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OPTIMAL INVESTMENT AND CONSUMPTION STRATEGIES FOR AN INVESTOR WITH STOCHASTIC ECONOMIC FACTOR IN A DEFAULTABLE MARKET

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Abstract. This paper considers the issue of optimal investment and consumption strategies for an investor with stochastic economic factor in a defaultable market. In our model, the price process is composed of a money market account and a default-free risky asset, assuming they rely on a stochastic economic factor described by a diffusion process. A defaultable perpetual bond is depicted by the reduced-form model, and both the default risk premium and the default intensity of it rely on the stochastic economic factor. Our goal is to maximize the infinite horizon expected discounted power utility of the consumption. Applying the dynamic programming principle, we derive the Hamilton–Jacobi–Bellman (HJB) equations and analyze them using the so-called sub-super solution method to prove the existence and uniqueness of their classical solutions. Next, we use a verification theorem to derive the explicit formula for optimal investment and consumption strategies. Finally, we provide a sensitivity analysis.

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1. Introduction

Merton was accredited as the forerunner of research on the issue of optimal investment and consumption (see [10–12]). In Merton's portfolio optimization problem, the investor allocated wealth dynamically between a riskless asset and a risky asset and chose a consumption ratio for the goal of maximizing the total expected discount utility of the consumption. Since then, the issue of default-free portfolio optimization has always been subjects under a great deal of investigation (see, *e.g.*, [6, 7, 13–15, 18]) and references therein. Additionally, one interesting promotion was initiated by Pang and Hussain [16], who considered the Merton-type portfolio optimization problem with complete memory over a finite time horizon, which was described as a stochastic control problem within the finite time frame, wherein the state was to evolve by following a process controlled by a stochastic process with memory.

Over the past decades, the domains of mathematical finance and financial engineering have developed rapidly thanks to the achievement of complex quantitative methods in helping professionals manage financial risks. As known to all, the credit risk is one of the fundamental factors of financial risks. Therefore, the research on credit

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risk has drawn attentions from broad scholars. Particularly, the portfolio optimization problem with defaultable securities has become a topic of interest. Bélanger *et al.* [1] provided a framework for the pricing of defaultable bonds and derivatives. Although this framework failed to provide effective settings for obtaining the result of the structural model, it had enough versatility such as to cover most structural models, thereby highlighting the commonality between the reduced form and the structural model. Bielecki and Jang [2] studied the optimal allocation problem related to a credit-risky asset by maximizing the expected CRRA utility of terminal wealth. Lakner and Liang [9] analyzed the optimal investment strategy for a defaultable corporate bond and a money market account in a continuous time model. Bo *et al.* [4] studied such an optimal portfolio problem with a stochastic factor: the representative investor may dynamically choose a consumption ratio and allocate his/her wealth in a defaultable perpetual bond, a default-free risky asset, and a money market account; the considered utility function is log utility. Subsequently, Bo *et al.* [5] generalized the above result to the circumstance of power utility. Rizal *et al.* [20] studied the reduced-form model for optimal investment of a defaultable corporate bond under the market risks (the risks of credit and inflation). Shen and Siu [21] proposed a new risk-based method to solve the optimal asset allocation problem with default risk, in which an investor's goal is to choose an optimal portfolio of the financial securities (a money market account, an ordinary share and a defaultable security) so as to minimize the risk metric of a portfolio.

With the development of society, high-yield corporate bonds have grown increasingly alluring to investors in today's financial markets. Compared to default-free bonds or stocks, corporate bonds promise higher risk-adjusted returns, making them frequently sought after. However, such allure comes with heightened risks. For instance, the default of Enron stands as one of the largest in the history of defaultable securities, which indirectly led to Kmart's subsequent bankruptcy. In another case, American International Group (AIG) underestimated the risks associated with Credit Default Swap (CDS) contracts and employed a flawed risk management approach, resulting in its collapse in 2008. These events inflicted substantial losses upon investors. Thus, in the real world, determining the optimal investment portfolio involving defaultable securities becomes a crucial area of research. We propose a general framework for managing credit risk in the default market to address these issues. Our approach involves modeling a perpetual defaultable bond using the reduced-form model and optimizing the investment portfolio from the perspective of the investor. The insights derived from our research can be applied to effectively manage portfolios that involve default risk. Another motivation for our work is the classical Merton portfolio optimization problem, along with subsequent enhancements proposed by other scholars. However, they usually consider a constant interest rate. In reality, we are well aware that the interest rate is subject to fluctuations, even within the banking sector. Moreover, the interest rate variation may correlate with price volatility in risk assets and the bond market. For instance, the Federal Reserve often adjusts the benchmark interest rate based on the performance of the U.S. stock market. As a result, our paper introduces the stochastic interest rate, acknowledging their variability in response to the stochastic economic factor.

This paper considers the issue of optimal investment and consumption strategies for an investor with a defaultable perpetual bond, where the money market account and the default-free risky asset are both associated with the stochastic economic factor. In other words, we consider the case of stochastic interest rate, and the drift and diffusion coefficients of the stochastic differential equation (SDE) obeyed by the default-free risky asset are both random. Here we intend to maximize the infinite horizon expected discounted power utility of the consumption. Additionally, our stochastic economic factor is characterized by a more general diffusion process rather than including the mere drift term (whose diffusion coefficient is a constant) as in the research by Bo *et al.* [5]. It follows that the pre-default and post-default HJB equations are all nonlinear. In order to study the classical solutions of the HJB equations, we use the so-called sub-super solution approach to derive the explicit formula for the optimal Markov control strategy. Since the diffusion term of the stochastic economic factor process is not constant, a classical result of Fleming and Pang [7] on sub-super solution approach cannot be directly applied. However, under the condition that the diffusion term is bounded, we find that the result can be generalized. Finally, the corresponding value function is derived.

In one recently published paper, Shen and Yin [22], an optimal investment and risk control problem for an insurer subject to a stochastic economic factor in a Lévy market is considered. The current paper and Shen and Yin [22] are distinct in financial and mathematical aspects, despite both utilizing dynamic programming to investigate stochastic control problems. First of all, they differ in terms of the research subjects. This paper explores the optimal investment and consumption for an investor, whereas Shen and Yin [22] delve into the optimal investment and risk control for an insurer. Secondly, the types of risks considered in either paper are different. The current paper takes into account the risk of default in a bond, specifically a perpetual defaultable bond, to find the optimal value for investing in this bond. On the other hand, Shen and Yin [22] focus on the insurer's risk, specifically the claim risk. In other words, this paper examines scenarios where an investor can invest in a perpetual defaultable bond, while Shen and Yin [22] assume an insurer is not allowed to invest in such bonds. Thirdly, the financial applications vary between the two papers. The findings of this paper can be broadly applied to the portfolio management of investors facing default risk. On the other hand, the conclusions in Shen and Yin [22] are only applicable to the portfolio management of insurance companies that do not involve defaultable bonds. Fourthly, since this paper addresses an optimization problem in the context of the default market, the value function is divided into pre-default and post-default states. Consequently, the HJB equation is also separated into pre-default and post-default, and the classical sub-super solution method is employed to establish the existence of classical solutions to the HJB equations. In contrast, an optimization problem in the Lévy market is considered in Shen and Yin [22], where the model of the stochastic economic factor is characterized by the Lévy processes (while the current paper employs a diffusion process). As a result, the HJB equation in Shen and Yin [22] becomes a fully nonlinear partial integro-differential equation, and thus, it lacks the classical solution. The approach taken in Shen and Yin [22] involves using the theory of viscosity solution to seek the viscosity solution.

In summary, this paper makes three significant contributions compared to the existing literature. Firstly, we consider the scenario with the stochastic interest rate, adding complexity to the research. Although Fleming and Pang [7] as well as Pang [14] have examined the investment portfolio problem with default-free under the stochastic interest rate, there is scarce literature addressing the investment portfolio problem with default under the stochastic interest rate. Secondly, we characterize the stochastic economic factor using a more general diffusion process, expanding the scope of our analysis. Thirdly, the HJB equations for both the pre-default and post-default states are fully nonlinear, making the direct application of standard existence and uniqueness results unfeasible. In response, we provide a novel sufficient condition, namely, the boundedness of the diffusion term in the stochastic economic factor, to ensure the existence of a unique solution to the HJB equation. In the proof, we ingeniously define a new function (refer to Thm. 3.7) and utilize the Intermediate Value Theorem to derive the solution to the HJB equation.

The remainder of this paper is organized in the following way. In Section 2, we present the model and derive the dynamics of the wealth process. In Section 3, we discuss the portfolio optimization problem with default risk under power utility. In Section 4, under certain conditions, we provide the verification theorem proof. Finally, in Section 5, we conduct a sensitivity analysis to study the impact of some important parameters on the optimal control strategies, and then illustrate the effects of the wealth and stochastic economic factor on the value function. The final section gives concluding remarks.

2. Formulation of the model

Throughout this paper, $\mathbb{R}_+$, $\mathbb{R}$ and $\mathbb{N}$ denote the family of nonnegative real numbers, real numbers and nonnegative integers, respectively. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete real-world probability space. This space also supports a 2-dimensional standard Brownian motion $(W_t, \widetilde{W}_t)_{t\geq 0}$ and a nontrivial random time τ. Suppose that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the augmented natural filtration of the Brownian motion.

2.1. The stochastic economic factor model and price processes

The stochastic economic factor model. In reality, we usually need to model the dynamics of macroeconomic variables (such as economic growth, interest rate and price index). Obviously, these factors will affect the asset price process. Therefore, it is necessary to take stochastic economic factors into account when modeling the price process. We use a diffusion process $(Y_t)_{t\geq 0}$ to describe the stochastic economic factor process. The dynamics of $(Y_t)_{t\geq 0}$ is driven by the following SDE: $Y_0 = y \in \mathbb{R}$ and

$$\mathrm{d}Y_t = \beta(Y_t)\,\mathrm{d}t + \theta(Y_t)\Big(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\Big), \tag{1}$$

where $\beta(\cdot) \in C^1(\mathbb{R})$, $\theta(\cdot)$ is Lipschitz continuous, and the correlation coefficient $\rho \in (-1, 1)$. We further assume that $-c_2 \leq \frac{\partial \beta(y)}{\partial y} \leq -c_1 < 0$ and $c_4 \leq \theta(y) \leq c_3$ for all $y \in \mathbb{R}$. Here c_1, c_2, c_3 and c_4 are four positive constants.

Remark 2.1. For all $T \geq 0$, let $t \in [0, T]$. Clearly, by virtue of Theorem 1.1 in Friedman [8], we know that (1) has a unique, strong solution Y_t. Moreover, by Theorem 2.3 in Friedman [8], it is not hard to verify that $\mathbb{E}|Y_t|^4 < \infty$.

Price processes. In the financial market, there are two assets available for investment, a money market account $(B_t)_{t\geq 0}$ and a default-free risky asset $(S_t)_{t\geq 0}$. The dynamics of $(B_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are given by

$$\mathrm{d}B_t = r(Y_t)B_t\,\mathrm{d}t, \tag{2}$$

$$\mathrm{d}S_t = \mu(Y_t)S_t\,\mathrm{d}t + \sigma(Y_t)S_t\,\mathrm{d}W_t, \tag{3}$$

where $0 < r(\cdot) \leq M$ is a C^1-function, $\mu(\cdot)$ and $\sigma(\cdot) > 0$ are C^1-functions. Here M is a positive constant. We also assume $\mu(y) > r(y)$ for all $y \in \mathbb{R}$. The initial conditions are $B_0 = 1$ and $S_0 > 0$.

2.2. A defaultable perpetual bond

In this subsection, we give the reduced-form model for a perpetual defaultable bond (see [3]). In other words, we only focus on the modeling of default time, while the value of the firm's assets and its capital structure are not modeled at all. Since the credit events are specified in terms of some exogenously specified jump process, the reduced-form model is henceforth referred to as the intensity-based model.

For $t \geq 0$, define a default indicator process $(Z_t)_{t\geq 0}$ by $Z_t = \mathbf{1}_{\{\tau \leq t\}}$, $Z_0 = 0$. Let $\mathcal{D}_t = \sigma(Z_s; 0 \leq s \leq t)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ with $t \geq 0$. We denote $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, then τ is a $\mathbb{G}$ stopping time. We further assume that all the filtrations satisfy the usual conditions. In the light of the reduced-form framework for an intensity-based defaultable market (see [5], Def. 2.1), we suppose that τ has a positive $\mathbb{F}$-adapted intensity process $(\lambda_t)_{t\geq 0}$. Thus, for $t \geq 0$,

$$M_t := Z_t - \int_{(0, t\wedge\tau]} \lambda_s\,\mathrm{d}s = Z_t - \int_0^t (1 - Z_s)\lambda_s\,\mathrm{d}s \tag{4}$$

is a $(\mathbb{P}, \mathbb{G})$-martingale. Moreover, we assume that $(\lambda_t)_{t\geq 0}$ and the default risk premium process $(1/\eta_t)_{t\geq 0}$ $(0 < \eta_t \leq 1)$ rely on a stochastic economic factor process $(Y_t)_{t\geq 0}$ (see [5, 18]), that is, there exit a nonnegative measurable $\lambda(\cdot)$ and a measurable $(0, 1]$-valued $\eta(\cdot)$ such that $\lambda_t = \lambda(Y_t)$, $\eta_t = \eta(Y_t)$, $t \geq 0$. Here we also assume that $\sup_{y\in\mathbb{R}} \lambda(y) \leq C$ and $\hat{\eta} := \inf_{y\in\mathbb{R}} \eta(y) > 0$, $C > 0$ is a constant.

Remark 2.2. It should be pointed out that in order to verify that (4) is a martingale, we need the following two results. The first is the conditional survival probability, which is given by

$$S_t := \mathbb{P}(\tau > t|\mathcal{F}_t) = \exp\bigg(-\int_0^t \lambda_s\,\mathrm{d}s\bigg).$$

The second is the conditional expectation $\mathbb{E}(\mathbf{1}_{\{\tau>t\}}Y|\mathcal{G}_t)$, which is given by

$$\mathbb{E}\big(\mathbf{1}_{\{\tau>t\}}Y|\mathcal{G}_t\big) = \mathbb{P}\{\tau > t|\mathcal{G}_t\}\frac{\mathbb{E}\big(\mathbf{1}_{\{\tau>t\}}Y|\mathcal{F}_t\big)}{\mathbb{P}\{\tau > t|\mathcal{F}_t\}},$$

where Y is a $\mathcal{G}$-measurable random variable. For more details, see Bielecki and Rutkowski [3], Chapter 5.

Now we use $(P_t)_{t\geq 0}$ to represent the cum-dividend price of a defaultable perpetual bond that pays constant coupon $\hat{C} > 0$ per unit time. Denote $\xi \in (0,1]$ as the constant loss rate when a default occurs. We adopt a similar method to Bo *et al.* [4], that is, directly randomize the market parameters in the dynamics. Therefore, the dynamics of $(P_t)_{t\geq 0}$ is given by

$$\mathrm{d}P_t = r(Y_t)P_t\,\mathrm{d}t + \xi\lambda(Y_t)P_t\hat{Z}_t(1/\eta(Y_t)-1)\,\mathrm{d}t - \hat{Z}_t\hat{C}\,\mathrm{d}t - \xi P_{t^-}\,\mathrm{d}M_t, \tag{5}$$

where $\hat{Z}_t := 1 - Z_t = \mathbf{1}_{\{\tau>t\}}$.

2.3. The dynamics of the wealth process

For $t \geq 0$, let X_t be the total wealth at time t. We denote k_t and l_t as the t-time proportions in the wealth X_t of $(P_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ respectively. Then the t-time proportion in the wealth X_t of $(B_t)_{t\geq 0}$ is $1-k_t-l_t$. Additionally, we assume that an investor can choose a consumption ratio c_t at time $t\geq 0$. Therefore, using the self-financing strategy, the wealth process $(X_t)_{t\geq 0}$ is given by the following dynamics

$$\begin{aligned}\mathrm{d}X_t = {} & X_t\Big[r(Y_t) - c_t + (\mu(Y_t) - r(Y_t))l_t + \xi k_t\hat{Z}_t\lambda(Y_t)(1/\eta(Y_t)-1)\Big]\,\mathrm{d}t\\ & + l_t\sigma(Y_t)X_t\,\mathrm{d}W_t - \xi k_{t^-}X_{t^-}\,\mathrm{d}M_t, \qquad X_0 = x > 0. \end{aligned}\tag{6}$$

Moreover, by the Itô formula with jumps (see [19], Thm. 37), we have

$$\begin{aligned}X_t = {} & x\exp\bigg\{\int_0^t\bigg[r(Y_s) - c_s + (\mu(Y_s)-r(Y_s))l_s + \xi k_s\hat{Z}_s\frac{\lambda(Y_s)}{\eta(Y_s)}\bigg]\,\mathrm{d}s\bigg\}\\ & \times\exp\bigg\{\int_0^t l_s\sigma(Y_s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^t l_s^2\sigma^2(Y_s)\,\mathrm{d}s\bigg\}\prod_{s\leq t}(1-\xi k_{s^-}\Delta Z_s).\end{aligned}$$

Noting that $\Delta Z_t \in \{0,1\}$ for all $t\geq 0$, to ensure that $X_t > 0$ a.s., we need assume that $k_t < \frac{1}{\xi}$.

Remark 2.3. The main difference between our model and the existing ones in the literature is that the money market account $(B_t)_{t\geq 0}$, the default-free risky asset $(S_t)_{t\geq 0}$, the default process $(\lambda_t)_{t\geq 0}$ and the default risk premium $(1/\eta_t)_{t\geq 0}$ all rely on a stochastic economic factor process. In addition, the SDE obeyed by our stochastic economic factor process contains not only drift term, but also nonconstant diffusion term. Therefore, we will deploy some new technologies to deal with these more complex situations.

3. The optimal portfolio problem under power utility

In this section, we use stochastic optimal control theory to find the optimal allocation pair $(k_t, l_t; t\geq 0)$ and the optimal consumption ratio $(c_t; t\geq 0)$ for an investor whose objective is to maximize the infinite horizon expected discounted power utility of the consumption.

Now, we define the admissible control set $\mathcal{A}(\mathbb{G})$.

Definition 3.1. A *càdlàg* $\mathbb{G}$-adapted stochastic control strategy $(k_t, l_t, c_t)_{t\geq 0}$ is in the admissible control set $\mathcal{A}(\mathbb{G})$, if for all $t\geq 0$,

$$k_t \in [0, 1/\xi), \quad l_t \in \mathbb{R}, \quad c_t \in \mathbb{R}_+.$$

Remark 3.2. The constraint condition $k_t \in [0,1/\xi)$ refers to avoiding bankruptcy. That is, each element belonging to $\mathcal{A}(\mathbb{G})$ is a bankruptcy avoiding portfolio. In other words, if a *càdlàg* $\mathbb{G}$-adapted stochastic control strategy $(k_t, l_t, c_t)_{t\geq 0} \in \mathcal{A}(\mathbb{G})$, then the wealth $X_t > 0$ for all $t\geq 0$.

In this paper, the utility function is given by $U(x) = \frac{1}{\gamma}x^\gamma$, where $\gamma \in (0,1)$ is the risk-aversion parameter. For an initial pair $(x,y,z) \in \mathbb{R}_+ \times \mathbb{R} \times \{0,1\}$ and an admissible control strategy $\Xi := (k_\cdot, l_\cdot, c_\cdot)$, we define the performance criterion as

$$
\begin{aligned}
J(x,y,z;u) &= \mathbb{E}_{x,y,z}\left[\int_0^\infty e^{-\alpha t} U(c_t X_t)\,\mathrm{d}t\right] \\
&:= \mathbb{E}\left[\int_0^\infty e^{-\alpha t} U(c_t X_t)\,\mathrm{d}t \,|\, X_0 = x, Y_0 = y, Z_0 = z\right],
\end{aligned}
$$

where $\alpha > 0$ is the discount factor. The value function is defined by

$$
V(x,y,z) = \sup_{\Xi \in \mathcal{A}(\mathbb{G})} J(x,y,z;u). \tag{7}
$$

The optimal portfolio problem is to find an admissible control strategy $\Xi^* = (k^*_\cdot, l^*_\cdot, c^*_\cdot)$ that attains the value function $V(x,y,z)$. The control Ξ^* is called an optimal control strategy.

3.1. The pre-default and post-default value functions and HJB equations

In this subsection, we introduce the definitions of pre-default and post-default value functions and the corresponding HJB equations.

By (7), the pre-default value function is defined by

$$
V^0(x,y) := V(x,y,0).
$$

Instead, under the post-default case, the value function is given by

$$
V^1(x,y) := V(x,y,1).
$$

We assume that $V^0(x,y)$ and $V^1(x,y)$ are all $C^{2,2}$. Applying the dynamic programming principle, we deduce that the following pre-default and post-default HJB equations associated with $V^0(x,y)$ and $V^1(x,y)$ respectively:

$$
\begin{aligned}
\alpha V^0(x,y) &= r(y)x\frac{\partial V^0(x,y)}{\partial x} + \beta(y)\frac{\partial V^0(x,y)}{\partial y} + \frac{1}{2}\theta^2(y)\frac{\partial^2 V^0(x,y)}{\partial y^2} \\
&\quad + \lambda(y)\sup_{k\in[0,1/\xi)}\left[\frac{k\xi x}{\eta(y)}\frac{\partial V^0(x,y)}{\partial x} + V^1(x - xk\xi, y) - V^0(x,y)\right] \\
&\quad + \sup_{l\in\mathbb{R}}\left[(\mu(y) - r(y))lx\frac{\partial V^0(x,y)}{\partial x} + \rho\theta(y)\sigma(y)lx\frac{\partial^2 V^0(x,y)}{\partial x\partial y} + \frac{1}{2}l^2\sigma^2(y)x^2\frac{\partial^2 V^0(x,y)}{\partial x^2}\right] \\
&\quad + \sup_{c\in\mathbb{R}_+}\left[\frac{1}{\gamma}(cx)^\gamma - cx\frac{\partial V^0(x,y)}{\partial x}\right],
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
\alpha V^1(x,y) &= r(y)x\frac{\partial V^1(x,y)}{\partial x} + \beta(y)\frac{\partial V^1(x,y)}{\partial y} + \frac{1}{2}\theta^2(y)\frac{\partial^2 V^1(x,y)}{\partial y^2} \\
&\quad + \sup_{l\in\mathbb{R}}\left[(\mu(y) - r(y))lx\frac{\partial V^1(x,y)}{\partial x} + \rho\theta(y)\sigma(y)lx\frac{\partial^2 V^1(x,y)}{\partial x\partial y} + \frac{1}{2}l^2\sigma^2(y)x^2\frac{\partial^2 V^1(x,y)}{\partial x^2}\right] \\
&\quad + \sup_{c\in\mathbb{R}_+}\left[\frac{1}{\gamma}(cx)^\gamma - cx\frac{\partial V^1(x,y)}{\partial x}\right].
\end{aligned} \tag{9}
$$

Since $V(x,y,z)$ is homogeneous in x (see [7], Lem. 2.3), the post-default value function admits the form

$$V^1(x,y) = \frac{1}{\gamma}x^\gamma W^1(y), \quad (x,y) \in \mathbb{R}_+ \times \mathbb{R},$$

where $W^1(y)$ satisfies that

$$\begin{aligned} 0 &= \frac{1}{2}\theta^2(y)\frac{\partial^2 W^1(y)}{\partial y^2} + \left[\beta(y) + \frac{\gamma\rho\theta(y)(\mu(y)-r(y))}{(1-\gamma)\sigma(y)}\right]\frac{\partial W^1(y)}{\partial y} \\ &\quad + \left[\frac{\gamma(\mu(y)-r(y))^2}{2(1-\gamma)\sigma^2(y)} + \gamma r(y) - \alpha\right]W^1(y) + \frac{\gamma\rho^2\theta^2(y)}{2(1-\gamma)}\frac{\left[\frac{\partial W^1(y)}{\partial y}\right]^2}{W^1(y)} + (1-\gamma)\big[W^1(y)\big]^{\frac{\gamma}{\gamma-1}}. \end{aligned} \tag{10}$$

Similarly, the pre-default value function is given by

$$V^0(x,y) = \frac{1}{\gamma}x^\gamma W^0(y), \quad (x,y) \in \mathbb{R}_+ \times \mathbb{R},$$

where $W^0(y)$ obeys that

$$\begin{aligned} \frac{\alpha}{\gamma}W^0(y) &= \left[r(y) - \frac{\lambda(y)}{\gamma}\right]W^0(y) + \frac{\beta(y)}{\gamma}\frac{\partial W^0(y)}{\partial y} + \frac{\theta^2(y)}{2\gamma}\frac{\partial^2 W^0(y)}{\partial y^2} \\ &\quad + \lambda(y)\sup_{k\in[0,1/\xi)}\left[\frac{k\xi}{\eta(y)}W^0(y) + \frac{1}{\gamma}(1-k\xi)^\gamma W^1(y)\right] \\ &\quad + \sup_{l\in\mathbb{R}}\left[(\mu(y)-r(y))lW^0(y) + \rho\theta(y)\sigma(y)l\frac{\partial W^0(y)}{\partial y} + \frac{1}{2}(\gamma-1)l^2\sigma^2(y)W^0(y)\right] \\ &\quad + \sup_{c\in\mathbb{R}_+}\left[\frac{1}{\gamma}c^\gamma - cW^0(y)\right]. \end{aligned} \tag{11}$$

It is not hard to verify that

$$\begin{aligned} k^*(y) &:= \arg\sup_{k\in[0,1/\xi)}\left[\frac{k\xi}{\eta(y)}W^0(y) + \frac{1}{\gamma}(1-k\xi)^\gamma W^1(y)\right] = \frac{1}{\xi}\left\{1 - \left[\frac{W^0(y)}{\eta(y)W^1(y)}\right]^{\frac{1}{\gamma-1}}\right\}, \\ l^*(y) &:= \arg\sup_{l\in\mathbb{R}}\left[(\mu(y)-r(y))lW^0(y) + \rho\theta(y)\sigma(y)l\frac{\partial W^0(y)}{\partial y}\right. \\ &\qquad \left. + \frac{1}{2}(\gamma-1)l^2\sigma^2(y)W^0(y)\right]\frac{\mu(y)-r(y)}{(1-\gamma)\sigma^2(y)} + \frac{\rho\theta(y)\frac{\partial W^0(y)}{\partial y}}{(1-\gamma)\sigma(y)W^0(y)}, \\ c^*(y) &:= \arg\sup_{c\in\mathbb{R}_+}\left[\frac{1}{\gamma}c^\gamma - cW^0(y)\right] = \big[W^0(y)\big]^{\frac{1}{\gamma-1}}. \end{aligned}$$

Substituting the above controls into (11) gives

$$\begin{aligned} 0 &= \frac{1}{2}\theta^2(y)\frac{\partial^2 W^0(y)}{\partial y^2} + \left[\beta(y) + \frac{\gamma\rho\theta(y)(\mu(y)-r(y))}{(1-\gamma)\sigma(y)}\right]\frac{\partial W^0(y)}{\partial y} \\ &\quad + \left[\frac{\gamma(\mu(y)-r(y))^2}{2(1-\gamma)\sigma^2(y)} + \gamma r(y) - \lambda(y) - \alpha + \gamma\frac{\lambda(y)}{\eta(y)}\right]W^0(y) + \frac{\gamma\rho^2\theta^2(y)}{2(1-\gamma)}\frac{\left[\frac{\partial W^0(y)}{\partial y}\right]^2}{W^0(y)} \\ &\quad + (1-\gamma)\left\{1 + \lambda(y)\left(\frac{1}{\eta(y)}\right)^{\frac{\gamma}{\gamma-1}}\big[W^1(y)\big]^{\frac{1}{\gamma-1}}\right\}\big[W^0(y)\big]^{\frac{\gamma}{\gamma-1}}. \end{aligned} \tag{12}$$

We will show that

$$\hat{V}(x,y,z) := \frac{1}{\gamma}x^{\gamma}\big[zW^1(y) + (1-z)W^0(y)\big] \tag{13}$$

is the classical solution to the HJB equation associated with the value function $V(x,y,z)$ for $(x,y,z) \in \mathbb{R}_+ \times \mathbb{R} \times \{0,1\}$, where $W^1(y)$ and $W^0(y)$ are classical solutions to (10) and (12), respectively. In other words, if (10) and (12) admit classical solutions $W^1(y)$ and $W^0(y)$, respectively, then $\hat{V}$ is the classical solution to the HJB equation associated with the value function V.

3.2. Solutions to HJB equations

In this subsection, we analyze existence and uniqueness of the global classical solutions of HJB equations (10) and (12) and then we will obtain $\hat{V}$ is the classical solution to the HJB equation associated with the value function V.

Since $W^1(y) > 0$ and $W^0(y) > 0$ for all $y \in \mathbb{R}$, we can define

$$W^1(y) = e^{u^1(y)} \quad \text{and} \quad W^0(y) = e^{u^0(y)}.$$

Differentiating yields

$$\begin{aligned}
\frac{\partial W^1(y)}{\partial y} &= e^{u^1(y)}\frac{\partial u^1(y)}{\partial y}, \quad \frac{\partial W^0(y)}{\partial y} = e^{u^0(y)}\frac{\partial u^0(y)}{\partial y},\\
\frac{\partial^2 W^1(y)}{\partial y^2} &= e^{u^1(y)}\left[\frac{\partial u^1(y)}{\partial y}\right]^2 + e^{u^1(y)}\frac{\partial^2 u^1(y)}{\partial y^2},\\
\frac{\partial^2 W^0(y)}{\partial y^2} &= e^{u^0(y)}\left[\frac{\partial u^0(y)}{\partial y}\right]^2 + e^{u^0(y)}\frac{\partial^2 u^0(y)}{\partial y^2}.
\end{aligned}$$

Substituting the above derivatives into (10) and (12) yield that

$$\begin{aligned}
0 = {} & \frac{1}{2}\theta^2(y)\frac{\partial^2 u^1(y)}{\partial y^2} + \frac{1-\big(1-\rho^2\big)\gamma}{2(1-\gamma)}\theta^2(y)\left[\frac{\partial u^1(y)}{\partial y}\right]^2 + \left[\beta(y) + \frac{\gamma\rho\theta(y)(\mu(y)-r(y))}{(1-\gamma)\sigma(y)}\right]\frac{\partial u^1(y)}{\partial y}\\
& + \frac{\gamma(\mu(y)-r(y))^2}{2(1-\gamma)\sigma^2(y)} + \gamma r(y) - \alpha + (1-\gamma)e^{\frac{u^1(y)}{\gamma-1}},
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
0 = {} & \frac{1}{2}\theta^2(y)\frac{\partial^2 u^0(y)}{\partial y^2} + \frac{1-\big(1-\rho^2\big)\gamma}{2(1-\gamma)}\theta^2(y)\left[\frac{\partial u^0(y)}{\partial y}\right]^2 + \left[\beta(y) + \frac{\gamma\rho\theta(y)(\mu(y)-r(y))}{(1-\gamma)\sigma(y)}\right]\frac{\partial u^0(y)}{\partial y}\\
& + \frac{\gamma(\mu(y)-r(y))^2}{2(1-\gamma)\sigma^2(y)} + \gamma r(y) - \lambda(y) - \alpha + \gamma\frac{\lambda(y)}{\eta(y)} + (1-\gamma)\left[1 + \lambda(y)\left(\frac{1}{\eta(y)}\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{u^1(y)}{1-\gamma}}\right] e^{\frac{u^0(y)}{\gamma-1}}.
\end{aligned} \tag{15}$$

By (13), we have

$$\hat{V}(x,y,z) = \frac{1}{\gamma}x^{\gamma}\left[ze^{u^1(y)} + (1-z)e^{u^0(y)}\right].$$

Therefore, to get $\hat{V}$, it suffices to seek the solutions $u^1(y)$ and $u^0(y)$ to (14) and (15), respectively. In the next section, we shall show that the value function V given by (7) is nothing but $\hat{V}$. Consequently, we devote to prove the existence of classical solutions to (14) and (15) in the remainder of this subsection. Due to the fact that (14) and (15) are fully nonlinear PDE, the method of subsolution and supersolution will be used. Refer to Fleming and Pang [7], Pao [17] and Walter [23] for details. To simplify our analysis, we let

$$\mathcal{L}u = \frac{1}{2}\theta^2(y)\frac{\partial^2 u}{\partial y^2} + \frac{1-\big(1-\rho^2\big)\gamma}{2(1-\gamma)}\theta^2(y)\left(\frac{\partial u}{\partial y}\right)^2\left[\beta(y) + \frac{\gamma\rho\theta(y)(\mu(y)-r(y))}{(1-\gamma)\sigma(y)}\right]\frac{\partial u}{\partial y}.$$

We first consider (14). Define

$$h(y,u) = Q(y) - \alpha + (1-\gamma)e^{\frac{u}{\gamma-1}},$$

where $Q(y) := \frac{\gamma(\mu(y)-r(y))^2}{2(1-\gamma)\sigma^2(y)} + \gamma r(y)$. According to the definition of subsolution (supersolution), u is a subsolution (supersolution) of (14) if and only if

$$-\mathcal{L}u \le (\ge) h(y,u).$$

In what follows, we assume that the following conditions hold for all $y \in \mathbb{R}$.

$(A1)$ There exists positive constants $\underline{\epsilon}$, $\bar{\epsilon}$ such that $\underline{\epsilon} \le \alpha - Q(y) \le \bar{\epsilon}$.
$(A2)$ There exists a constant $c_5 > 0$ such that $0 < \frac{\mu(y)-r(y)}{\sigma(y)} \le c_5$.
$(A3)$ There exists a constant $d_1 > 0$ such that

$$0 < d_1 < \min\left\{ \frac{(1-\gamma)c_1}{[1-(1-\rho^2)\gamma]c_3^2}, \quad \frac{\alpha - Q(y)}{c_3^2} \right\},$$

and

$$\left|\beta(0) + \frac{\gamma\rho c_3 c_5}{1-\gamma}\right| < \sqrt{\frac{2[(1-\gamma)c_1 - (1-(1-\rho^2)\gamma)c_3^2 d_1](\alpha - Q(y) - c_3^2 d_1)}{(1-\gamma)d_1}}.$$

$(A4)$ $\gamma < \hat{\eta}$.

Lemma 3.3. *Suppose that* $(A1)$ *holds. Then any constant* $\bar{C} \le (\gamma-1)\log\frac{\alpha - Q(y)}{1-\gamma} =: C_Q(y)$ *is a subsolution of* (14)*. In particular, any constant* $\bar{C}' \le (\gamma-1)\log\frac{\bar{\epsilon}}{1-\gamma} =: C_{\bar{\epsilon}}$ *is a subsolution of* (14)*.*

Proof. By a direct computation, it is easy to verify that

$$h(y, C_Q(y)) = 0.$$

For any constant $\bar{C} \le C_Q(y)$, we have

$$-\mathcal{L}\bar{C} = 0 = h(y, C_Q(y)).$$

Moreover, $h(y,u)$ is a decreasing function of u, so we have $h(y, C_Q(y)) \le h(y, \bar{C})$. Thus,

$$-\mathcal{L}\bar{C} \le h\big(y, \bar{C}\big),$$

which implies that $\bar{C} \le C_Q(y)$ is a subsolution of (14) according to the definition of subsolution. Clearly, since $C_{\bar{\epsilon}} \le C_Q(y)$, then any constant $\bar{C}' \le C_{\bar{\epsilon}} \le C_Q(y)$ is a subsolution of (14). The proof is complete. □

Lemma 3.4. *Let* $(A1)$*–*$(A3)$ *hold. Then there exist a positive constant* $d_2 > (\gamma-1)\log\frac{\underline{\epsilon}}{1-\gamma} \ge C_Q(y)$ *such that*

$$\bar{u}^1(y) := d_1 y^2 + d_2 \tag{16}$$

is a supersolution of (14)*.*

Proof. Since $-c_2 \le \frac{\partial\beta(y)}{\partial y} \le -c_1$ for all $y \in \mathbb{R}$, we have

$$\beta(0) - c_2 y \le \beta(y) \le \beta(0) - c_1 y. \tag{17}$$

If $y \ge 0$, by virtue of (16), (17) and $(A2)$, we can obtain

$$-\mathcal{L}\bar{u}^1 \ge 2\left[c_1 d_1 - \frac{1-(1-\rho^2)\gamma}{1-\gamma}c_3^2 d_1^2\right]y^2 - 2d_1\left(\beta(0) + \frac{\gamma\rho c_3 c_5}{1-\gamma}\right)y - c_3^2 d_1 =: I.$$

So we only need to show that

$$I - Q(y) + \alpha \geq (1-\gamma)e^{\frac{d_1 y^2 + d_2}{\gamma - 1}}. \tag{18}$$

From $(A3)$, we may easily observe that

$$c_1 d_1 - \frac{1-(1-\rho^2)\gamma}{1-\gamma} c_3^2 d_1^2 > 0$$

and

$$4d_1^2\left(\beta(0) + \frac{\gamma\rho c_3 c_5}{1-\gamma}\right)^2 - 8\left[c_1 d_1 - \frac{1-(1-\rho^2)\gamma}{1-\gamma}c_3^2 d_1^2\right](\alpha - Q(y) - c_3^2 d_1) < 0.$$

Therefore

$$2\left[c_1 d_1 - \frac{1-(1-\rho^2)\gamma}{1-\gamma}c_3^2 d_1^2\right]y^2 - 2d_1\left(\beta(0) + \frac{\gamma\rho c_3 c_5}{1-\gamma}\right)y - c_3^2 d_1 - Q(y) + \alpha > 0. \tag{19}$$

This implies that we can take $d_2 > (\gamma - 1)\log\frac{\epsilon}{1-\gamma} \geq C_Q(y)$ large enough such that (18) holds. In other words, we have

$$-\mathcal{L}\bar{u}^1 \geq h(y, \bar{u}^1). \tag{20}$$

When $y < 0$, similarly, we also have (20) because $\frac{\gamma\rho\theta(y)(\mu(y)-r(y))}{(1-\gamma)\sigma(y)} \geq 0$ and $c_2 \geq c_1 > 0$ for all $y \in \mathbb{R}$. The proof is therefore complete. $\square$

Remark 3.5. It should be noted that the parameters satisfying the above conditions are nonempty. For example,

$$(r(y), \mu(y), \sigma(y), c_1, c_2, c_3, c_5, d_1, \rho, \gamma, \alpha, \beta(0)) = (0.05, 0.1, 1, 0.2, 0.3, 0.6, 0.06, 0.1, 0.5, 0.3, 1, 0.5)$$

is a set of feasible parameters.

In order to obtain a solution of (14), we need the following lemma. To proceed, we let

$$H_1(y,\nu,p) := -\frac{1-(1-\rho^2)\gamma}{1-\gamma}p^2 - 2\left[\frac{\beta(y)}{c_3^2} + \frac{\gamma\rho(\mu(y)-r(y))}{c_3(1-\gamma)\sigma(y)}\right]p - \frac{2(Q(y)-\alpha)}{c_3^2} - \frac{2(1-\gamma)}{c_3^2}e^{\frac{\nu}{\gamma-1}},$$

and

$$H_2(y,\nu,p) := -\frac{1-(1-\rho^2)\gamma}{1-\gamma}p^2 - 2\left[\frac{\beta(y)}{c_4^2} + \frac{\gamma\rho(\mu(y)-r(y))}{c_4(1-\gamma)\sigma(y)}\right]p - \frac{2(Q(y)-\alpha)}{c_4^2} - \frac{2(1-\gamma)}{c_4^2}e^{\frac{\nu}{\gamma-1}}.$$

Lemma 3.6. *Suppose that* $(A1)-(A3)$ *hold. Then*

$$\frac{\partial^2 Z(y)}{\partial y^2} = H_1\left(y, Z(y), \frac{\partial Z(y)}{\partial y}\right) \tag{21}$$

and

$$\frac{\partial^2 Z(y)}{\partial y^2} = H_2\left(y, Z(y), \frac{\partial Z(y)}{\partial y}\right) \tag{22}$$

have solutions $Z_1(y)$ *and* $Z_2(y)$*, respectively.*

Proof. Similar to Lemma 3.4, it is not difficult to find a subsolution and a supersolution of equation (21). Let $(Z_L(y), Z_U(y))$ be a pair of ordered subsolution and a supersolution of equation (21). Clearly, $H_1(y,\nu,p)$ is strictly increasing with respect to ν. Next we let $J_1 = [n_1, n_2]$ be an arbitrary finite interval on the real line, $y_n := \max\{|n_1|, |n_2|\}$ and $\Psi := \max\{\sup_{y\in J_1} |Z_L(y)|, \sup_{y\in J_1} |Z_U(y)|\}$. Moreover, according to (17), we have $|\beta(y)| \le |\beta(0)| + c_2|y|$ for $y \in \mathbb{R}$. Thus, for all $y \in J_1$ and $|\nu| \le 3\Psi$, we have

$$|H_1(y,\nu,p)| \le \frac{2-\gamma}{1-\gamma}p^2 + \left[\frac{|\beta(0)| + c_2 y_n}{c_3^2} + \frac{\gamma\rho c_5}{c_3(1-\gamma)}\right]^2 + \frac{4\alpha}{c_3^2} + \frac{2}{c_3^2}e^{\frac{3\Psi}{1-\gamma}}.$$

Thus, equation (21) has a solution $Z_1(y)$ follows from Theorem 3.8 in Fleming and Pang [7] immediately.

In a similar way, we can show that equation (22) has a solution $Z_2(y)$. □

Theorem 3.7. *Assume that* (A1)−(A3) *hold. Then* (14) *admits a solution* u_*^1*, and* $C_{\bar{\epsilon}} \le u_*^1(y) \le \bar{u}^1(y)$*, where* $C_{\bar{\epsilon}}$ *and* $\bar{u}^1(y)$ *are defined in Lemmas* 3.3 *and* 3.4*, respectively.*

Proof. Let

$$H(y,\nu,p) := -\frac{1-(1-\rho^2)\gamma}{1-\gamma}p^2 - 2\left[\frac{\beta(y)}{\theta^2(y)} + \frac{\gamma\rho(\mu(y)-r(y))}{(1-\gamma)\sigma(y)\theta(y)}\right]p - \frac{2(Q(y)-\alpha)}{\theta^2(y)} - \frac{2(1-\gamma)}{\theta^2(y)}e^{\frac{\nu}{\gamma-1}},$$

then (14) can be rewritten as

$$\frac{\partial^2 u^1(y)}{\partial y^2} = H\left(y, u^1(y), \frac{\partial u^1(y)}{\partial y}\right).$$

By Lemma 3.6, we have

$$\frac{\partial^2 Z_1(y)}{\partial y^2} = H_1\left(y, Z_1(y), \frac{\partial Z_1(y)}{\partial y}\right) \quad \text{and} \quad \frac{\partial^2 Z_2(y)}{\partial y^2} = H_2\left(y, Z_2(y), \frac{\partial Z_2(y)}{\partial y}\right).$$

Since $\theta(y)$ is bounded for all $y \in \mathbb{R}$, we obtain $H_2(y,\nu,p) \le H(y,\nu,p) \le H_1(y,\nu,p)$. Define

$$\begin{aligned} F(\pi,y) := {} & \pi\frac{\partial^2 Z_1(y)}{\partial y^2} + (1-\pi)\frac{\partial^2 Z_2(y)}{\partial y^2} \\ & - H\left(y, \pi Z_1(y) + (1-\pi)Z_2(y), \pi\frac{\partial Z_1(y)}{\partial y} + (1-\pi)\frac{\partial Z_2(y)}{\partial y}\right), \quad 0 \le \pi \le 1. \end{aligned}$$

Then,

$$\begin{aligned} F(0,y) &= \frac{\partial^2 Z_2(y)}{\partial y^2} - H\left(y, Z_2(y), \frac{\partial Z_2(y)}{\partial y}\right) \\ &\le \frac{\partial^2 Z_2(y)}{\partial y^2} - H_2\left(y, Z_2(y), \frac{\partial Z_2(y)}{\partial y}\right) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} F(1,y) &= \frac{\partial^2 Z_1(y)}{\partial y^2} - H\left(y, Z_1(y), \frac{\partial Z_1(y)}{\partial y}\right) \\ &\ge \frac{\partial^2 Z_1(y)}{\partial y^2} - H_1\left(y, Z_1(y), \frac{\partial Z_1(y)}{\partial y}\right) \\ &= 0. \end{aligned}$$

Therefore, we must have a $\pi^* \in [0,1]$ such that $F(\pi^*, y) = 0$. In other words, there exists a $u_*^1(y) := \pi^* Z_1(y) + (1-\pi^*)Z_2(y)$ such that

$$\frac{\partial^2 u_*^1(y)}{\partial y^2} = H\left(y, u_*^1(y), \frac{\partial u_*^1(y)}{\partial y}\right).$$

If this conclusion is false, then for any $\pi \in [0,1]$ there must exist a $y_0 \in \mathbb{R}$ satisfying $F(\pi, y_0) \neq 0$. Since $F(\pi, y_0)$ is continuous in $\pi \in [0,1]$ and $F(0, y_0) < 0$ and $F(1, y_0) > 0$, it is easily known that there is a $\pi^\circ \in (0,1)$ such that $F(\pi^\circ, y_0) = 0$, which gives a contradiction. By virtue of Lemmas 3.3 and 3.4, $(C_{\overline{\epsilon}}, \bar{u}^1(y))$ is a pair of ordered subsolution and supersolution of (14), so we have $C_{\overline{\epsilon}} \le u_*^1(y) \le \bar{u}^1(y)$. The proof is therefore complete. □

We now consider (15). With the help of Theorem 3.7, we define

$$j(y,u) = Q(y) - \alpha - \lambda(y) + \gamma\frac{\lambda(y)}{\eta(y)} + (1-\gamma)\left[1 + \lambda(y)\left(\frac{1}{\eta(y)}\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{u_*^1(y)}{1-\gamma}}\right] e^{\frac{u}{\gamma-1}}. \tag{23}$$

Lemma 3.8. *Suppose that* (A1) *and* (A4) *hold. Then any constant $\widetilde{C} \le C_Q(y)$ is a subsolution of* (15).

Proof. Since $\widetilde{C}$ is a constant, we have $-\mathcal{L}\widetilde{C} = 0$. So we only need to show that $j(y, \widetilde{C}) \ge 0$. By (23), it is easy to verify that

$$j\left(y, \widetilde{C}\right) = Q(y) - \alpha + (1-\gamma)e^{\frac{\widetilde{C}}{\gamma-1}} + \lambda(y)\left[\frac{\gamma}{\eta(y)} - 1 + (1-\gamma)\left(\frac{1}{\eta(y)}\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{u_*^1(y)-\widetilde{C}}{1-\gamma}}\right].$$

Following some basic calculations, it is not hard to verify that

$$Q(y) - \alpha + (1-\gamma)e^{\frac{\widetilde{C}}{\gamma-1}} \ge 0, \quad e^{\frac{u_*^1(y)-\widetilde{C}}{1-\gamma}} \ge 1 \quad \text{and} \quad \frac{\gamma}{\eta(y)} - 1 + (1-\gamma)\left(\frac{1}{\eta(y)}\right)^{\frac{\gamma}{\gamma-1}} \ge 0.$$

So we have $j(y, \widetilde{C}) \ge 0$. Therefore, $\widetilde{C}$ is a subsolution of (15). □

Lemma 3.9. *Let the conditions* (A1)*–*(A4) *be satisfied. Then there exists a constant $d_3 > d_2$ such that*

$$\bar{u}^0(y) := d_1 y^2 + d_3 \tag{24}$$

is a supersolution of (15).

Proof. To ensure that $\bar{u}^0(y)$ is a supersolution of (15), we only need to show that

$$-\mathcal{L}\bar{u}^0 \ge j\left(y, \bar{u}^0\right). \tag{25}$$

Using a similar method in Lemma 3.4, we have $-\mathcal{L}\bar{u}^0 \ge I$ for $y \in \mathbb{R}$. In order to obtain (25), it is sufficient to show that

$$I - Q(y) + \alpha \ge -\lambda(y) + \gamma\frac{\lambda(y)}{\eta(y)} + (1-\gamma)\left[1 + \lambda(y)\left(\frac{1}{\eta(y)}\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{u_*^1(y)}{1-\gamma}}\right] e^{\frac{d_1 y^2 + d_3}{\gamma-1}}. \tag{26}$$

By $(A4)$, we have

$$-\lambda(y) + \gamma\frac{\lambda(y)}{\eta(y)} < 0.$$

Moreover, we can find a sufficiently large $d_3 > d_2$ such that

$$(1-\gamma)\left[1 + \lambda(y)\left(\frac{1}{\eta(y)}\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{u_*^1(y)}{1-\gamma}}\right] e^{\frac{d_1 y^2 + d_3}{\gamma-1}} < \varepsilon,$$

where $\varepsilon > 0$ is arbitrarily. Combined with (19), this implies (26). Therefore, the proof is complete. □

Remark 3.10. Going back to Remark 3.5, if we choose $\hat{\eta} = 0.4$ and the values of other parameters remain unchanged, this is a set of feasible parameters. Therefore, the set of parameters which satisfies $(A1)-(A4)$ is also nonempty.

To derive a solution of (15), similar to the method of obtaining the solution of (14), define

$$\bar{H}_1(y,\nu,q) := -\frac{1-\left(1-\rho^2\right)\gamma}{1-\gamma}q^2 - 2\left[\frac{\beta(y)}{c_3^2} + \frac{\gamma\rho(\mu(y)-r(y))}{(1-\gamma)\sigma(y)c_3}\right]q - \frac{2(Q(y)-\alpha)}{c_3^2} + \frac{2\left[\lambda(y)-\gamma\frac{\lambda(y)}{\eta(y)}\right]}{c_4^2} - \frac{2(1-\gamma)}{c_3^2}\left[1+\lambda(y)\left(\frac{1}{\eta(y)}\right)^{\frac{\gamma}{\gamma-1}}e^{\frac{u_*^1(y)}{1-\gamma}}\right]e^{\frac{\nu}{\gamma-1}},$$

and

$$\bar{H}_2(y,\nu,q) := -\frac{1-(1-\rho^2)\gamma}{1-\gamma}q^2 - 2\left[\frac{\beta(y)}{c_4^2} + \frac{\gamma\rho(\mu(y)-r(y))}{(1-\gamma)\sigma(y)c_4}\right]q - \frac{2(Q(y)-\alpha)}{c_4^2} + \frac{2\left[\lambda(y)-\gamma\frac{\lambda(y)}{\eta(y)}\right]}{c_3^2} - \frac{2(1-\gamma)}{c_4^2}\left[1+\lambda(y)\left(\frac{1}{\eta(y)}\right)^{\frac{\gamma}{\gamma-1}}e^{\frac{u_*^1(y)}{1-\gamma}}\right]e^{\frac{\nu}{\gamma-1}}.$$

Lemma 3.11. *Suppose that* $(A1)-(A4)$ *hold. Then*

$$\frac{\partial^2\bar{Z}(y)}{\partial y^2} = \bar{H}_1\left(y,\bar{Z}(y),\frac{\partial\bar{Z}(y)}{\partial y}\right) \tag{27}$$

and

$$\frac{\partial^2\bar{Z}(y)}{\partial y^2} = \bar{H}_2\left(y,\bar{Z}(y),\frac{\partial\bar{Z}(y)}{\partial y}\right) \tag{28}$$

have solutions $\bar{Z}_1(y)$ *and* $\bar{Z}_2(y)$, *respectively.*

Proof. Similar to Lemmas 3.8 and 3.9, it is not difficult to find a subsolution and a supersolution of equation (27). Let $(\bar{Z}_L(y),\bar{Z}_U(y))$ be a pair of ordered subsolution and a supersolution of equation (27). Clearly, $\bar{H}_1(y,\nu,q)$ is strictly increasing with respect to ν. Let $J_2 = [m_1,m_2]$ be an arbitrary finite interval on $\mathbb{R}$, $y_m := \max\{|m_1|,|m_2|\}$ and $\Phi := \max\{\sup_{y\in J_2}|\bar{Z}_L(y)|, \sup_{y\in J_2}|\bar{Z}_U(y)|\}$. Therefore, for all $y \in J_2$ and $|\nu| \le 3\Phi$, we can show that

$$|\bar{H}_1(y,\nu,q)| \le \frac{2-\gamma}{1-\gamma}q^2 + \left[\frac{|\beta(0)|+c_2y_m}{c_3^2} + \frac{\gamma\rho c_5}{(1-\gamma)c_3}\right]^2 + \frac{4\alpha}{c_3^2} + \frac{4C}{c_4^2} + \frac{2}{c_3^2}\left(1+Ce^{\frac{\bar{Z}_U(y)}{1-\gamma}}\right)e^{\frac{3\Phi}{1-\gamma}}.$$

Thus, equation (27) has a solution $\bar{Z}_1(y)$ follows from Theorem 3.8 in Fleming and Pang [7] directly. Moreover, we can prove that equation (28) has a solution $\bar{Z}_2(y)$ in a similar way. $\square$

Theorem 3.12. *Let* $(A1)-(A4)$ *hold. Then* (15) *possesses a solution* $u_*^0(y)$ *such that* $C_{\overline{\epsilon}} \le u_*^0(y) \le \bar{u}^0(y)$, *where* $\bar{u}^0(y)$ *is as defined in* (24).

Proof. Let

$$\bar{H}(y,\nu,q) := -\frac{1-(1-\rho^2)\gamma}{1-\gamma}q^2 - 2\left[\frac{\beta(y)}{\theta^2(y)} + \frac{\gamma\rho(\mu(y)-r(y))}{(1-\gamma)\sigma(y)\theta(y)}\right]q - \frac{2(Q(y)-\alpha)}{\theta^2(y)} + \frac{2\left[\lambda(y)-\gamma\frac{\lambda(y)}{\eta(y)}\right]}{\theta^2(y)} - \frac{2(1-\gamma)}{\theta^2(y)}\left[1+\lambda(y)\left(\frac{1}{\eta(y)}\right)^{\frac{\gamma}{\gamma-1}}e^{\frac{u_*^1(y)}{1-\gamma}}\right]e^{\frac{\nu}{\gamma-1}},$$

then (15) can be represented as

$$\frac{\partial^2 u^0(y)}{\partial y^2} = \bar{H}\left(y, u^0(y), \frac{\partial u^0(y)}{\partial y}\right).$$

By Lemma 3.11, we have

$$\frac{\partial^2 \bar{Z}_1(y)}{\partial y^2} = \bar{H}_1\left(y, \bar{Z}_1(y), \frac{\partial \bar{Z}_1(y)}{\partial y}\right) \quad \text{and} \quad \frac{\partial^2 \bar{Z}_2(y)}{\partial y^2} = \bar{H}_2\left(y, \bar{Z}_2(y), \frac{\partial \bar{Z}_2(y)}{\partial y}\right).$$

Since $\theta(y)$ is bounded for all $y \in \mathbb{R}$, we have $\bar{H}_2(y, \nu, p) \leq \bar{H}(y, \nu, p) \leq \bar{H}_1(y, \nu, p)$. In the similar way as in the proof of Theorem 3.7, we must have a $u^0_*(y)$ such that

$$\frac{\partial^2 u^0_*(y)}{\partial y^2} = \bar{H}\left(y, u^0_*(y), \frac{\partial u^0_*(y)}{\partial y}\right).$$

According to Lemmas 3.8 and 3.9 and noting that the constant $C_{\overline{\epsilon}} \leq C_Q(y)$, $(C_{\overline{\epsilon}}, \bar{u}^0(y))$ is a pair of ordered subsolution and supersolution of (15), so we have $C_{\overline{\epsilon}} \leq u^0_*(y) \leq \bar{u}^0(y)$. The proof is therefore complete. □

Remark 3.13. Noting that $u^0_*(y) \in C^2$ for $y \in J$, $J = [y_1, y_2]$ is an arbitrary finite interval. Let $y_m := \max\{|y_1|, |y_2|\}$ and $\Upsilon := \max\{\sup_{y\in J} |\bar{u}^0(y)|, |C_Q(y)|\}$. Hence, for all $y \in J$ and $|\nu| \leq 3\Upsilon$, we have

$$|\bar{H}(y, \nu, q)| \leq \frac{2-\gamma}{1-\gamma} q^2 + \left[\frac{|\beta(0)| + c_2 y_m}{c_4^2} + \frac{\gamma\rho c_5}{(1-\gamma)c_4}\right]^2 + \frac{4\alpha}{c_4^2} + \frac{4C}{c_4^2} + \frac{2}{c_4^2}\left(1 + Ce^{\frac{d_1 y_m^2 + d_2}{1-\gamma}}\right)e^{\frac{3\Upsilon}{1-\gamma}}.$$

In the light of Lemma 3.4 in Fleming and Pang [7] and the arbitrariness of J, we may conclude that

$$\left|\frac{\partial u^0_*(y)}{\partial y}\right| \leq \Lambda \tag{29}$$

for $y \in \mathbb{R}$, where Λ is a constant.

Remark 3.14. As already mentioned, by virtue of Theorems 3.7 and 3.12, it is easy to see that

$$\hat{V}(x, y, z) = \frac{1}{\gamma} x^\gamma \left[z e^{u^1_*(y)} + (1-z) e^{u^0_*(y)}\right] \tag{30}$$

is a classical solution of the HJB equation associated with the value function V.

At the end of this subsection, we give a $\mathbb{G}$-adapted stochastic control strategy $(k^*_t, l^*_t, c^*_t)_{t\geq 0}$ expressed as

$$k^*_t = k^*(Y_t) = \begin{cases} \frac{1}{\xi}\left[1 - \left(\frac{e^{u^0_*(Y_t)}}{\eta(Y_t) e^{u^1_*(Y_t)}}\right)^{\frac{1}{\gamma-1}}\right], & 0 \leq t < \tau, \\ 0, & t \geq \tau, \end{cases} \tag{31}$$

$$l^*_t = l^*(Y_t) = \frac{\mu(Y_t) - r(Y_t)}{(1-\gamma)\sigma^2(Y_t)} + \frac{\rho\theta(Y_t)\frac{\partial u^0_*}{\partial y}(Y_t)}{(1-\gamma)\sigma(Y_t)}, \quad t \geq 0, \tag{32}$$

and

$$c^*_t = c^*(Y_t) = \begin{cases} e^{\frac{u^0_*(Y_t)}{\gamma-1}}, & 0 \leq t < \tau, \\ e^{\frac{u^1_*(Y_t)}{\gamma-1}}, & t \geq \tau, \end{cases} \tag{33}$$

where u^1_* and u^0_* are classical solutions to (14) and (15), respectively. According to Definition 3.1, it is not hard verify that $(k^*_t, l^*_t, c^*_t)_{t\geq 0} \in \mathcal{A}(\mathbb{G})$.

4. The verification theorem

In this section, we will provide a verification theorem, in which the stochastic control strategy $(k_t^*, l_t^*, c_t^*)_{t\geq 0}$ given by (31)–(33) is optimal and $\hat{V}$ given by (30) is just the value function V defined by (7). To proceed, we need the following lemmas.

Lemma 4.1. *Denote $(X_t^*)_{t\geq 0}$ as the wealth process satisfying (6) with $(k_t, l_t, c_t)_{t\geq 0}$ replaced by $(k_t^*, l_t^*, c_t^*)_{t\geq 0}$. Let*

$$c_6 := M + \frac{c_5^2 + c_3 c_5 \Lambda}{1-\gamma} + C\left(\frac{1}{\hat{\eta}} - 1\right).$$

Then for all $0 \leq t \leq T$, we have

$$\mathbb{E}_x\left(\sup_{0\leq s\leq t} |X_s^*|^2\right) \leq 4x^2 e^{\bar{\Lambda} T}, \tag{34}$$

where

$$\bar{\Lambda} := 2c_6 T + 32\left[C + \left(\frac{c_5 + c_3\Lambda}{1-\gamma}\right)^2\right].$$

Moreover, for all $\alpha > c_6$,

$$\lim_{T\to\infty} e^{-\alpha T}\mathbb{E}_x(X_T^*) = 0. \tag{35}$$

Proof. In the light of (6), we have

$$\mathrm{d}X_t^* = X_t^*\Big[r(Y_t) - c_t^* + (\mu(Y_t) - r(Y_t))l_t^* + \xi k_t^* \hat{Z}_t \lambda(Y_t)(1/\eta(Y_t) - 1)\Big]\,\mathrm{d}t + l_t^*\sigma(Y_t)X_t^*\,\mathrm{d}W_t - \xi k_{t-}^* X_{t-}^* \hat{Z}_{t-}\,\mathrm{d}M_t.$$

For each $T \geq 0$, define

$$M_T^* := \int_0^T l_t^*\sigma(Y_t)X_t^*\,\mathrm{d}W_t - \int_0^T \xi k_{t-}^* X_{t-}^* \hat{Z}_{t-}\,\mathrm{d}M_t,$$

and a random function

$$f(x) = \Big[r(Y_t) - c_t^* + (\mu(Y_t) - r(Y_t))l_t^* + \xi k_t^* \hat{Z}_t \lambda(Y_t)(1/\eta(Y_t) - 1)\Big]x, \quad x > 0.$$

Therefore,

$$X_T^* = x + M_T^* + \int_0^T f(X_t^*)\,\mathrm{d}t. \tag{36}$$

It is easy to see that

$$c_7 \leq \frac{\partial f}{\partial x} \leq c_6, \quad f(0) = 0,$$

where

$$c_7 = -\frac{c_3 c_5 \Lambda}{1-\gamma} - e^{\frac{C_Q(y)}{\gamma - 1}}.$$

Noting that $X_t^* > 0$ a.s. for all $t \geq 0$. For $n \in \mathbb{N}$, define

$$\tau_n = \inf\{t \geq 0; X_t^* \geq n\}.$$

By (4), the Cauchy–Schwarz inequality, and the Burkhölder–Davis–Gundy inequality, for $0 < T < \infty$, we have

$$\mathbb{E}_x\left(\sup_{0\leq t\leq T\wedge\tau_n} |X_t^*|^2\right) \leq 2\mathbb{E}_x\left(\sup_{0\leq t\leq T\wedge\tau_n} |x + M_t^*|^2\right) + 2c_6 T\mathbb{E}_x\int_0^{T\wedge\tau_n} |X_t^*|^2\,\mathrm{d}t$$

$$
\begin{aligned}
&\leq 4x^2 + 2c_6 T\mathbb{E}_x \int_0^{T\wedge\tau_n} |X_t^*|^2 \,\mathrm{d}t + 8\mathbb{E}_x \left[\sup_{0\leq t\leq T\wedge\tau_n} \left| \int_0^t l_s^* \sigma(Y_s) X_s^* \,\mathrm{d}W_s \right|^2 \right] \\
&\quad + 8\mathbb{E}_x \left[\sup_{0\leq t\leq T\wedge\tau_n} \left| \int_0^t \xi k_{s-}^* X_{s-}^* \hat{Z}_{s-} \,\mathrm{d}M_s \right|^2 \right] \\
&\leq 4x^2 + 2c_6 T\mathbb{E}_x \int_0^{T\wedge\tau_n} |X_t^*|^2 \,\mathrm{d}t + 32\mathbb{E}_x \int_0^{T\wedge\tau_n} |l_t^*|^2 \sigma^2(Y_t) |X_t^*|^2 \,\mathrm{d}t \\
&\quad + 32\mathbb{E}_x \left[\sum_{0\leq t\leq T\wedge\tau_n} \xi^2 |k_{t-}^*|^2 |X_{t-}^*|^2 \Delta Z_t \right] \\
&\leq 4x^2 + 2c_6 T\mathbb{E}_x \int_0^{T\wedge\tau_n} |X_t^*|^2 \,\mathrm{d}t + 32\mathbb{E}_x \int_0^{T\wedge\tau_n} |l_t^*|^2 \sigma^2(Y_t) |X_t^*|^2 \,\mathrm{d}t \\
&\quad + 32\mathbb{E}_x \left[\int_0^{T\wedge\tau_n} \xi^2 \lambda(Y_t) |k_t^*|^2 |X_t^*|^2 \,\mathrm{d}t \right] \leq 4x^2 + \bar{\Lambda} \int_0^{T\wedge\tau_n} \mathbb{E}_x \left(\sup_{s\leq t} |X_s^*|^2 \right) \mathrm{d}t.
\end{aligned}
$$

Clearly, $\tau_n \to \infty$ holds as $n \to \infty$. Hence, using the Fatou Lemma, the Monotone Convergence Theorem, and the Gronwall inequality, we can obtain

$$
\mathbb{E}_x \left(\sup_{0\leq s\leq t} |X_s^*|^2 \right) \leq 4x^2 e^{\bar{\Lambda} T}, \quad \forall t \in [0, T].
$$

This also implies that $(M_t^*)_{t\geq 0}$ is a $(\mathbb{P}, \mathbb{G})$-adapted *càdlàg* martingale. In addition, by (36), we have, for $0 \leq t \leq T$,

$$
\mathbb{E}_x(X_t^*) \leq x + \int_0^t c_6 \mathbb{E}_x(X_s^*) \,\mathrm{d}s.
$$

Then, by virtue of the Gronwall inequality, we may easily observe that

$$
\mathbb{E}_x(X_T^*) \leq x e^{c_6 T}.
$$

Therefore, for all $\alpha > c_6$, we have

$$
\lim_{T\to\infty} e^{-\alpha T} \mathbb{E}_x(X_T^*) = 0.
$$

The proof is complete. $\square$

Lemma 4.2. *Assume that*

$$
d_1 \in \left(0, \frac{c_1}{K c_3^2} \right), \tag{37}
$$

where $K > 8$ is a constant. Then for

$$
\alpha > \frac{K d_1 |\beta(0)|^2}{2c_1 - 2K d_1 c_3^2} + K d_1 c_3^2 := c_8, \tag{38}
$$

there exists a constant $\widetilde{\Lambda}$ such that

$$
e^{-\alpha T} \mathbb{E}_y \left(e^{K d_1 Y_T^2} \right) \leq \widetilde{\Lambda} \tag{39}
$$

for all $T \geq 0$.

Proof. Let $g(y) = Kd_1y^2$. Define a sequence of functions $\{g_N(y), N = 1, 2, 3, \cdots\}$ such that

$$\begin{cases} g_N \in C^\infty; \quad 0 \le g_N(y) \le N; \quad \left|\frac{\partial g_N(y)}{\partial y}\right| \le \bar{M}; \quad \left|\frac{\partial^2 g_N(y)}{\partial y^2}\right| \le \bar{M}; \\ g_{N_1}(y) \le g_{N_2}(y) \le g(y), \ \forall y \in \mathbb{R}, \ N_1 < N_2; \ \lim_{N\to\infty} g_N(y) = g(y), \ \forall y \in \mathbb{R}, \end{cases}$$

where $\bar{M}$, N_1 and N_2 are three positive constants. Next we will show that

$$e^{-\alpha T}\mathbb{E}_y\Big[e^{g_N(Y_T)}\Big] \le \widetilde{\Lambda},$$

where $\widetilde{\Lambda}$ is a constant which is independent of N, $\bar{M}$ and T. According to Theorem 5.6.1 of Friedman [8], we have

$$\mathbb{E}_y\Big[e^{g_N(Y_T)}\Big] \in C^{1,2}([0,\infty), \mathbb{R}),$$

and, if we define

$$\varphi(T, y) = e^{-\alpha T}\mathbb{E}_y\Big[e^{g_N(Y_T)}\Big],$$

then φ is a solution of the problem

$$\begin{cases} \frac{\partial \varphi}{\partial T} = \frac{1}{2}\theta^2(y)\frac{\partial^2\varphi}{\partial y^2} + \beta(y)\frac{\partial\varphi}{\partial y} - \alpha\varphi =: \widetilde{\mathcal{L}}\varphi, \\ \varphi(0, y) = e^{g_N(y)}. \end{cases} \tag{40}$$

Let $\bar{\varphi}(y) = e^{g(y)}$. Then

$$\frac{\partial\bar{\varphi}}{\partial y} = 2Kd_1ye^{Kd_1y^2}, \quad \frac{\partial^2\bar{\varphi}}{\partial y^2} = 2Kd_1e^{Kd_1y^2} + 4K^2d_1^2y^2e^{Kd_1y^2}.$$

So

$$\begin{aligned} -\widetilde{\mathcal{L}}\bar{\varphi} &= -\frac{1}{2}\theta^2(y)\frac{\partial^2\bar{\varphi}}{\partial y^2} - \beta(y)\frac{\partial\bar{\varphi}}{\partial y} + \alpha\bar{\varphi} = \big[-\theta^2(y)Kd_1 - 2\theta^2(y)K^2d_1^2y^2 - 2\beta(y)Kd_1y + \alpha\big]e^{Kd_1y^2} \\ &\ge \big[-c_3^2Kd_1 - 2c_3^2K^2d_1^2y^2 + 2c_1Kd_1y^2 - 2\beta(0)Kd_1y + \alpha\big]e^{Kd_1y^2} =: \bar{g}(y)e^{Kd_1y^2}. \end{aligned}$$

Since $\bar{g}(y)$ is quadratic, by (37) and (38), it is not hard to verify that $\bar{g}(y) > 0$. Therefore, $-\widetilde{\mathcal{L}}\bar{\varphi} > 0$. This implies that $\bar{\varphi}(y) = e^{Kd_1y^2}$ is a supersolution of (40). That is

$$\begin{cases} \frac{\partial\bar{\varphi}}{\partial T} \ge \frac{1}{2}\theta^2(y)\frac{\partial^2\bar{\varphi}}{\partial y^2} + \beta(y)\frac{\partial\bar{\varphi}}{\partial y} - \alpha\bar{\varphi}, \\ \bar{\varphi}(0, y) = e^{Kd_1y^2}. \end{cases}$$

Define $\zeta(T, y) = \varphi(T, y) - \bar{\varphi}(y)$, then it satisfies

$$\begin{cases} \frac{\partial\zeta}{\partial T} \le \frac{1}{2}\theta^2(y)\frac{\partial^2\zeta}{\partial y^2} + \beta(y)\frac{\partial\zeta}{\partial y} - \alpha\zeta, \\ \zeta(0, y) \le 0. \end{cases} \tag{41}$$

By definitions of φ, $\bar{\varphi}$ and ζ, we can get

$$\lim_{|y|\to\infty} \zeta(T, y) = -\infty.$$

Let $T_1 \ge T$ and $T_0 \in [0, T_1]$. If the maximum point of $\zeta(T, y)$ is located at (T_0, y_0) and $\zeta(T_0, y_0) > 0$, then we have $T_0 > 0$, and

$$\frac{\partial\zeta}{\partial y}(T_0, y_0) = 0, \quad \frac{\partial^2\zeta}{\partial y^2}(T_0, y_0) \le 0, \quad \frac{\partial\zeta}{\partial T}(T_0, y_0) \ge 0.$$

This contradicts (41). Then for all y and T, we must have $\zeta(T,y) \le 0$. Thus,

$$e^{-\alpha T}\mathbb{E}_y\left[e^{g_N(Y_T)}\right] \le e^{Kd_1y^2} =: \widetilde{\Lambda},$$

Clearly, $\widetilde{\Lambda}$ is a constant which is independent of N, $\bar{M}$ and T. Finally, by the Monotone Convergence Theorem, the required assertion (39) follows immediately. □

Lemma 4.3. *Suppose that* (37) *holds. If* Y_t *satisfies* (1)*, then*

$$\mathbb{E}_y\left(\sup_{0\le t\le T} e^{2d_1Y_t^2}\right) < \infty. \tag{42}$$

Proof. Let $\phi(Y_t) = e^{2d_1Y_t^2}$. By the Itô formula, we have

$$\begin{aligned} \mathrm{d}\phi(Y_t) &= \left[4d_1Y_t\beta(Y_t) + 8d_1^2Y_t^2\theta^2(Y_t) + 2d_1\theta^2(Y_t)\right]\phi(Y_t)\,\mathrm{d}t + 4d_1Y_t\theta(Y_t)\phi(Y_t)\left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right) \\ &\le \left[4d_1\left(2d_1c_3^2 - c_1\right)Y_t^2 + 4d_1\beta(0)Y_t + 2d_1c_3^2\right]\phi(Y_t)\,\mathrm{d}t + \mathrm{d}m_t, \end{aligned}$$

where

$$m_t = 4d_1\int_0^t Y_s\theta(Y_s)\phi(Y_s)\left(\rho\,\mathrm{d}W_s + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_s\right).$$

From (37), we may easily observe that $2d_1c_3^2 - c_1 < 0$. Thus, $4d_1(2d_1c_3^2-c_1)Y_t^2 + 4d_1\beta(0)Y_t + 2d_1c_3^2$ is upper-bounded. We might as well assume that U_B is an upper bound. So we have

$$\phi(Y_t) \le \phi(y) + \int_0^t U_B\phi(Y_s)\,\mathrm{d}s + m_t. \tag{43}$$

By Lemma 4.2, we can derive that, for $\forall\varepsilon > 0$ and $0 \le s \le t$,

$$\mathbb{E}_y\left[Y_s^2\phi^2(Y_s)\right] = \mathbb{E}_y\left[Y_s^2e^{4d_1Y_s^2}\right] \le \frac{1}{2}\left[\mathbb{E}_y|Y_s|^4 + \mathbb{E}_y\left(e^{8d_1Y_s^2}\right)\right] \le \frac{1}{2}\mathbb{E}_y|Y_s|^4 + \frac{1}{2}\mathbb{E}_y\left[e^{(8+\varepsilon)d_1Y_s^2}\right].$$

Moreover, by Remark 2.1 and Lemma 4.2, we have

$$\mathbb{E}_y|Y_s|^4 < \infty, \quad \mathbb{E}_y\left[e^{(8+\varepsilon)d_1Y_s^2}\right] < \infty.$$

This means that

$$\mathbb{E}_y\left[Y_s^2\phi^2(Y_s)\right] < \infty, \quad 0 \le s \le t.$$

So we have

$$16d_1^2\int_0^t \mathbb{E}_y\left[Y_s^2\theta^2(Y_s)\phi^2(Y_s)\right]\mathrm{d}s \le 16c_3^2d_1^2\int_0^t \mathbb{E}_y\left[Y_s^2\phi^2(Y_s)\right]\mathrm{d}s < \infty.$$

Therefore, m_t is a martingale. Applying the Doob martingale inequality, we can obtain

$$\mathbb{E}_y\left[\sup_{0\le t\le T} m_t^2\right] \le 4\mathbb{E}_y m_T^2 \le \Upsilon < \infty,$$

where Υ is a positive constant. Going back to (43), we may easily observe that

$$\mathbb{E}_y\left[\sup_{0\le t\le T}\phi(Y_t)\right] \le \phi(y) + \Upsilon^{\frac{1}{2}} + \int_0^T U_B\mathbb{E}_y\left[\sup_{0\le s\le t}\phi(Y_s)\right]\mathrm{d}t.$$

Then the Gronwall inequality gives

$$\mathbb{E}_y\left[\sup_{0\le t\le T}\phi(Y_t)\right] \le \left[\phi(y) + \Upsilon^{\frac{1}{2}}\right]e^{U_BT},$$

and hence the required assertion (42) follows. □

Now, we present the verification theorem.

Theorem 4.4 (Verification Theorem)**.** *Let the conditions* $(A1)-(A4)$ *be satisfied, and* $0<\gamma\le\frac{1}{2}$*. For* $(x,y,z)\in\mathbb{R}_+\times\mathbb{R}\times\{0,1\}$*, define*

$$\hat{V}(x,y,z)=\frac{1}{\gamma}x^{\gamma}e^{zu_*^1(y)+(1-z)u_*^0(y)}=\frac{1}{\gamma}x^{\gamma}\Big[ze^{u_*^1(y)}+(1-z)e^{u_*^0(y)}\Big], \tag{44}$$

where u_*^1 *and* u_*^0 *are classical solutions to* (14) *and* (15)*, respectively.*

(i) *For all admissible control strategies* $(k_t,l_t,c_t)_{t\ge0}\in\mathcal{A}(\mathbb{G})$*, it holds that*

$$\hat{V}(x,y,z)\ge\mathbb{E}_{x,y,z}\int_0^{\infty}e^{-\alpha t}U(c_tX_t)\,\mathrm{d}t. \tag{45}$$

(ii) *If the conditions* (37) *and* (38) *hold, then, for all* $\alpha>c_6$*, we have*

$$V(x,y,z)\coloneqq\mathbb{E}_{x,y,z}\int_0^{\infty}e^{-\alpha t}U(c_t^*X_t^*)\,\mathrm{d}t=\hat{V}(x,y,z). \tag{46}$$

Proof. To simplify our analysis, we denote

$$\hat{V}_x\coloneqq\frac{\partial\hat{V}}{\partial x},\quad \hat{V}_{xx}\coloneqq\frac{\partial^2\hat{V}}{\partial x^2},\quad \hat{V}_{xy}\coloneqq\frac{\partial^2\hat{V}}{\partial x\partial y}.$$

For all $(k_t,l_t,c_t)_{t\ge0}\in\mathcal{A}(\mathbb{G})$, by the Itô formula with jumps, we can get

$$\begin{aligned}\mathrm{d}\hat{V}(X_t,Y_t,Z_t)&=\hat{V}_xX_t\Big[r(Y_t)-c_t+(\mu(Y_t)-r(Y_t))l_t+\xi k_t\hat{Z}_t\frac{\lambda(Y_t)}{\eta(Y_t)}\Big]\,\mathrm{d}t\\&\quad+\Big[\frac{1}{2}\hat{V}_{xx}l_t^2\sigma^2(Y_t)X_t^2+\hat{V}_y\beta(Y_t)+\hat{V}_{xy}X_t\rho l_t\theta(Y_t)\sigma(Y_t)+\frac{1}{2}\hat{V}_{yy}\theta^2(Y_t)\Big]\,\mathrm{d}t\\&\quad+\Big[\hat{V}(X_t-\xi k_tX_t\hat{Z}_t,Y_t,Z_t+1)-\hat{V}(X_t,Y_t,Z_t)\Big]\hat{Z}_t\lambda(Y_t)\,\mathrm{d}t+\mathrm{d}N_t,\end{aligned}$$

where $(N_t)_{t\ge0}$ is a $(\mathbb{P},\mathbb{G})$-adapted *càdlàg* local martingale defined by

$$\begin{aligned}N_t\coloneqq&\int_0^t\hat{V}_x(X_s,Y_s,Z_s)l_s\sigma(Y_s)X_s\,\mathrm{d}W_s-\int_0^t\hat{V}_y(X_s,Y_s,Z_s)\theta(Y_s)\Big(\rho\,\mathrm{d}W_s+\sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_s\Big)\\&+\int_0^t\Big[\hat{V}\Big(X_{s^-}-\xi k_{s^-}X_{s^-}\hat{Z}_{s^-},Y_{s^-},Z_{s^-}+1\Big)-\hat{V}(X_{s^-},Y_{s^-},Z_{s^-})\Big]\,\mathrm{d}M_s.\end{aligned}$$

Noting that

$$Z_t\hat{Z}_t=0,\quad \hat{Z}_t\hat{Z}_t=\hat{Z}_t,\quad \Big(1-\xi k_t\hat{Z}_t\Big)^{\gamma}\hat{Z}_t=(1-\xi k_t)^{\gamma}\hat{Z}_t.$$

Therefore, by (44), we have

$$\begin{aligned}\mathrm{d}\hat{V}(X_t,Y_t,Z_t)-\mathrm{d}N_t=Z_tX_t^{\gamma}e^{u_*^1(Y_t)}\Bigg[&r(Y_t)-c_t+(\mu(Y_t)-r(Y_t))l_t+\frac{1}{2}(\gamma-1)l_t^2\sigma^2(Y_t)\\&+\frac{1}{\gamma}(\beta(Y_t)+\gamma\rho l_t\theta(Y_t)\sigma(Y_t))\frac{\partial u_*^1}{\partial y}(Y_t)+\frac{1}{2\gamma}\theta^2(Y_t)\Big(\frac{\partial u_*^1}{\partial y}(Y_t)\Big)^2\\&+\frac{1}{2\gamma}\theta^2(Y_t)\frac{\partial^2u_*^1}{\partial y^2}(Y_t)\Bigg]\,\mathrm{d}t+\hat{Z}_tX_t^{\gamma}e^{u_*^0(Y_t)}\Bigg[r(Y_t)-c_t+(\mu(Y_t)-r(Y_t))l_t\end{aligned}$$

$$+ \xi k_t \frac{\lambda(Y_t)}{\eta(Y_t)} + \frac{1}{2}(\gamma-1) l_t^2 \sigma^2(Y_t) + \frac{1}{\gamma}(\beta(Y_t) + \gamma\rho l_t \theta(Y_t)\sigma(Y_t)) \frac{\partial u_*^0}{\partial y}(Y_t)$$
$$+ \frac{1}{2\gamma}\theta^2(Y_t)\left(\frac{\partial u_*^0}{\partial y}(Y_t)\right)^2 + \frac{1}{2\gamma}\theta^2(Y_t)\frac{\partial^2 u_*^0}{\partial y^2}(Y_t)\Bigg] \mathrm{d}t$$
$$+ \frac{1}{\gamma}\hat{Z}_t X_t^\gamma \left[(1-\xi k_t)^\gamma e^{u_*^1(Y_t)} - e^{u_*^0(Y_t)}\right]\lambda(Y_t)\,\mathrm{d}t.$$

Substituting $W^1 = e^{u_*^1}$ and $W^0 = e^{u_*^0}$ into the above equation, and using

$$\mathrm{d}\left[e^{-\alpha t}\hat{V}(X_t, Y_t, Z_t)\right] = e^{-\alpha t}\,\mathrm{d}\hat{V}(X_t, Y_t, Z_t) - \alpha e^{-\alpha t}\hat{V}(X_t, Y_t, Z_t)\,\mathrm{d}t,$$

equations (10), (11) and (13), it is not hard to verify that

$$\begin{aligned}
& e^{-\alpha T}\hat{V}(X_T, Y_T, Z_T) - \hat{V}(x,y,z) - \int_0^T e^{-\alpha T}\,\mathrm{d}N_t \\
&= \int_0^T e^{-\alpha t}\,\mathrm{d}\hat{V}(X_t, Y_t, Z_t) - \int_0^T \alpha e^{-\alpha t}\hat{V}(X_t, Y_t, Z_t)\,\mathrm{d}t - \int_0^T e^{-\alpha t}\,\mathrm{d}N_t \\
&= \int_0^T e^{-\alpha t}\Bigg\{ Z_t X_t^\gamma \Bigg[\left(r(Y_t) - c_t + (\mu(Y_t) - r(Y_t))l_t + \frac{1}{2}(\gamma-1)l_t^2\sigma^2(Y_t)\right) W^1(y) \\
&\quad + \frac{1}{\gamma}(\beta(Y_t) + \gamma\rho l_t\theta(Y_t)\sigma(Y_t))\frac{\partial W^1}{\partial y}(Y_t) + \frac{1}{2\gamma}\theta^2(Y_t)\frac{\partial^2 W^1}{\partial y^2}(Y_t)\Bigg] \\
&\quad + \hat{Z}_t X_t^\gamma \Bigg[\left(r(Y_t) - c_t + (\mu(Y_t) - r(Y_t))l_t + \xi k_t\frac{\lambda(Y_t)}{\eta(Y_t)} + \frac{1}{2}(\gamma-1)l_t^2\sigma^2(Y_t)\right) W^0(y) \\
&\quad + \frac{1}{\gamma}(\beta(Y_t) + \gamma\rho l_t\theta(Y_t)\sigma(Y_t))\frac{\partial W^0}{\partial y}(Y_t) + \frac{1}{2\gamma}\theta^2(Y_t)\frac{\partial^2 W^0}{\partial y^2}(Y_t) - \frac{\lambda(Y_t)}{\gamma}W^0(Y_t)\Bigg] \\
&\quad + \hat{Z}_t X_t^\gamma \frac{\lambda(Y_t)}{\gamma}(1-\xi k_t)^\gamma W^1(Y_t) - \alpha\hat{V}(X_t, Y_t, Z_t)\Bigg\}\,\mathrm{d}t \\
&\le \int_0^T e^{-\alpha t}\left[Z_t X_t^\gamma\left(\frac{\alpha}{\gamma}W^1(Y_t) - \frac{1}{\gamma}c_t^\gamma\right) + \hat{Z}_t X_t^\gamma\left(\frac{\alpha}{\gamma}W^0(Y_t) - \frac{1}{\gamma}c_t^\gamma\right) - \alpha\hat{V}(X_t, Y_t, Z_t)\right]\mathrm{d}t \\
&= \int_0^T e^{-\alpha t}\left[\frac{\alpha}{\gamma}X_t^\gamma\left(Z_t W^1(Y_t) + \hat{Z}_t W^0(Y_t)\right) - \frac{1}{\gamma}X_t^\gamma\left(Z_t c_t^\gamma + \hat{Z}_t c_t^\gamma\right) - \alpha\hat{V}(X_t, Y_t, Z_t)\right]\mathrm{d}t \\
&= -\int_0^T e^{-\alpha t}U(c_t X_t)\,\mathrm{d}t.
\end{aligned}$$

Thus, for each $T > 0$, we have

$$\hat{V}(x,y,z) \ge \mathbb{E}_{x,y,z}\int_0^{T\wedge\tau_R} e^{-\alpha t}U(c_t X_t)\,\mathrm{d}t + \mathbb{E}_{x,y,z}\left[e^{-\alpha T\wedge\tau_R}\hat{V}(X_{T\wedge\tau_R}, Y_{T\wedge\tau_R}, Z_{T\wedge\tau_R})\right], \tag{47}$$

where τ_R is the exit time of the $(X_t, Y_t)_{t\ge 0}$ from $\{(x,y)\in\mathbb{R}^2 : x^2 + y^2 \le R^2\}$. Clearly, $\tau_R \to \infty$ holds as $R \to \infty$. So

$$\hat{V}(x,y,z) \ge \mathbb{E}_{x,y,z}\int_0^T e^{-\alpha t}U(c_t X_t)\,\mathrm{d}t.$$

This implies that the required assertion (45) follows by letting $T \to \infty$.

On the other hand, note that $(k_t^*, l_t^*, c_t^*)_{t\geq 0} \in \mathcal{A}(\mathbb{G})$, instead of (47), we obtain

$$\hat{V}(x,y,z) = \mathbb{E}_{x,y,z} \int_0^{T\wedge\tau_R} e^{-\alpha t} U(c_t^* X_t^*)\, \mathrm{d}t + \mathbb{E}_{x,y,z}\Big[e^{-\alpha T\wedge\tau_R} \hat{V}\big(X_{T\wedge\tau_R}^*, Y_{T\wedge\tau_R}, Z_{T\wedge\tau_R}\big)\Big]. \tag{48}$$

By (44), Theorems 3.7 and 3.12, and $\bar{u}^1 < \bar{u}^0$, we have

$$\begin{aligned} e^{-\alpha T\wedge\tau_R} \hat{V}\big(X_{T\wedge\tau_R}^*, Y_{T\wedge\tau_R}, Z_{T\wedge\tau_R}\big) &= \frac{1}{\gamma} e^{-\alpha T\wedge\tau_R} \big(X_{T\wedge\tau_R}^*\big)^\gamma e^{Z_{T\wedge\tau_R} u_*^1(Y_{T\wedge\tau_R}) + (1-Z_{T\wedge\tau_R}) u_*^0(Y_{T\wedge\tau_R})} \\ &\leq \frac{1}{\gamma}\big(X_{T\wedge\tau_R}^*\big)^\gamma e^{\bar{u}^0(Y_{T\wedge\tau_R})} = \frac{1}{\gamma}\big(X_{T\wedge\tau_R}^*\big)^\gamma e^{d_1 Y_{T\wedge\tau_R}^2 + d_3} \\ &\leq \Lambda_1 \Bigg(\sup_{0\leq t\leq T} |X_t^*|^{2\gamma} + \sup_{0\leq t\leq T} e^{2d_1 Y_t^2}\Bigg), \end{aligned}$$

where Λ_1 is a constant independent of T and R. Therefore, by (34) and (42),

$$\mathbb{E}_{x,y,z}\Big[e^{-\alpha T\wedge\tau_R} \hat{V}\big(X_{T\wedge\tau_R}^*, Y_{T\wedge\tau_R}, Z_{T\wedge\tau_R}\big)\Big] < \infty.$$

Similarly, we can show that

$$\mathbb{E}_{x,y,z}\Big[e^{-\alpha T} \hat{V}(X_T^*, Y_T, Z_T)\Big] \leq \Lambda_2 e^{-\alpha T} \mathbb{E}_{x,y,z}\Big[(X_T^*)^{2\gamma} + e^{2d_1 Y_T^2}\Big],$$

where Λ_2 is a constant independent of T and R. Moreover, we note that for each $0 < \gamma < \frac{1}{2}$, $x^{2\gamma} \leq 1 + x$, $\forall x > 0$. Then

$$\mathbb{E}_{x,y,z}(X_T^*)^{2\gamma} \leq 1 + \mathbb{E}_{x,y,z}(X_T^*).$$

Denote $\hat{\alpha} = \max\{c_6, c_8\}$ and $\delta = \alpha - \hat{\alpha}$. Then, from $\alpha > c_6$ and (38), we obtain $\delta > 0$, and $\frac{\delta}{2} + \hat{\alpha} > c_8$. So

$$\mathbb{E}_{x,y,z}\Big[e^{-\alpha T} \hat{V}(X_T^*, Y_T, Z_T)\Big] \leq \Lambda_2 e^{-\alpha T}[1 + \mathbb{E}_{x,y,z}(X_T^*)] + \Lambda_2 e^{-\frac{\delta}{2}T} e^{-\left(\frac{\delta}{2}+\hat{\alpha}\right)T} \mathbb{E}_{x,y,z}\Big(e^{2d_1 Y_T^2}\Big).$$

Therefore, by virtue of Lemmas 4.1 and 4.2, we have

$$\limsup_{T\to\infty} \mathbb{E}_{x,y,z}\Big[e^{-\alpha T} \hat{V}(X_T^*, Y_T, Z_T)\Big] \leq 0.$$

Applying the Lebesgue Dominated Convergence Theorem to (48), we have

$$\hat{V}(x,y,z) = \mathbb{E}_{x,y,z} \int_0^T e^{-\alpha t} U(c_t^* X_t^*)\, \mathrm{d}t + \mathbb{E}_{x,y,z}\Big[e^{-\alpha T} \hat{V}(X_T^*, Y_T, Z_T)\Big].$$

This implies that

$$\hat{V}(x,y,z) \leq \mathbb{E}_{x,y,z} \int_0^\infty e^{-\alpha t} U(c_t^* X_t^*)\, \mathrm{d}t = V(x,y,z).$$

Finally, by using (45), we can easily get (46). The proof is therefore complete. □

5. Sensitivity analysis

In this section, we analyze the impact of some important parameters on the optimal control strategy $(k_t^*, c_t^*)_{t\geq 0}$, and then study the effects of the wealth and stochastic economic factor on the value function $\hat{V}$ obtained in Theorem 4.4.

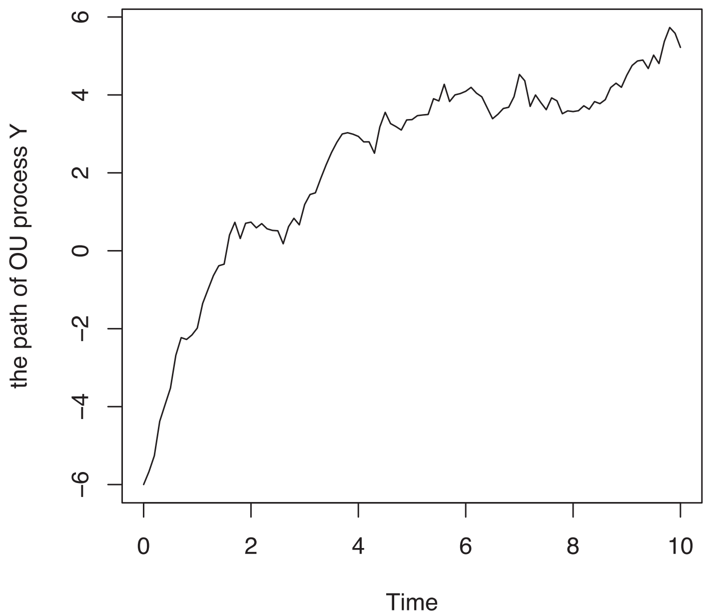

Figure 1. The path of OU process Y_t with $T = 10$.

In order to conduct the sensitivity analysis, we interpreted the model as follows. For the stochastic economic factor model, we take OU process as an example, that is,

$$\mathrm{d}Y_t = (2 - 0.5Y_t)\,\mathrm{d}t + \left(0.9 + 0.1e^{-2Y_t^2}\right)\mathrm{d}\widetilde{W}_t, \quad Y_0 = -6 \in \mathbb{R}, \tag{49}$$

where $\theta(y) = 0.9 + 0.1e^{-2y^2} \leq 1$. Therefore, $c_1 = c_2 = 0.5$, $c_3 = 1$, $\rho = 0$ and $\beta(0) = 2$. Clearly, there exists a unique solution Y_t. The price dynamics of the default-free risky asset is given by

$$\mathrm{d}S_t = \mu(Y_t)S_t\,\mathrm{d}t + \sigma(Y_t)S_t\,\mathrm{d}W_t, \qquad S_0 > 0,$$

where the appreciation rate $\mu(y) = 0.01e^{-y^2} + 0.07$, and the volatility $\sigma(y) = \sqrt{0.0225 + e^{0.02y}}$. Moreover, we assume that the interest rate $r(y) = 0.01e^{-y^2} + 0.01$, and $\frac{1}{\eta(y)} = 1 + 0.5e^{-0.5y^2}$.

For great convenience, some parameters are given as follows: $\alpha = 0.92$, $\gamma = 0.10$, $C = 0.02$ and $K = 10$.

Figure 1 shows us the path of OU process Y_t with $Y_0 = -6$, $\beta(y) = 2 - 0.5y$, $\theta(y) = 0.9 + 0.1e^{-2y^2}$ and $T = 10$. From Figure 1, we see that the value of Y_t increases from the initial value $Y_0 = -6$ to 0, and then stays above the level 0. From (49), it is easy to obtain $\mathbb{E}(Y_T) = 4 - 10e^{-0.5T} \approx 3.93$. Therefore, the value of Y_t fluctuates up and down around 3.93 for the second half of the time period due to the mean-reverting property of OU process, which is also well illustrated in Figure 1.

By virtue of Lemma 4.2, we have $d_1 \in (0, 0.05)$. Thus, we can choose $d_1 = 0.014$ and $c_5 = 0.4$. Recall the definitions $Q(y)$ and $C_Q(y)$, we have $Q(y) \leq 0.0109$ and $C_Q(y) \leq -0.0039$. Then it is not hard to verify that the conditions $(A1)$–$(A4)$, (18), (26), (38) and $\alpha > c_6$ are satisfied. By (18) and (26), we can deduce that $d_2 \geq 0.0282$ and $d_3 \geq 0.0362$. Consequently,

$$-0.0039 \leq u_*^1(y) \leq 0.014y^2 + 0.0282, \tag{50}$$

and

$$-0.0039 \leq u_*^0(y) \leq 0.014y^2 + 0.0362, \tag{51}$$

where u_*^1 and u_*^0 are classical solutions to (14) and (15), respectively.

By the way, based on our parameter settings, we observe that as the interest rate $r(y)$ increases, $Q(y)$ also increases, which in turn leads to possibly smaller values for d_2 and d_3. Consequently, $\bar{u}^1(y)$ and $\bar{u}^0(y)$ might attain smaller values, thereby narrowing down the range of possible values for $u_*^1(y)$ and $u_*^0(y)$. Moreover, from equations (52) and (53) below, it becomes apparent that the range of the optimal control strategies k_t^* and c_t^* will also shrink. In essence, as the interest rate rises, the investor tends to reduce his or her investments in the defaultable bond and decrease his or her consumption. Conversely, the investor will increase his or her investments in the money market account. This behavior is also consistent with our intuition.

5.1. The optimal control strategy $(k_t^*, c_t^*)_{t\geq 0}$

In this subsection, we investigate the parametric sensitivity of the optimal control strategy $(k_t^*, c_t^*)_{t\geq 0}$. Here we only consider the pre-default case.

We begin by discussing the parametric sensitivity in the optimal control k^* for the defaultable bond. For $0 \leq t < \tau$, by (31),

$$k_t^* = k^*(Y_t) = \frac{1}{\xi}\left[1 - \left(\frac{e^{u_*^0(Y_t)}}{\eta(Y_t)e^{u_*^1(Y_t)}}\right)^{\frac{1}{\gamma-1}}\right].$$

Then we infer by (50) and (51),

$$\frac{1}{\xi}\left[1 - \left(\frac{e^{-0.014y^2-0.0321}}{\eta(y)}\right)^{\frac{1}{\gamma-1}}\right] \leq k_t^* \leq \frac{1}{\xi}\left[1 - \left(\frac{e^{0.014y^2+0.0401}}{\eta(y)}\right)^{\frac{1}{\gamma-1}}\right] =: \bar{k}_t^*, \tag{52}$$

where $\frac{1}{\eta(y)} = 1 + 0.5e^{-0.5y^2}$. Since the lower and upper bounds of k^* are all completely characterized by the loss rate ξ, risk aversion parameter γ and stochastic economic factor y, we only consider the upper bound $\bar{k}^*$. First of all, we study the relationship between the risk aversion parameter γ and the upper bound $\bar{k}^*$. We fix $\xi = 0.5$. Figure 2 shows that $\bar{k}^*$ increases as the risk aversion parameter increases. This behavior is supported by the economic interpretation of γ. The smaller γ implies that an investor is more risk averse. Therefore, when γ decreases, an investor reduces his/her investment proportion of the defaultable bond. In addition, Figure 2 is symmetrical about y-axis. If γ is fixed, we vary the stochastic economic factor y from 0 to 6, the upper bound $\bar{k}^*$ decreases first and then increases.

The Figure 3 displays the relationship between the loss rate ξ, stochastic economic factor y and the upper bound $\bar{k}^*$. For a fixed y or $1/\eta(y)$, $\bar{k}^*$ decreases while ξ increases. In other words, a higher loss rate may lead to high losses, so an investor will naturally minimize his/her investment proportion of the defaultable bond. Similarly, Figure 3 is also symmetrical about y-axis. If ξ is fixed, we vary y from 0 to 6, the upper bound $\bar{k}^*$ decreases first and then increases.

We now investigate the parametric sensitivity of the optimal consumption ratio c_t^* given by (33). Recall that

$$c_t^* = c^*(Y_t) = e^{\frac{u_*^0(Y_t)}{\gamma-1}}, \quad 0 \leq t < \tau.$$

By (51), we have

$$\underline{c}^* := e^{\frac{0.014y^2+0.0362}{\gamma-1}} \leq c_t^* \leq e^{\frac{0.0039}{1-\gamma}} =: \bar{c}^*, \quad 0 \leq t < \tau. \tag{53}$$

In the left figure in Figure 4 depicts the lower bound $\underline{c}^*$ *versus* the risk aversion parameter γ and stochastic economic factor y. These observations are supported by the economic interpretation of γ. For instance, if the stochastic economic factor y is fixed, for a large γ, a low risk averse investor will increase his/her investment proportion of the defaultable bond and thus will reduce his/her consumption. However, the risk aversion parameter γ affects the upper bound $\bar{c}^*$ in an opposite way, which is well illustrated in the right figure in Figure 4.

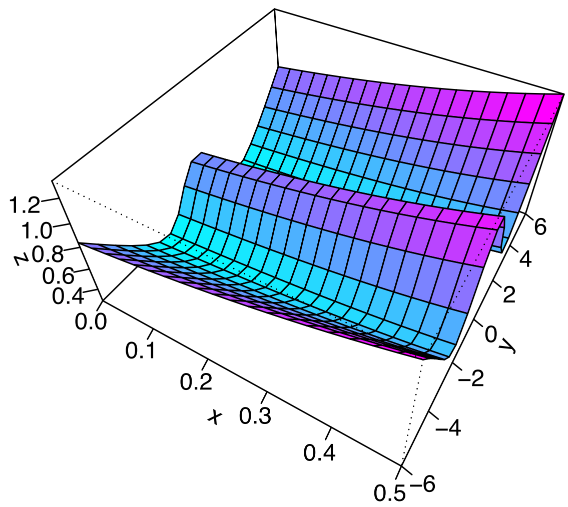

FIGURE 2. The upper bound $\bar{k}^*$ (z-axis) *versus* the risk aversion parameter γ (x-axis) and stochastic economic factor y (y-axis).

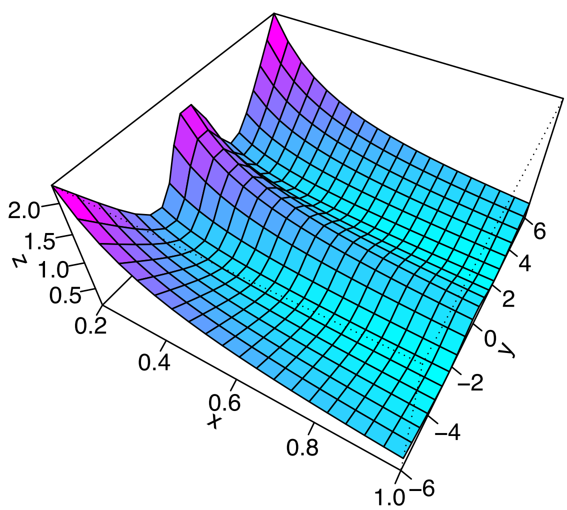

FIGURE 3. The upper bound $\bar{k}^*$ (z-axis) *versus* the loss rate ξ (x-axis) and stochastic economic factor y (y-axis).

5.2. The value function $\hat{V}$

In this subsection, we analyze the value function $\hat{V}(x, y, z)$ obtained in Theorem 4.4. In the light of (44), (50) and (51), we can easily obtain the lower and upper bounds of $\hat{V}$.

$$\text{VDB} := \frac{1}{\gamma} x^{\gamma} e^{-0.0039} \leq \hat{V}(x, y, z) \leq \frac{1}{\gamma} x^{\gamma} e^{0.014y^2 + 0.0362} =: \text{VUB}. \tag{54}$$

The left figure in Figure 5 depicts the upper bound of $\hat{V}$ *versus* the wealth x and stochastic economic factor y. If we vary y from 0 to 10, it is shown that as y increases, the upper bound of $\hat{V}$ moves up. Moreover, as the

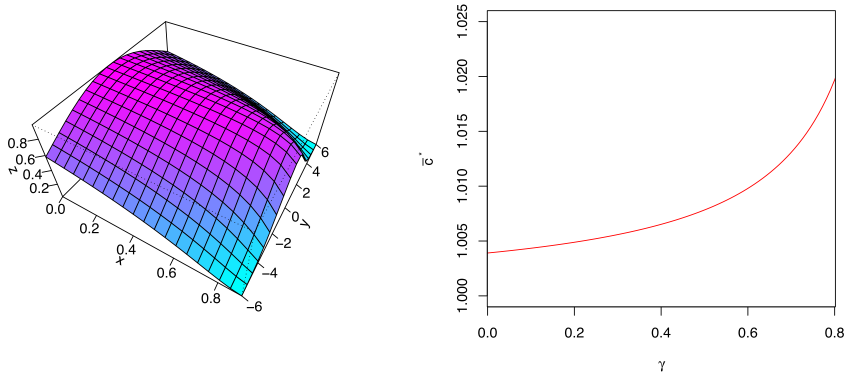

FIGURE 4. *Left*: lower bound $\underline{c}^*$ (z-axis) *versus* the risk aversion parameter γ (x-axis) and stochastic economic factor y (y-axis). *Right*: upper bound $\bar{c}^*$ *versus* the risk aversion parameter γ as $\gamma \in [0, 0.80]$.

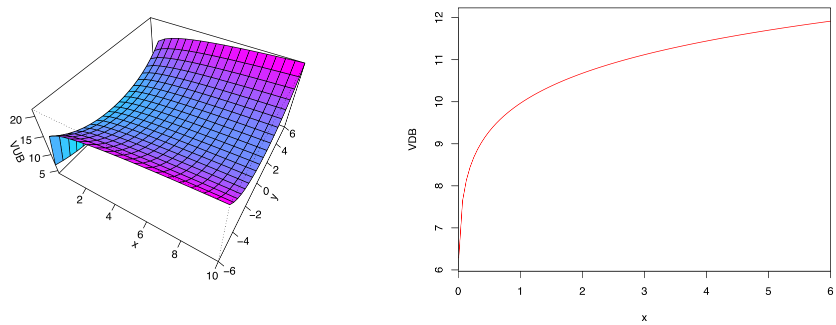

FIGURE 5. *Left*: upper bound VUB *versus* the wealth x and stochastic economic factor y. *Right*: lower bound VDB *versus* the wealth x.

wealth x increases, the upper bound VUB increases. As is shown in the right figure of Figure 5, the relationship between the lower bound of $\hat{V}$ and the wealth x is similar.

6. Conclusions

In this paper, we have considered the optimal investment and consumption strategies for an investor with stochastic economic factor in a default market under the infinite time horizon. An investor allocates his/her wealth dynamically in a perpetual defaultable bond, a money market account, and a default-free risky asset and chooses a consumption rate. The goal is to maximize the expected discount power utility of the consumption. We have generalized the existing model to a more general circumstance, assuming all financial securities rely on a stochastic economic factor process which is described by a diffusion process related to a default-free risky asset. The optimal investment and consumption strategies has been obtained through analysis on the classical solutions of the corresponding HJB equations by the so-called sub-super solution method. Finally, we have analyzed the sensitivity to parameters of the optimal control strategies and the value function through numerical simulation. An interesting extension is probably a Lévy diffusion, which is used to describe the stochastic economic factor process, with the interdependence introduced between the default time and the risky asset.

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