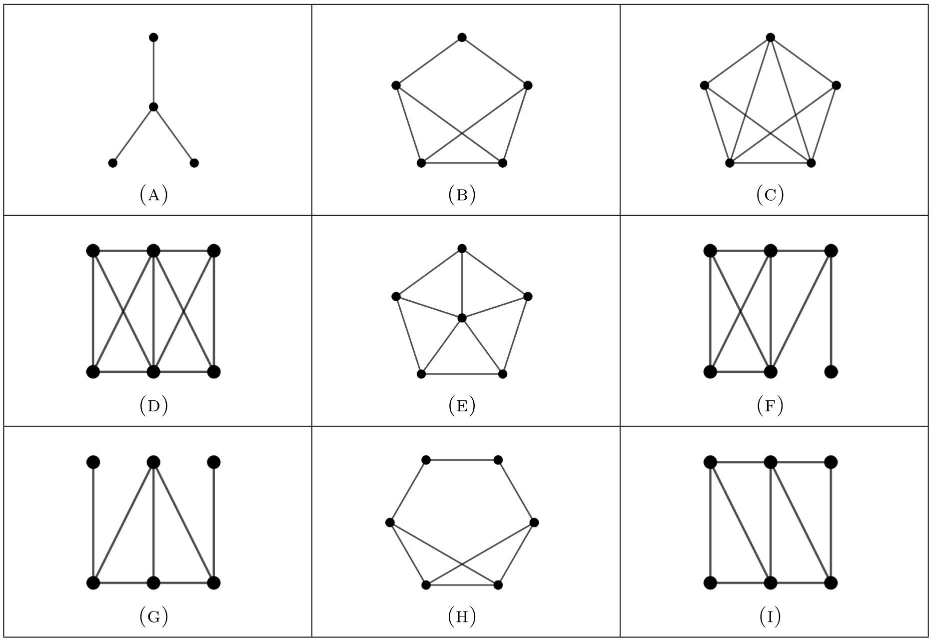

FIGURE 6. The nine minimal graphs that are not line graphs.

Proof. Suppose that $H = C_n^k$ is a non-conformable power of cycle which is neither a cycle nor a complete graph. From Theorem 3.4, C_n^k satisfies that, n is odd and $n < 3(k+1)$. As C_n^k is not the complete graph, $n > 2k+1$. Since n is odd, $n \geq 2k+3$.

(1) If k is even, then $3(k+1)$ is odd. Since $n < 3(k+1) = 3k+3$, $n \leq 3k+1$ and we conclude that $2k+3 \leq n \leq 3k+1$. Then, n belongs to the range $2k+3 = 2k+(2+1) \leq n \leq 2k+\left(2\left(\frac{k}{2}\right)+1\right) = 3k+1$, or $n = 2k+2j+1$ with $j \in \{1,\ldots,\frac{k}{2}\}$. Let $C_n^k = C_{2k+2j+1}^k$ be the power of cycle with $j \in \{1,\ldots,\frac{k}{2}\}$. We provide the forbidden induced subgraph $H[S]$ of $H = C_n^k$ induced by the set $S = \{v_0, v_1, v_k, v_{k+2j}, v_{2k+2j}\}$ corresponding to a Beineke graph in Figure 6b. From the definition of power of cycle:
   - $N(v_0) = \{v_{k+2j+1},\ldots,v_{2k+2j}\} \cup \{v_1, v_2, \ldots, v_k\}$ and so:
     - vertex v_0 is adjacent to v_1, v_k and v_{2k+2j} in $H[S]$;
     - vertex v_0 is not adjacent to v_{k+2j} in H.
   - $N(v_1) = \{v_{k+2j+2},\ldots,v_{2k+2j}, v_0\} \cup \{v_2, v_3, \ldots, v_k, v_{k+1}\}$ and so:
     - vertex v_1 is adjacent to v_0, v_k and v_{2k+2j} in $H[S]$;
     - vertex v_1 is not adjacent to v_{k+2j} in H.
   - $N(v_k) = \{v_0,\ldots,v_{k-1}\} \cup \{v_{k+1},\ldots,v_{2k}\}$ and so:
     - vertex v_k is adjacent to v_0, v_1 and v_{k+2j} in in $H[S]$;
     - vertex v_k is not adjacent to v_{2k+2j} in H.
   - $N(v_{k+2j}) = \{v_{2j},\ldots,v_{k+2j-1}\} \cup \{v_{k+2j+1},\ldots,v_{2k+2j}\}$ and so:
     - vertex v_{k+2j} is adjacent to v_k and v_{2k+2j} in $H[S]$;
     - vertex v_{k+2j} is not adjacent to v_0 and v_1 in H.

   Hence, $H[S]$ is isomorphic to a Beineke graph depicted in Figure 6b and so there is a forbidden induced subgraph of C_n^k with 5 vertices. From Beineke's characterization [3], C_n^k is not a line graph. For the convenience of the reader we offer in Figure 7 one example with $k = 2$ and $n = 2 \cdot 2 + 2j + 1$ with $j \in \{1\}$; and two examples with $k = 4$ and $n = 2 \cdot 4 + 2j + 1$ with $j \in \{1, 2\}$.

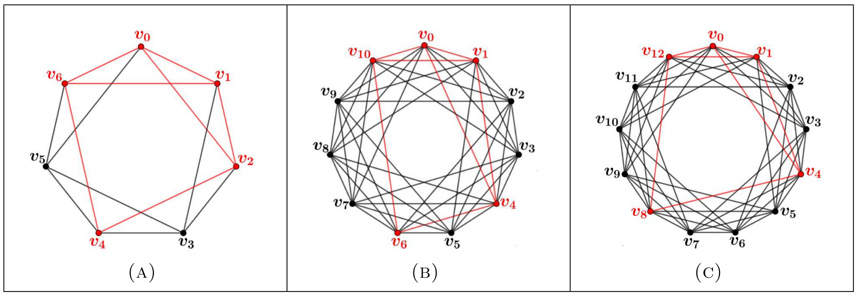

FIGURE 7. The power of cycle graph $H = C_n^k$ with k even, $j \in \{1, \ldots, \frac{k}{2}\}$ and $n = 2k+2j+1$. In (7a), $H = C_7^2$; in (7b), $H = C_{11}^4$; and in (7c), $H = C_{13}^4$. In each case, we depict the forbidden induced subgraph $H[S]$ (edges in red) isomorphic to a Beineke graph depicted in Figure 6b, where $S = \{v_0, v_1, v_k, v_{k+2j}, v_{2k+2j}\}$.

(2) If k is odd, then $3(k+1)$ is even. Since $n < 3(k+1) = 3k+3$, $n \leq 3k+2$ and $2k+3 \leq n \leq 3k+2$. Then, $n = 2k+2j+1$ with $j \in \{1, \ldots, \frac{k-1}{2}, \frac{k+1}{2}\}$.
  (a) Suppose that $j \in \{1, \ldots, \frac{k-1}{2}\}$. Consider the power of cycle $C_{2k+2j+1}^k$. We provide $H[S]$ the forbidden induced subgraph for line graphs, subgraph of C_n^k induced by the set $S = \{v_0, v_1, v_k, v_{k+2j}, v_{2k+2j}\}$ corresponding to a Beineke graph in Figure 6b. From the definition of power of cycle:
    – $N(v_0) = \{v_{k+2j+1}, \ldots, v_{2k+2j}\} \cup \{v_1, v_2, \ldots, v_k\}$ and so:
      • vertex v_0 is adjacent to v_1, v_k and v_{2k+2j} in $H[S]$;
      • vertex v_0 is not adjacent to v_{k+2j} in H.
    – $N(v_1) = \{v_{k+2j+2}, \ldots, v_{2k+2j}, v_0\} \cup \{v_2, v_3, \ldots, v_k, v_{k+1}\}$ and so:
      • vertex v_1 is adjacent to v_0, v_k and v_{2k+2j} in $H[S]$;
      • vertex v_1 is not adjacent to v_{k+2j} in H.
    – $N(v_k) = \{v_0, \ldots, v_{k-1}\} \cup \{v_{k+1}, \ldots, v_{2k}\}$ and so:
      • vertex v_k is adjacent to v_0, v_1 and v_{k+2j} in $H[S]$;
      • vertex v_k is not adjacent to v_{2k+2j} in H.
    – $N(v_{k+2j}) = \{v_{2j}, \ldots, v_{k+2j-1}\} \cup \{v_{k+2j+1}, \ldots, v_{2k+2j}\}$ and so:
      • vertex v_{k+2j} is adjacent to v_k and v_{2k+2j} in $H[S]$;
      • vertex v_{k+2j} is not adjacent to v_0 and v_1 in H.

  Hence, $H[S]$ is isomorphic to a Beineke graph depicted in Figure 6b and so there is a forbidden induced subgraph of C_n^k with 5 vertices. From the Beineke's characterization [3], C_n^k is not a line graph. For the convenience of the reader we offer in Figure 8 an example with $k = 3$ and $n = 2 \cdot 3 + 2j + 1$ with $j \in \{1\}$.
  (b) Suppose that $j = \frac{k+1}{2}$. Consider the power of cycle C_{3k+2}^k. We provide $H[S]$ the forbidden induced subgraph for line graphs, subgraph of C_n^k induced by the set $S = \{v_0, v_1, v_k, v_{2k}, v_{2k+1}, v_{3k+1}\}$ corresponding to a Beineke graph in Figure 6h. From the definition of power of cycle:
    – $N(v_0) = \{v_{2k+2}, \ldots, v_{3k+1}\} \cup \{v_1, v_2, \ldots, v_k\}$ and so:
      • vertex v_0 is adjacent to v_1, v_k and v_{3k+1} and in $H[S]$;
      • vertex v_0 is not adjacent to v_{2k} and v_{2k+1} in H.
    – $N(v_1) = \{v_{2k+3}, \ldots, v_{3k+1}, v_0\} \cup \{v_2, v_3, \ldots, v_k, v_{k+1}\}$ and so:
      • vertex v_1 is adjacent to v_0, v_k and v_{3k+1} in $H[S]$;
      • vertex v_1 is not adjacent to v_{2k} and v_{2k+1} in H.
    – $N(v_{2k}) = \{v_k, v_{k+1}, \ldots, v_{2k-1}\} \cup \{v_{2k+1}, \ldots, v_{3k}\}$ and so:
      • vertex v_{2k} is adjacent to v_k and v_{2k+1} in $H[S]$;
      • vertex v_{2k} is not adjacent to v_0, v_1 and v_{3k+1} in H.
    – $N(v_{2k+1}) = \{v_{k+1}, v_{k+2}, \ldots, v_{2k}\} \cup \{v_{2k+2}, \ldots, v_{3k+1}\}$ and so:

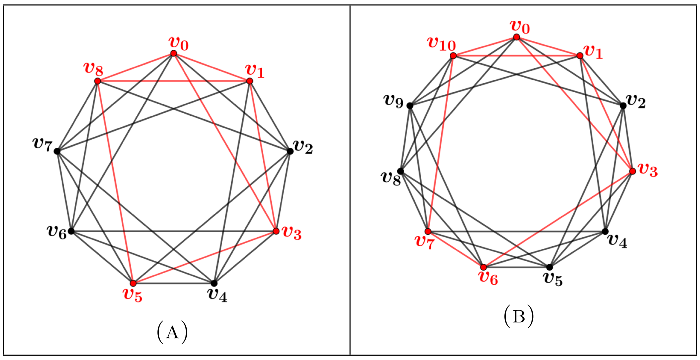

FIGURE 8. The power of cycle graph $H = C_n^k$ with k odd, $j \in \{1, \ldots, \frac{k-1}{2}, \frac{k+1}{2}\}$, $n = 2k+2j+1$ and the corresponding $H[S]$ isomorphic to a forbidden induced subgraph of Beineke (edges in red). In (8a), $H = C_9^3$, $S = \{v_0, v_1, v_k, v_{k+2j}, v_{2k+2j}\}$ and a Beineke graph is depicted in Figure 6b. In (8b), $H = C_{11}^3$, $S = \{v_0, v_1, v_k, v_{2k}, v_{2k+1}, v_{3k+1}\}$ and a Beineke graph is depicted in Figure 6h.

- vertex v_{2k+1} is adjacent to v_{2k} and v_{3k+1} in $H[S]$;
- vertex v_{2k+1} is not adjacent to v_0, v_k and v_1 in H.

Hence, there is a forbidden induced subgraph of C_n^k with 6 vertices and from Beineke's characterization [3], C_n^k is not a line graph. For the convenience of the reader we offer in Figure 8 an example with $k = 3$ and $n = 2 \cdot 3 + 2j + 1$ with $j = \frac{3+1}{2} = 2$.

□

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