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RESTRAINED {2}-DOMINATION IN GRAPHS

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Abstract. A restrained {2}-dominating function (R{2}DF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that : (i) $f(N[v]) \geq 2$ for all $v \in V$, where $N[v]$ is the set containing v and all vertices adjacent to v; (ii) the subgraph induced by the vertices assigned 0 under f has no isolated vertices. The weight of an R{2}DF is the sum of its function values over all vertices, and the restrained {2}-domination number $\gamma_{r\{2\}}(G)$ is the minimum weight of an R{2}DF on G. In this paper, we initiate the study of the restrained {2}-domination number. We first prove that the problem of computing this parameter is NP-complete, even when restricted to bipartite graphs. Then we give various bounds on this parameter. In particular, we establish upper and lower bounds on the restrained {2}-domination number of a tree T in terms of the order, the numbers of leaves and support vertices.

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1. Introduction

In this paper, we only consider finite simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. For a vertex $v \in V$, the set $N(v)$ (or $N_G(v)$ to refer to G) denotes the set of vertices adjacent to v while $N[v]$ (or $N_G[v]$) is the set $N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$, and the *maximum degree* in G is denoted by $\Delta = \Delta(G)$. The *girth* of G, denoted by $g(G)$, is the minimum length of a cycle in G. The subgraph of G induced by a set of vertices S is denoted by $G[S]$. A *path* joining two vertices x and y is called a (x, y)-path. The *diameter* of a connected graph G, denoted diam(G), is the length of the shortest path between the most distanced vertices. A *diametral path* of a graph G is a shortest path whose length is equal to diam(G).

As usually, the *path*, *cycle* and the *complete graph* of order n are denoted by P_n, C_n and K_n, respectively. A *tree* is a connected acyclic graph. A star of order $n \geq 2$ is the tree $K_{1,n-1}$ in which at least $n-1$ vertices have degree one.

A set $S \subseteq V$ is a *restrained dominating set*, abbreviated RD-set, if every vertex in $V \setminus S$ has at least one neighbor in S and another one in $V \setminus S$. The *restrained domination number* $\gamma_r(G)$ of a graph G is the minimum cardinality of an RD-set in G. A restrained dominating set of cardinality $\gamma_r(G)$ is called a $\gamma_r(G)$-set. Restrained

Keywords. Restrained {2}-dominating function, restrained {2}-domination number, restrained domination number, NP-completeness.

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domination was introduced by Telle and Proskurowski [16], albeit indirectly, as a vertex partitioning problem, and widely studied later by several authors. For more details we refer the reader to the recent book chapter by Hattingh and Joubert [9]. The restrained version of some domination parameters have been studied in literature (see for example [2, 5, 14, 15, 17–19]).

In 2000, Harary and Haynes [10] introduced the concept of double domination in graphs. A set $S \subseteq V$ is a *double dominating set,* or DD-set for short, if for every vertex $v \in V$, $|N[v] \cap S| \geq 2$, that is, v is in S and has at least one neighbor in S or v is in $V \setminus S$ and has at least two neighbors in S. The minimum cardinality of a DD-set is the *double domination number* $dd(G)$ of G. In 2008, Kala and Nirmala Vasantha introduced the restrained version of DD-sets, where a *restrained double dominating set,* or RDD-set for short, of G is a DD-set S such that $G[V \setminus S]$ has no isolated vertices. The minimum cardinality of an RDD-set $dd_r(G)$ of G is the *restrained double domination number.*

In 1991, Domke *et al.* [7] introduced the concept of $\{2\}$-dominating functions as follows. For a graph G, a $\{2\}$-*dominating function* is a function $f : V \longrightarrow \{0,1,2\}$ having the property that for every vertex $v \in V$, $f(N[v]) \geq 2$. The weight of a $\{2\}$-dominating function is the sum $\omega(f) = f(V) = \sum_{v\in V} f(v)$, and the minimum weight of a $\{2\}$-dominating function f is the $\{2\}$-*domination number*, denoted $\gamma_{\{2\}}(G)$. For any $\{2\}$-dominating function f of G, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2$. Since these three sets determine f, we can equivalently write $f = (V_0, V_1, V_2)$, and we observe that $f(V) = |V_1| + 2|V_2|$. Further results on $\{2\}$-dominating functions can be found in [3, 4].

In this paper, we are interested in studying the restrained version of $\{2\}$-dominating functions. A *restrained* $\{2\}$-*dominating function* on a graph G, abbreviated R$\{2\}$DF, is a $\{2\}$-dominating function $f = (V_0, V_1, V_2)$ such that $G[V_0]$ has no isolated vertex. The *restrained* $\{2\}$-*domination number* $\gamma_{r\{2\}}(G)$ of G is the minimum weight of an R$\{2\}$DF on G. A $\gamma_{r\{2\}}(G)$-function f is an R$\{2\}$DF of G with $f(V) = \gamma_{r\{2\}}(G)$. It is straightforward to see that if D is a $dd_r(G)$-set, then the function $f = (V \setminus D, D, \emptyset)$ is an R$\{2\}$DF of G and thus

$$\gamma_{r\{2\}}(G) \leq dd_r(G). \tag{1}$$

We start this paper by showing that the decision problem for the restrained $\{2\}$-domination number is NP-complete even when restricted to bipartite graphs. Then various bounds on the restrained $\{2\}$-domination number are presented. In particular, we establish upper and lower bounds on the restrained $\{2\}$-domination number of a tree T in terms of the order, the numbers of leaves and support vertices.

2. Complexity result

We mainly show in this section that the decision problem for the restrained $\{2\}$-domination number is NP-complete for bipartite graphs. For this purpose, consider the following decision problem.

Restrained $\{2\}$-domination number problem (R$\{2\}$DN)

Instance: A nonempty bipartite graph G and a positive integer k.

Question: Is $\gamma_{r\{2\}}(G) \leq k$?

The NP-completeness of R$\{2\}$DN problem is shown by transforming the 3-SAT problem to it in polynomial time. Recall that the 3-SAT problem specified below has been proven to be NP- complete in [8].

3-SAT problem

Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for each $j \in \{1, 2, \ldots, m\}$.

Question: Is there a truth assignment for U that satisfies all the clauses in $\mathcal{C}$?

Theorem 2.1. *Problem R$\{2\}$DN is NP-complete for bipartite graphs.*

Proof. Problem R$\{2\}$DN belongs to $\mathcal{NP}$ since checking that a given function $f : V \to \{0,1,2\}$ on a bipartite graph has weight at most k and is an R$\{2\}$DF can be done in polynomial time. Now let us show how to transform

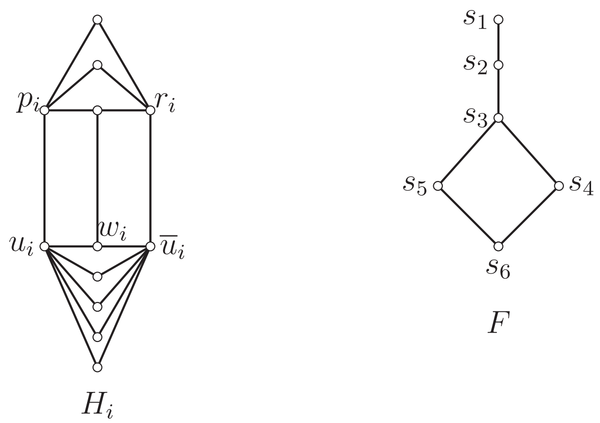

FIGURE 1. The graphs H_i and F.

any instance of 3-SAT into an instance G of R{2}DN so that one of them has a solution if and only if the other one has a solution. Let $U = \{u_1, u_2, \dots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \dots, C_m\}$ be an arbitrary instance of 3-SAT.

We will build a bipartite graph G and a positive integer k such that $\mathscr{C}$ is satisfiable if and only if $\gamma_{r\{2\}}(G) \leq k$. The graph G is built as follows.

For each $i \in \{1, 2, \dots, n\}$, we associate to the variable $u_i \in U$ a bipartite graph H_i, as depicted in Figure 1. For each $j \in \{1, 2, \dots, m\}$, we associate to the clause $C_j = \{p_j, q_j, r_j\} \in \mathscr{C}$ a single vertex c_j to which a set of edges $E_j = \{c_j p_j, c_j q_j, c_j r_j\}$ is added. Finally, we add the graph F depicted in Figure 1 by connecting s_1 to every vertex c_j. Set $k = 4n+4$. Clearly, G is a bipartite graph of order $12n+m+6$. Figure 2 provides an example of the bipartite graph G built from the instance $(U, \mathscr{C})$, where $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \overline{u_3}\}, C_2 = \{\overline{u_1}, u_2, u_4\}$ and $C_3 = \{\overline{u_2}, u_3, u_4\}$.

We shall prove that $\gamma_{r\{2\}}(G) = 4n + 4$ if and only if there is a truth assignment for U satisfying each clause in $\mathscr{C}$. To that end, we will need the following two claims.

Claim 2.2. $\gamma_{r\{2\}}(G) \geq 4n+4$. *Moreover, if* $\gamma_{r\{2\}}(G) = 4n+4$, *then for any* $\gamma_{r\{2\}}(G)$*-function* $f = (V_0, V_1, V_2)$, $f(V(H_i)) = 4, \{u_i, \overline{u_i}\} \cap V_1 = \emptyset$, $|\{u_i, \overline{u_i}\} \cap V_2| = 1$ *for each* i, $f(c_j) = 0$ *for each* j, $f(s_2) = f(s_6) = 2$ *and* $f(x) = 0$ *for any other vertex* $x \in V(F)$.

Proof of Claim 1. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(G)$-function. It is easy to note that $f(V(H_i)) \geq 4$ for each $i \in \{1, 2, \dots, n\}$ and $f(V(F)) + \sum_{j=1}^{m} f(c_j) \geq 4$. Therefore,

$$\gamma_{r\{2\}}(G) = f(V(G)) = \sum_{i=1}^{n} f(V(H_i)) + \left(f(V(F)) + \sum_{j=1}^{m} f(c_j) \right) \geq 4n + 4.$$

Suppose now that $\gamma_{r\{2\}}(G) = 4n+4$. Then we must have$f(V(H_i)) = 4$ for each i and $f(V(F)) + \sum_{j=1}^{m} f(c_j) = 4$. If $f(s_1) = 1$, then for the remaining vertices of F, we need that $f(N[s_3]) \cup \{s_6\}) \geq 4$ which leads to a contradiction. Hence $f(s_1) \neq 1$. Now assume that $f(s_1) = 2$. If $f(s_2) = 0$, then $f(s_3) = 0$ and so $f(s_4) + f(s_5) + f(s_6) \geq 3$ which again results in a contradiction. Hence $f(s_2) \neq 0$, and thus the remaining vertices of F will fulfill $f(s_3)+ f(s_4) + f(s_5) + f(s_6) \geq 3$, a contradiction too. Hence $f(s_1) = 0$. A similar argument as above shows that if $f(s_2) \neq 2$, then $f(V(F)) + \sum_{j=1}^{m} f(c_j) > 4$, leading a contradiction. Hence $f(s_2) = 2$. Therefore, $f(s_6) = 2$ and each remaining vertex of F as well as every vertex c_j must be assigned a 0.

To complete the proof of the claim, we shall show that for each i, exactly one of u_i and $\overline{u_i}$ is assigned 2 and that $\{u_i, \overline{u_i}\} \cap V_1 = \emptyset$. Assume first that $f(u_i) = f(\overline{u_i}) = 2$ for some $i \in \{1, \dots, n\}$. Since $f(V(H_i)) = 4$, it

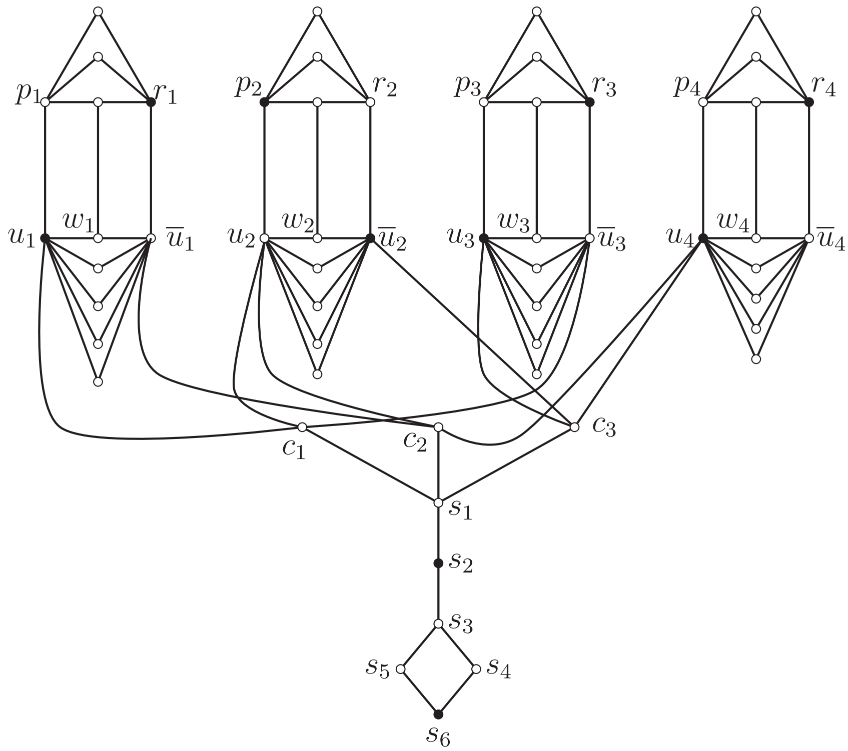

FIGURE 2. The graph G.

follows that $f(x) = 0$ for each $x \in V(H_i) \setminus \{u_i, \overline{u_i}\}$. But then the third neighbor of w_i has no neighbor assigned a non-zero value which leads to a contradiction. Therefore, $|\{u_i, \overline{u_i}\} \cap V_2| \leq 1$. Now if $|\{u_i, \overline{u_i}\} \cap V_2| = \emptyset$, then the four common neighbors of u_i and $\overline{u_i}$ different from w_i will be assigned non-zero values (2 or 1 depending on the values assigned to u_i and $\overline{u_i}$). But in any case, we will definitely get the contradiction that $f(V(H_i)) > 4$. Consequently, $|\{u_i, \overline{u_i}\} \cap V_2| = 1$. Assume now, without loss of generality, that $f(u_i) = 1$. Then by the previous fact $f(\overline{u_i}) = 2$ and thus the four common neighbors of u_i and $\overline{u_i}$ different from w_i will be assigned non-zero values leading to the contradiction $f(V(H_i)) > 4$. Hence $\{u_i, \overline{u_i}\} \cap V_1 = \emptyset$. ♦ □

Claim 2.3. *$\gamma_{r\{2\}}(G) = 4n + 4$ if and only if $\mathscr{C}$ is satisfiable.*

Proof of Claim 2. Suppose that $\gamma_{r\{2\}}(G) = 4n + 4$ and let f be a $\gamma_{r\{2\}}(G)$-function. By Claim 2.2, at most one of $f(u_i)$ and $f(\overline{u_i})$ is 2 for each $i \in \{1, 2, \ldots, n\}$. Define a mapping $t : U \longrightarrow \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if } f(u_i) = 2, \\ F & \text{otherwise,} \end{cases} \tag{2}$$

for $i = 1, \ldots, n$. We now show that t is a satisfying truth assignment for $\mathscr{C}$. It is sufficient to show that every clause in $\mathscr{C}$ is satisfied by t. To this end, we arbitrarily choose a clause $C_j \in \mathscr{C}$ for some $j \in \{1, \ldots, m\}$.

By Claim 2.2, $f(s_1) = f(c_j) = 0$. Hence, there exists some $i \in \{1, \ldots, n\}$ such that c_j is adjacent to either u_i with $f(u_i) = 2$ or to $\overline{u_i}$ with $f(\overline{u_i}) = 2$. If the first situation occurs, then by (2), $t(u_i) = T$, implying that the clause C_j containing such a literal u_i is satisfied by t. Moreover, if the latter situation occurs, $t(u_i) = F$ by (2). Thus, t assigns $\overline{u_i}$ the truth value T, that is, t satisfies the clause C_j containing such a literal $\overline{u_i}$. By the arbitrariness of j, we have shown that t satisfies all clauses in $\mathscr{C}$, that is, $\mathscr{C}$ is satisfiable.

Conversely, suppose that $\mathscr{C}$ is satisfiable, and let $t : U \to \{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. We construct a subset D of vertices of G as follows. If $t(u_i) = T$, then put the vertices u_i and r_i in D while if $t(u_i) = F$, then put the vertices $\overline{u_i}$ and p_i in D. Hence in any case, $|D| = 2n$. Now, define the function g on $V(G)$ by $g(x) = 2$ for every $x \in D$, $g(s_2) = g(s_6) = 2$ and $g(y) = 0$ for the remaining vertices. Since t is a satisfying truth assignment for $\mathscr{C}$, the corresponding vertex c_j in G is adjacent to at least one vertex in D. Also, one can easily check that g is an R{2}DF on G of weight $4n + 4$ and thus $\gamma_{r\{2\}}(G) \leq 4n + 4$. By Claim 2.2, $\gamma_{r\{2\}}(G) \geq 4n + 4$, and therefore $\gamma_{r\{2\}}(G) = 4n + 4$. ♦
This completes the proof. □

3. Exact values for paths and cycles

As shown in the previous section, since the decision problem for the restrained {2}-domination number is NP-complete even for bipartite graphs, it is natural to consider specific graphs for which the exact value can be computed. In this section, we determine the restrained {2}-domination number for paths and cycles. The following result will be used.

Proposition 3.1 ([6])**.** *Let G be a connected graph of order n. Then $\gamma_r(G) = n$ if and only if G is a star.*

Since $\gamma_{r\{2\}}(G) \geq \gamma_r(G)$ for any graph G, Proposition 3.1 leads that $\gamma_{r\{2\}}(P_2) = 2$ and $\gamma_{r\{2\}}(P_3) = 3$. For paths of order at least four, we have the following proposition.

Proposition 3.2. *If $n \geq 4$, then $\gamma_{r\{2\}}(P_n) = \lceil \frac{2n+4}{3} \rceil$. More precisely,*

$$\gamma_{r\{2\}}(P_n) = \begin{cases} \frac{2n+6}{3} & \text{if } \ n \equiv 0 \pmod 3 \\ \frac{2n+4}{3} & \text{if } \ n \equiv 1 \pmod 3 \\ \frac{2n+5}{3} & \text{if } \ n \equiv 2 \pmod 3). \end{cases}$$

Proof. Let $P_n = v_1 v_2 \dots v_n$. Define the function f on P_n as follows: if $n \equiv 1 \pmod 3$, then let $f(v_{3i+1}) = 2$ for $0 \leq i \leq \frac{n-1}{3}$ and $f(x) = 0$ otherwise; if $n \equiv 2 \pmod 3$, then let $f(v_n) = 1$, $f(v_{3i+1}) = 2$ for $0 \leq i \leq \frac{n-2}{3}$ and $f(x) = 0$ otherwise; finally if $n \equiv 0 \pmod 3$, then let $f(v_n) = f(v_{n-1}) = 1$, $f(v_{3i+1}) = 2$ for $0 \leq i \leq \frac{n-3}{3}$ and $f(x) = 0$ otherwise. Clearly f is an R{2}DF of P_n of weight $\lceil \frac{2n+4}{3} \rceil$ as outlined for each case. Hence $\gamma_{r\{2\}}(P_n) \leq \lceil \frac{2n+4}{3} \rceil$.

To prove the inverse inequality, we proceed by induction on n. Since the result is immediate for $n \in \{4, 5, 6, 7\}$, we assume that $n \geq 8$ and let f be a $\gamma_{r\{2\}}(P_n)$-function. We only consider the case $n \equiv 1 \pmod 3$, and the same argument can be applied for the two remaining cases. Obviously, $f(v_n) \geq 1$. If $f(v_n) = 2$, then we have $f(v_{n-1}) = f(v_{n-2}) = 0$ and thus $f(v_{n-3}) = 2$. In this case, the function f restricted to $P' = P_n - \{v_n, v_{n-1}, v_{n-2}\}$ is an R{2}DF of P_{n-3} and by the induction hypothesis on P' we get

$$\gamma_{r\{2\}}(P_n) = \omega(f) \geq 2 + \frac{2(n-3)+4}{3} = \frac{2n+4}{3}.$$

Now, let $f(v_n) = 1$. Then $f(v_{n-1}) \geq 1$. If $f(v_{n-1}) = 2$, then we have $f(v_{n-2}) = f(v_{n-3}) = 0$ and thus $f(v_{n-4}) = 2$. In this case, the function g defined on $P' = P_n - \{v_n, v_{n-1}, v_{n-2}\}$ by $g(v_{n-3}) = 1$ and $g(x) = f(x)$ otherwise, is an R{2}DF of P_{n-3} and by using as above the induction hypothesis we obtain $\gamma_{r\{2\}}(P_n) \geq \frac{2n+4}{3}$. Hence we can assume that $f(v_{n-1}) = 1$. Clearly, $f(v_{n-2}) \neq 0$. If $f(v_{n-2}) = 2$, then we have $f(v_{n-3}) = f(v_{n-4}) = 0$ and $f(v_{n-5}) = 2$, and thus the function g defined on $P' = P_n - \{v_n, v_{n-1}, v_{n-2}\}$ by $g(v_{n-3}) = g(v_{n-4}) = 1$ and $g(x) = f(x)$ otherwise, is an R{2}DF of P_{n-3}. The desired bound is therefore obtained after applying the induction on P'. Finally, if $f(v_{n-2}) = 1$, then $f(v_{n-3}) \geq 1$. Note that if $f(v_{n-3}) = 1$, then $f(v_{n-4}) \neq 0$. In either case, the function g defined on $P' = P_n - \{v_n, v_{n-1}, v_{n-2}\}$ by $g(v_{n-3}) = 2$ and $g(x) = f(x)$ otherwise, is an R{2}DF of P_{n-3} and the desired bound is obtained by induction. Consequently, $\gamma_{r\{2\}}(P_n) = \frac{2n+4}{3}$. □

The proof of the next result is analogously to that of Proposition 3.2.

Proposition 3.3. *If $n \geq 4$, then*

$$\gamma_{r\{2\}}(C_n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod 3 \\ \frac{2n+4}{3} & \text{if } n \equiv 1 \pmod 3 \\ \frac{2n+5}{3} & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

Proof. Let $C_n = v_1v_2 \dots v_nv_1$. Define the function f on C_n as follows: if $n \equiv 0 \pmod 3$, then let $f(v_{3i+1}) = 2$ for $0 \leq i \leq \frac{n-3}{3}$ and $f(x) = 0$ otherwise; if $n \equiv 1 \pmod 3$, then let $f(v_{3i+1}) = 2$ for $0 \leq i \leq \frac{n-1}{3}$ and $f(x) = 0$ otherwise; finally if $n \equiv 2 \pmod 3$, then let $f(v_n) = 1$, $f(v_{3i+1}) = 2$ for $0 \leq i \leq \frac{n-2}{3}$ and $f(x) = 0$ otherwise. Clearly f is an R{2}DF of C_n with the desired weight in each case, whence the upper bound follows.

To prove the lower bound in each case, we proceed by induction on n. Since the result is immediate for $n \in \{4, 5, 6, 7\}$, we assume that $n \geq 8$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(C_n)$-function. If $V_0 = \emptyset$, then we have $\omega(f) = n$ and the bounds are immediate. Hence let $V_0 \neq \emptyset$, and assume, without loss of generality, that $f(v_2) = f(v_3) = 0$. Then we have $f(v_1) = f(v_4) = 2$. Consider the restriction of f on the vertices of $V(C_n) \setminus \{v_2, v_3, v_4\}$. Certainly, such a function has weight $\omega(f) - 2$ and it is an R{2}DF of the cycle of order $n-3$ obtained from C_n by removing vertices v_2, v_3, v_4 and adding the edge v_1v_5. Using the induction hypothesis, the desired lower bound is obtained according to the case of n. This completes the proof. □

4. Bounds

In this section we present various bounds on the restrained {2}-domination number.

4.1. Bounds in terms of the order, size, girth and diameter

We begin by presenting two simple bounds.

Proposition 4.1. *Let G be a graph of order n with no isolated vertex with maximum degree Δ. The following hold.*
(i) $\left\lceil \frac{2n}{\Delta+1} \right\rceil \leq \gamma_{r\{2\}}(G) \leq n$. *Both bounds are attained*
(ii) If $\delta(G) \geq 3$, then $\gamma_{r\{2\}}(G) \leq n-2$, and the bound is sharp for K_4.
(iii) If G has a non-support vertex v of degree at least three, then $\gamma_{r\{2\}}(G) \leq n-1$.

Proof. (i) The upper bound follows from the fact that since G has no isolated vertex, assigning a 1 to every vertex of G provides an R{2}DF of weight n. To prove the lower bound, let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(G)$-function. Let $V_0^1 \subseteq V_0$ be the set of vertices with label 0 having a neighbor in V_2 and let $V_0^2 = V_0 \setminus V_0^1$. Then $\left|V_0^1\right| \leq \Delta \left|V_2\right|$. Moreover, since each vertex in V_1 must have a neighbor in $V_1 \cup V_2$ and each vertex in V_0^2 has at least two neighbors in V_1, we get $|V_0^2| \leq \frac{(\Delta-1)|V_1|}{2}$. Hence $2|V_0| = 2|V_0^1| + 2|V_0^2| \leq 2\Delta|V_2| + (\Delta-1)|V_1|$, and thus

$$\begin{aligned} 2n &= 2|V_0| + 2|V_1| + 2|V_2| \\ &\leq (2\Delta|V_2| + (\Delta-1)|V_1|) + 2|V_1| + 2|V_2| \\ &= (\Delta+1)|V_1| + 2(\Delta+1)|V_2| \\ &= (\Delta+1)\gamma_{r\{2\}}(G), \end{aligned}$$

and the lower bound follows.
The upper bound is attained for stars of order at least two, while the lower bound is attained for cycles C_n of order n with $n \equiv 0$ (see Prop. 3.3).

(ii) Let v and u be two adjacent vertices of G. Then assigning to u and v a 0 and a 1 to any other vertex of G provides an R{2}DF of G of weight $n-2$, and hence $\gamma_{r\{2\}R}(G) \leq n-2$.

(iii) Let u be a neighbor of v and $u' \in N(u) \setminus \{v\}$. Then assigning to u and v a 0, a 2 to u' and a 1 to any other vertex of G provides an R{2}DF of G of weight $n-1$, and hence $\gamma_{r\{2\}R}(G) \leq n-1$.

□

Next we present a lower bound on $\gamma_{r\{2\}}(G)$ in terms of the order and size of the graph.

Theorem 4.2. *Let G be a connected graph of order $n \geq 2$ and size m. Then*

$$\gamma_{r\{2\}}(G) \geq \min\left\{\frac{5n-2m}{4}, 2n-\frac{4m}{3}, \frac{3n+1-2m}{2}\right\}$$

Proof. Let $k = \min\left\{\frac{5n-2m}{4}, 2n-\frac{4m}{3}, \frac{3n-2m}{2}\right\}$. The result is immediate for $n \in \{2,3,4,5\}$. Hence assume that $n \geq 6$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(G)$-function. Let m_i be the number of edges in $G[V_i]$ for each $i \in \{0,1,2\}$, and $m_{i,j} = |[V_i, V_j]|$ be the number of edges between sets V_i and V_j, with $0 \leq i < j \leq 2$. Suppose first that $V_2 = \emptyset$. Since the induced subgraph $G[V_i]$ has no isolated vertices, $m_i \geq \frac{|V_i|}{2}$ for each $i \in \{0,1\}$. Moreover, since each vertex in V_0 has at least two neighbors in V_1, $m_{0,1} \geq 2|V_0|$. It follows that

$$m = m_0 + m_1 + m_{0,1} \geq n/2 + 2|V_0| = n/2 + 2(n - |V_1|) = \frac{5n}{2} - 2\gamma_{r\{2\}}(G),$$

leading to $\gamma_{r\{2\}}(G) \geq \frac{5n-2m}{4} \geq k$.

In the following, we can assume that $V_2 \neq \emptyset$. If $V_1 = \emptyset$, then f is a restrained Roman dominating function of G (a variant of Roman dominating functions), for which it is observed in [1] that $\gamma_{r\{2\}}(G) = \omega(f) \geq 2n - \frac{4m}{3} \geq k$. Hence we assume that $V_1 \neq \emptyset$. Let V_0^1 be the set of vertices in V_0 having a neighbor in V_2 and let $V_0^2 = V_0 \setminus V_0^1$. As above, $m_0 \geq |V_0|/2$. Since each vertex in V_0^2 has at least two neighbors in V_1 and each vertex in V_0^1 has a neighbor in V_2, we have $|[V_0^1, V_2]| \geq |V_0^1|$ and $|[V_0^2, V_1]| \geq 2|V_0^2|$. Now, let V_1^1 be the set of vertices in V_1 having a neighbor in V_2, V_1^2 be the set of vertices in $V_1 \setminus V_1^1$ having a neighbor in V_1^1, V_1^3 be the set of vertices in $V_1 \setminus (V_1^1 \cup V_1^2)$ having a neighbor in V_1^2 and so on. Let this process end in kth step (note that V_1 is a finite set) and let $V_1^{k+1} = V_1 \setminus \cup_{i=1}^k V_1^i$. Since each vertex in V_1 has a neighbor in $V_1 \cup V_2$, we have,

$$\begin{aligned} |[V_1, V_2]| + |[V_1, V_1]| &\geq |[V_1^1, V_2]| + |[V_1^2, V_1^1]| + \cdots + |[V_1^k, V_1^{k-1}]| + |[V_1^{k+1}, V_1^{k+1}]| \\ &\geq |V_1^1| + |V_1^2| + \cdots + |V_1^k| + |V_1^{k+1}|/2 \\ &\geq \textstyle\sum_{i=1}^{k+1} |V_1^i|/2 = |V_1|/2. \end{aligned}$$

It follows that

$$\begin{aligned} m &\geq m_0 + |[V_0^1, V_2]| + |[V_0^2, V_1]| + |[V_1, V_1]| + |[V_1, V_2]| \\ &\geq |V_0|/2 + |V_0^1| + 2|V_0^2| + |V_1|/2 \\ &\geq \tfrac{3|V_0|}{2} + |V_1|/2 \\ &= \tfrac{3n}{2} - \tfrac{3|V_2|}{2} - \tfrac{3|V_1|}{2} + |V_1|/2 \\ &= \tfrac{3n}{2} - 2|V_2| - |V_1| + |V_2|/2 \\ &\geq \tfrac{3n}{2} - \gamma_{r\{2\}}(G) + \tfrac{1}{2} \\ &= \tfrac{3n+1}{2} - \gamma_{r\{2\}}(G), \end{aligned}$$

and so $\gamma_{r\{2\}}(G) \geq \frac{3n+1}{2} - m \geq k$. This completes the proof. □

Proposition 4.3. *For any connected graph G of order n with minimum degree at least two and girth $d \geq 6$ such that G is not a cycle,*

$$\gamma_{r\{2\}}(G) \leq \begin{cases} n - \frac{d}{3} & \text{if } n \equiv 0 \pmod 3 \\ n - \frac{d-4}{3} & \text{if } n \equiv 1 \pmod 3 \\ n - \frac{d-5}{3} & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

Proof. Let $d = g(G)$, and let $C = (x_1 x_2 \ldots x_d x_1)$ be a cycle of G on d vertices. Since $G \neq C$, we may assume that x_1 has a neighbor $w_1 \in V(G) \setminus V(C)$. Let $w_2 \in N(w_1) \setminus \{x_1\}$. Observe that since $d \geq 6$, each vertex in $V(G) \setminus V(C)$ has at most one neighbor in $V(C)$. Let f be a $\gamma_{r\{2\}}(C)$-function and define the function g on G by $g(x) = f(x)$ for $x \in V(C)$ and $g(x) = 1$ otherwise. Clearly g is an R{2}DF on G of weight $\gamma_{r\{2\}}(C) + n - d$. Applying Proposition 3.3, the desired result follows. □

Proposition 4.4. *Let G be a connected triangle-free graph of order n with $\delta(G) \geq 2$. Then*

$$\gamma_{r\{2\}}(G) \leq n + 1 - \left\lfloor \frac{\mathrm{diam}(G)}{3} \right\rfloor.$$

Proof. Let $P = v_0v_1 \dots v_d$ be a diametral path of G ($d = \mathrm{diam}(G)$) and let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(P)$-function. Define the function $g : V(G) \to \{0, 1, 2\}$ by $g(x) = f(x)$ for $x \in V(P)$ and $g(x) = 1$ otherwise. Since G is triangle-free and P is a diametral path, each vertex in $V(G) \setminus V(P)$ has at most one neighbor in $V(P)$. Therefore, g is an R{2}DF of G of weight $\gamma_{r\{2\}}(P) + n - (d + 1)$. Applying Proposition 3.2, we obtain

$$\gamma_{r\{2\}}(G) \leq n - (d + 1) + \left\lceil \frac{2d + 6}{3} \right\rceil = n + 1 - \left\lfloor \frac{d}{3} \right\rfloor.$$

□

The following example of graphs shows that the bound in Proposition 4.4 is asymptotically reached. Consider the graph G obtained from a path $P_k = v_1v_2 \dots v_k$ with $k \equiv 2 \pmod 3$ and $k \geq 5$, by adding two new vertices x and y by attaching x to v_1 and v_3 and attaching y to v_k and v_{k-2}. Note that G has order $k+2$ and diameter $k-1$. Moreover, G is triangle-free with minimum degree two having a restrained {2}-domination number equals number $\frac{2k+5}{3} + 1 = \frac{2k+8}{3}$ while the bound in Proposition 4.4 gives the value $\frac{2k+11}{3}$.

4.2. Bounds in terms of the restrained domination number

In this subsection, we establish a relationship involving the restrained {2}-domination number with the restrained domination number.

Theorem 4.5. *Let G be a connected graph of order $n \geq 2$ different from the star $K_{1,n-1}$. Then*

$$\gamma_r(G) + 1 \leq \gamma_{r\{2\}}(G) \leq 2\gamma_r(G).$$

Proof. The upper bound follows from the fact that for any $\gamma_r(G)$-set D, the function $g = (V \setminus D, \emptyset, D)$ is an R{2}DF of G of weight $2\gamma_r(G)$.

To prove the lower bound, let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(G)$-function. If $V_2 \neq \emptyset$, then clearly $V_1 \cup V_2$ is an RD-set of G and so $\gamma_r(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| - 1 = \gamma_{r\{2\}}(G) - 1$. Hence we assume that $V_2 = \emptyset$. Then $V_1 \neq \emptyset$ and each vertex in V_0 (if any) has at least two neighbors in V_1. Moreover, every vertex $u \in V_1$ must have a neighbor in V_1; otherwise $f(N[u]) = 1$ contradicting f is an R{2}DF of G. Now if $V_0 = \emptyset$, then $\gamma_{r\{2\}}(G) = n$ and since G is not a star we get from Proposition 3.1 that $\gamma_{r\{2\}}(G) \geq \gamma_r(G) + 1$. So assume that $V_0 \neq \emptyset$, and let $v \in V_1$ be a vertex having a neighbor in V_0. Then $V_1 \setminus \{v\}$ is an RD-set of G leading to $\gamma_r(G) \leq \gamma_{r\{2\}}(G) - 1$. This completes the proof. □

The next result provides a necessary and sufficient condition for connected graphs G such that $\gamma_{r\{2\}}(G) = 2\gamma_r(G)$.

Proposition 4.6. *Let G be a nontrivial connected graph G. Then $\gamma_{r\{2\}}(G) = 2\gamma_r(G)$ if and only if there exits a $\gamma_{r\{2\}}(G)$-function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$.*

Proof. Clearly, if $\gamma_{r\{2\}}(G) = 2\gamma_r(G)$, then for any $\gamma_r(G)$-set D, the function $g = (V \setminus D, \emptyset, D)$ is an R{2}DF on G of weight $2\gamma_r(G)$, implying that g is a $\gamma_{r\{2\}}(G)$-function such that $V_1 = \emptyset$.

Conversely, if $f = (V_0, V_1, V_2)$ is a $\gamma_{r\{2\}}(G)$-function such that $V_1 = \emptyset$, then V_2 is an RD-set of G, and thus, $2\gamma_r(G) \leq 2|V_2| = \gamma_{r\{2\}}(G)$. The equality follows from Theorem 4.5, and the proof is complete. □

Restricted to connected graphs with minimum degree at least two, we establish a necessary and sufficient condition for such graphs reaching the lower bound of Theorem 4.5.

Proposition 4.7. *Let G be a nontrivial connected graph with minimum degree at least two. Then* $\gamma_{r\{2\}}(G) = \gamma_r(G) + 1$ *if and only if either* $\gamma_r(G) = 1$ *or* $dd_r(G) = \gamma_r(G) + 1$.

Proof. Assume first that $\gamma_{r\{2\}}(G) = \gamma_r(G) + 1$, and let $f(V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(G)$-function. Since $V_1 \cup V_2$ is an RD-set of G,

$$\gamma_r(G) + |V_2| \leq |V_1| + 2|V_2| = \gamma_{r\{2\}}(G) = \gamma_r(G) + 1.$$

Hence, $|V_2| \leq 1$. If $|V_2| = 0$, then $V_1 \neq \emptyset$, and since each vertex of V_0 has at least two neighbors in V_1, and each vertex of V_1 also has a neighbor in V_1, we deduce that V_1 is an RDD-set of G. Hence, $dd_r(G) \leq |V_1| = \gamma_{r\{2\}}(G) = \gamma_r(G) + 1$ and since $\gamma_{r\{2\}}(G) \leq dd_r(G)$ by (1), we conclude that $dd_r(G) = \gamma_r(G) + 1$. Hence we can assume in the next that $|V_2| = 1$, say $V_2 = \{v\}$. We claim that $V_1 = \emptyset$. Suppose to the contrary that $V_1 \neq \emptyset$, and let $u \in V_1$. By definition $N(u) \cap (V_1 \cup V_2) \neq \emptyset$. If u has a neighbor in V_0, then the set $(V_1 \cup \{v\}) \setminus \{u\}$ is an RD-set of G implying that

$$\begin{aligned}\gamma_r(G) &\leq |(V_1 \cup \{v\}) \setminus \{u\}| = |V_1| \\ &= |V_1| + 2\,|V_2| - 2\,|V_2| = \gamma_{r\{2\}}(G) - 2,\end{aligned}$$

a contradiction. Therefore, u has no neighbor in V_0. But since $\delta(G) \geq 2$ and $|V_2| = 1$, u must have a neighbor w in V_1. The same argument used above for u shows that w has no neighbor in V_0. In this case, one can easily see that $(V_1 \cup \{v\}) \setminus \{u, w\}$ is an RD-set of G, and hence

$$\begin{aligned}\gamma_r(G) &\leq |(V_1 \cup \{v\}) \setminus \{u, w\}| = |V_1| - 1 \\ &= |V_1| - 1 + 2\,|V_2| - 2\,|V_2| = \gamma_{r\{2\}}(G) - 3,\end{aligned}$$

a contradiction too, which completes the proof of the claim. Therefore $V_1 = \emptyset$, and thus $N[v] = V(G)$, *i.e.* $\gamma_r(G) = 1$.

Conversely, assume that G fulfills either $\gamma_r(G) = 1$ or $dd_r(G) = \gamma_r(G) + 1$. If $\gamma_r(G) = 1$, then it follows from Theorem 4.5 that $\gamma_{r\{2\}}(G) = \gamma_r(G) + 1$, while if $dd_r(G) = \gamma_r(G) + 1$, then by (1) and Theorem 4.5 we have $\gamma_{r\{2\}}(G) = \gamma_r(G) + 1$ and the proof is complete. □

5. Trees

In this section, we establish an upper and two lower bounds on the restrained {2}-domination number in trees. Since we are restricting to the class of trees, we give some additional definitions and notations. A *leaf* is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf, and a *strong support vertex* is a support vertex with at least two leaf neighbors. An end support vertex is a support vertex with at most one non-leaf neighbor. For any tree T, let $s(T)$ and $\ell(T)$ denote the number of support vertices and leaves of T, respectively. A *double star* $DS_{p,q}$ is a tree containing exactly two vertices that are not leaves, one of which is adjacent to p leaves and the other to q leaves. We are also considering *rooted trees* distinguished by one vertex r called the *root*. For a vertex $v \neq r$ in a rooted tree T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $w \neq v$ such that the unique (r, w)-path contains v. The *maximal subtree* at v denoted by T_v is the subtree of T induced by v and all its descendants.

Proposition 5.1. *Let T be a tree of order $n \geq 3$. Then*

$$\gamma_{r\{2\}}(T) \leq \left\lceil \frac{2n + 3(s(T) + \ell(T)) - 8}{3} \right\rceil.$$

The bound is attained for paths P_n when $n \equiv 1 \pmod 3$ and for the trees T_1 and T_2 illustrated in Figure 3.

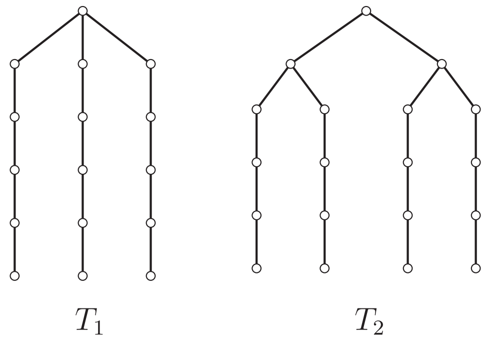

FIGURE 3. Two trees attaining the bound in Proposition 5.1.

Proof. We proceed by induction on the order n of T. Since the result is valid for all trees of order $n \in \{3, 4\}$, let $n \geq 5$ and suppose that the result holds for every nontrivial tree T of order less than n. Assume that T is a tree of order n. If $\text{diam}(T) = 2$, then T is a star $K_{1,n-1}$, where $s(T) = 1, \ell(T) = n-1, \gamma_{r\{2\}}(T) = n$, and so $n < \lceil \frac{2n+3(1+(n-1))-8}{3} \rceil$. If $\text{diam}(T) = 3$, then T is a double star $DS_{r,s}$ for some integers $r, s \geq 1$. In this case, $s(T) = 2, \ell(T) = r + s$ and $\gamma_{r\{2\}}(T) \leq n < \lceil \frac{5n-8}{3} \rceil = \lceil \frac{2n+3(2+(r+s))-8}{3} \rceil$. If T is a path, then Proposition 3.2 leads to the desired result. Hence we may assume that $\text{diam}(T) \geq 4$ and T is not a path.
Assume that T has an end strong support vertex, say u, and let v be the unique non-leaf neighbor of u in T. Let T' be the tree obtained from T by deleting u and its leaf neighbors, and let $f = (V_0, V_1, V_2)$ be a has $\gamma_{r\{2\}}(T')$-function. Note that $\deg_T(u) \geq 3$ and T' has order at least three, where $|V(T')| = n - \deg_T(u)$, $s(T') \leq s(T)$ and $\ell(T') \leq \ell(T) - (\deg_T(u) - 1) + 1$. Moreover, one can easily check that the function $g = (V_0, V_1 \cup (N_T[u] \setminus \{v\}), V_2)$ is an R{2}DF of T and hence $\gamma_{r\{2\}}(T) \leq \gamma_{r\{2\}}(T') + \deg_T(u)$. It follows from the induction hypothesis on T' that

$$\begin{aligned}
\gamma_{r\{2\}}(T) &\leq \gamma_{r\{2\}}(T') + \deg_T(u) \\
&\leq \lceil \tfrac{2(n-\deg_T(u))+3s(T')+3\ell(T')-8}{3} \rceil + \deg_T(u) \\
&\leq \lceil \tfrac{2n-2\deg_T(u)+3s(T)+3(\ell(T)-(\deg_T(u)-1)+1)-8}{3} \rceil + \deg_T(u) \\
&= \lceil \tfrac{2n+3s(T)+3\ell(T)-(2\deg_T(u)-6)-8}{3} \rceil \\
&\leq \lceil \tfrac{2n+3s(T)+3\ell(T)-8}{3} \rceil,
\end{aligned}$$

as desired. Hence we can assume T has no end strong support vertex. Suppose that $\text{diam}(T) = 4$. If $n = 5$, then $T = P_5$ and $\gamma_{r\{2\}}(T) = 5 = \lceil \frac{2n+3s(T)+3\ell(T)-8}{3} \rceil$, while if $n \geq 6$, then $s(T) + \ell(T) \in \{n-1, n\}$ and thus $\gamma_{r\{2\}}(T) = n < \frac{5n-11}{3}$ as desired.

In the following we can assume that $\text{diam}(T) \geq 5$. Let $v_1 v_2 \ldots v_d$ be a diametral path in T and root T at v_d. Since T has no end strong support vertex, $\deg_T(v_2) = \deg_T(v_{d-1}) = 2$ and every child of v_3 is either a leaf or a support vertex of degree 2. We consider two cases.

Case 1. $\deg_T(v_3) \geq 3$.

Let $T' = T - V(T_{v_2})$. Then $|V(T')| = n - 2, s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. It is easy to see that $\gamma_{r\{2\}}(T) \leq \gamma_{r\{2\}}(T') + 2$, and by the inductive hypothesis, we obtain

$$\begin{aligned}
\gamma_{r\{2\}}(T) &\leq \gamma_{r\{2\}}(T') + 2 \\
&\leq \lceil \tfrac{2(n-2)+3(s(T')+\ell(T'))-8}{3} \rceil + 2 \\
&= \lceil \tfrac{2n+3(s(T)+\ell(T))-18}{3} \rceil + 2 \\
&< \lceil \tfrac{2n+3(s(T)+\ell(T))-8}{3} \rceil.
\end{aligned}$$

Case 2. $\deg_T(v_3) = 2$.

We distinguish the following subcases.

Subcase 2.1. $\deg_T(v_4) \geq 3$.

Let $T' = T - V(T_{v_3})$ and f be a $\gamma_{r\{2\}}(T')$-function. Then $|V(T')| = n - 3 \geq 3, s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Since f can be extended to an R{2}DF of T by assigning 1 to v_1, v_2, v_3, we have $\gamma_{r\{2\}}(T) \leq \gamma_{r\{2\}}(T') + 3$. By the induction hypothesis, we get

$$\begin{aligned}\gamma_{r\{2\}}(T) &\leq \gamma_{r\{2\}}(T') + 3\\ &\leq \lceil \tfrac{2(n-3)+3(s(T')+\ell(T'))-8}{3} \rceil + 3\\ &= \lceil \tfrac{2n+3(s(T)+\ell(T))-20}{3} \rceil + 3\\ &< \lceil \tfrac{2n+s(T)+4\ell(T)-8}{3} \rceil.\end{aligned}$$

Subcase 2.2. $\deg_T(v_4) = 2$.

Assume that $\deg_T(v_5) \geq 3$. Let $T' = T - V(T_{v_4})$ and f be a $\gamma_{r\{2\}}(T')$-function. Then $|V(T')| = n - 4 \geq 3, s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. As above, f can be extended to an R{2}DF of T by assigning 1 to v_1, v_2, v_3, v_4, and thus $\gamma_{r\{2\}}(T) \leq \gamma_{r\{2\}}(T') + 4$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{r\{2\}}(T) &\leq \gamma_{r\{2\}}(T') + 4\\ &\leq \lceil \tfrac{2(n-4)+3(s(T')+\ell(T'))-8}{3} \rceil + 4\\ &= \lceil \tfrac{2n+3(s(T)+\ell(T))-22}{3} \rceil + 4\\ &< \lceil \tfrac{2n+3(s(T)+\ell(T))-8}{3} \rceil.\end{aligned}$$

In the following, we can assume that $\deg_T(v_5) = 2$. Let k be the greatest integer such that $\deg_T(v_k) \geq 3$ and $\deg_T(v_i) = 2$ for all $i \in \{2, \ldots, k-1\}$ (note that T is not a path). Consider the tree $T' = T - V(T_{v_{k-1}})$. Clearly, $T_{v_{k-1}}$ is a path of order $k-1$, $|V(T')| = n - (k-1) \geq 3$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Moreover, for any $\gamma_{r\{2\}}(T')$-function f and $\gamma_{r\{2\}}(P_{k-1})$-function h define the function g on $V(T)$ by $g(x) = f(x)$ if $x \in V(T')$ and $g(x) = h(x)$ if $x \in V(T_{v_{k-1}})$. Then g is an R{2}DF of T leading to $\gamma_{r\{2\}}(T) \leq \gamma_{r\{2\}}(T') + \gamma_{r\{2\}}(P_{k-1})$. Using the induction hypothesis on T' and Proposition 3.2 for the path P_{k-1}, we can check that the upper bound easily follows, which completes the proof. ☐

Our next result is a lower bound on the restrained {2}-domination number of trees different from stars.

Theorem 5.2. *For every tree T of diameter at least three, order n, with $\ell(T)$ leaves and $s(T)$ support vertices, we have* $\gamma_{r\{2\}}(T) \geq \frac{2n+\ell(T)-2s(T)+6}{3}$.

Proof. We proceed by induction on n. If $\mathrm{diam}(T) = 3$, then T is a double star for which $\frac{2n+\ell(T)-2s(T)+6}{3} = \frac{2n+(n-2)-4+6}{3} = n = \gamma_{r\{2\}}(T)$. Hence assume that $\mathrm{diam}(T) \geq 4$, and thus $n \geq 5$. Assume that the result holds for every tree T' of order $n' < n$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(T)$-function.

Suppose first that T has a strong support vertex of T, say x, and let y and z be two leaf neighbors of x. Consider the tree $T' = T - \{y\}$. Then $n' = n - 1, \ell(T') = \ell(T) - 1$ and $s(T') = s(T)$. Also, $y, z \in V_1 \cup V_2$. Now, if $f(x) \neq 0$, then the minimality of f implies that $f(y) = f(z) = 1$ while if $f(x) = 0$, then $f(y) = f(z) = 2$. Hence in either case, the restriction of f to T' is an R{2}DF of T', and thus $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 1$. By the induction hypothesis on T' we have

$$\begin{aligned}\gamma_{r\{2\}}(T) &\geq \gamma_{r\{2\}}(T') + 1\\ &\geq \tfrac{2n'+\ell(T')-2s(T')+6}{3} + 1\\ &= \tfrac{2(n-1)+\ell(T)-1-2s(T)+9}{3} = \tfrac{2n+\ell(T)-2s(T)+6}{3}.\end{aligned}$$

Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

If $\mathrm{diam}(T) = 4$, then T is a tree where every vertex, except possibly its center v, is either a leaf or a support vertex. If $n = 5$, then $T = P_5$, and so $\gamma_{r\{2\}}(T) = 5 = \lceil \frac{14}{3} \rceil = \lceil \frac{2n+\ell(T)-2s(T)+6}{3} \rceil$. Hence we assume that $n \geq 6$. Now, if v is a support vertex, then $s(T) = \ell(T) = \frac{n}{2}$ and thus $\gamma_{r\{2\}}(T) = n \geq \frac{2n+\ell(T)-2s(T)+6}{3}$, while if v is not a support vertex, then $s(T) = \ell(T) = \frac{n-1}{2}$ and so $\gamma_{r\{2\}}(T) = n-1 \geq \frac{2n+\ell(T)-2s(T)+6}{3}$. Therefore, the result is valid when $\mathrm{diam}(T) = 4$.

Hence assume that $\mathrm{diam}(T) \geq 5$ and let $v_1v_2\dots v_d$ be a diametral path in T and root T at v_d. Since T has no strong support vertex, we have $\deg_T(v_2) = \deg_T(v_{d-1}) = 2$. Moreover, every child of v_3 is either a leaf or a support vertex of degree 2. Consider the following two cases.

Case 1. $\deg_T(v_3) \geq 3$.

First assume that v_3 is a support vertex. Let x be the unique leaf neighbor of v_3, and let $T' = T - V(T_{v_2})$. Then $\ell(T') = \ell(T) - 1$ and $s(T') = s(T) - 1$. If $f(v_3) = 0$ and $\big(N_T(v_3) \setminus \{x, v_2\}\big) \cap V_0 \neq \emptyset$, then $f(x) = 2$, $f(v_1) + f(v_2) = 2$ and so the restriction of f to $V(T')$ is an R{2}DF of T', yielding $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2$. Now if $\big(N_T(v_3) \setminus \{x, v_2\}\big) \cap V_0 = \emptyset$, then $f(v_2) = 0$ and $f(x) = f(v_1) = 2$. In this case, the function g defined on $V(T')$ by $g(v_3) = g(x) = 1$ and $g(z) = f(z)$, otherwise, is an R{2}DF of T', yielding again $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2$. In either case, by the induction hypothesis, it follows that

$$\begin{aligned}\gamma_{r\{2\}}(T) &\geq \gamma_{r\{2\}}(T') + 2\\ &\geq \tfrac{2n'+\ell(T)-1-2(s(T)-1)+6}{3} + 2\\ &= \tfrac{2(n-2)+\ell(T)-2s(T)+9}{3} > \tfrac{2n+\ell(T)-2s(T)+6}{3}.\end{aligned}$$

Now, assume that v_3 is not a support vertex. Let x be a child of v_3 other than v_2, and let y be the unique leaf neighbor of x. Consider the tree $T' = T - V(T_{v_2})$, where $\ell(T') = \ell(T) - 1$ and $s(T') = s(T) - 1$. It is easy to see that $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 1$, and by the induction hypothesis we get

$$\begin{aligned}\gamma_{r\{2\}}(T) &\geq \gamma_{r\{2\}}(T') + 1\\ &\geq \tfrac{2n'+\ell(T)-1-2(s(T)-1)+6}{3} + 1\\ &= \tfrac{2(n-2)+\ell(T)-1-2s(T)+2+9}{3} = \tfrac{2n+\ell(T)-2s(T)+6}{3}.\end{aligned}$$

Case 2. $\deg_T(v_3) = 2$.

We distinguish the following subcases.

Subcase 2.1. $\deg_T(v_4) \geq 3$.

Assume first that $f(v_4) = 2$. Then $f(v_1) = 2, f(v_3) = f(v_2) = 0$, and the function f restricted to the tree $T' = T - V(T_{v_3})$ is an R{2}DF leading to $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2$. Since $\deg_T(v_4) \geq 3$, $\ell(T') = \ell(T) - 1, s(T') = s(T) - 1$ and by the induction hypothesis we obtain

$$\begin{aligned}\gamma_{r\{2\}}(T) &\geq \gamma_{r\{2\}}(T') + 2\\ &\geq \tfrac{2n'+\ell(T)-1-2(s(T)-1)+6}{3} + 2\\ &= \tfrac{2(n-3)+\ell(T)-1-2s(T)+2+12}{3}\\ &= \tfrac{2n+\ell(T)-2s(T)+7}{3} > \tfrac{2n+\ell(T)-2s(T)+6}{3}.\end{aligned}$$

Assume now that $f(v_4) = 1$. Then $f(v_3) = f(v_2) = f(v_1) = 1$. Consider the tree $T' = T - V(T_{v_3})$, and observe that the restriction of f to $V(T')$ with $f(v_4) = 2$ is an R{2}DF of T' yielding $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2$. Using the induction as above we get the desired bound.

Finally, assume that $f(v_4) = 0$. Note that v_3 may be assigned a 0. But in any case $f(v_2) \neq 0$ (because of v_3) and thus $f(v_1) \neq 0$. Moreover, we observe that $f(v_3) + f(v_2) + f(v_1) = 3$. In this case, by considering the tree $T' = T - \{v_1\}$, the restriction of f to T' is an R{2}DF of T', yielding $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 1$. Using the induction and the fact that $\ell(T') = \ell(T)$ and $s(T') = s(T)$, we obtain

$$\begin{aligned}\gamma_{r\{2\}}(T) &\geq \gamma_{r\{2\}}(T') + 1\\ &\geq \tfrac{2n'+\ell(T')-2s(T')+6}{3} + 1\\ &= \tfrac{2(n-1)+\ell(T)-2s(T)+9}{3}\\ &= \tfrac{2n+\ell(T)-2s(T)+7}{3} > \tfrac{2n+\ell(T)-2s(T)+6}{3}.\end{aligned}$$

Subcase 2.2. $\deg_T(v_4) = 2$.

If $f(v_2) = 0$, then $f(v_1) = f(v_4) = 2$ and $f(v_3) = 0$. By considering the tree $T' = T - V(T_{v_3})$, we have $\ell(T') = \ell(T), s(T') \leq s(T)$ and the restriction of f to $V(T')$ is an R{2}DF of T', leading to $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2$. By the induction hypothesis, we get

$$\begin{aligned}\gamma_{r\{2\}}(T) &\geq \gamma_{r\{2\}}(T') + 2 \\ &\geq \tfrac{2n'+\ell(T')-2s(T')+6}{3} + 2 \\ &\geq \tfrac{2(n-3)+\ell(T)-2s(T)+6}{3} + 2 \\ &= \tfrac{2n+\ell(T)-2s(T)+6}{3}.\end{aligned}$$

Hence we can assume in the next that $f(v_2) \neq 0$. Then $f(v_1) \geq 1$. Note that v_3 may be assigned a 0. But in either case, $f(v_1) + f(v_2) + f(v_3) \geq 3$. In this case, let $T' = T - \{v_1\}$ and note that $\ell(T') = \ell(T), s(T') = s(T)$. Since the restriction of f to $V(T')$ is an R{2}DF of T', it follows from the induction that

$$\begin{aligned}\gamma_{r\{2\}}(T) &\geq \gamma_{r\{2\}}(T') + 1 \\ &\geq \tfrac{2n'+\ell(T')-2s(T')+6}{3} + 1 \\ &= \tfrac{2(n-1)+\ell(T)-2s(T)+6}{3} + 1 \\ &> \tfrac{2n+\ell(T)-2s(T)+6}{3}.\end{aligned}$$

and this completes the proof.

□

Our next result shows that $\frac{2n+4}{3}$ is also a lower bound for any tree T of order $n \geq 4$. It is worth mentioning that this new lower bound is better than that of Theorem 5.2 for all trees T with $2s(T) > l(T) + 2$. In addition, we will provide a characterization of all trees T of order n with $\gamma_{r\{2\}}(T) = \frac{2n+4}{3}$.

For a graph G, let

$$W_G = \{u \in V(G) \mid \text{there exists a } \gamma_{r\{2\}}(G)\text{-function } f \text{ such that } f(u) = 2\}.$$

Define the family $\mathcal{T}$ of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees such that $T_1 = P_1$ and $T = T_k$. If $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

Operation $\mathcal{O}_1$: If $u \in W_{T_i}$, then $\mathcal{O}_1$ adds a path $P_3 : abc$ attached at u by an edge ua to obtain T_{i+1}.

Operation $\mathcal{O}_2$: If $u \in W_{T_i}$, then $\mathcal{O}_2$ adds the tree illustrated in Figure 5 attached at u by an edge us to obtain T_{i+1}.

Proposition 5.3. *For any tree T in $\mathcal{T}$, $\gamma_{r\{2\}}(T) \leq \frac{2n(T)+4}{3}$.*

Proof. Let $T \in \mathcal{T}$. Then T is obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees such that $T_1 = P_1$, $T = T_k$ and if $k \geq 2$, then T_{i+1} is obtained recursively from T_i by one of the operations $\mathcal{O}_1$ and $\mathcal{O}_2$. We proceed by induction on k. The property is true for $T_1 = P_1$. Suppose the property is true for all trees of $\mathcal{T}$ constructed with $k - 1 \geq 0$ operations. Let $T = T_k$ with $k \geq 2$, and let f be a $\gamma_{r\{2\}}(T_{k-1})$-function such that $f(u) = 2$. If T is obtained from T_{k-1} by Operation $\mathcal{O}_1$, then $n(T) = n(T_{k-1}) + 3$ and f can be extended to an R{2}DF of T by assigning 0 to a and b and 2 to c. It follows from the induction hypothesis that $\gamma_{r\{2\}}(T) \leq \omega(f) + 2 \leq \frac{2n(T_{k-1})+4}{3} + 2 = \frac{2n(T)+4}{3}$. Now, if T is obtained from T_{k-1} by Operation $\mathcal{O}_2$, then $n(T) = n(T_{k-1}) + 6$ and f can be extended to an R{2}DF of T by assigning 0 to s and s' and a 1 to the remaining vertices. It follows from the induction hypothesis that $\gamma_{r\{2\}}(T) \leq \omega(f) + 4 \leq \frac{2n(T_{k-1})+4}{3} + 4 = \frac{2n(T)+4}{3}$. □

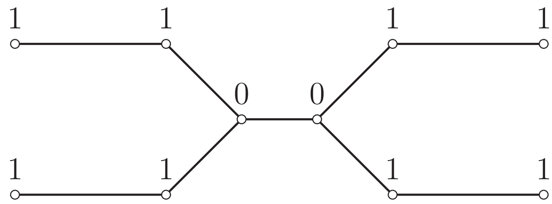

FIGURE 4. Tree T_3 of order 10 with its unique $\gamma_{r\{2\}}(T_3)$-function of weight 8.

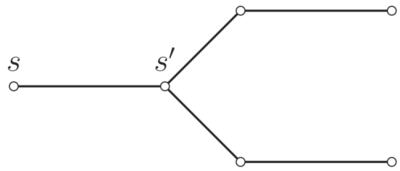

FIGURE 5. Tree used in Operation $\mathcal{O}_2$.

Theorem 5.4. *If T is a tree of order $n \notin \{2, 3\}$, then*

$$\gamma_{r\{2\}}(T) \geq \frac{2n+4}{3},$$

with equality if and only if $T \in \mathcal{T} \cup \{T_3, K_{1,3}\}$.

Proof. We proceed by induction on n. If $n = 1$, then $T = P_1$ and $\gamma_{r\{2\}}(T) = 2 = \frac{2n+4}{3}$. If $n = 4$, then $T \in \{P_4, K_{1,3}\} \subset \mathcal{T} \cup \{T_3, K_{1,3}\}$ and clearly $\gamma_{r\{2\}}(T) = 4 = \frac{2n(T)+4}{3}$. These establish the base cases. Now, since for stars $K_{1,n-1}$ of order $n \geq 5$, we have $\gamma_{r\{2\}}(K_{1,n-1}) = n > \frac{2n+4}{3}$, we can assume that $\text{diam}(T) \geq 3$ and $n \geq 5$. If $\text{diam}(T) = 3$, then T is a double star $DS_{p,q}$ and clearly $\gamma_{r\{2\}}(S_{p,q}) = n > \frac{2n+4}{3}$. Hence, in the following, we may assume that $\text{diam}(T) \geq 4$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(T)$-function.

If any support vertex, say x, of T is adjacent to two or more leaves, say y and z, then let T' be the tree obtained from T by removing y. Note that if $f(x) \neq 0$, then $f(y) = f(z) = 1$, while if $f(x) = 0$, then $f(y) = f(z) = 2$. Hence in either case, the restriction of f to $V(T')$ is an R{2}DF of T', leading to $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 1$. Using the induction on T', we obtain

$$\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 1 \geq \frac{2n'+4}{3} + 1 = \frac{2(n-1)+7}{3} > \frac{2n+4}{3}.$$

Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf. If $\text{diam}(T) = 4$, then every vertex of T except possibly the center vertex, say v, is either a leaf or a support vertex. Now, if $n = 5$, then $T = P_5$, and so by Proposition 3.2, $\gamma_{r\{2\}}(T) = \frac{2n+5}{3} > \frac{2n+4}{3}$. Thus let $n \geq 6$. If v is a support vertex of T, then $\gamma_{r\{2\}}(T) = n > \frac{2n+4}{3}$ while if v is not a support vertex, then T is a healthy spider with $n \geq 7$ and $\gamma_{r\{2\}}(T) = n - 1 \geq \frac{2n+4}{3}$ with equality if and only if $n = 7$. In this case, $T \in \mathcal{T}$ since it is obtained from P_1 applying operation $\mathcal{O}_2$. Therefore, in the following we can assume that $\text{diam}(T) \geq 5$.

We now root T at a vertex q of maximum eccentricity $\text{diam}(T)$. Let z be a leaf at maximum distance from q, v be the parent of z, u be the parent of v, w be the parent of u and d be the parent of w in the rooted tree. Clearly, $\deg_T(v) = 2$ and every child of u is either a leaf or a support vertex of degree 2. Note that $\deg_T(d) \geq 2$ (since $\text{diam}(T) \geq 5$) and $z \notin V_0$. We consider two cases.

Case 1. $\deg_T(u) \geq 3$.

First assume that u is a support vertex. Let x be the unique leaf neighbor of u, and let $T' = T - V(T_v)$.

Observe that if $f(u) \neq 0$, then $f(z) + f(v) = 2$ and thus the restriction of f to $V(T')$ is an R{2}DF of T', leading to $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2$. Hence let us assume that $f(u) = 0$. If $\big(N_T(u) \setminus \{x, v\}\big) \cap V_0 \neq \emptyset$, then $f(x) = 2, f(z) + f(v) = 2$ and thus the restriction of f to $V(T')$ is an R{2}DF of T', yielding again $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2$. If $\big(N_T(u) \setminus \{x, v\}\big) \cap V_0 = \emptyset$, then $f(v) = 0$ and $f(x) = f(z) = 2$, and thus the function g defined on $V(T')$ by $g(u) = g(x) = 1$ and $g(z) = f(z)$, otherwise, is an R{2}DF of T' implying that $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2$. Therefore, in either case, by the induction hypothesis we get

$$\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 2 \geq \frac{2n' + 4}{3} + 2 = \frac{2(n-2) + 4}{3} + 2 > \frac{2n + 4}{3}\cdot$$

Now let us suppose that u is not a support vertex. Let x be a child of u besides v and let y be the leaf neighbor of x. If $f(u) \neq 0$, then $f(v) = f(z) = 1$ and so f restricted to $T' = T - \{z\}$ is an R{2}DF of T', yielding $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 1$. Using the induction hypothesis we obtain

$$\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 1 \geq \frac{2n' + 4}{3} + 1 = \frac{2(n-1) + 7}{3} > \frac{2n + 4}{3}\cdot$$

Therefore assume that $f(u) = 0$, and consider the following two subcases.

Subcase 1.1. $f(v) = 0$.

Then $f(z) = 2$. Now, if u has a neighbor assigned 0 other than v, then the restriction of f on the tree $T' = T - \{v, z\}$ is an R{2}DF of T'. By induction on T', we have

$$\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 2 \geq \frac{2n' + 4}{3} + 2 = \frac{2(n-2) + 10}{3} > \frac{2n + 4}{3}\cdot$$

Hence we can assume that v is the unique vertex assigned 0 adjacent to u. Since $\deg_T(u) \geq 3$ and every child s of u with $s \neq v$ is a support vertex of degree two, we deduce that both s and its leaf neighbor are assigned 1. In this case, let $T' = T - V(T_u)$. If $n(T') = 3$, then it is easy to verify that $\gamma_{r\{2\}}(T) = n - 1 > \frac{2n+4}{3}$. Let $n(T') \geq 4$. Since the restriction of f to $V(T')$ is an R{2}DF of T', $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2(\deg_T(u) - 1)$. By the induction hypothesis we have

$$\begin{aligned}\gamma_{r\{2\}}(T) &\geq \gamma_{r\{2\}}(T') + 2(\deg_T(u) - 1)\\ &\geq \tfrac{2n'+4}{3} + 2(\deg_T(u) - 1)\\ &= \tfrac{2(n-2\deg_T(u)+1)+4}{3} + 2(\deg_T(u) - 1)\\ &= \tfrac{2n+4}{3} + \tfrac{2\deg_T(u)-4}{3} > \tfrac{2n+4}{3}.\end{aligned}$$

Subcase 1.2. $f(v) \neq 0$.

According to Subcase 1.1, we assume that no child of u is assigned a 0. Now, if $f(x) = 2$, then $f(y) = 1$ and the restriction of f on the tree $T' = T - \{y\}$ is an R{2}DF of T', and by applying the induction hypothesis on T' we obtain $\gamma_{r\{2\}}(T) > \frac{2n+4}{3}$. Hence we can assume that every child of u is assigned a 1 under f. Therefore, all leaves in T_u are also assigned a 1 under f, and thus w must be assigned a 0 because of f is an R{2}DF.

Now, if w has a neighbor assigned 0 other than u, then consider the tree $T' = T - V(T_u)$. Note that T' has order $n' \geq 4$. Since the restriction of f to $V(T')$ is an R{2}DF of T', we have $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2(\deg_T(u) - 1)$. As seen in Subcase 1.1, using the induction on T', we get $\gamma_{r\{2\}}(T) > \frac{2n+4}{3}$. In the next, we can assume that u is the unique vertex assigned 0 adjacent to w. If $\deg_T(u) \geq 4$, then by considering the tree $T' = T - \{z, v\}$ we see that f restricted to $V(T')$ is an R{2}DF of T', and using the induction we get $\gamma_{r\{2\}}(T) > \frac{2n+4}{3}$. Hence we assume that $\deg_T(u) = 3$. We need to examine several situations about vertex w.

(i)- If w is a support vertex with leaf neighbor w', then $f(w') = 2$. In this case, let $T' = T - V(T_u)$. Clearly, the restriction of f to $V(T')$ by reassigning w and w' the value 1 is an R{2}DF of T' leading to $\gamma_{r\{2\}}(T') \le \gamma_{r\{2\}}(T) - 4$. It follows from the inductive hypothesis that

$$\gamma_{r\{2\}}(T) \ge \gamma_{r\{2\}}(T') + 4 \ge \frac{2n'+4}{3} + 4 = \frac{2(n-5)+16}{3} > \frac{2n+4}{3}\cdot$$

(ii)- Assume that w has a child w^* which is a support vertex of degree two and let v^* be the leaf neighbor of w^*. Then $f(w^*)+f(v^*) \ge 2$. In this case, consider the tree $T' = T-(V(T_u)\cup\{w^*, v^*\})$, where $n' \ge 3$. If $n' = 3$, then clearly T is precisely the tree T_3 shown in Figure 4 belonging to $\mathcal{T}\cup\{T_3, K_{1,3}\}$, where $\gamma_{r\{2\}}(T_3) = \frac{2n+4}{3}$. Thus let $n' \ge 4$, and define the function g on $V(T')$ by $g(w) = 1$ and $g(t) = f(t)$ otherwise. Then g is an R{2}DF of T' of weight $\gamma_{r\{2\}}(T) - 5$, and thus $\gamma_{r\{2\}}(T') \le \gamma_{r\{2\}}(T) - 5$. By the induction hypothesis, we get

$$\gamma_{r\{2\}}(T) \ge \gamma_{r\{2\}}(T') + 5 \ge \frac{2n'+4}{3} + 5 = \frac{2(n-7)+19}{3} > \frac{2n+4}{3}\cdot$$

(iii)- Assume that w has a child w^* of degree at least two such that every child of w^* is a support vertex of degree two. We note that w^* cannot be a support vertex because it would have degree at least three and since it plays the same role as u, such a situation was already discussed in the beginning of Case 1. In addition, since it is assumed that u is the unique vertex assigned 0 adjacent to w, all vertices in T_{w^*} are assigned a 1 under f. Now, if $\deg_T(w^*) \ge 3$, then again w^* plays the same role as w and w^* is assigned a non-zero value, such a situation has already been considered before Subcase 1.1. Hence we deduce that $\deg_T(w^*) = 2$, that is T_{w^*} is a path P_3 having w^* as a leaf. Let $T' = T - (V(T_u) \cup V(T_{w^*}))$, and note that $n' \ge 4$ (otherwise $n' = 3$ and thus the diametral path would have end vertices z and the other leaf in T_{w^*}). Define the function g on $V(T')$ by $g(w) = 1$ and $g(t) = f(t)$ otherwise. Then g is an R{2}DF of T' of weight $\gamma_{r\{2\}}(T) - 4 - 3 + 1$, and thus $\gamma_{r\{2\}}(T') \le \gamma_{r\{2\}}(T) - 6$. Using the induction hypothesis, we obtain

$$\gamma_{r\{2\}}(T) \ge \gamma_{r\{2\}}(T') + 6 \ge \frac{2n'+4}{3} + 6 = \frac{2(n-5-3)+4}{3} + 6 = \frac{2n+4}{3} + \frac{2}{3} > \frac{2n+4}{3}\cdot$$

(iv)- Finally, assume that $\deg_T(w) = 2$. Note that because of $f(u) = f(w) = 0$ we deduce that $f(d) = 2$. Consider the tree $T' = T - V(T_w)$, and note that $n' \ge 2$. One can easily see that if $n' = 2$, then $\gamma_{r\{2\}}(T) = n - 1 = 7$ while if $n' = 3$, then $\gamma_{r\{2\}}(T) = n - 1 = 8$, and in any case $\gamma_{r\{2\}}(T) > \frac{2n+4}{3}$. Hence we can assume that $n' \ge 4$. Since u is the unique vertex assigned 0 adjacent to w, the restriction of f to $V(T')$ is an R{2}DF of T', and thus $\gamma_{r\{2\}}(T') \le \gamma_{r\{2\}}(T) - 4$. By the induction hypothesis we obtain

$$\gamma_{r\{2\}}(T) \ge \gamma_{r\{2\}}(T') + 4 \ge \frac{2n'+4}{3} + 4 = \frac{2(n-6)+16}{3} = \frac{2n+4}{3}\cdot$$

If further $\gamma_{r\{2\}}(T) = \frac{2n+4}{3}$, then $\gamma_{r\{2\}}(T') = \frac{2n'+4}{3}$ and that the restriction of f to $V(T')$ is a $\gamma_{r\{2\}}(T')$-function under which vertex d is assigned a 2, that is d belongs to $W_{T'}$. It follows from the induction hypothesis that $T' \in \mathcal{T}$. Therefore $T \in \mathcal{T}$ since it can be obtained from T by applying Operation $\mathcal{O}_2$. Moreover, since $T \in \mathcal{T}$, Proposition 5.3 implies that $\gamma_{r\{2\}}(T) \le \frac{2n+4}{3}$ and so $\gamma_{r\{2\}}(T) = \frac{2n+4}{3}$.

Case 2. $\deg_T(u) = 2$.

We distinguish the following subcases.

Subcase 2.1. $\deg_T(w) \ge 3$.

Assume first that $f(w) = 2$. Then we have $f(u) = f(v) = 0$ and $f(z) = 2$ and the restriction of f on the tree $T' = T - V(T_u)$ is an R{2}DF of T' leading to $\gamma_{r\{2\}}(T') \le \gamma_{r\{2\}}(T) - 2$. By induction on T', we get

$$\gamma_{r\{2\}}(T) \ge \gamma_{r\{2\}}(T') + 2 \ge \frac{2n'+4}{3} + 2 = \frac{2(n-3)+4}{3} + 2 = \frac{2n+4}{3}\cdot$$

If further$\gamma_{r\{2\}}(T) = \frac{2n+4}{3}$, then $\gamma_{r\{2\}}(T') = \frac{2n'+4}{3}$ and that the restriction of f to $V(T')$ is a $\gamma_{r\{2\}}(T')$-function under which vertex w is assigned a 2, that is w belongs to $W_{T'}$. It follows from the induction hypothesis that $T' \in \mathcal{T}$. Now since T can be obtained from T by applying operation $\mathcal{O}_1$ we have $T \in \mathcal{T}$. Moreover, since $T \in \mathcal{T}$, Proposition 5.3 implies that $\gamma_{r\{2\}}(T) \leq \frac{2n+4}{3}$ and so $\gamma_{r\{2\}}(T) = \frac{2n+4}{3}$.
Now if $f(w) = 1$, then we must have $f(u) = f(v) = f(z) = 1$, and by reassigning the vertices z, v, u, w the values $2, 0, 0, 2$ we would be in the preceding situation. Finally, we can assume that $f(w) = 0$. Then either $f(u) = 0$ and thus $f(v) = 2$, $f(z) = 1$ or $f(u) \neq 0$ and thus $f(v) = f(z) = 1$. In either case, the restriction of f to the tree $T' = T - \{z\}$ is an R{2}DF of T' yielding $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 1$. The induction hypothesis leads to

$$\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 1 \geq \frac{2n' + 4}{3} + 1 = \frac{2(n-1) + 4}{3} + 1 > \frac{2n + 4}{3}.$$

Subcase 2.2. $\deg_T(w) = 2$.
If $n = 6$, then $T = P_6$ and by Proposition 3.2, $\gamma_{r\{2\}}(T) > \frac{2n+4}{3}$. If $n = 7$, then since d cannot be adjacent to two leaves by our earlier assumption), we deduce that $T = P_7$. In this case, by Proposition 3.2, $\gamma_{r\{2\}}(P_7) = \frac{2n+4}{3}$, and one can easily see that $T \in \mathcal{T}$ because it can be obtained from $T_1 = P_1$ by applying twice Operation $\mathcal{O}_1$. Hence we assume that $n \geq 8$. If $f(v) = 0$, then $f(z) = f(w) = 2$ and $f(u) = 0$, and thus the restriction of f on the tree $T' = T - V(T_u)$ is an R{2}DF of T', yielding $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2$. Applying the induction hypothesis we get $\gamma_{r\{2\}}(T) \geq \frac{2n+4}{3}$. Using the same argument as Subcase 2.1, we can see that the equality holds if and only if $T \in \mathcal{T}$. Hence let us assume that $f(v) \neq 0$. Then $f(z) + f(v) + f(u) \geq 3$, and clearly the restriction of f on the tree $T' = T - \{z\}$ is an R{2}DF of T' of weight $\gamma_{r\{2\}}(T) - 1$ leading to $\gamma_{r\{2\}}(T) > \frac{2n+4}{3}$ by applying the induction hypothesis. This completes the proof.

□

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References

[1] H. Abdollahzadeh Ahangar and S.R. Mirmehdipour, Bounds on the restrained Roman domination number of a graph. *Commun. Comb. Optim.* **1** (2016) 75–82.

[2] J. Amjadi, B. Samadi and L. Volkmann, Total restrained Roman domination. *Commun. Comb. Optim.* **8** (2023) 575–587.

[3] A. Cabrera-Martínez and A. Conchado Peiro, On the {2}-domination number of graphs. *AIMS Math.* **7** (2022) 10731–10743.

[4] F. Bonomo, B. Brešar, L.N. Grippo, M. Milanič and M.D. Safe, Domination parameters with number 2: Interrelations and algorithmic consequences. *Discrete Appl. Math.* **235** (2018) 23–50.

[5] X. Chen, J. Liu and J. Meng, Total restrained domination in graphs. *Comput. Math. Appl.* **62** (2011) 2892–2898.

[6] G.S. Domke, J.H.T. Hattingh, S.T. Hedetniemi, R.C. Laskar and L.R. Markus, Restrained domination in graphs. *Discrete Math.* **203** (1999) 61–69.

[7] G.S. Domke, S.T. Hedetniemi, R.C. Laskar and G.H. Fricke, Relationships between Integer and Fractional Parameters of Graphs, In Vol. 2 of *Graph theory, combinatorics, and applications: proceedings of the sixth quadrennial international conference on the theory and applications of graphs.* Western Michigan University, John Wiley and Sons Inc. (1991) 371–387.

[8] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completness. Freeman, San Francisco (1979).

[9] J.H. Hattingh and E.J. Joubert, Restrained and total restrained domination in graphs, edited by T.W. Haynes, S.T. Hedetniemi and M.A. Henning. In: *Topics in domination in graphs.* Springer International Publishing (2020).

[10] F. Harary and T.W. Haynes, Double domination in graphs. *Ars Combin.* **55** (2000) 201–213.

[11] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamental of Domination in Graphs. Marcel Deker, New York (1998).

[12] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics. Marcel Dekker, Inc., New York (1998).

[13] R. Kala and T.R. Nirmala Vasantha, Restrained double domination number of a graph. *AKCE Int. J. Graphs Comb.* **5** (2008) 73–82.

[14] B. Samadi, M. Alishahi, I. Masoumi and D.A. Mojdeh, Restrained Italian domination in graphs. *RAIRO: OR* **55** (2021) 319–332.

[15] B. Samadi, N. Soltankhah, H. Abdollahzadeh Ahangar, M. Chellali, D.A. Mojdeh, S.M. Sheikholeslami and J.C. Valenzuela-Tripodoro, Restrained condition on double Roman dominating functions. *Appl. Math. Comput.* **438** (2023) 127554.

[16] J.A. Telle and A. Proskurowski, Algorithms for vertex partitioning problems on partial *k-trees. SIAM J. Discrete Math.* **10** (1997) 529–550.

[17] L. Volkmann, Remarks on the restrained Italian domination number in graphs. *Commun. Comb. Optim.* **8** (2023) 183–191.

[18] L. Volkmann, Restrained double Italian domination in graphs. *Commun. Comb. Optim.* **8** (2023) 1–11.

[19] C. Xi and J. Yue, The restrained double Roman domination in graphs. *Bull. Malays. Math. Sci. Soc.* **46** (2023) 6.