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SUFFICIENT CONDITIONS FOR GRAPHS TO HAVE STRONG PARITY FACTORS

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Abstract. A graph G has a strong parity factor F if for every subset $X \subseteq V(G)$ with $|X|$ even, G contains a spanning subgraph F such that $\delta(F) \geq 1$, $d_F(u) \equiv 1 \pmod 2$ for any $u \in X$, and $d_F(v) \equiv 0 \pmod 2$ for any $v \in V(G) \setminus X$. In this article, we first provide a size condition for a graph having a strong parity factor. Then we put forward a toughness condition to guarantee that a graph has a strong parity factor.

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1. Introduction

In this article, we deal only with finite and undirected graphs without multiple edges or loops. Let G denote a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices in G is called its order and denoted by $n = |V(G)|$. The number of edges in G is called its size and denoted by $e(G) = |E(G)|$. The degree of a vertex v, denoted by $d_G(v)$, is the number of edges incident to the vertex v. Let $\delta(G)$ denote the minimum degree in G. Let S and T be two disjoint vertex subsets of G. We use $e_G(S,T)$ to denote the number of edges of G joining S to T. When $S = \{v\}$, we write $e_G(v,T)$ for $e_G(\{v\},T)$. We denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph formed from G by deleting the vertices in S and their incident edges. Let $c(G)$ denote the number of connected components in G. The complement of a graph G is a graph $\overline{G}$ with the same vertex set as G, in which any two distinct vertices are adjacent if and only if they are nonadjacent in G. The complete graph of order n is denoted by K_n.

The toughness $t(G)$ of a graph G was first defined by Chvátal [3] as follows.

$$t(G) = \min\left\{\frac{|X|}{c(G-X)} : X \subseteq V(G), c(G-X) \geq 2\right\},$$

if G is not complete; otherwise, $t(G) = +\infty$.

Let G_1 and G_2 be two vertex disjoint graphs. We use $G_1 \cup G_2$ to denote the disjoint union of G_1 and G_2. For any nonnegative integer t, let tG denote the disjoint union of t copies of G. The join $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 by an edge.

Keywords. Graph, size, toughness, strong parity factor.

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Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(v) \leq f(v)$ and $g(v) \equiv f(v) \pmod 2$ for any $v \in V(G)$. A (g, f)-parity factor of a graph G is a spanning subgraph F such that $d_F(v) \equiv f(v) \pmod 2$ and $g(v) \leq d_F(v) \leq f(v)$ for any $v \in V(G)$. For two positive integers a and b with $a \leq b$ and $a \equiv b \pmod 2$, if $g(v) = a$ and $f(v) = b$ for any $v \in V(G)$, then a (g, f)-parity factor is called an (a, b)-parity factor. An (a, b)-parity factor is called a $(1, b)$-odd factor if $a = 1$. Note that a $(1, b)$-odd factor is an extension of a perfect matching. A (g, f)-parity factor is called an f-factor if $g(v) = f(v)$ for any $v \in V(G)$. If $f(v) = k$ for any $v \in V(G)$, then an f-factor is a k-factor. For any real function h on $V(G)$ and any subset $S \subseteq V(G)$, let $h(S) = \sum_{v \in S} h(v)$.

In mathematical literature, the study on factors and parity factors of graphs attracted much attention. Some sufficient conditions for graphs with k-factors were derived by Niessen and Randerath [15], Nishimura [16], Enomoto *et al.* [5]. Lots of researchers [6, 8, 17, 20, 21, 25, 26, 28, 29, 32, 33, 35] proved some conditions related to neighborhood, toughness, binding number, etc., for a graph with a $(1, 2)$-factor. Much effort has been devoted to finding sufficient conditions for the existence of (a, b)-factors in graphs by using various graphic parameters such as toughness [9], Fan-type condition [14], stability number [11] and others [7, 18, 19, 23, 24, 27, 31, 34].

Amahashi [1] gave a criterion for a graph to possess a $(1, b)$-odd factor. Cui and Kano [4] provided a neighborhood condition for a graph with a $(1, b)$-odd factor. Kim *et al.* [10] established a connection between eigenvalues and $(1, b)$-odd factors in graphs. Zhou and Liu [30] showed two sufficient conditions for the existence of $(1, b)$-odd factors in graphs. Liu and Lu [12] obtained a degree condition for a graph to possess an (a, b)-parity factor. Yang *et al.* [22] developed a sufficient condition for the existence of (a, b)-parity factors in graphs in view of the independence number and the connectivity.

A graph G has a strong parity factor F if for every subset $X \subseteq V(G)$ with $|X|$ even, G contains a spanning subgraph F such that $\delta(F) \geq 1$, $d_F(u) \equiv 1 \pmod 2$ for any $u \in X$, and $d_F(v) \equiv 0 \pmod 2$ for any $v \in V(G) \setminus X$. Bujtás *et al.* [2] introduced the concept of strong parity factor and derived some sufficient conditions for graphs with this property. Lu *et al.* [13] presented a characterization for graphs to possess strong parity factors, which is shown in the following.

Theorem 1.1 ([13]). *A graph G has strong parity factors if and only if*

$$c(G - S) \leq \sum_{v \in S} d_G(v) - 2|S| + 1$$

for any $S \subseteq V(G)$.

In fact, it's complicated that one determines a graph having strong parity factors by Theorem 1.1. Hence, for computational reasons we want to present other sufficient conditions to guarantee that a graph has strong parity factors. In this article, we proceed to investigate strong parity factors in graphs, and pose two sufficient conditions for graphs possessing strong parity factor *via* the size and the toughness, respectively. Our main results will be shown in the following.

Theorem 1.2. *Let G be a connected graph of order $n \geq \max\left\{\delta^2 + 3\delta + 3, \frac{(\delta^2-5)^2}{6(\delta-2)} + \delta + 3\right\}$ with minimum degree $\delta \geq 3$. If*

$$e(G) > \binom{n - \delta(\delta - 2) - 1}{2} + \delta(\delta(\delta - 2) + 1),$$

then G has a strong parity factor.

Theorem 1.3. *Let G be a connected graph with minimum degree $\delta \geq 3$. If its toughness*

$$t(G) \geq \frac{1}{\delta - 2},$$

then G has a strong parity factor.

2. The proof of Theorem 1.2

In this section, we present a proof of Theorem 1.2, which provides a size condition for a graph to have a strong parity factor.

Proof of Theorem 1.2. Suppose that a connected graph G has no strong parity factor. Then it follows from Theorem 1.1 that there exists some nonempty subset S of $V(G)$ such that

$$c(G-S) \geq \sum_{v\in S} d_G(v) - 2|S| + 2 \geq (\delta-2)|S|+2.$$

Let $|S| = s$ and $c(G-S) = t$. Then

$$t \geq (\delta-2)s+2$$

and G is a spanning subgraph of $G_s^1 = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{(\delta-2)s+2}})$, where $n_1, n_2, \ldots, n_{(\delta-2)s+2}$ are positive integers with $n_1 \geq n_2 \geq \ldots \geq n_{(\delta-2)s+2}$ and $\sum_{i=1}^{(\delta-2)s+2} n_i = n-s$. Thus,

$$e(G) \leq e(G_s^1). \tag{2.1}$$

The following proof will be divided into two cases by the value of s.

Case 1. $s \leq \delta$.

Note that the minimum degree of G equals δ and G is a spanning subgraph of $G_s^1 = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{(\delta-2)s+2}})$. Obviously, the minimum degree of G_s^1 is at least δ, and so $n_{(\delta-2)s+2} \geq \delta+1-s$. If $n_i \geq \delta+2-s$ for some $i \in \{2, 3, \ldots, (\delta-2)s+2\}$, then let $G_s^2 = K_s \vee (K_{n_1+1} \cup K_{n_2} \cup \cdots \cup K_{n_i-1} \cup \cdots \cup K_{n_{(\delta-2)s+2}})$. Thus, we infer

$$e\big(G_s^2\big) = e\big(G_s^1\big) - (n_i-1) + n_1 = e\big(G_s^1\big) + (n_1-n_i) + 1 > e\big(G_s^1\big).$$

We proceed the above procedure until $n_2 = n_3 = \ldots = n_{(\delta-2)s+2} = \delta+1-s$ in G_s^1. Then we obtain a new graph, which is denoted by

$$G_s^3 = K_s \vee \big(K_{n-s-((\delta-2)s+1)(\delta+1-s)} \cup ((\delta-2)s+1)K_{\delta+1-s}\big).$$

Thus, we possess

$$e\big(G_s^1\big) \leq e\big(G_s^3\big) \tag{2.2}$$

with equality if and only if $n_2 = n_3 = \ldots = n_{(\delta-2)s+2} = \delta+1-s$.

Note that $e(G_s^3) = \binom{n-((\delta-2)s+1)(\delta+1-s)}{2} + s((\delta-2)s+1)(\delta+1-s) + ((\delta-2)s+1)\binom{\delta+1-s}{2}$. By a directed calculation, we get

$$\binom{n-\delta(\delta-2)-1}{2} + \delta(\delta(\delta-2)+1) - e(G_s^3) = \frac{\delta-s}{2}\varphi(n), \tag{2.3}$$

where $\varphi(n) = (2(\delta-2)s-2\delta+6)n + (\delta-2)^2s^3 - (\delta-2)(\delta^2-5)s^2 - (\delta-2)(2\delta+6)s + \delta^3 - 2\delta^2 + \delta - 8$.

Note that $s \geq 1$ and $\delta \geq 3$. Then we have $2(\delta-2)s - 2\delta + 6 \geq 2(\delta-2) - 2\delta + 6 = 2$. Together with $n \geq \frac{(\delta^2-5)^2}{6(\delta-2)} + \delta + 3$, we possess

$$\begin{aligned}
\varphi(n) &= (2(\delta-2)s-2\delta+6)n + (\delta-2)^2s^3 \\
&\quad - (\delta-2)(\delta^2-5)s^2 - (\delta-2)(2\delta+6)s + \delta^3 - 2\delta^2 + \delta - 8 \\
&\geq (2(\delta-2)s-2\delta+6)\left(\frac{(\delta^2-5)^2}{6(\delta-2)} + \delta + 3\right) + (\delta-2)^2s^3
\end{aligned}$$

$$\begin{aligned}&-(\delta-2)(\delta^2-5)s^2-(\delta-2)(2\delta+6)s+\delta^3-2\delta^2+\delta-8\\ &=(\delta-2)^2s^3-(\delta-2)(\delta^2-5)s^2+\frac{(\delta^2-5)^2}{3}s\\ &\quad+(-\delta+3)\frac{(\delta^2-5)^2}{3(\delta-2)}+\delta^3-4\delta^2+\delta+10\\ &:=f(s).\end{aligned} \tag{2.4}$$

It is straightforward to check that the derivative function of $f(x)$ is

$$f'(x)=3(\delta-2)^2x^2-2(\delta-2)(\delta^2-5)x+\frac{(\delta^2-5)^2}{3}\cdot$$

Then the symmetry axis of $f'(x)$ is

$$x=\frac{\delta^2-5}{3(\delta-2)}\in(1,\delta)$$

by $\delta\geq 3$. This yields that

$$f'(x)\geq f'\left(\frac{\delta^2-5}{3(\delta-2)}\right)=3(\delta-2)^2\left(\frac{\delta^2-5}{3(\delta-2)}\right)^2-2(\delta-2)(\delta^2-5)\frac{\delta^2-5}{3(\delta-2)}+\frac{(\delta^2-5)^2}{3}=0$$

for $x\in(1,\delta)$, which implies that $f(x)$ is increasing in the interval $[1,\delta]$. Thus,

$$f(s)\geq f(1)=\frac{\delta^4-3\delta^3+2\delta^2+1}{3(\delta-2)}\geq\frac{2\delta^2+1}{3(\delta-2)}>0 \tag{2.5}$$

for $s\in[1,\delta]$ and $\delta\geq 3$. It follows from (2.3) to (2.5) and $s\leq\delta$ that

$$e(G_s^3)\leq\binom{n-\delta(\delta-2)-1}{2}+\delta(\delta(\delta-2)+1). \tag{2.6}$$

By virtue of (2.1), (2.2) and (2.6), we deduce

$$e(G)\leq e(G_s^1)\leq e(G_s^3)\leq\binom{n-\delta(\delta-2)-1}{2}+\delta(\delta(\delta-2)+1),$$

which is a contradiction to the condition that $e(G)>\binom{n-\delta(\delta-2)-1}{2}+\delta(\delta(\delta-2)+1)$.

Case 2. $s\geq\delta+1$.

Recall that $G_s^1=K_s\vee(K_{n_1}\cup K_{n_2}\cup\cdots\cup K_{n_{(\delta-2)s+2}})$. Let $G_s^4=K_s\vee(K_{n-(\delta-1)s-1}\cup((\delta-2)s+1)K_1)$. We verify the following claims.

Claim 1. $e(G_s^1)\leq e(G_s^4)$.

Proof. If $n_i=1$ for all $i\in\{2,3,\dots,(\delta-2)s+2\}$, then we have $G_s^1=G_s^4$. If $n_i\geq 2$ for some $i\in\{2,3,\dots,(\delta-2)s+2\}$, then we create a new graph G_s^5 obtained from G_s^1 by deleting (n_i-1) vertices in K_{n_i} and adding (n_i-1) vertices to K_{n_1} by joining the (n_i-1) vertices to the vertices in $V(K_{n_1})\cup S$ so that $G[S]$ and all connected components of G_s^5-S are complete graphs. Thus, we possess

$$\begin{aligned}e(G_s^5)&=e(G_s^1)-(n_i-1)s-(n_i-1)-\binom{n_i-1}{2}+(n_i-1)(n_1+s)+\binom{n_i-1}{2}\\ &=e(G_s^1)+(n_1-1)(n_i-1)>e(G_s^1).\end{aligned}$$

We proceed the above procedure until $n_2 = n_3 = \cdots = n_{(\delta-2)s+2} = 1$ in G_s^1. Thus, we derive

$$G_s^4 = K_s \vee (K_{n-(\delta-1)s-1} \cup ((\delta-2)s+1)K_1)$$

and

$$e(G_s^1) \leq e(G_s^4).$$

We complete the proof of Claim 1. □

Claim 2. $e(G_s^4) \leq \binom{n-\delta(\delta-2)-1}{2} + \delta(\delta(\delta-2)+1)$.

Proof. Note that $e(G_s^4) = \binom{n-s(\delta-2)-1}{2} + s(s(\delta-2)+1)$. By a directed computation, we obtain

$$\binom{n-\delta(\delta-2)-1}{2} + \delta(\delta(\delta-2)+1) - e(G_s^4) = \frac{s-\delta}{2}\Phi(n), \tag{2.7}$$

where $\Phi(n) = 2(\delta-2)n - \delta(\delta-2)s - \delta^3 + 2\delta^2 - 3\delta + 4$. If $s \geq \delta+7$, then it follows from $\delta \geq 3$ and $n \geq (\delta-1)s+2$ that

$$\begin{aligned}
\Phi(n) &= 2(\delta-2)n - \delta(\delta-2)s - \delta^3 + 2\delta^2 - 3\delta + 4 \\
&\geq 2(\delta-2)((\delta-1)s+2) - \delta(\delta-2)s - \delta^3 + 2\delta^2 - 3\delta + 4 \\
&= (\delta^2 - 4\delta + 4)s - \delta^3 + 2\delta^2 + \delta - 4 \\
&\geq (\delta^2 - 4\delta + 4)(\delta+7) - \delta^3 + 2\delta^2 + \delta - 4 \\
&= (\delta-3)(5\delta-8) \\
&\geq 0.
\end{aligned}$$

Together with (2.7), we possess

$$e(G_s^4) \leq \binom{n-\delta(\delta-2)-1}{2} + \delta(\delta(\delta-2)+1).$$

If $\delta+1 \leq s \leq \delta+6$, then it follows from $\delta \geq 3$ and $n \geq \delta^2 + 3\delta + 3$ that

$$\begin{aligned}
\Phi(n) &= 2(\delta-2)n - \delta(\delta-2)s - \delta^3 + 2\delta^2 - 3\delta + 4 \\
&\geq 2(\delta-2)(\delta^2+3\delta+3) - \delta(\delta-2)(\delta+6) - \delta^3 + 2\delta^2 - 3\delta + 4 \\
&= 3\delta - 8 \\
&> 0.
\end{aligned}$$

Combining this with (2.7), we have

$$e(G_s^4) < \binom{n-\delta(\delta-2)-1}{2} + \delta(\delta(\delta-2)+1).$$

Claim 2 is proved. □

According to (2.1), Claims 1 and 2, we derive

$$e(G) \leq e\big(G_s^1\big) \leq e(G_s^4) \leq \binom{n-\delta(\delta-2)-1}{2} + \delta(\delta(\delta-2)+1),$$

which is a contradiction to the condition that $e(G) > \binom{n-\delta(\delta-2)-1}{2} + \delta(\delta(\delta-2)+1)$. This completes the proof of Theorem 1.2. □

3. The proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3, which establish a connection between toughness and a strong parity factor in a graph.

Proof of Theorem 1.3. Suppose, to the contrary, that a connected graph G has no strong parity factor. In terms of Theorem 1.1, we possess

$$c(G-S) \geq \sum_{v \in S} d_G(v) - 2|S| + 2 \geq (\delta-2)|S| + 2 \geq 2 \tag{3.1}$$

for some vertex subset S of G.

Obviously, $S \neq \emptyset$. Otherwise, it follows from (3.1) that $c(G) \geq 2$, which contradicts the condition that G is connected. If G is a complete graph, then by (3.1) and $\delta \geq 3$ we possess

$$1 = c(G-S) \geq \sum_{v \in S} d_G(v) - 2|S| + 2 \geq (\delta-2)|S| + 2 \geq (\delta-2) + 2 = \delta \geq 3,$$

which is a contradiction. In what follows, we assume that G is not a complete graph.

By virtue of (3.1) and the definition of $t(G)$, we deduce

$$t(G) \leq \frac{|S|}{c(G-S)} \leq \frac{|S|}{(\delta-2)|S|+2} < \frac{1}{\delta-2},$$

which contradicts the condition that $t(G) \geq \frac{1}{\delta-2}$. This completes the proof of Theorem 1.3. □

4. Concluding remarks

Bujtás *et al.* [2] introduced the concept of strong parity factor and conjectured that every 2-edge-connected graph with minimum degree at least three has a strong parity factor. Lu *et al.* [13] showed a characterization for a graph to possess a strong parity factor and confirmed the above conjecture for 3-edge-connected graph. In this paper, we put forward two sufficient conditions for a graph to have a strong parity factor with respect to the size or the toughness. Indeed, there are very few results on strong parity factors in graphs. Various parameters of different graphs have different computational complexity. For different graphs we want to present distinct parameter conditions to guarantee that a graph has strong parity factors. Hence, we need to investigate other sufficient conditions. Consequently, we present two open problems as the end of this paper.

Problem 4.1. Investigate other sufficient conditions for the existence of strong parity factors in graphs.

Problem 4.2. Find some properties (for instance, Hamiltonicity, spectral radius and matching number etc.) of graphs with strong parity factors.

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Conflict of Interest. The authors declare that they have no conflicts of interest to this work.

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