RESTRAINED $\{2\}$-DOMINATION IN GRAPHS

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Abstract. A restrained $\{2\}$-dominating function (R$\{2\}$DF) on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ such that: (i) $f(N[v]) \geq 2$ for all $v \in V$, where $N[v]$ is the set containing $v$ and all vertices adjacent to $v$; (ii) the subgraph induced by the vertices assigned 0 under $f$ has no isolated vertices. The weight of an R$\{2\}$DF is the sum of its function values over all vertices, and the restrained $\{2\}$-domination number $\gamma_r(\{2\})(G)$ is the minimum weight of an R$\{2\}$DF on $G$. In this paper, we initiate the study of the restrained $\{2\}$-domination number. We first prove that the problem of computing this parameter is NP-complete, even when restricted to bipartite graphs. Then we give various bounds on this parameter. In particular, we establish upper and lower bounds on the restrained $\{2\}$-domination number of a tree $T$ in terms of the order, the numbers of leaves and support vertices.

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1. Introduction

In this paper, we only consider finite simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The order of $G$ is $n = n(G) = |V|$. For a vertex $v \in V$, the set $N(v)$ (or $N_G(v)$ to refer to $G$) denotes the set of vertices adjacent to $v$ while $N[v]$ (or $N_G[v]$) is the set $N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is deg$_G(v) = |N(v)|$, and the maximum degree in $G$ is denoted by $\Delta = \Delta(G)$. The girth of $G$, denoted by $g(G)$, is the minimum length of a cycle in $G$. The subgraph of $G$ induced by a set of vertices $S$ is denoted by $G[S]$. A path joining two vertices $x$ and $y$ is called a $(x, y)$-path. The diameter of a connected graph $G$, denoted diam$(G)$, is the length of the shortest path between the most distanced vertices. A diametral path of a graph $G$ is a shortest path whose length is equal to diam$(G)$.

As usually, the path, cycle and the complete graph of order $n$ are denoted by $P_n$, $C_n$ and $K_n$, respectively. A tree is a connected acyclic graph. A star of order $n \geq 2$ is the tree $K_{1,n-1}$ in which at least $n - 1$ vertices have degree one.

A set $S \subseteq V$ is a restrained dominating set, abbreviated RD-set, if every vertex in $V \setminus S$ has at least one neighbor in $S$ and another one in $V \setminus S$. The restrained domination number $\gamma_r(G)$ of a graph $G$ is the minimum cardinality of an RD-set in $G$. A restrained dominating set of cardinality $\gamma_r(G)$ is called a $\gamma_r(G)$-set. Restricted

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domination was introduced by Telle and Proskurowski [16], albeit indirectly, as a vertex partitioning problem, and widely studied later by several authors. For more details we refer the reader to the recent book chapter by Hattingh and Joubert [9]. The restrained version of some domination parameters have been studied in literature (see for example [2, 5, 14, 15, 17–19]).

In 2000, Harary and Haynes [10] introduced the concept of double domination in graphs. A set \( S \subseteq V \) is a double dominating set, or DD-set for short, if for every vertex \( v \in V \), \(|N[v] \cap S| \geq 2\), that is, \( v \) is in \( S \) and has at least one neighbor in \( S \) or \( v \) is in \( V \setminus S \) and has at least two neighbors in \( S \). The minimum cardinality of a DD-set is the double domination number \( dd(G) \) of \( G \). In 2008, Kala and Nirmala Vasantha introduced the restrained version of DD-sets, where a restrained double dominating set, or RDD-set for short, of \( G \) is a DD-set \( S \) such that \( G[V \setminus S] \) has no isolated vertices. The minimum cardinality of an RDD-set \( dd_r(G) \) of \( G \) is the restrained double domination number.

In 1991, Domke et al. [7] introduced the concept of \( \{2\}\)-dominating functions as follows. For a graph \( G \), a \( \{2\}\)-dominating function is a function \( f : V \rightarrow \{0,1,2\} \) having the property that for every vertex \( v \in V \), \( f(N[v]) \geq 2 \). The weight of a \( \{2\}\)-dominating function is the sum \( \omega(f) = f(V) = \sum_{v \in V} f(v) \), and the minimum weight of a \( \{2\}\)-dominating function \( f \) is the \( \{2\}\)-domination number, denoted \( \gamma_{\{2\}}(G) \). For any \( \{2\}\)-dominating function \( f \) of \( G \), let \( V_i = \{v \in V \mid f(v) = i\} \) for \( i = 0, 1, 2 \). Since these three sets determine \( f \), we can equivalently write \( f = (V_0, V_1, V_2) \), and we observe that \( f(V) = |V_1| + 2|V_2| \). Further results on \( \{2\}\)-dominating functions can be found in [3, 4].

In this paper, we are interested in studying the restrained version of \( \{2\}\)-dominating functions. A restrained \( \{2\}\)-dominating function on a graph \( G \), abbreviated \( R\{2\}\)DF, is a \( \{2\}\)-dominating function \( f = (V_0, V_1, V_2) \) such that \( G[V_0] \) has no isolated vertex. The restrained \( \{2\}\)-domination number \( \gamma_{r\{2\}}(G) \) of \( G \) is the minimum weight of an \( R\{2\}\)DF on \( G \). A \( \gamma_{r\{2\}}(G) \)-function \( f \) is an \( R\{2\}\)DF of \( G \) with \( f(V) = \gamma_{r\{2\}}(G) \). It is straightforward to see that if \( D \) is a \( dd_r(G) \)-set, then the function \( f = (V \setminus D, D, \emptyset) \) is an \( R\{2\}\)DF of \( G \) and thus

\[
\gamma_{r\{2\}}(G) \leq dd_r(G). \tag{1}
\]

We start this paper by showing that the decision problem for the restrained \( \{2\}\)-domination number is \( \text{NP} \)-complete even when restricted to bipartite graphs. Then various bounds on the restrained \( \{2\}\)-domination number are presented. In particular, we establish upper and lower bounds on the restrained \( \{2\}\)-domination number of a tree \( T \) in terms of the order, the numbers of leaves and support vertices.

2. Complexity Result

We mainly show in this section that the decision problem for the restrained \( \{2\}\)-domination number is \( \text{NP} \)-complete for bipartite graphs. For this purpose, consider the following decision problem.

**Restrainted \( \{2\}\)-domination number problem (R\{2\}DN)**

**Instance:** A nonempty bipartite graph \( G \) and a positive integer \( k \).

**Question:** Is \( \gamma_{r\{2\}}(G) \leq k \)?

The \( \text{NP} \)-completeness of \( R\{2\}\)DN problem is shown by transforming the 3-SAT problem to it in polynomial time. Recall that the 3-SAT problem specified below has been proven to be \( \text{NP} \)-complete in [8].

**3-SAT problem**

**Instance:** A collection \( C = \{C_1, C_2, \ldots, C_m\} \) of clauses over a finite set \( U \) of variables such that \( |C_j| = 3 \) for each \( j \in \{1, 2, \ldots, m\} \).

**Question:** Is there a truth assignment for \( U \) that satisfies all the clauses in \( C \)?

**Theorem 2.1.** Problem \( R\{2\}\)DN is \( \text{NP} \)-complete for bipartite graphs.

**Proof.** Problem \( R\{2\}\)DN belongs to \( \mathcal{NP} \) since checking that a given function \( f : V \rightarrow \{0, 1, 2\} \) on a bipartite graph has weight at most \( k \) and is an \( R\{2\}\)DF can be done in polynomial time. Now let us show how to transform
any instance of 3-SAT into an instance $G$ of R\{2\}DN so that one of them has a solution if and only if the other one has a solution. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3-SAT.

We will build a bipartite graph $G$ and a positive integer $k$ such that $\mathcal{C}$ is satisfiable if and only if $\gamma_{r\{2\}}(G) \leq k$. The graph $G$ is built as follows.

For each $i \in \{1, 2, \ldots, n\}$, we associate to the variable $u_i \in U$ a bipartite graph $H_i$, as depicted in Figure 1. For each $j \in \{1, 2, \ldots, m\}$, we associate to the clause $C_j = \{p_j, q_j, r_j\} \in \mathcal{C}$ a single vertex $c_j$ to which a set of edges $E_j = \{c_j p_j, c_j q_j, c_j r_j\}$ is added. Finally, we add the graph $F$ depicted in Figure 1 by connecting $s_1$ to every vertex $c_j$. Set $k = 4n + 4$. Clearly, $G$ is a bipartite graph of order $12n + m + 6$. Figure 2 provides an example of the bipartite graph $G$ built from the instance $(U, \mathcal{C})$, where $U = \{u_1, u_2, u_3, u_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \overline{u_3}\}$, $C_2 = \{\overline{u_1}, u_2, u_4\}$ and $C_3 = \{\overline{u_2}, u_3, u_4\}$.

We shall prove that $\gamma_{r\{2\}}(G) = 4n + 4$ if and only if there is a truth assignment for $U$ satisfying each clause in $\mathcal{C}$. To that end, we will need the following two claims.

Claim 2.2. $\gamma_{r\{2\}}(G) \geq 4n + 4$. Moreover, if $\gamma_{r\{2\}}(G) = 4n + 4$, then for any $\gamma_{r\{2\}}(G)$-function $f = (V_0, V_1, V_2)$, $f(V(H_i)) = 4, \{u_i, \overline{u_i}\} \cap V_1 = \emptyset, |\{u_i, \overline{u_i}\} \cap V_2| = 1$ for each $i$, $f(c_j) = 0$ for each $j$, $f(\overline{s_2}) = f(s_6) = 2$ and $f(x) = 0$ for any other vertex $x \in V(F)$.

Proof of Claim 1. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(G)$-function. It is easy to note that $f(V(H_i)) \geq 4$ for each $i \in \{1, 2, \ldots, n\}$ and $f(V(F)) + \sum_{j=1}^{m} f(c_j) \geq 4$. Therefore,

$$\gamma_{r\{2\}}(G) = f(V(G)) = \sum_{i=1}^{n} f(V(H_i)) + \left( f(V(F)) + \sum_{j=1}^{m} f(c_j) \right) \geq 4n + 4.$$

Suppose now that $\gamma_{r\{2\}}(G) = 4n + 4$ then we must have $f(V(H_i)) = 4$ for each $i$ and $f(V(F)) + \sum_{j=1}^{m} f(c_j) = 4$. If $f(s_1) = 1$, then for the remaining vertices of $F$, we need that $f(N[s_3]) \cup \{s_6\}) \geq 4$ which leads to a contradiction. Hence $f(s_1) \neq 1$. Now assume that $f(s_1) = 2$. If $f(s_2) = 0$, then $f(s_3) = 0$ and so $f(s_4) + f(s_5) + f(s_6) \geq 3$ which again results in a contradiction. Hence $f(s_2) \neq 0$, and thus the remaining vertices of $F$ will fulfill $f(s_3) + f(s_4) + f(s_5) + f(s_6) \geq 3$, a contradiction too. Hence $f(s_1) = 0$. A similar argument as above shows that if $f(s_2) \neq 2$, then $f(V(F)) + \sum_{j=1}^{m} f(c_j) > 4$, leading a contradiction. Hence $f(s_2) = 2$. Therefore, $f(s_6) = 2$ and each remaining vertex of $F$ as well as every vertex $c_j$ must be assigned a 0.

To complete the proof of the claim, we shall show that for each $i$, exactly one of $u_i$ and $\overline{u_i}$ is assigned 2 and that $\{u_i, \overline{u_i}\} \cap V_1 = \emptyset$. Assume first that $f(u_i) = f(\overline{u_i}) = 2$ for some $i \in \{1, \ldots, n\}$. Since $f(V(H_i)) = 4$, it
follows that \( f(x) = 0 \) for each \( x \in V(H_i) \setminus \{u_i, \overline{u_i}\} \). But then the third neighbor of \( w_i \) has no neighbor assigned a non-zero value which leads to a contradiction. Therefore, \(|\{u_i, \overline{u_i}\} \cap V_2| \leq 1\). Now if \(|\{u_i, \overline{u_i}\} \cap V_2| = 0\), then the fourth common neighbors of \( u_i \) and \( \overline{u_i} \) different from \( w_i \) will be assigned non-zero values (2 or 1 depending on the values assigned to \( u_i \) and \( \overline{u_i} \)). But in any case, we will definitely get the contradiction \( f(V(H_i)) > 4 \).

Consequently, \(|\{u_i, \overline{u_i}\} \cap V_1| = 0\).

\[\Box \] 

**Claim 2.3.** \( \gamma_{r\{2\}}(G) = 4n + 4 \) if and only if \( \mathcal{C} \) is satisfiable.

**Proof of Claim 2.** Suppose that \( \gamma_{r\{2\}}(G) = 4n + 4 \) and let \( f \) be a \( \gamma_{r\{2\}}(G) \)-function. By Claim 2.2, at most one of \( f(u_i) \) and \( f(\overline{u_i}) \) is 2 for each \( i \in \{1, 2, \ldots, n\} \). Define a mapping \( t : U \rightarrow \{T, F\} \) by

\[
t(u_i) = \begin{cases} 
T & \text{if } f(u_i) = 2, \\
F & \text{otherwise},
\end{cases}
\]

for \( i = 1, \ldots, n \). We now show that \( t \) is a satisfying truth assignment for \( \mathcal{C} \). It is sufficient to show that every clause in \( \mathcal{C} \) is satisfied by \( t \). To this end, we arbitrarily choose a clause \( C_j \in \mathcal{C} \) for some \( j \in \{1, \ldots, m\} \).

By Claim 2.2, \( f(s_1) = f(c_j) = 0 \). Hence, there exists some \( i \in \{1, \ldots, n\} \) such that \( c_j \) is adjacent to either \( u_i \) with \( f(u_i) = 2 \) or to \( \overline{u_i} \) with \( f(\overline{u_i}) = 2 \). If the first situation occurs, then by (2), \( t(u_i) = T \), implying that the clause \( C_j \) containing such a literal \( u_i \) is satisfied by \( t \). Moreover, if the latter situation occurs, \( t(u_i) = F \) by (2). Thus, \( t \) assigns \( \overline{u_i} \) the truth value \( T \), that is, \( t \) satisfies the clause \( C_j \) containing such a literal \( \overline{u_i} \). By the arbitrariness of \( j \), we have shown that \( t \) satisfies all clauses in \( \mathcal{C} \), that is, \( \mathcal{C} \) is satisfiable.
Conversely, suppose that $\mathcal{C}$ is satisfiable, and let $t : U \rightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset $D$ of vertices of $G$ as follows. If $t(u_i) = T$, then put the vertices $u_i$ and $r_i$ in $D$ while if $t(u_i) = F$, then put the vertices $\overline{u}_i$ and $p_i$ in $D$. Hence in any case, $|D| = 2n$. Now, define the function $g$ on $V(G)$ by $g(x) = 2$ for every $x \in D$, $g(s_2) = g(v_6) = 2$ and $g(y) = 0$ for the remaining vertices. Since $t$ is a satisfying truth assignment for $\mathcal{C}$, the corresponding vertex $c_j$ in $G$ is adjacent to at least one vertex in $D$. Also, one can easily check that $g$ is an $R\{2\}$DF on $G$ of weight $4n + 4$ and thus $\gamma_{r\{2\}}(G) \leq 4n + 4$. By Claim 2.2, $\gamma_{r\{2\}}(G) \geq 4n + 4$, and therefore $\gamma_{r\{2\}}(G) = 4n + 4$. ♦

This completes the proof.

3. Exact Values for Paths and Cycles

As shown in the previous section, since the decision problem for the restrained $\{2\}$-domination number is NP-complete even for bipartite graphs, it is natural to consider specific graphs for which the exact value can be computed. In this section, we determine the restrained $\{2\}$-domination number for paths and cycles. The following result will be used.

**Proposition 3.1** ([6]). Let $G$ be a connected graph of order $n$. Then $\gamma_r(G) = n$ if and only if $G$ is a star.

Since $\gamma_{r\{2\}}(G) \geq \gamma_r(G)$ for any graph $G$, Proposition 3.1 leads that $\gamma_{r\{2\}}(P_2) = 2$ and $\gamma_{r\{2\}}(P_3) = 3$. For paths of order at least four, we have the following proposition.

**Proposition 3.2.** If $n \geq 4$, then $\gamma_{r\{2\}}(P_n) = \left\lceil \frac{2n+4}{3} \right\rceil$. More precisely,

$$\gamma_{r\{2\}}(P_n) = \begin{cases} 
\frac{2n+6}{3} & \text{if } n \equiv 0 \pmod{3} \\
\frac{2n+4}{3} & \text{if } n \equiv 1 \pmod{3} \\
\frac{2n+5}{3} & \text{if } n \equiv 2 \pmod{3}.
\end{cases}$$

**Proof.** Let $P_n = v_1v_2\ldots v_n$. Define the function $f$ on $P_n$ as follows: if $n \equiv 1 \pmod{3}$, then let $f(v_{3i+1}) = 2$ for $0 \leq i \leq \frac{n-1}{3}$ and $f(x) = 0$ otherwise; if $n \equiv 2 \pmod{3}$, then let $f(v_n) = 1$, $f(v_{3i+1}) = 2$ for $0 \leq i \leq \frac{n-2}{3}$ and $f(x) = 0$ otherwise; finally if $n \equiv 0 \pmod{3}$, then let $f(v_n) = f(v_{n-1}) = 1$, $f(v_{3i+1}) = 2$ for $0 \leq i \leq \frac{n-3}{3}$ and $f(x) = 0$ otherwise. Clearly $f$ is an $R\{2\}$DF of $P_n$ of weight $\left\lceil \frac{2n+4}{3} \right\rceil$ as outlined for each case. Hence $\gamma_{r\{2\}}(P_n) \leq \left\lceil \frac{2n+4}{3} \right\rceil$.

To prove the inverse inequality, we proceed by induction on $n$. Since the result is immediate for $n \in \{4, 5, 6, 7\}$, we assume that $n \geq 8$ and let $f$ be a $\gamma_{r\{2\}}(P_n)$-function. We only consider the case $n \equiv 1 \pmod{3}$, and the same argument can be applied for the two remaining cases. Obviously, $f(v_n) \geq 1$. If $f(v_n) = 2$, then we have $f(v_{n-1}) = f(v_{n-2}) = 0$ and thus $f(v_{n-3}) = 2$. In this case, the function $f$ restricted to $P' = P_n - \{v_n, v_{n-1}, v_{n-2}\}$ is an $R\{2\}$DF of $P_{n-3}$ and by the induction hypothesis on $P'$ we get

$$\gamma_{r\{2\}}(P_n) = \omega(f) \geq 2 + \frac{2(n-3) + 4}{3} = \frac{2n + 4}{3}.$$ 

Now, let $f(v_n) = 1$. Then $f(v_{n-1}) \geq 1$. If $f(v_{n-1}) = 2$, then we have $f(v_{n-2}) = f(v_{n-3}) = 0$ and thus $f(v_{n-4}) = 2$. In this case, the function $g$ defined on $P' = P_n - \{v_n, v_{n-1}, v_{n-2}\}$ by $g(v_{n-3}) = 1$ and $g(x) = f(x)$ otherwise, is an $R\{2\}$DF of $P_{n-3}$ and by using as above the induction hypothesis we obtain $\gamma_{r\{2\}}(P_n) \geq \frac{2n+4}{3}$. Hence we can assume that $f(v_{n-1}) = 1$. Clearly, $f(v_{n-2}) \neq 0$. If $f(v_{n-2}) = 2$, then we have $f(v_{n-3}) = f(v_{n-4}) = 0$ and $f(v_{n-5}) = 2$, and thus the function $g$ defined on $P' = P_n - \{v_n, v_{n-1}, v_{n-2}\}$ by $g(v_{n-3}) = g(v_{n-4}) = 1$ and $g(x) = f(x)$ otherwise, is an $R\{2\}$DF of $P_{n-3}$. The desired bound is therefore obtained after applying the induction on $P'$. Finally, if $f(v_{n-2}) = 1$, then $f(v_{n-3}) \geq 1$. Note that if $f(v_{n-3}) = 1$, then $f(v_{n-4}) \neq 0$. In either case, the function $g$ defined on $P' = P_n - \{v_n, v_{n-1}, v_{n-2}\}$ by $g(v_{n-3}) = 2$ and $g(x) = f(x)$ otherwise, is an $R\{2\}$DF of $P_{n-3}$ and the desired bound is obtained by induction. Consequently, $\gamma_{r\{2\}}(P_n) = \frac{2n+4}{3}$. ♦

The proof of the next result is analogously to that of Proposition 3.2.
Proposition 3.3. If \( n \geq 4 \), then

\[
\gamma_{r(2)}(C_n) = \begin{cases} 
\frac{2n}{\Delta+1} & \text{if } n \equiv 0 \pmod{3} \\
\frac{2n+4}{\Delta+3} & \text{if } n \equiv 1 \pmod{3} \\
\frac{2n+5}{\Delta+4} & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

Proof. Let \( C_n = v_1v_2 \ldots v_nv_1 \). Define the function \( f \) on \( C_n \) as follows: if \( n \equiv 0 \pmod{3} \), then let \( f(v_{3i+1}) = 2 \) for \( 0 \leq i \leq \frac{n-3}{3} \) and \( f(x) = 0 \) otherwise; if \( n \equiv 1 \pmod{3} \), then let \( f(v_{3i+1}) = 2 \) for \( 0 \leq i \leq \frac{n-1}{3} \) and \( f(x) = 0 \) otherwise; finally if \( n \equiv 2 \pmod{3} \), then let \( f(v_n) = 1 \), \( f(v_{3i+1}) = 2 \) for \( 0 \leq i \leq \frac{n-2}{3} \) and \( f(x) = 0 \) otherwise. Clearly \( f \) is an \( R \{2\} \) DF of \( C_n \) with the desired weight in each case, whence the upper bound follows.

To prove the lower bound in each case, we proceed by induction on \( n \). Since the result is immediate for \( n \in \{4,5,6,7\} \), we assume that \( n \geq 8 \). Let \( f = (V_0,V_1,V_2) \) be a \( \gamma_{r(2)}(C_n) \)-function. If \( V_0 = \emptyset \), then we have \( \omega(f) = n \) and the bounds are immediate. Hence let \( V_0 \neq \emptyset \), and assume, without loss of generality, that \( f(v_2) = f(v_3) = 0 \). Then we have \( f(v_1) = f(v_4) = 2 \). Consider the restriction of \( f \) on the vertices of \( V(C_n) \setminus \{v_2,v_3,v_4\} \). Certainly, such a function has weight \( \omega(f) - 2 \) and it is an \( R \{2\} \) DF of the cycle of order \( n-3 \) obtained from \( C_n \) by removing vertices \( v_2,v_3,v_4 \) and adding the edge \( v_1v_5 \). Using the induction hypothesis, the desired lower bound is obtained according to the case of \( n \). This completes the proof. \( \square \)

4. Bounds

In this section we present various bounds on the restrained \( \{2\} \)-domination number.

4.1. Bounds in terms of the order, size, girth and diameter

We begin by presenting two simple bounds.

Proposition 4.1. Let \( G \) be a graph of order \( n \) with no isolated vertex with maximum degree \( \Delta \). The following hold,

(i) \[ \left[ \frac{2n}{\Delta+1} \right] \leq \gamma_{r(2)}(G) \leq n. \] Both bounds are attained

(ii) If \( \delta(G) \geq 3 \), then \( \gamma_{r(2)}(G) \leq n - 2 \), and the bound is sharp for \( K_4 \).

(iii) If \( G \) has a non-support vertex \( v \) of degree at least three, then \( \gamma_{r(2)}(G) \leq n - 1 \).

Proof. (i) The upper bound follows from the fact that since \( G \) has no isolated vertex, assigning a 1 to every vertex of \( G \) provides an \( R \{2\} \) DF of weight \( n \). To prove the lower bound, let \( f = (V_0,V_1,V_2) \) be a \( \gamma_{r(2)}(G) \)-function. Let \( V_1' \subseteq V_1 \) be the set of vertices with label 0 having a neighbor in \( V_2 \) and let \( V_0^2 = V_0 \setminus V_1' \). Then \( |V_1'| \leq \Delta |V_2| \). Moreover, since each vertex in \( V_1 \) must have a neighbor in \( V_1 \cup V_2 \) and each vertex in \( V_0^2 \) has at least two neighbors in \( V_1 \), we get \( |V_0^2| \leq \frac{(\Delta - 1)|V_1|}{2} \). Hence \( 2|V_0| = 2|V_0^2| + 2|V_0^1| \leq 2\Delta |V_2| + (\Delta - 1)|V_1| \), and thus

\[
2n = 2|V_0| + 2|V_1| + 2|V_2| \\
\leq (2\Delta |V_2| + (\Delta - 1)|V_1|) + 2|V_1| + 2|V_2| \\
= (\Delta + 1)|V_1| + 2(\Delta + 1)|V_2| \\
= (\Delta + 1)\gamma_{r(2)}(G),
\]

and the lower bound follows.

The upper bound is attained for stars of order at least two, while the lower bound is attained for cycles \( C_n \) of order \( n \) with \( n \equiv 0 \) (see Prop. 3.3).

(ii) Let \( v \) and \( u \) be two adjacent vertices of \( G \). Then assigning to \( u \) and \( v \) a 0 and a 1 to any other vertex of \( G \) provides an \( R \{2\} \) DF of \( G \) of weight \( n - 2 \), and hence \( \gamma_{r(2)}(G) \leq n - 2 \).

(iii) Let \( u \) be a neighbor of \( v \) and \( u' \in N(u) \setminus \{v\} \). Then assigning to \( u \) and \( v \) a 0, a 2 to \( u' \) and a 1 to any other vertex of \( G \) provides an \( R \{2\} \) DF of \( G \) of weight \( n - 1 \), and hence \( \gamma_{r(2)}(G) \leq n - 1 \). \( \square \)
Next we present a lower bound on $\gamma_{r(2)}(G)$ in terms of the order and size of the graph.

**Theorem 4.2.** Let $G$ be a connected graph of order $n \geq 2$ and size $m$. Then

$$\gamma_{r(2)}(G) \geq \min\left\{ \frac{5n - 2m}{4}, 2n - \frac{4m}{3}, \frac{3n + 1 - 2m}{2} \right\}$$

**Proof.** Let $k = \min\left\{ \frac{5n - 2m}{4}, 2n - \frac{4m}{3}, \frac{3n + 1 - 2m}{2} \right\}$. The result is immediate for $n \in \{2, 3, 4, 5\}$. Hence assume that $n \geq 6$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{r(2)}(G)$-function. Let $m_i$ be the number of edges in $G[V_i]$ for each $i \in \{0, 1, 2\}$, and $m_{ij} = |[V_i, V_j]|$ be the number of edges between sets $V_i$ and $V_j$, with $0 \leq i < j \leq 2$. Suppose first that $V_2 = \emptyset$. Since the induced subgraph $G[V_1]$ has no isolated vertices, $m_i \geq \frac{|V_i|}{2}$ for each $i \in \{0, 1\}$. Moreover, since each vertex in $V_0$ has at least two neighbors in $V_1$, $m_{0,1} \geq 2|V_0|$. It follows that

$$m = m_0 + m_1 + m_{0,1} \geq n/2 + 2|V_0| = n/2 + 2(n - |V_1|) = \frac{5n}{2} - 2\gamma_{r(2)}(G),$$

leading to $\gamma_{r(2)}(G) \geq \frac{5n - 2m}{4} \geq k$.

In the following, we can assume that $V_2 \neq \emptyset$. If $V_1 = \emptyset$, then $f$ is a restrained Roman dominating function of $G$ (a variant of Roman dominating functions), for which it is observed in [1] that $\gamma_{r(2)}(G) = \omega(f) \geq 2n - \frac{4m}{3} \geq k$. Hence we assume that $V_1 \neq \emptyset$. Let $V_0^1$ be the set of vertices in $V_0$ having a neighbor in $V_2$ and let $V_0^2 = V_0 \setminus V_0^1$. As above, $m_0 \geq |V_0|/2$. Since each vertex in $V_0^0$ has at least two neighbors in $V_1$ and each vertex in $V_0^1$ has a neighbor in $V_2$, we have $|V_0^0, V_2| \geq |V_0^0|$ and $|V_0^2, V_2| \geq 2|V_0^2|$. Now, let $V_1^1$ be the set of vertices in $V_1$ having a neighbor in $V_2$, $V_1^2$ be the set of vertices in $V_1 \setminus V_1^1$ having a neighbor in $V_1^1$, $V_1^3$ be the set of vertices in $V_1 \setminus (V_1^1 \cup V_1^2)$ having a neighbor in $V_1^2$ and so on. Let this process end in $k$th step (note that $V_1$ is a finite set) and let $V_1^{k+1} = V_1 \cup \cup_{i=1}^{k+1} V_1^i$. Since each vertex in $V_1$ has a neighbor in $V_1 \cup V_2$, we have,

$$|V_1, V_2| + |V_1, V_1| \geq |V_1^1, V_2| + |V_1^1, V_1^1| + \cdots + |V_1^{k+1}, V_1^{k+1}| + |V_1^{k+1}, V_1^{k+1}| \geq |V_1^1| + |V_1^2| + \cdots + |V_1^k| + |V_1^{k+1}|/2 \geq \sum_{i=1}^{k+1} |V_1^i|/2 = |V_1|/2.$$ 

It follows that

$$m \geq m_0 + |V_0^0, V_2| + |V_0^2, V_1| + |V_1, V_1| + |V_1, V_2| \geq |V_0|/2 + |V_0|/2 + 2|V_0^2| + |V_1|/2 \geq \frac{3|V_0|}{2} + \frac{|V_1|}{2} \geq \frac{3n - 3|V_2| - 3|V_1|}{2} \geq \frac{3n - 2|V_2| - |V_1| + |V_2|}{2} \geq \frac{3n - 2}{2} - \gamma_{r(2)}(G) + \frac{1}{2} \geq \frac{3n + 1}{2} - \gamma_{r(2)}(G),$$

and so $\gamma_{r(2)}(G) \geq \frac{3n + 1}{2} - m \geq k$. This completes the proof. \hfill $\Box$

**Proposition 4.3.** For any connected graph $G$ of order $n$ with minimum degree at least two and girth $d \geq 6$ such that $G$ is not a cycle,

$$\gamma_{r(2)}(G) \leq \begin{cases} n - \frac{d}{3} & \text{if } n \equiv 0 \pmod{3} \\ n - \frac{d - 2}{3} & \text{if } n \equiv 1 \pmod{3} \\ n - \frac{d - 1}{3} & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

**Proof.** Let $d = g(G)$, and let $C = (x_1, x_2, \ldots, x_d, x_1)$ be a cycle of $G$ on $d$ vertices. Since $G \neq C$, we may assume that $x_1$ has a neighbor $w_1 \in V(G) \setminus V(C)$. Let $w_2 \in N(w_1) \setminus \{x_1\}$. Observe that since $d \geq 6$, each vertex in $V(G) \setminus V(C)$ has at most one neighbor in $V(C)$. Let $f$ be a $\gamma_{r(2)}(C)$-function and define the function $g$ on $G$ by $g(x) = f(x)$ for $x \in V(C)$ and $g(x) = 1$ otherwise. Clearly $g$ is an $R\{2\}$DF on $G$ of weight $\gamma_{r(2)}(C) + n - d$. Applying Proposition 3.3, the desired result follows. \hfill $\Box$
Proposition 4.4. Let $G$ be a connected triangle-free graph of order $n$ with $\delta(G) \geq 2$. Then

$$
\gamma_{r\{2\}}(G) \leq n + 1 - \left\lfloor \frac{\text{diam}(G) - 1}{3} \right\rfloor.
$$

Proof. Let $P = v_0v_1 \ldots v_d$ be a diametral path of $G$ ($d = \text{diam}(G)$) and let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(P)$-function. Define the function $g : V(G) \to \{0, 1, 2\}$ by $g(x) = f(x)$ for $x \in V(P)$ and $g(x) = 1$ otherwise. Since $G$ is triangle-free and $P$ is a diametral path, each vertex in $V(G) \setminus V(P)$ has at most one neighbor in $V(P)$. Therefore, $g$ is an $R\{2\}$DF of $G$ of weight $\gamma_{r\{2\}}(P) + n - (d + 1)$. Applying Proposition 3.2, we obtain

$$
\gamma_{r\{2\}}(G) \leq n - (d + 1) + \left\lfloor \frac{2d + 6}{3} \right\rfloor = n + 1 - \left\lfloor \frac{d}{3} \right\rfloor.
$$

\hfill \Box

The following example of graphs shows that the bound in Proposition 4.4 is asymptotically reached. Consider the graph $G$ obtained from a path $P_k = v_1v_2 \ldots v_k$ with $k \equiv 2 \pmod{3}$ and $k \geq 5$, by adding two new vertices $x$ and $y$ by attaching $x$ to $v_1$ and $v_3$ and attaching $y$ to $v_k$ and $v_{k-2}$. Note that $G$ has order $k + 2$ and diameter $k - 1$. Moreover, $G$ is triangle-free with minimum degree two having a restrained $\{2\}$-set number equals number $2k + 1 = 2k + 1$ while the bound in Proposition 4.4 gives the value $2k + 1$.

4.2. Bounds in terms of the restrained domination number

In this subsection, we establish a relationship involving the restrained $\{2\}$-domination number with the restrained domination number.

Theorem 4.5. Let $G$ be a connected graph of order $n \geq 2$ different from the star $K_{1,n-1}$. Then

$$
\gamma_r(G) + 1 \leq \gamma_{r\{2\}}(G) \leq 2\gamma_r(G).
$$

Proof. The upper bound follows from the fact that for any $\gamma_r(G)$-set $D$, the function $g = (V \setminus D, \emptyset, D)$ is an $R\{2\}$DF of $G$ of weight $2\gamma_r(G)$.

To prove the lower bound, let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(G)$-function. If $V_2 \neq \emptyset$, then clearly $V_1 \cup V_2$ is an $R$-set of $G$ and so $\gamma_r(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| - 1 = \gamma_{r\{2\}}(G) - 1$. Hence we assume that $V_2 = \emptyset$. Then $V_1 \neq \emptyset$ and each vertex in $V_0$ (if any) has at least two neighbors in $V_1$. Moreover, every vertex $v \in V_1$ must have a neighbor in $V_1$; otherwise $f(N[v]) = 1$ contradicting $f$ is an $R\{2\}$DF of $G$. Now if $V_0 = \emptyset$, then $\gamma_{r\{2\}}(G) = n$ and since $G$ is not a star we get from Proposition 3.1 that $\gamma_{r\{2\}}(G) \geq \gamma_r(G) + 1$. So assume that $V_0 \neq \emptyset$, and let $v \in V_1$ be a vertex having a neighbor in $V_0$. Then $V_1 \setminus \{v\}$ is an $R$-set of $G$ leading to $\gamma_r(G) \leq \gamma_{r\{2\}}(G) - 1$. This completes the proof. \hfill \Box

The next result provides a necessary and sufficient condition for connected graphs $G$ such that $\gamma_{r\{2\}}(G) = 2\gamma_r(G)$.

Proposition 4.6. Let $G$ be a nontrivial connected graph $G$. Then $\gamma_{r\{2\}}(G) = 2\gamma_r(G)$ if and only if there exists a $\gamma_{r\{2\}}(G)$-function $f = (V_0, V_1, V_2)$ such that $V_1 = \emptyset$.

Proof. Clearly, if $\gamma_{r\{2\}}(G) = 2\gamma_r(G)$, then for any $\gamma_r(G)$-set $D$, the function $g = (V \setminus D, \emptyset, D)$ is an $R\{2\}$DF on $G$ of weight $2\gamma_r(G)$, implying that $g$ is a $\gamma_{r\{2\}}(G)$-function such that $V_1 = \emptyset$.

Conversely, if $f = (V_0, V_1, V_2)$ is a $\gamma_{r\{2\}}(G)$-function such that $V_1 = \emptyset$, then $V_2$ is an $R$-set of $G$ and thus, $2\gamma_r(G) \leq 2|V_2| = \gamma_{r\{2\}}(G)$. The equality follows from Theorem 4.5, and the proof is complete. \hfill \Box

Restricted to connected graphs with minimum degree at least two, we establish a necessary and sufficient condition for such graphs reaching the lower bound of Theorem 4.5.
Proposition 4.7. Let $G$ be a nontrivial connected graph with minimum degree at least two. Then $\gamma_r(2)(G) = \gamma_r(G) + 1$ if and only if either $\gamma_r(G) = 1$ or $dd_r(G) = \gamma_r(G) + 1$.

Proof. Assume first that $\gamma_r(2)(G) = \gamma_r(G) + 1$, and let $f(V_0, V_1, V_2)$ be a $\gamma_r(2)$-function. Since $V_1 \cup V_2$ is an RD-set of $G$,

$$\gamma_r(G) + |V_2| \leq |V_1| + 2|V_2| = \gamma_r(2)(G) = \gamma_r(G) + 1.$$ 

Hence, $|V_2| \leq 1$. If $|V_2| = 0$, then $V_1 \neq \emptyset$, and since each vertex of $V_0$ has at least two neighbors in $V_1$, and each vertex of $V_1$ also has a neighbor in $V_1$, we deduce that $V_1$ is an RDD-set of $G$. Hence, $dd_r(G) \leq |V_1| = \gamma_r(2)(G) = \gamma_r(G) + 1$ and since $\gamma_r(2)(G) = dd_r(G)$ by (1), we conclude that $dd_r(G) = \gamma_r(G) + 1$. Hence we can assume in the next that $|V_2| = 1$, say $V_2 = \{v\}$. We claim that $V_1 = \emptyset$. Suppose to the contrary that $V_1 \neq \emptyset$, and let $u \in V_1$. By definition $N(u) \cap (V_1 \cup V_2) \neq \emptyset$. If $u$ has a neighbor in $V_0$, then the set $(V_1 \cup \{v\}) \setminus \{u\}$ is an RD-set of $G$ implying that

$$\gamma_r(G) \leq |(V_1 \cup \{v\}) \setminus \{u\}| = |V_1| = |V_2| + 2|V_2| - 2|V_2| = \gamma_r(2)(G) - 2,$$

a contradiction. Therefore, $u$ has no neighbor in $V_0$. But since $\delta(G) \geq 2$ and $|V_2| = 1$, $u$ must have a neighbor $w$ in $V_1$. The same argument used above for $u$ shows that $w$ has no neighbor in $V_0$. In this case, one can easily see that $(V_1 \cup \{v\}) \setminus \{u, w\}$ is an RD-set of $G$, and hence

$$\gamma_r(G) \leq |(V_1 \cup \{v\}) \setminus \{u, w\}| = |V_1| - 1 = |V_1| - 1 + 2|V_2| - 2|V_2| = \gamma_r(2)(G) - 3,$$

a contradiction too, which completes the proof of the claim. Therefore $V_1 = \emptyset$, and thus $N[v] = V(G)$, i.e. $\gamma_r(G) = 1$.

Conversely, assume that $G$ fulfills either $\gamma_r(G) = 1$ or $dd_r(G) = \gamma_r(G) + 1$. If $\gamma_r(G) = 1$, then it follows from Theorem 4.5 that $\gamma_r(2)(G) = \gamma_r(G) + 1$, while if $dd_r(G) = \gamma_r(G) + 1$, then by (1) and Theorem 4.5 we have $\gamma_r(2)(G) = \gamma_r(G) + 1$ and the proof is complete. \qed

5. Trees

In this section, we establish an upper and two lower bounds on the restrained $\{2\}$-domination number in trees. Since we are restricting to the class of trees, we give some additional definitions and notations. A leaf is a vertex of degree 1, a support vertex is a vertex adjacent to a leaf, and a strong support vertex is a support vertex with at least two leaf neighbors. An end support vertex is a support vertex with at most one non-leaf neighbor. For any tree $T$, let $s(T)$ and $\ell(T)$ denote the number of support vertices and leaves of $T$, respectively. A double star $DS_{p,q}$ is a tree containing exactly two vertices that are not leaves, one of which is adjacent to $p$ leaves and the other to $q$ leaves. We are also considering rooted trees distinguished by one vertex $r$ called the root. For a vertex $v \neq r$ in a rooted tree $T$, the parent of $v$ is the neighbor of $v$ on the unique $(r,v)$-path, while a child of $v$ is any other neighbor of $v$. A descendant of $v$ is a vertex $w \neq v$ such that the unique $(r,w)$-path contains $v$. The maximal subtree at $v$ denoted by $T_v$ is the subtree of $T$ induced by $v$ and all its descendants.

Proposition 5.1. Let $T$ be a tree of order $n \geq 3$. Then

$$\gamma_r(2)(T) \leq \left\lceil \frac{2n + 3(s(T) + \ell(T)) - 8}{3} \right\rceil.$$

The bound is attained for paths $P_n$ when $n \equiv 1 \pmod 3$ and for the trees $T_1$ and $T_2$ illustrated in Figure 3.
Assume that Since \( T \) is a tree of order \( n \). If \( \text{diam}(T) = 2 \), then \( T \) is a star \( K_{1,n-1} \), where \( s(T) = 1, \ell(T) = n - 1, \gamma_{r_2}(T) = n \), and so \( n < \left\lceil \frac{2n+3(1+(n-1)) - 8}{3} \right\rceil \). If \( \text{diam}(T) = 3 \), then \( T \) is a double star \( D_{r,s} \) for some integers \( r, s \geq 1 \). In this case, \( s(T) = 2, \ell(T) = r + s \) and \( \gamma_{r_2}(T) \leq n < \left\lceil \frac{5n-8}{3} \right\rceil = \left\lceil \frac{2n+3(2+(r+s)) - 8}{3} \right\rceil \). If \( T \) is a path, then Proposition 3.2 leads to the desired result. Hence we may assume that \( \text{diam}(T) \geq 4 \) and \( T \) is not a path. Assume that \( T \) has an end strong support vertex, say \( u \), and let \( v \) be the unique non-leaf neighbor of \( u \) in \( T \). Let \( T' \) be the tree obtained from \( T \) by deleting \( u \) and its leaf neighbors, and let \( f = (V_0, V_1, V_2) \) be a has \( \gamma_{r_2}(T') \)-function. Note that \( \deg_T(u) \geq 3 \) and \( T' \) has order at least three, where \( |V(T')| = n - \deg_T(u), s(T') \leq s(T) \) and \( \ell(T') \leq \ell(T) - (\deg_T(u) - 1) + 1 \). Moreover, one can easily check that the function \( g = (V_0, V_1 \cup (N_T[u] \setminus \{v\}), V_2) \) is an \( R(2) \)DF of \( T \) and hence \( \gamma_{r_2}(T) \leq \gamma_{r_2}(T') + \deg_T(u) \). It follows from the induction hypothesis on \( T' \) that

\[
\gamma_{r_2}(T) \leq \gamma_{r_2}(T') + \deg_T(u) \\
\leq \left\lceil \frac{2(n - \deg_T(u) + 3s(T') + 3\ell(T') - 8)}{3} \right\rceil + \deg_T(u) \\
\leq \left\lceil \frac{2n - 2 \deg_T(u) + 3s(T) + 3(\ell(T) - (\deg_T(u) - 1) - 1) - 8}{3} \right\rceil + \deg_T(u) \\
= \left\lceil \frac{2n + 3s(T) + 3\ell(T) - (2 \deg_T(u) - 6) - 8}{3} \right\rceil.
\]

as desired. Hence we can assume \( T \) has no end strong support vertex. Suppose that \( \text{diam}(T) = 4 \). If \( n = 5 \), then \( T = P_5 \) and \( \gamma_{r_2}(T) = 5 = \left\lceil \frac{2n + 3s(T) + 3\ell(T) - 8}{3} \right\rceil \), while if \( n \geq 6 \), then \( s(T) + \ell(T) \in \{n - 1, n\} \) and thus \( \gamma_{r_2}(T) = n < \frac{5n+11}{3} \) as desired.

In the following we can assume that \( \text{diam}(T) \geq 5 \). Let \( v_1v_2\ldots v_d \) be a diametral path in \( T \) and root \( T \) at \( v_d \). Since \( T \) has no end strong support vertex, \( \deg_T(v_2) = \deg_T(v_{d-1}) = 2 \) and every child of \( v_3 \) is either a leaf or a support vertex of degree 2. We consider two cases.

**Case 1.** \( \deg_T(v_3) \geq 3 \).

Let \( T' = T - V(T_{v_2}) \). Then \( |V(T')| = n - 2, s(T') = s(T) - 1 \) and \( \ell(T') = \ell(T) - 1 \). It is easy to see that \( \gamma_{r_2}(T) \leq \gamma_{r_2}(T') + 2 \), and by the inductive hypothesis, we obtain

\[
\gamma_{r_2}(T) \leq \gamma_{r_2}(T') + 2 \\
\leq \left\lceil \frac{2(n - 2) + 3s(T') + 3\ell(T') - 8}{3} \right\rceil + 2 \\
= \left\lceil \frac{2n + 3s(T) + 3(\ell(T) - 18)}{3} \right\rceil + 2 \\
< \left\lceil \frac{2n + 3s(T) + 3(\ell(T) - 18)}{3} \right\rceil.
\]

**Case 2.** \( \deg_T(v_3) = 2 \).

We distinguish the following subcases.

**Figure 3.** Two trees attaining the bound in Proposition 5.1.
Subcase 2.1. $\deg_T(v_4) \geq 3$.
Let $T' = T - V(T_{v_4})$ and $f$ be a $\gamma_{r\{2\}}(T')$-function. Then $|V(T')| = n - 3 \geq 3$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Since $f$ can be extended to an $R\{2\}$DF of $T$ by assigning $1$ to $v_1, v_2, v_3$, we have $\gamma_{r\{2\}}(T) \leq \gamma_{r\{2\}}(T') + 3$. By the induction hypothesis, we get

$$\gamma_{r\{2\}}(T) \leq \gamma_{r\{2\}}(T') + 3$$

Subcase 2.2. $\deg_T(v_k) = 2$.
Assume that $\deg_T(v_k) \geq 3$. Let $T' = T - V(T_{v_k})$ and $f$ be a $\gamma_{r\{2\}}(T')$-function. Then $|V(T')| = n - 4 \geq 3$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. As above, $f$ can be extended to an $R\{2\}$DF of $T$ by assigning $1$ to $v_1, v_2, v_3, v_4$, and thus $\gamma_{r\{2\}}(T) \leq \gamma_{r\{2\}}(T') + 4$. By the induction hypothesis, we have

$$\gamma_{r\{2\}}(T) \leq \gamma_{r\{2\}}(T') + 4$$

In the following, we can assume that $\deg_T(v_k) = 2$. Let $k$ be the greatest integer such that $\deg_T(v_k) \geq 3$ and $\deg_T(v_i) = 2$ for all $i \in \{2, \ldots, k - 1\}$ (note that $T$ is not a path). Consider the tree $T' = T - V(T_{v_{k-1}})$. Clearly, $T_{v_{k-1}}$ is a path of order $k - 1$, $|V(T')| = n - (k - 1) \geq 3$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Moreover, for any $\gamma_{r\{2\}}(T')$-function $f$ and $\gamma_{r\{2\}}(P_{k-1})$-function $h$ define the function $g$ on $V(T)$ by $g(x) = f(x)$ if $x \in V(T')$ and $g(x) = h(x)$ if $x \in V(T_{v_{k-1}})$. Then $g$ is an $R\{2\}$DF of $T$ leading to $\gamma_{r\{2\}}(T) \leq \gamma_{r\{2\}}(T') + \gamma_{r\{2\}}(P_{k-1})$. Using the induction hypothesis on $T'$ and Proposition 3.2 for the path $P_{k-1}$, we can check that the upper bound easily follows, which completes the proof.

Our next result is a lower bound on the restrained $\{2\}$-domination number of trees different from stars.

**Theorem 5.2.** For every tree $T$ of diameter at least three, order $n$, with $\ell(T)$ leaves and $s(T)$ support vertices, we have $\gamma_{r\{2\}}(T) \geq \frac{2n + \ell(T) - 2s(T) + 6}{3}$.

**Proof.** We proceed by induction on $n$. If $\text{diam}(T) = 3$, then $T$ is a double star for which $\frac{2n + \ell(T) - 2s(T) + 6}{3} = \frac{2n + (n-2) - 4 + 6}{3} = n = \gamma_{r\{2\}}(T)$. Hence assume that $\text{diam}(T) \geq 4$, and thus $n \geq 5$. Assume that the result holds for every tree $T'$ of order $n' < n$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{r\{2\}}(T')$-function.

Suppose first that $T$ has a strong support vertex of $T$, say $x$, and let $y$ and $z$ be two leaf neighbors of $x$. Consider the tree $T' = T - \{y\}$. Then $n' = n - 1$, $\ell(T') = \ell(T) - 1$ and $s(T') = s(T)$. Also, $y, z \in V_1 \cup V_2$. Now, if $f(x) \neq 0$, then the minimality of $f$ implies that $f(y) = f(z) = 1$ while if $f(x) = 0$, then $f(y) = f(z) = 2$. Hence in either case, the restriction of $f$ to $T'$ is an $R\{2\}$DF of $T'$, and thus $\gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 1$. By the induction hypothesis on $T'$ we have

$$\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 1$$

$$\geq \frac{2n + \ell(T') - 2s(T') + 6}{3} + 1$$

$$= \frac{2(n-1) + \ell(T) - 1 - 2s(T) + 9}{3} = \frac{2n + \ell(T) - 2s(T) + 6}{3}.$$ 

Henceforth, we can assume that every support vertex of $T$ is adjacent to exactly one leaf.

If $\text{diam}(T) = 4$, then $T$ is a tree where every vertex, except possibly its center $v$, is either a leaf or a support vertex. If $n = 5$, then $T = P_5$, and so $\gamma_{r\{2\}}(T) = 5 = \lceil \frac{14}{3} \rceil = \lceil \frac{2n + \ell(T) - 2s(T) + 6}{3} \rceil$. Hence we assume that $n \geq 6$. Now, if $v$ is a support vertex, then $s(T) = \ell(T) = \frac{n}{2}$ and thus $\gamma_{r\{2\}}(T) = n \geq \frac{2n + \ell(T) - 2s(T) + 6}{3}$, while if $v$ is not
a support vertex, then \( s(T) = \ell(T) = \frac{n-1}{2} \) and so \( \gamma_r(2)(T) = n - 1 \geq \frac{2n + \ell(T) - 2s(T) + 6}{3} \). Therefore, the result is valid when \( \text{diam}(T) = 4 \).

Hence assume that \( \text{diam}(T) \geq 5 \) and let \( v_1v_2 \ldots v_d \) be a diametral path in \( T \) and root \( T \) at \( v_d \). Since \( T \) has no strong support vertex, we have \( \deg_T(v_3) = \deg_T(v_d-1) = 2 \). Moreover, every child of \( v_3 \) is either a leaf or a support vertex of degree 2. Consider the following two cases.

**Case 1.** \( \deg_T(v_3) \geq 3 \).

First assume that \( v_3 \) is a support vertex. Let \( x \) be the unique leaf neighbor of \( v_3 \), and let \( T' = T - V(T_{v_3}) \). Then \( \ell(T') = \ell(T) - 1 \) and \( s(T') = s(T) - 1 \). If \( f(v_3) = 0 \) and \( (N_T(v_3) \setminus \{x, v_2\}) \cap V_0 \neq \emptyset \), then \( f(x) = 2 \), \( f(v_1) + f(v_2) = 2 \) and so the restriction of \( f \) to \( V(T') \) is an \( R(2) \) DF of \( T' \), yielding \( \gamma_r(2)(T') \leq \gamma_r(2)(T) - 2 \).

Now if \( (N_T(v_3) \setminus \{x, v_2\}) \cap V_0 = \emptyset \), then \( f(v_2) = 0 \) and \( f(x) = f(v_1) = 2 \). In this case, the function \( g \) defined on \( V(T') \) by \( g(v_3) = g(x) = 1 \) and \( g(z) = f(z) \), otherwise, is an \( R(2) \) DF of \( T' \), yielding again \( \gamma_r(2)(T') \leq \gamma_r(2)(T) - 2 \). In either case, by the induction hypothesis, it follows that

\[
\gamma_r(2)(T) \geq \gamma_r(2)(T') + 2 \geq \frac{2n + \ell(T') - 2s(T') + 6}{3} + 2 \geq \frac{2(n-2) + \ell(T' - 2s(T) + 9}{3} \geq \frac{2n + \ell(T) - 2s(T) + 6}{3},
\]

Now, assume that \( v_3 \) is not a support vertex. Let \( x \) be a child of \( v_3 \) other than \( v_2 \), and let \( y \) be the unique leaf neighbor of \( x \). Consider the tree \( T' = T - V(T_{v_3}) \), where \( \ell(T') = \ell(T) - 1 \) and \( s(T') = s(T) - 1 \). It is easy to see that \( \gamma_r(2)(T') \leq \gamma_r(2)(T) - 1 \), and by the induction hypothesis we get

\[
\gamma_r(2)(T) \geq \gamma_r(2)(T') + 1 \geq \frac{2n + \ell(T) - 2s(T) - 1 + 6}{3} + 1 \geq \frac{2(n-2) + \ell(T) - 2s(T) + 2 + 9}{3} = \frac{2n + \ell(T) - 2s(T) + 6}{3}.
\]

**Case 2.** \( \deg_T(v_3) = 2 \).

We distinguish the following subcases.

**Subcase 2.1.** \( \deg_T(v_d) \geq 3 \).

Assume first that \( f(v_d) = 2 \). Then \( f(v_1) = 2, f(v_3) = f(v_2) = 0 \), and the function \( f \) restricted to the tree \( T' = T - V(T_{v_3}) \) is an \( R(2) \) DF leading to \( \gamma_r(2)(T') \leq \gamma_r(2)(T) - 2 \). Since \( \deg_T(v_d) \geq 3 \), \( \ell(T') = \ell(T) - 1, s(T') = s(T) - 1 \) and by the induction hypothesis we obtain

\[
\gamma_r(2)(T) \geq \gamma_r(2)(T') + 2 \geq \frac{2n + \ell(T') - 2s(T') + 6}{3} + 2 \geq \frac{2(n-3) + \ell(T) - 2s(T) + 2 + 12}{3} \geq \frac{2n + \ell(T) - 2s(T) + 6}{3}.
\]

Assume now that \( f(v_d) = 1 \). Then \( f(v_3) = f(v_2) = f(v_1) = 1 \). Consider the tree \( T' = T - V(T_{v_3}) \), and observe that the restriction of \( f \) to \( V(T') \) with \( f(v_4) = 2 \) is an \( R(2) \) DF of \( T' \) yielding \( \gamma_r(2)(T') \leq \gamma_r(2)(T) - 2 \). Using the induction as above we get the desired bound.

Finally, assume that \( f(v_d) = 0 \). Note that \( v_3 \) may be assigned a 0. But in any case \( f(v_2) \neq 0 \) (because of \( v_3 \) and thus \( f(v_1) \neq 0 \). Moreover, we observe that \( f(v_3) + f(v_2) + f(v_1) = 3 \). In this case, by considering the tree \( T' = T - \{v_1\} \), the restriction of \( f \) to \( T' \) is an \( R(2) \) DF of \( T' \), yielding \( \gamma_r(2)(T') \leq \gamma_r(2)(T) - 1 \). Using the induction and the fact that \( \ell(T') = \ell(T) \) and \( s(T') = s(T) \), we obtain

\[
\gamma_r(2)(T) \geq \gamma_r(2)(T') + 1 \geq \frac{2n + \ell(T') - 2s(T') + 6}{3} + 1 \geq \frac{2(n-1) + \ell(T) - 2s(T) + 9}{3} \geq \frac{2n + \ell(T) - 2s(T) + 6}{3}.
\]
Subcase 2.2. \( \deg_T(v_4) = 2 \).

If \( f(v_2) = 0 \), then \( f(v_1) = f(v_4) = 2 \) and \( f(v_3) = 0 \). By considering the tree \( T' = T - V(T_{v_4}) \), we have \( \ell(T') = \ell(T) \), \( s(T') \leq s(T) \) and the restriction of \( f \) to \( V(T') \) is an \( R\{2\} \)DF of \( T' \), leading to \( \gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2 \). By the induction hypothesis, we get

\[
\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 2 \\
\geq \frac{2n + \ell(T') - 2s(T') + 2}{3} + 2 \\
\geq \frac{2n - 3 + \ell(T) - 2s(T) + 2}{3} + 2 \\
= \frac{2n + \ell(T) - 2s(T) + 6}{3}.
\]

Hence we can assume in the next that \( f(v_2) \neq 0 \). Then \( f(v_1) \geq 1 \). Note that \( v_3 \) may be assigned a 0. But in either case, \( f(v_1) + f(v_2) + f(v_3) \geq 3 \). In this case, let \( T' = T - \{v_1\} \) and note that \( \ell(T') = \ell(T) \), \( s(T') = s(T) \). Since the restriction of \( f \) to \( V(T') \) is an \( R\{2\} \)DF of \( T' \), it follows from the induction that

\[
\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 1 \\
\geq \frac{2n + \ell(T') - 2s(T') + 1}{3} + 1 \\
= \frac{2n - 3 + \ell(T) - 2s(T) + 1}{3} + 1 \\
> \frac{2n + \ell(T) - 2s(T) + 6}{3}.
\]

and this completes the proof.

\( \square \)

Our next result shows that \( \frac{2n+4}{3} \) is also a lower bound for any tree \( T \) of order \( n \geq 4 \). It is worth mentioning that this new lower bound is better than that of Theorem 5.2 for all trees \( T \) with \( 2s(T) > \ell(T) + 2 \). In addition, we will provide a characterization of all trees \( T \) of order \( n \) with \( \gamma_{r\{2\}}(T) = \frac{2n+4}{3} \).

For a graph \( G \), let

\[
W_G = \{u \in V(G) \mid \text{there exists a } \gamma_{r\{2\}}(G)\text{-function } f \text{ such that } f(u) = 2\}.
\]

Define the family \( \mathcal{T} \) of unlabeled trees \( T \) that can be obtained from a sequence \( T_1, T_2, \ldots, T_k \) (\( k \geq 1 \)) of trees such that \( T_1 = P_1 \) and \( T = T_k \). If \( k \geq 2 \), then \( T_{i+1} \) can be obtained recursively from \( T_i \) by one of the following operations.

Operation \( O_1 \): If \( u \in W_{T_i} \), then \( O_1 \) adds a path \( P_3 : abc \) attached at \( u \) by an edge \( ua \) to obtain \( T_{i+1} \).

Operation \( O_2 \): If \( u \in W_{T_i} \), then \( O_2 \) adds the tree illustrated in Figure 5 attached at \( u \) by an edge \( au \) to obtain \( T_{i+1} \).

Proposition 5.3. For any tree \( T \) in \( \mathcal{T} \), \( \gamma_{r\{2\}}(T) \leq \frac{2n(T) + 4}{3} \).

Proof. Let \( T \in \mathcal{T} \). Then \( T \) is obtained from a sequence \( T_1, T_2, \ldots, T_k \) (\( k \geq 1 \)) of trees such that \( T_1 = P_1 \), \( T = T_k \) and if \( k \geq 2 \), then \( T_{i+1} \) is obtained recursively from \( T_i \) by one of the operations \( O_1 \) and \( O_2 \). We proceed by induction on \( k \). The property is true for \( T_1 = P_1 \). Suppose the property is true for all trees of \( \mathcal{T} \) constructed with \( k - 1 \geq 0 \) operations. Let \( T = T_k \) with \( k \geq 2 \), and let \( f \) be a \( \gamma_{r\{2\}}(T_{k-1}) \)-function such that \( f(u) = 2 \). If \( T \) is obtained from \( T_{k-1} \) by Operation \( O_1 \), then \( n(T) = n(T_{k-1}) + 3 \) and \( f \) can be extended to an \( R\{2\} \)DF of \( T \) by assigning 0 to \( a \) and \( b \) and 2 to \( c \). It follows from the induction hypothesis that \( \gamma_{r\{2\}}(T) \leq \omega(f) + 2 \leq \frac{2n(T_{k-1})+4}{3} + 2 = \frac{2n(T) + 4}{3} \). Now, if \( T \) is obtained from \( T_{k-1} \) by Operation \( O_2 \), then \( n(T) = n(T_{k-1}) + 6 \) and \( f \) can be extended to an \( R\{2\} \)DF of \( T \) by assigning 0 to \( s \) and \( s' \) and a 1 to the remaining vertices. It follows from the induction hypothesis that \( \gamma_{r\{2\}}(T) \leq \omega(f) + 4 \leq \frac{2n(T_{k-1})+4}{3} + 4 = \frac{2n(T) + 4}{3} \). \( \square \)
Figure 4. Tree $T_3$ of order 10 with its unique $\gamma_{r_{\{2\}}}(T_3)$-function of weight 8.

Figure 5. Tree used in Operation $O_2$.

\textbf{Theorem 5.4.} If $T$ is a tree of order $n \notin \{2, 3\}$, then

$$\gamma_{r_{\{2\}}}(T) \geq \frac{2n + 4}{3},$$

with equality if and only if $T \in T \cup \{T_3, K_{1,3}\}$.

\textbf{Proof.} We proceed by induction on $n$. If $n = 1$, then $T = P_1$ and $\gamma_{r_{\{2\}}}(T) = 2 = \frac{2n+4}{3}$. If $n = 4$, then $T \in \{P_4, K_{1,3}\} \subset T \cup \{T_3, K_{1,3}\}$ and clearly $\gamma_{r_{\{2\}}}(T) = 4 = \frac{2n(T)+4}{3}$. These establish the base cases. Now, since for stars $K_{1,n-1}$ of order $n \geq 5$, we have $\gamma_{r_{\{2\}}}(K_{1,n-1}) = n > \frac{2n+4}{3}$, we can assume that $\text{diam}(T) \geq 3$ and $n \geq 5$. If $\text{diam}(T) = 3$, then $T$ is a double star $DS_{p,q}$ and clearly $\gamma_{r_{\{2\}}}(DS_{p,q}) = n > \frac{2n+4}{3}$. Hence, in the following, we may assume that $\text{diam}(T) \geq 4$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{r_{\{2\}}}(T)$-function.

If any support vertex, say $x$, of $T$ is adjacent to two or more leaves, say $y$ and $z$, then let $T'$ be the tree obtained from $T$ by removing $y$. Note that if $f(x) \neq 0$, then $f(y) = f(z) = 1$, while if $f(x) = 0$, then $f(y) = f(z) = 2$. Hence in either case, the restriction of $f$ to $V(T')$ is an $R\{2\}$DF of $T'$, leading to $\gamma_{r_{\{2\}}}(T') \leq \gamma_{r_{\{2\}}}(T) - 1$. Using the induction on $T'$, we obtain

$$\gamma_{r_{\{2\}}}(T) \geq \gamma_{r_{\{2\}}}(T') + 1 \geq \frac{2n' + 4}{3} + 1 = \frac{2(n - 1) + 7}{3} > \frac{2n + 4}{3}.$$

Henceforth, we can assume that every support vertex of $T$ is adjacent to exactly one leaf. If $\text{diam}(T) = 4$, then every vertex of $T$ except possibly the center vertex, say $v$, is either a leaf or a support vertex. Now, if $n = 5$, then $T = P_5$, and so by Proposition 3.2, $\gamma_{r_{\{2\}}}(T) = 2n + 5 > 2n + 4$. Thus let $n \geq 6$. If $v$ is a support vertex of $T$, then $\gamma_{r_{\{2\}}}(T) = n > \frac{2n+4}{3}$ while if $v$ is not a support vertex, then $T$ is a healthy spider with $n \geq 7$ and $\gamma_{r_{\{2\}}}(T) = n - 1 \geq \frac{2n+4}{3}$ with equality if and only if $n = 7$. In this case, $T \in T$ since it is obtained from $P_1$ applying operation $O_2$. Therefore, in the following we can assume that $\text{diam}(T) \geq 5$.

We now root $T$ at a vertex $q$ of maximum eccentricity $\text{diam}(T)$. Let $z$ be a leaf at maximum distance from $q$, $v$ be the parent of $z$, $u$ be the parent of $v$, $w$ be the parent of $u$ and $d$ be the parent of $w$ in the rooted tree. Clearly, $\text{deg}_T(v) = 2$ and every child of $u$ is either a leaf or a support vertex of degree 2. Note that $\text{deg}_T(d) \geq 2$ (since $\text{diam}(T) \geq 5$) and $z \notin V_0$. We consider two cases.

\textbf{Case 1.} $\text{deg}_T(u) \geq 3$.

First assume that $u$ is a support vertex. Let $x$ be the unique leaf neighbor of $u$, and let $T' = T - V(T_v)$.
Observe that if \( f(u) \neq 0 \), then \( f(z) + f(v) = 2 \) and thus the restriction of \( f \) to \( V(T') \) is an \( R\{2\} \)-DF of \( T' \), leading to \( \gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2 \). Hence let us assume that \( f(u) = 0 \). If \((N_T(u) \setminus \{v, w\}) \cap V_0 \neq \emptyset\), then \( f(x) = 2, f(z) + f(v) = 2 \) and thus the restriction of \( f \) to \( V(T') \) is an \( R\{2\} \)-DF of \( T' \), yielding again \( \gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2 \). If \((N_T(u) \setminus \{x, v\}) \cap V_0 = \emptyset\), then \( f(v) = 0 \) and \( f(x) = f(z) = 2 \), and thus the function \( g \) defined on \( V(T') \) by \( g(u) = g(x) = 1 \) and \( g(z) = f(z) \), otherwise, is an \( R\{2\} \)-DF of \( T' \) implying that \( \gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2 \). Therefore, in either case, by the induction hypothesis we get

\[
\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 2 \geq \frac{2n' + 4}{3} + 2 = \frac{2(n - 2) + 4}{3} + 2 > \frac{2n + 4}{3}.
\]

Now let us suppose that \( u \) is not a support vertex. Let \( x \) be a child of \( u \) besides \( v \) and let \( y \) be the leaf neighbor of \( x \). If \( f(u) \neq 0 \), then \( f(v) = f(z) = 1 \) and so \( f \) restricted to \( T' = T - \{z\} \) is an \( R\{2\} \)-DF of \( T' \), yielding \( \gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 1 \). Using the induction hypothesis we obtain

\[
\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 1 \geq \frac{2n' + 4}{3} + 1 = \frac{2(n - 1) + 7}{3} > \frac{2n + 4}{3}.
\]

Therefore assume that \( f(u) = 0 \), and consider the following two subcases.

**Subcase 1.1.** \( f(v) = 0 \).

Then \( f(z) = 2 \). Now, if \( u \) has a neighbor assigned 0 other than \( v \), then the restriction of \( f \) on the tree \( T' = T - \{v, z\} \) is an \( R\{2\} \)-DF of \( T' \). By induction on \( T' \), we have

\[
\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 2 \geq \frac{2n' + 4}{3} + 2 = \frac{2(n - 2) + 10}{3} > \frac{2n + 4}{3}.
\]

Hence we can assume that \( v \) is the unique vertex assigned 0 adjacent to \( u \). Since \( \deg_T(u) \geq 3 \) and every child \( s \) of \( u \) with \( s \neq v \) is a support vertex of degree two, we deduce that both \( s \) and its leaf neighbor are assigned 1. In this case, let \( T' = T - V(T_u) \). If \( n(T') = 3 \), then it is easy to verify that \( \gamma_{r\{2\}}(T) = n - 1 > \frac{2n + 4}{3} \). Let \( n(T') \geq 4 \). Since the restriction of \( f \) to \( V(T') \) is an \( R\{2\} \)-DF of \( T' \), \( \gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2(\deg_T(u) - 1) \). By the induction hypothesis we have

\[
\gamma_{r\{2\}}(T) \geq \gamma_{r\{2\}}(T') + 2(\deg_T(u) - 1)
\]

\[
\geq \frac{2n' + 4}{3} + 2(\deg_T(u) - 1)
\]

\[
= \frac{2n - 2\deg_T(u) + 1 + 4}{3} + 2(\deg_T(u) - 1)
\]

\[
= \frac{2n + 4}{3} + \frac{2\deg_T(u) - 4}{3} > \frac{2n + 4}{3}.
\]

**Subcase 1.2.** \( f(v) \neq 0 \).

According to Subcase 1.1, we assume that no child of \( u \) is assigned a 0. Now, if \( f(x) = 2 \), then \( f(y) = 1 \) and the restriction of \( f \) on the tree \( T' = T - \{y\} \) is an \( R\{2\} \)-DF of \( T' \), and by applying the induction hypothesis on \( T' \) we obtain \( \gamma_{r\{2\}}(T) > \frac{2n + 4}{3} \). Hence we can assume that every child of \( u \) is assigned a 1 under \( f \). Therefore, all leaves in \( T_u \) are also assigned a 1 under \( f \), and thus \( u \) must be assigned a 0 because of \( f \) is an \( R\{2\} \)-DF.

Now, if \( w \) has a neighbor assigned 0 other than \( u \), then consider the tree \( T' = T - V(T_u) \). Note that \( T' \) has order \( n' \geq 4 \). Since the restriction of \( f \) to \( V(T') \) is an \( R\{2\} \)-DF of \( T' \), we have \( \gamma_{r\{2\}}(T') \leq \gamma_{r\{2\}}(T) - 2(\deg_T(u) - 1) \). As seen in Subcase 1.1, using the induction on \( T' \), we get \( \gamma_{r\{2\}}(T) > \frac{2n + 4}{3} \).

In the next, we can assume that \( u \) is the unique vertex assigned 0 adjacent to \( w \). If \( \deg_T(u) \geq 4 \), then by considering the tree \( T' = T - \{z, w\} \) we see that \( f \) restricted to \( V(T') \) is an \( R\{2\} \)-DF of \( T' \), and using the induction we get \( \gamma_{r\{2\}}(T) > \frac{2n + 4}{3} \). Hence we assume that \( \deg_T(u) = 3 \). We need to examine several situations about vertex \( w \).

(i) If \( w \) is a support vertex with leaf neighbor \( w' \), then \( f(w') = 2 \). In this case, let \( T' = T - V(T_u) \).

Clearly, the restriction of \( f \) to \( V(T') \) by reassigning \( w \) and \( w' \) the value 1 is an \( R\{2\} \)-DF of \( T' \) leading.
to $\gamma_{r(2)}(T') \leq \gamma_{r(2)}(T) - 4$. It follows from the inductive hypothesis that

$$\gamma_{r(2)}(T) \geq \gamma_{r(2)}(T') + 4 \geq \frac{2n' + 4}{3} + 4 = \frac{2(n - 5) + 16}{3} > \frac{2n + 4}{3}. $$

(ii)- Assume that $w$ has a child $w^*$ which is a support vertex of degree two and let $v^*$ be the leaf neighbor of $w^*$. Then $f(w^*) + f(v^*) \geq 2$. In this case, consider the tree $T' = T - (V(T_u) \cup \{w^*, v^*\})$, where $\nu' \geq 3$. If $\nu' = 3$, then clearly $T$ is precisely the tree $T_3$ shown in Figure 4 belonging to $T \cup \{T_3, K_{1,3}\}$, where $\gamma_{r(2)}(T_3) = \frac{2n + 4}{3}$. Thus let $\nu' \geq 4$, and define the function $g$ on $V(T')$ by $g(w) = 1$ and $g(t) = f(t)$ otherwise. Then $g$ is an $R\{2\}$DF of $T'$ of weight $\gamma_{r(2)}(T) - 5$, and thus $\gamma_{r(2)}(T') \leq \gamma_{r(2)}(T) - 5$. By the induction hypothesis, we get

$$\gamma_{r(2)}(T) \geq \gamma_{r(2)}(T') + 5 \geq \frac{2\nu' + 4}{3} + 5 = \frac{2(n - 7) + 19}{3} > \frac{2n + 4}{3}. $$

(iii)- Assume that $w$ has a child $w^*$ of degree at least two such that every child of $w^*$ is a support vertex of degree two. We note that $w^*$ cannot be a support vertex because it would have degree at least three and since it plays the same role as $u$, such a situation was already discussed in the beginning of Case 1. In addition, since it is assumed that $u$ is the unique vertex assigned 0 adjacent to $w$, all vertices in $T_{w^*}$ are assigned a 1 under $f$. Now, if $\deg_f(w^*) \geq 3$, then again $w^*$ plays the same role as $u$ and $w^*$ is assigned a zero-value, such a situation has already been considered before Subcase 1.1. Hence we deduce that $\deg_f(w^*) = 2$, that is $T_{w^*}$ is a path $P_3$ having $w^*$ as a leaf. Let $T' = T - (V(T_u) \cup V(T_{w^*}))$, and note that $\nu' \geq 4$ (otherwise $\nu' = 3$ and thus the diameter path would have end vertices $z$ and the other leaf in $T_{w^*}$). Define the function $g$ on $V(T')$ by $g(w) = 1$ and $g(t) = f(t)$ otherwise. Then $g$ is an $R\{2\}$DF of $T'$ of weight $\gamma_{r(2)}(T) - 4 - 3 + 1$, and thus $\gamma_{r(2)}(T') \leq \gamma_{r(2)}(T) - 6$. Using the induction hypothesis, we obtain

$$\gamma_{r(2)}(T) \geq \gamma_{r(2)}(T') + 6 \geq \frac{2\nu' + 4}{3} + 6 = \frac{2(n - 5 - 3) + 4}{3} + 6 = \frac{2n + 4}{3} + \frac{2}{3} > \frac{2n + 4}{3}. $$

(iv)- Finally, assume that $\deg_f(w) = 2$. Note that because of $f(u) = f(w) = 0$ we deduce that $f(d) = 2$. Consider the tree $T' = T - V(T_w)$, and note that $\nu' \geq 2$. One can easily see that if $\nu' = 2$, then $\gamma_{r(2)}(T) = n - 1 = 7$ while if $\nu' = 3$, then $\gamma_{r(2)}(T) = n - 1 = 8$, and in any case $\gamma_{r(2)}(T) > \frac{2n + 4}{3}$. Hence we can assume that $\nu' \geq 4$. Since $u$ is the unique vertex assigned 0 adjacent to $w$, the restriction of $f$ to $V(T')$ is an $R\{2\}$DF of $T'$, and thus $\gamma_{r(2)}(T') \leq \gamma_{r(2)}(T) - 4$. By the induction hypothesis we obtain

$$\gamma_{r(2)}(T) \geq \gamma_{r(2)}(T') + 4 \geq \frac{2\nu' + 4}{3} + 4 = \frac{2(n - 6) + 16}{3} = \frac{2n + 4}{3}. $$

If further $\gamma_{r(2)}(T) = \frac{2n + 4}{3}$, then $\gamma_{r(2)}(T') = \frac{2n + 4}{3}$ and that the restriction of $f$ to $V(T')$ is a $\gamma_{r(2)}(T')$-function under which vertex $d$ is assigned a 2, that is $d$ belongs to $W_{T'}$. It follows from the induction hypothesis that $T' \in \mathcal{T}$. Therefore $T \in \mathcal{T}$ since it can be obtained from $T$ by applying Operation $O_2$. Moreover, since $T \in \mathcal{T}$, Proposition 5.3 implies that $\gamma_{r(2)}(T) \leq \frac{2n + 4}{3}$ and so $\gamma_{r(2)}(T) = \frac{2n + 4}{3}$.

**Case 2.** $\deg_f(u) = 2$.

We distinguish the following subcases.

**Subcase 2.1.** $\deg_f(w) \geq 3$.

Assume first that $f(w) = 2$. Then we have $f(u) = f(v) = 0$ and $f(z) = 2$ and the restriction of $f$ on the tree $T' = T - V(T_u)$ is an $R\{2\}$DF of $T'$ leading to $\gamma_{r(2)}(T') \leq \gamma_{r(2)}(T) - 2$. By induction on $T'$, we get

$$\gamma_{r(2)}(T) \geq \gamma_{r(2)}(T') + 2 \geq \frac{2\nu' + 4}{3} + 2 = \frac{2(n - 3) + 4}{3} + 2 = \frac{2n + 4}{3}. $$
If further $\gamma_{r(2)}(T) = \frac{2n+4}{3}$, then $\gamma_{r(2)}(T') = \frac{2n+4}{3}$ and that the restriction of $f$ to $V(T')$ is a $\gamma_{r(2)}(T')$-function under which vertex $w$ is assigned a 2, that is $w$ belongs to $W_{T'}$. It follows from the induction hypothesis that $T' \in \mathcal{T}$. Now since $T$ can be obtained from $T'$ by applying operation $O_1$, we have $T \in \mathcal{T}$. Moreover, since $T \in \mathcal{T}$, Proposition 5.3 implies that $\gamma_{r(2)}(T) \leq \frac{2n+4}{3}$ and so $\gamma_{r(2)}(T) = 2\frac{n+4}{3}$.

Now if $f(w) = 1$, then we must have $f(u) = f(v) = f(z) = 1$, and by reassigning the vertices $z, v, u, w$ the values 2, 0, 0, 2 we would be in the preceding situation. Finally, we can assume that $f(w) = 0$. Then either $f(u) = 0$ and thus $f(v) = 2$, $f(z) = 1$ or $f(u) \neq 0$ and thus $f(v) = f(z) = 1$. In either case, the restriction of $f$ to the tree $T' = T - \{z\}$ is an $R(2)$-DF of $T'$ yielding $\gamma_{r(2)}(T') \leq \gamma_{r(2)}(T) - 1$. The induction hypothesis leads to

$$\gamma_{r(2)}(T) \geq \gamma_{r(2)}(T') + 1 \geq \frac{2n' + 4}{3} + 1 = \frac{2(n - 1) + 4}{3} + 1 > \frac{2n + 4}{3}.$$

**Subcase 2.2.** $\deg_T(w) = 2$.

If $n = 6$, then $T = P_6$ and by Proposition 3.2, $\gamma_{r(2)}(T) > \frac{2n+4}{3}$. If $n = 7$, then since $d$ cannot be adjacent to two leaves by our earlier assumption), we deduce that $T = P_7$. In this case, by Proposition 3.2, $\gamma_{r(2)}(P_7) = \frac{2n+4}{3}$, and one can easily see that $T \in \mathcal{T}$ because it can be obtained from $T_1 = P_1$ by applying twice Operation $O_1$. Hence we assume that $n \geq 8$. If $f(v) = 0$, then $f(z) = f(w) = 2$ and $f(u) = 0$, and thus the restriction of $f$ on the tree $T' = T - V(T_u)$ is an $R(2)$-DF of $T'$, yielding $\gamma_{r(2)}(T') \leq \gamma_{r(2)}(T) - 2$. Applying the induction hypothesis we get $\gamma_{r(2)}(T) \geq \frac{2n+4}{3} + 1$. Using the same argument as Subcase 2.1, we can see that the equality holds if and only if $T \in \mathcal{T}$. Hence let us assume that $f(v) \neq 0$. Then $f(z) + f(v) + f(u) \geq 3$, and clearly the restriction of $f$ on the tree $T' = T - \{z\}$ is an $R(2)$-DF of $T'$ of weight $\gamma_{r(2)}(T) - 1$ leading to $\gamma_{r(2)}(T) > \frac{2n+4}{3}$ by applying the induction hypothesis. This completes the proof.

\[\square\]

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**Data Availability.** There is no data associated with this article.

**References**


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