REDUCIBLE 3-CRITICAL GRAPHS

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Abstract. In this paper, we provide further insights into the reducibility of 3-critical graphs. A graph is 3-critical if it is class two, that is, has chromatic index 4, and the removal of any one edge renders a graph with chromatic index 3. We consider two types of reductions: the suppression of a 2-connected subgraph into a single edge; and the suppression of a 3-connected subgraph into a single vertex. That is, in cases where the 2- or 3-connected subgraph is cubic or strictly subcubic. We show that every cyclically 2-connected 3-critical graph can be reduced to a smaller 3-critical graph using this method. We also prove that every 3-critical graph which is cyclically \( k \)-connected with \( k = 2 \) or \( k = 3 \), contains a 3-critical minor which is cyclically \( k \)-connected with \( k \geq 4 \).

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1. Introduction

A strictly subcubic graph has maximum degree three and minimum degree less than three. A subcubic class two graph is a subcubic graph which cannot have its edges coloured with three colours, such that no two adjacent edges have the same colour. If it can be coloured in this way with three colours, then it is class one. A 3-critical graph is a subcubic class two graph such that the removal of any one edge renders a class one graph. Cubic class two graphs are known as snarks. No snark can be 3-critical, since the minimum number of edges that need to be removed from a snark is at least two for any snark [5]. Formal definitions to follow. Snarks do, however, necessarily contain (more than one) 3-critical subgraph [1]. Snarks have long been of particular interest in graph theory, largely for the fact that many major problems in graph theory are easily solvable for graphs which are not snarks. Tutte’s 5-flow conjecture [13] and the cycle double cover conjecture [12] are major examples of these problems.

For our purposes, we define semi-graphs. A semi-graph \( G = (V,E) \) which consists of a set of vertices \( V = V(G) \) and a set of edges and semi-edges \( E = E(G) \subseteq [V]^2 \cup [V]^1 \), where \( [V]^1 \) denotes the subset of \( i \)-element subsets of \( V \). Note that if \( E \) contains no elements from \( [V]^1 \) then \( G \) is simply a graph. The 2-element subsets of \( V \) in \( E \) are called edges, as expected, while the 1-element sets are called semi-edges. We denote the semi-edge \( \{u \} \) as \( (u) \) and we say that vertex \( u \) is incident to \( (u) \), and that edge \( uv \) is adjacent to \( (u) \). Furthermore, we define a join between two semi-edges \( (u) \) and \( (v) \) as the removal of semi-edges \( (u) \) and \( (v) \), and the addition of edge \( uv \). A semi-edge \( (u) \) and a vertex \( v \) may also join to form an edge \( uv \), with \( (u) \) being removed. We

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define a split of an edge $uv$ as the removal of $uv$ and the addition of semi-edges $(u)$ and $(v)$. The degree of a vertex $v$ in a semi-graph $G$ is defined as the combined total of edges and semi-edges incident to $v$. Thus a cubic semi-graph is a semi-graph with each vertex having degree 3. Essentially, semi-edges behave like edges except that they are associated with one vertex instead of two vertices, with each vertex having at most one semi-edge.

We say that a graph $G$ contains a semi-graph $G'$ if $V(G') \subseteq V(G)$, $uv \in E(G')$ implies that $uv \in E(G)$, for every semi-edge $(u) \in E(G')$ there is an edge $uv \in E(G)$, and for every vertex $u \in V(G')$ the degree of $u$ in $G$ is greater than or equal to the degree of $u$ in $G'$ (for example, the trivial semi-graph with one vertex and one semi-edge is contained in $K_2$). We also say that $G'$ is a semi-subgraph of $G$. The cyclic connectivity of a graph is the minimum cardinality of an edge cut-set, such that each resulting component contains a cycle. Such an edge-cut we call a cyclical edge cut. Note as well that throughout this paper, if we refer to the removal of a vertex, or a subgraph, we mean that the incident edges are split and remain as semi-edges.

Let $G = (V,E)$ be a semi-graph. A $k$-edge-colouring, $f$, of $G$ is a mapping from the set of edges of $G$ to a set of $k$ colours. That is, $f : E \rightarrow \{1, \ldots, k\}$. $f$ is a proper $k$-edge-colouring of $G$ if no two adjacent elements in $E$ are mapped to the same colour. By Vizing’s theorem ([4], Thm. 6.2), if $G$ is a graph and $f$ is a proper colouring then the smallest possible value of $k$ is $\Delta$ or $\Delta + 1$, where $\Delta$ is the maximum degree of any vertex in $G$. The smallest possible value of $k$ is called the chromatic index. If the chromatic index is $\Delta$, then we say that $G$ is class one, or $\Delta$-edge-colourable. Otherwise we say that $G$ is class two, or ($\Delta + 1$)-edge-colourable.

Given a $k$-edge-colouring $f$, we call the set $f^{-1}(i)$ a colour class, for each $i \in \{1, \ldots, k\}$. The resistance of $G$, denoted as $r(G)$, is defined as the min$\{|f^{-1}(i)| : f$ is a proper $(\Delta + 1)$-edge-colouring of $G$ and $f^{-1}(i)$ is a colour class$\}$. That is, the minimum number of edges that can be removed from a graph such that the resulting graph is $\Delta$-edge-colourable [9]. As alluded to, if $G$ is a snark, then $r(G) > 1$.

The Parity Lemma is a well-known result which is useful in the study of class two graphs. There are many slight variations in how the lemma has been stated. One way in which it can be stated is in the context of semi-graphs and semi-edges.

**Lemma 1** (The Parity Lemma [6, 7]). Let $G$ be a cubic semi-graph with $m$ semi-edges and let $f$ be a proper $3$-edge-colouring of $G$. If $m_i$ equals the number of semi-edges coloured $i$ by $f$ for $i = \{1, 2, 3\}$, then

$$m_1 \equiv m_2 \equiv m_3 \equiv m \mod 2.$$ 

Reductions of snarks, and also considerations of triviality within snarks, are notions which have been much studied [1,3]. Many insights into the structure of snarks have been drawn from these investigations. For example, in [10] Steffen considers “vertex-reductions” which involve the removal of two distinct vertices and subsequent addition of edges to restore 3-regularity. Steffen characterises snarks into those which are reducible to smaller snarks, those which are reducible to only class one graphs, and those which are reducible to either a snark or a class one graph. In fact, Steffen conjectures that the Petersen graph is the only snark which is not vertex-reducible to a smaller snark. In our opinion, this conjecture is closely related to the more well-known conjecture by Tutte which states that every snark has the Petersen as a minor.

A snark which has girth less than 5 or cyclic connectivity less than 4 is generally considered to be trivial. Although, we believe triviality in snarks should be regarded as a much deeper notion. In [1] we reasoned instead that snarks which can be easily reduced to smaller snarks with the same resistance, should be considered to contain some trivial substructure. This is partly because there exists many examples of cyclically 2- or 3-connected snarks which cannot be simply reduced to smaller snarks without decreasing resistance, and the reduction of resistance cannot be seen as trivial, since some complexity in the structure of the snark is contributing to that resistance being present. In [1], we characterised the resistance of a snark in terms of the 3-critical subgraphs. We showed essentially, that the complexity of the structure of a snark is contained only its 3-critical subgraphs, since the vertices and edges not contained in any 3-critical subgraph, do not contribute to resistance in any way. Thus, any snark which contains a subgraph not contained in any 3-critical subgraph, can also be easily reduced, with cyclic connectivity also increasing. For more literature on snark reductions, see as well [2,8,11].
One reason for this notion of triviality is that so-called trivial snarks can easily be reduced to smaller snarks by certain well-defined reductions [5]. Such methods of reductions have, however, not been considered in great detail for 3-critical graphs. Since 3-critical graphs are necessarily strictly subcubic, the aforementioned reduction of a cyclically 3-connected subgraph into a single vertex, fails to necessarily produce a smaller 3-critical graph. Here, the idea of cyclic connectivity less than four does not necessarily suggest any kind of triviality. We thus consider the study of reducing 3-critical graphs to smaller 3-critical graphs (not just smaller class two subcubic graphs) in particular, to be potentially insightful. Particularly with regards to the aforementioned conjecture by Tutte.

In this paper, we consider in particular two types of methods of subcubic graph reduction. That is, firstly, the suppression of a cyclically 2-connected subgraph into a single edge. That is, the removal of the 2-connected subgraph, and the joining of the two remaining semi-edges into a single edge. We show that every cyclically 2-connected 3-critical graph can be reduced to a smaller 3-critical graph by such a reduction, whether the 2-connected subgraph is cubic or strictly subcubic. Secondly, we also consider the suppression of a cyclically 3-connected subgraph into a single vertex, also in both the cubic and strictly subcubic cases. That is, the removal of the 3-connected subgraph, and the joining of the three remaining semi-edges to a single added vertex. Using results proven, we are able to prove that every cyclically k-connected 3-critical graph with k = 2 or k = 3 contains a minor which is 3-critical and cyclically k-connected with k ≥ 4.

2. Reductions

We begin our investigations into the reductions of 3-critical graphs by considering the cyclically 2-connected case, and the suppression of the 2-connected subgraph into a single edge.

**Proposition 1.** Let G be a cyclically 2-connected 3-critical graph. Let \{e₁, e₂\} be a cyclic edge cut-set of G which separates G into H and K, where H is a cubic semi-subgraph of G. Then the graph obtained by the suppression of H into an edge is 3-critical.

**Proof.** Split the edge e₁ into semi-edges h₁ and k₁, and split the edge e₂ into semi-edges h₂ and k₂, where h₁ and h₂ are in H and k₁ and k₂ are in K. Let H′ be the graph with semi-edges h₁ and h₂ joined to form h₃. Let K′ be the graph with semi-edges k₁ and k₂ joined to form k₃. Let f_H be a proper 3-edge-colouring of H. Then f_H(h₁) = f_H(h₂) by the Parity Lemma. Let f_K be a proper 3-edge-colouring of K. If f_K(k₁) = f_K(k₂), we could then ensure that f_K(k₁) = f_K′(k₂) = f_H(h₁) = f_H(h₂), and then combine the colourings f_H and f_K, by joining the semi-edges, to form a proper 3-edge-colouring of G, which is impossible. Therefore, f_K(k₁) ≠ f_K(k₂) for any proper 3-edge-colouring f_K of K. This implies that there exists no proper 3-edge-colouring of K′, so that K′ is not 3-edge-colourable.

Since K′ − k₃ is 3-edge-colourable, we have that r(K′) = 1. Let k ≠ k₃ be an edge in K′. We know that G − k is 3-edge-colourable. Let f_G be a proper 3-edge-colouring of G − k, which also then properly colours H with 3 colours. By the Parity Lemma, f_G(e₁) = f_G(e₂). Therefore, we can restrict f_G to K′ by joining semi-edges, so that K′ − k is 3-edge-colourable. Therefore, K′ is 3-critical. □

**Proposition 2.** Let G be a cyclically 2-connected 3-critical graph. Let \{e₁, e₂\} be a cyclic edge cut-set which separates G into H and K, where H and K are strictly subcubic subgraphs of G. Let H′ be the graph obtained by the suppression of H into an edge, and let K′ be the graph obtained by the suppression of K into an edge. Then exactly one of H′ or K′ is 3-critical.

**Proof.** Split the edge e₁ into semi-edges h₁ and k₁, and split the edge e₂ into semi-edges h₂ and k₂, where h₁ and h₂ are in H and k₁ and k₂ are in K. Let H′ be the graph with semi-edges h₁ and h₂ joined to form h₃. Let K′ be the graph with semi-edges k₁ and k₂ joined to form k₃. Assume both H′ and K′ are 3-edge-colourable. Let f_H and f_K be proper 3-edge-colourings of H′ and K′, respectively. We could then have a proper 3-edge-colouring f_H of H where f_H(h₁) = f_H(h₂). Similarly, we could have f_K of K with f_K(k₁) = f_K(k₂). We could then
ensure that \( f_K(k_1) = f_K(k_2) = f_H(h_1) = f_H(h_2) \) so that we could then combine \( f_H \) and \( f_K \), to form a proper 3-edge-colouring of \( G \), a contradiction. Therefore, at least one of \( H' \) or \( K' \) is not 3-edge-colourable.

Now, assume that both \( H' \) and \( K' \) are not 3-edge-colourable. We know that both \( H \) and \( K \) are 3-edge-colourable. Then every proper 3-edge-colouring of \( H \), say \( f_H \), is such that \( f_H(h_1) \neq f_H(h_2) \). Similarly, every proper 3-edge-colouring of \( K \), say \( f_K \), is such that \( f_K(k_1) \neq f_K(k_2) \). However, we could then ensure that \( f_H(h_1) \neq f_K(k_1) \) and \( f_H(h_2) \neq f_K(k_2) \), and then combine \( f_H \) and \( f_K \), to form a proper 3-edge-colouring of \( G \), a contradiction. Therefore, exactly one of \( H' \) and \( K' \) is 3-edge-colourable, and exactly one is not 3-edge-colourable.

Let \( H' \) be not 3-edge-colourable. Since \( H' \) is not 3-edge-colourable, every proper 3-edge-colouring of \( H \) must have \( h_1 \) and \( h_2 \) with different colours. Therefore,

1. there cannot exist a proper 3-edge-colouring of \( K \) with \( k_1 \) and \( k_2 \) having different colours.

Otherwise, we could find colourings \( f_H \) of \( H \) and \( f_K \) of \( K \) such that \( f_K(k_1) = f_H(h_1) \) and \( f_H(h_2) = f_K(k_2) \), and then combine these colourings to form a proper 3-edge-colouring of \( G \). Assume now that \( H' \) is not 3-critical. Then there exists an edge \( h \neq h_3 \) such that \( H' \) is not 3-edge-colourable. This implies that

2. every proper 3-edge-colouring of \( G - h \) has \( h_1 \) and \( h_2 \) with different colours.

However, \( G - h \) is 3-edge-colourable. Let \( f_G \) be a proper 3-edge-colouring of \( G - h \). If \( f_G(e_1) \neq f_G(e_2) \), then we could restrict the colouring to form a proper 3-edge-colouring of \( K \) with \( k_1 \) and \( k_2 \) having different colours, contradicting (1). Therefore, \( f_G(e_1) = f_G(e_2) \) for all proper 3-edge-colourings \( f_G \) of \( G - h \). Then we could restrict the colouring to form a proper 3-edge-colouring of \( G - h \) with \( h_1 \) and \( h_2 \) having the same colour, contradicting (2). Since neither \( f_G(e_1) = f_G(e_2) \) nor \( f_G(e_1) \neq f_G(e_2) \) can be true, we have a contradiction. Therefore, \( H' \) is 3-critical. \( \square \)

**Example 1.** The following is a 3-critical graph. As per Proposition 2, we see that splitting the edges \( e_1 \) and \( e_2 \), and joining the resulting semi-edges, renders exactly one 3-critical graph (which is the Petersen graph with one vertex and associated edges removed).

We continue with the 3-connected case.

**Proposition 3.** Let \( G \) be a cyclically 3-connected 3-critical graph. Let \( \{e_1, e_2, e_3\} \) be a cyclical edge cut-set which separates \( G \) into \( H \) and \( K \), where \( H \) is a cubic semi-subgraph of \( G \). Then the graph obtained by the suppression of \( H \) into a single vertex, is 3-critical.

**Proof.** Split the edge \( e_i \) into semi-edges \( h_i \) and \( k_i \) for each \( i \in \{1, 2, 3\} \). Let \( H' \) be the cubic graph with semi-edges \( h_1, h_2 \) and \( h_3 \) joined to added vertex \( v_h \). Let \( K' \) be the subcubic graph with semi-edges \( k_1, k_2 \) and \( k_3 \) joined to added vertex \( v_k \). \( H \) is 3-edge-colourable. Let \( f_H \) be a proper 3-edge-colouring of \( H \). Then \( f_H(h_1) \),
\(f_H(h_2)\) and \(f_H(h_3)\) are all distinct by the Parity Lemma. Let \(f_K\) be a proper 3-edge-colouring of \(K\). If \(f_K(k_1)\), \(f_K(k_2)\) and \(f_K(k_3)\) are all distinct, then we could ensure that \(f_H(h_i) = f_K(k_i)\) for each \(i \in \{1, 2, 3\}\), and then combine the colourings \(f_H\) and \(f_K\) to form a proper 3-edge-colouring of \(G\), which is impossible. Therefore, at least two of \(f_K(k_1)\), \(f_K(k_2)\) and \(f_K(k_3)\) are equal in every proper 3-edge-colouring of \(K\). It follows that \(K'\) is not 3-edge-colourable.

Since \(K' - v_k\) is 3-colourable, we have that \(r(K') = 1\). Assume that \(K'\) is not 3-critical. Then there exists some edge \(k\) in \(K\) such that \(K' - k\) is not 3-edge-colourable. \(G - k\) is 3-edge-colourable. Let \(f_G\) be a proper 3-edge-colouring of \(G - k\). Since \(H\) is cubic and properly coloured by \(f_G\), we have by the Parity Lemma that \(f_G(e_1), f_G(e_2)\) and \(f_G(e_3)\) are all distinct. (If without loss of generality \(k = k_1\), then we still have that \(f_G(e_2)\) and \(f_G(e_3)\) are distinct.) However, we may then restrict \(f_G\) to \(K\), and then extend to \(K'\), so that we have a proper 3-edge-colouring of \(K' - k\), which goes against our assumption. Therefore, there exist no such \(k\). Therefore, \(K'\) is 3-critical.

**Proposition 4.** Let \(G\) be a 3-critical graph. Let \(\{e_1, e_2, e_3\}\) be a cyclical edge cut-set which separates \(G\) into \(H\) and \(K\), where \(H\) and \(K\) are both strictly subcubic subgraphs of \(G\). Then either the graph obtained by the suppression of \(H\) into a single vertex, or the graph obtained by the suppression of \(K\) into a single vertex, contains a 3-critical minor.

**Proof.** Split the edge \(e_i\) into semi-edges \(h_i\) and \(k_i\) for each \(i \in \{1, 2, 3\}\). Let \(H'\) be the cubic graph with semi-edges \(h_1, h_2\) and \(h_3\) joined to added vertex \(v_h\). Let \(K'\) be the subcubic graph with semi-edges \(k_1, k_2\) and \(k_3\) joined to added vertex \(v_k\). We assume that both \(H'\) and \(K'\) are 3-colourable.

Let \(f_H\) be a proper 3-edge-colouring of \(H'\). Then \(f_H(h_1), f_H(h_2)\) and \(f_H(h_3)\) are all distinct, since they are all adjacent. Similarly, let \(f_K\) be a proper 3-edge-colouring of \(K\). If \(f_K(k_1), f_K(k_2)\) and \(f_K(k_3)\) are all distinct, then we could ensure that \(f_H(h_1) = f_K(k_1), f_H(h_2) = f_K(k_2)\) and \(f_H(h_3) = f_K(k_3)\). We could then combine the colourings \(f_H\) and \(f_K\) to form a proper 3-edge-colouring of \(G\), which is impossible. Therefore, at least one of \(H'\) or \(K'\) is not 3-edge-colourable.

Let \(H'\) be not 3-edge-colourable. Then \(H'\) contains a 3-critical subgraph, which is a minor of \(G\). □

**Remark 1.** We note that the 3-critical minor obtained at the end of the proof of Proposition 4, must contain the vertex \(v_h\). Also, we suspect that similar to Proposition 2, whichever of the graph \(H'\) and \(K'\) in the proof of Proposition 4 is not 3-colourable, is indeed 3-critical.

The following result follows from the above propositions.

**Theorem 1.** Let \(G\) be a cyclically \(k\)-connected 3-critical graph with \(k = 2\) or \(k = 3\). Then \(G\) contains a minor which is 3-critical and cyclically \(k\)-connected with \(k \geq 4\).

**Proof.** One of Propositions 1, 2, 3 or 4 may be applied to \(G\) in order to obtain a minor, distinct from \(G\), say \(G_1\), which is 3-critical. We may then apply the same process to get a minor of \(G_1\), to get \(G_2\). Since the process reduces the size of the graph, it must terminate. The process can only terminate when some \(G_i\) is 3-critical, and cyclically \(k\)-connected with \(k \geq 4\). □

**References**


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