RAIRO-Oper. Res. 58 (2024) 423–439
https://doi.org/10.1051/ro/2023165

RAIRO Operations Research
www.rairo-ro.org

GRACEFUL GAME ON SOME GRAPH CLASSES

Deise L. de Oliveira[1], Danilo Artigas[2], Simone Dantas[1], Luisa Frickes[1] and Atílio G. Luiz[3,*]

Abstract. A graceful labeling of a graph G with m edges consists in labeling the vertices of G with distinct integers from 0 to m such that each edge is uniquely identified by the absolute difference of the labels of its endpoints. In this work, we study the graceful labeling problem in the context of maker-breaker graph games. The Graceful Game was introduced by Tuza, in 2017, as a two-players game on a connected graph in which the players, Alice and Bob, take moves labeling the vertices with distinct integers from 0 to m. Players are constrained to use only legal labelings (moves), that is, after a move, all edge labels are distinct. Alice's goal is to obtain a graceful labeling for the graph, as Bob's goal is to prevent it from happening. In this work, we study winning strategies for Alice and Bob in graph classes: paths, complete graphs, cycles, complete bipartite graphs, caterpillars, trees, gear graphs, web graphs, prisms, hypercubes, 2-powers of paths, wheels and fan graphs.

Mathematics Subject Classification. 05C57, 05C78, 91A46.

Received December 9, 2022. Accepted October 8, 2023.

1. Introduction

Graph labeling is an area of graph theory whose main concern consists in determining the feasibility of assigning labels to the elements (vertices or edges) of a graph satisfying certain conditions. Usually, the labels are elements of a set that supports some kind of mathematical operation as, for example, the set of nonnegative integers. Since the 1960s [6], many contexts have emerged where it is required to label the vertices or edges of a given graph with numbers. Most of these problems, such as harmonious labelings [9] and $L(2,1)$-labelings [10], arose naturally from modeling of optimization problems on networks.

Formally, given a graph G and a set $L \subset \mathbb{R}$, a *vertex labeling* of G is a function $f\colon V(G) \to L$ that induces an edge labeling $g\colon E(G) \to \mathbb{R}$ in the following way: $g(uv)$ is a function of $f(u)$ and $f(v)$, for all $uv \in E(G)$, and g respects some specified restrictions. Depending on the function g chosen, on the restrictions that g is required to satisfy, or even on the chosen subset of labels L, many different types of graph labelings can be defined. We remark that, in this paper, all graphs $G = (V(G), E(G))$ are undirect, finite and simple, with $m = |E(G)|$.

One of the oldest and most studied graph labelings is the *graceful labeling*, named by Golomb [8] and initially introduced by Rosa [17] around 1966, which later inspired the creation of most graph labeling methods. A

Keywords. Graceful labeling, graph labeling, graceful game, combinatorial games.

[1] IME, Universidade Federal Fluminense, Niterói, RJ, Brazil.

[2] RIC, Universidade Federal Fluminense, Rio das Ostras, RJ, Brazil.

[3] Campus Quixadá, Universidade Federal do Ceará, Quixadá, CE, Brazil.

*Corresponding author: gomes.atilio@ufc.br

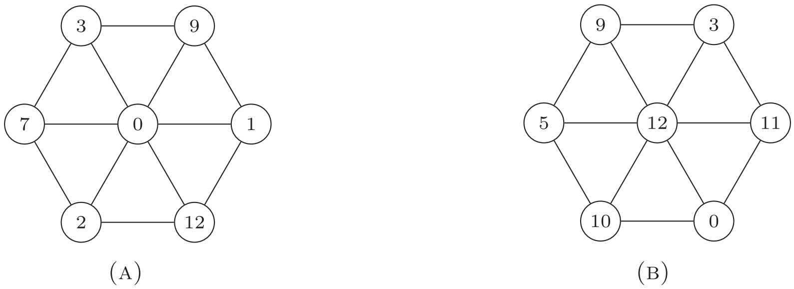

FIGURE 1. Two graceful labelings of the same graph. (a) G with graceful labeling $\overline{f}$. (b) Two graceful labelings of the same graph.

graceful labeling of a graph G with m edges is an injective function $f\colon V(G) \to \{0, 1, \ldots, m\}$ such that, when each edge $uv \in E(G)$ is assigned the (induced) label $g(uv) = |f(u) - f(v)|$, all induced edge labels are distinct. A graph G that admits a graceful labeling is called *graceful*. In Figure 1a, we show a graph with a graceful labeling. Given a graph G with m edges and a graceful labeling f, it is possible to obtain a second graceful labeling $\overline{f}$ of G, called *complementary labeling*, defined as $\overline{f}(v) = m - f(v)$, for all $v \in V(G)$. Figure 1b illustrates complementary labeling.

Labeled graphs are useful models for a wide range of applications such as: coding theory, crystallography, and many others. Although more than fifty years have passed since the first paper on graph labeling was introduced, there are few general results about graceful labeling. The most studied open problem related to graceful labelings is the *Graceful Tree Conjecture*, which states that all trees are graceful. This conjecture was posed by Rosa [17] in 1966 and, since then, this and many other labeling problems have been studied [9, 10, 15–18, 20].

From the vast literature on graph labeling [6], it is notorious that labeling problems are usually studied from the perspective of determining whether a given graph has a required labeling or not. The complexity of the problem of deciding if a graph G is graceful has not yet been determined.

An alternative perspective to study graph problems is to analyze them from the point of view of combinatorial games such as Clobber [4]. Hefetz *et al.* [12] surveyed the area of Positional games, including Maker-Breaker games. These games were determined to be PSPACE-complete by Schaefer [19], where the game was mentioned as POS CNF. On the other hand, only a few papers have been published on labeling games [1–3, 7, 11, 22]. Three of them deal with magic labelings [2, 7, 11], one of them considers the game version of $L(d, 1)$-labelings [3], and another one considers the game version of neighbor-sum-distinguishing edge labelings [1].

Recently, Tuza [22] surveyed the area and proposed new labeling games such as the graceful game studied in this work. While the number of published papers in labeling games has been scarce so far, in the related area of graph colorings, there is a track of research concerning the "game chromatic number" comprising more than fifty published papers (see Tuza and Zhu's survey [23].)

The *graceful game* is defined as follows. Alice and Bob alternately assign an unused label $f(v) \in \{0, \ldots, m\}$ to a previously unlabeled vertex v of a given graph G with m edges. We call a vertex of G *free* if it is not labeled yet. If both endpoints of an edge $uv \in E(G)$ are already labeled, then the *label* of the edge uv is defined as $|f(u) - f(v)|$. A labeling (move) is said to be *legal* if, after it, all edge labels are distinct. Players are constrained to use only legal labelings. Alice *wins* if graph G is gracefully labeled, and Bob *wins* if he can prevent this.

It is well known that not every graph is graceful [8]. Rosa [17] proved that Eulerian graphs such that $m \equiv 1, 2 \pmod 4$ are not graceful. For non-graceful graphs, for example K_n with $n \geq 5$ and C_n with $n \not\equiv 0, 3 \pmod 4$, it is immediate that Bob is the winner and, therefore, the game is completely determined for such graphs.

In this work, we investigated graph classes for which it is possible to obtain a graceful labeling, or those whose gracefulness was not determined yet. We implemented computational algorithms to analyse the game on small graphs. We applied backtracking technique which means that, at each player turn, we growed the tree of partial solutions, branching on the set of all possible moves. We pruned the branching process for a partial solution as soon as we can decide which player wins the game at that position. For the general case, a partial solution is a winning position for a player X if X can reach a winning position on its next move. A position where all vertices have a label is a winning position for Alice. A position where there exist unlabeled vertices and the player of the turn could not make a legal move is a winning position for Bob. The computational results obtained are described in the last section.

We investigate the graceful game on simple graphs and contribute to the graph labeling games research area surveyed by Tuza ([22], Sect. 2.1). We study winning strategies for Alice and Bob in the following families of graphs: paths, complete graphs, cycles, complete bipartite graphs, caterpillars, trees, gear graphs, web graphs, prisms, hypercubes, 2-powers of paths, wheel and fan graphs. Our results show structural properties implied by the graceful labeling constraints that can contribute to the study of the graceful labeling of graphs in which the gracefulness was not yet determined, for example powers of paths and web graphs.

2. The graceful game

Given a graph G, two adjacent vertices are called *neighbors* and $N(v)$ is the set of neighbors of $v \in V(G)$. The *distance* $d(u, v)$ between $u, v \in V(G)$ is the number of edges in a shortest path between u and v in G (if this path does not exist, $d(u, v) = \infty$).

Our next results show properties that are used throughout this work.

Lemma 1. *Alice can only use the label* 0 *(resp.* m*) to label a vertex* $v \in V(G)$ *if* v *is adjacent to every remaining free vertex or* v *is adjacent to a vertex already labeled by Bob with* m *(resp.* 0*).*

Proof. A graceful graph G must have an edge with induced label m and the only way to obtain it is by assigning labels 0 and m to two adjacent vertices. Thus, suppose Alice labels a vertex $v \in V(G)$ with 0 (resp. m), without Bob having already labeled any vertex with m (resp. 0), and there is a free vertex u not adjacent to v in G. On Bob's next move, he assigns label m (resp. 0) to u, preventing Alice from gracefully labeling the graph. □

Lemma 2. *If Bob assigns label* 0 *(resp.* m*) to a vertex* $v \in V(G)$*, such that* v *has only one free neighbor or there are two free vertices in* G *not adjacent to* v*, then Alice is forced to label a vertex adjacent to* v *with* m *(resp.* 0*).*

Proof. First, suppose Bob labels $v \in V(G)$ with 0 (resp. m) and v has exactly one free neighbor. If Alice does not assign m (resp. 0) to v's unique neighbor, then Bob can label it with a label $\ell \in \{1, 2, \ldots, m-1\}$. Similarly, for the case where there are two free vertices in G not adjacent to v, if Alice assigns $\ell \neq m$ (resp. 0) on any vertex, then Bob assigns m (resp. 0) to a vertex not adjacent to v. In both cases, Bob wins the game. □

Lemma 3. *If a graph* G *has two vertices* u *and* v *of degree* 1 *such that* $d(u, v) \geq 3$*, then Bob wins the graceful game on* G *when he starts.*

Proof. Bob starts the game on G by assigning label 0 to u, thus forcing Alice to assign label m to u's unique neighbor, to generate edge label m. So, Bob wins the game by assigning label 1 to v. □

Lemma 4. *If a graph* G *has two vertices* u *and* v *of degree* 1 *such that* $d(u, v) \geq 4$*, then Bob wins the graceful game on* G *no matter who starts.*

Proof. Suppose that Alice starts the game. By Lemma 1, Alice starts by assigning label $\ell \in \{2, \ldots, m-1\}$ to an arbitrary vertex $x \in V(G)$ (the case $\ell = 1$ is analogous to $\ell = m-1$ by complementary labeling). In the next move, Bob assigns label 0 to an unlabeled pendant vertex in the set $\{u, v\}$, that is farthest from x, say u, which forces Alice to assign label m to the neighbor of u so as to generate the edge label m. Bob assigns label 1 to v if it is unlabeled, or to its neighbor otherwise. Thus, Bob wins the game since it is not possible for Alice to generate the edge label $m-1$.

FIGURE 2. Graceful labelings of P_4.

Now, suppose that Bob starts the game by assigning label 0 to u. Alice is forced to assign label m to the neighbor of u so as to generate the edge label m. On the third move, Bob assigns label 1 to vertex v, thus, making it impossible for Alice to generate the edge label $m-1$. □

3. Main results

Next, we present our main results on the graceful game for several graph classes.

3.1. Paths

A *path graph* P_n is a connected graph on n vertices whose vertices can be arranged in a linear sequence $(v_0, v_1, \ldots, v_{n-1})$ in such a way that two vertices are adjacent if and only if they are consecutive in the linear sequence. Rosa [17] proved that all paths are graceful. In order to prove the proposition left by Tuza ([22], Prop. 2.5) and to cover the scenario of the graceful game on paths, Theorem 5 characterizes the graceful game for all paths.

Theorem 5. *Bob has a winning strategy for the graceful game on any path P_n, with $n \geq 4$, no matter who starts. Alice wins the graceful game on P_1 and P_2; and on P_3 the winner is the player who starts the game.*

Proof. Alice always wins on P_1 and P_2 since there is only one way of labeling these graphs (up to complementary labeling) and the obtained labeling is graceful. For $P_3 = (v_0, v_1, v_2)$, if Bob starts the game by assigning labels 0 or 2 to v_1, then Alice wins because it is always possible to create a graceful labeling of P_3. So, he labels v_1 with 1 and wins the game, since it is impossible to obtain the edge label 2. If Alice starts, then she labels v_0 with 1, winning the game. For $P_4 = (v_0, v_1, v_2, v_3)$, there are only two graceful labelings (illustrated at Fig. 2): $(1, 2, 0, 3)$ and $(2, 1, 3, 0)$ (they are complementary). If Alice starts and labels v_0 (resp. v_1) with $\ell \in \{1, 2\}$ (Lem. 1), then Bob wins by assigning 0 to v_1 (resp. v_0). If Bob starts the game, he assigns label 0 to v_3. Alice is forced to assign label 3 to v_2 and Bob wins by assigning label 1 to v_0. For P_n with $n \geq 5$, the result follows from Lemma 4. □

3.2. Complete graphs

A *complete graph* K_n is the simple graph with n vertices in which every pair of distinct vertices is connected by one edge. Golomb [8] proved that the complete graph K_n is graceful if and only if $n \leq 4$ [8]. The cases $n \in \{1, 2\}$ are treated on Theorem 5. In order to establish the remaining cases for K_n, we next prove the proposition left by Tuza ([22], Prop. 2.6).

Theorem 6 (Tuza [22])**.** *Alice wins the graceful game on K_3, and Bob wins on K_4, no matter who starts.*

Proof. Any graceful labeling of K_3 has both labels 0 and 3. If Bob starts, he assigns label 1 to a vertex. Alice is forced to assign label 0 or 3 to any vertex. If she assigns label 0 (resp. 3), then she obtains the edge label 1 (resp. 2). Next, Bob is forced to assign label 3 (resp. 0) to the last vertex (label 2 creates repeated edge label 1), and Alice wins. On K_4, no matter who starts, Bob assigns label 3 on his turn and wins since it is not possible to assign both 0 and 6 anymore. □

FIGURE 3. The graceful labelings of C_4. One is the complementary labeling of the other.

3.3. Cycles

The next class of graphs considered in this work are the cycles. A *cycle graph* C_n, with $n \geq 3$ vertices, is a connected simple graph such that all of its vertices can be arranged in a cyclic sequence $(v_0, v_1, \ldots, v_{n-1})$ such that two vertices are adjacent if and only if they are consecutive in the sequence. Rosa [17] proved that the cycle graph C_n is graceful if and only if $n \equiv 0, 3 \pmod 4$. Therefore, it is immediate that Bob is the winner when $n \not\equiv 0, 3 \pmod 4$.

Theorem 7. *Bob wins the graceful game on all C_n with $n \geq 4$, and Alice wins the graceful game on C_3.*

Proof. For $C_3 \cong K_3$, the result follows by Theorem 6. Let $C_4 = (v_0, v_1, v_2, v_3)$. By inspection, C_4 has only two graceful labelings: $(0, 4, 1, 2)$ and $(4, 0, 3, 2)$ (note that none of these graceful labelings has both labels 1 and 3 Fig. 3).

Thus, if Alice starts, then she is forced to label a vertex v with 2, because if she labels v with 1 (resp. 3), then Bob labels a neighbor of v with 3 (resp. 1) and she loses the game (it will always create repeated edge labels). Now, Bob labels the vertex $u \notin N(v)$ with 1 or 3, winning the game, since it is impossible to obtain the edge label m. Now, suppose that Bob starts the game assigning label 0 to v_1. By Lemma 1, Alice labels vertex v_2 (resp. v_0) with 4, creating the edge label 4. On the third move, Bob labels the vertex v_0 (resp. v_2) with 1, winning the game, since it is impossible to obtain the edge label 3. Let $C_n = (v_0, v_1, \ldots, v_{n-1})$, with $n > 4$ and $n \equiv 0, 3 \pmod 4$. If Alice starts, then w.l.o.g. she labels vertex v_0 with $\ell \in \{1, \ldots, m-1\}$ (Lem. 1). If $\ell \neq 1$ then Bob labels v_3 with 0, forcing Alice to label v_4 (resp. v_2) with m. Thus, Bob labels v_2 (resp. v_4) with 1, preventing Alice from creating edge label $m-1$, and wins the game. If $\ell = 1$ then Bob labels v_1 with 0, forcing Alice to label v_2 with m, and Bob wins the game. If Bob starts, then he assigns label 0 to vertex v_1. Alice is forced to assign label m to a neighbor of v_1, say v_2 (resp. v_0). Bob wins by assigning label 1 to v_0 (resp. v_2), preventing the creation of edge label $m-1$. □

3.4. Complete bipartite graphs

Next, we analyze the graceful game for complete bipartite graphs. A *bipartite graph* is a simple graph $G = (V(G), E(G))$ such that there exists a partition $P = (X, Y)$ of $V(G)$ such that every edge of $E(G)$ connects a vertex in X to a vertex in Y. A *complete bipartite graph* $K_{p,q}$ is a simple bipartite graph with bipartition (X, Y), in which each vertex of X is joined to every vertex of Y, with $p = |X|$ and $q = |Y|$. Note that $|V(K_{p,q})| = p + q$ and $|E(K_{p,q})| = pq$. All complete bipartite graphs are graceful [17].

Theorem 8. *Alice wins the graceful game on $K_{1,q}$ if and only if she starts.*

Proof. For $K_{1,0}$, $K_{1,1}$ and $K_{1,2}$, the result follows from Theorem 5. Now, let $K_{p,q}$ with $p = |X| = 1$ and $q = |Y| = m \geq 3$. First, if Alice starts the game, she assigns label 0 to the unique vertex in X, and wins the game since it is possible to generate different edge labels. If Bob starts, then he labels the unique vertex in X with $\ell \notin \{0, m\}$ and wins since he prevents the edge label m. □

Theorem 9. *Bob has a winning strategy for the graceful game on a complete bipartite graph $K_{p,q}$, for all $p, q \geq 2$.*

Proof. Let $K_{p,q}$, with $p, q \geq 2$, and bipartition $\{X, Y\}$. For $K_{2,2}$, the result follows from Theorem 7. Let $p = |X| = 2$ and $q = |Y| \geq 3$. If Bob starts, he labels a vertex in Y with 0. Alice is forced to assign label m to a vertex in X. Bob assigns label 1 to the last free vertex in X, and wins since he prevents the edge label $m-1$ from being created. If Alice starts and labels $v \in Y$ with $\ell \in \{1, \ldots, m-1\}$, then Bob assigns label 0 to a vertex in X. Alice is forced to label a vertex in Y with m. If $\ell = 1$ (resp. $m-1$), then Bob assigns label $m-2$ (resp. 2) to the unique free vertex $u \in X$ and he wins the game. If $\ell \neq 1$, then Bob assigns label $m-1$ to the last free vertex in X and wins, since he prevents the edge label $m-1$. Now, suppose that Alice starts by labeling $v \in X$ with $\ell \in \{1, \ldots, m-1\}$. If $\ell = 1$ (resp. $m-1$), then Bob assigns label 0 (resp. m) to a vertex in Y in the second move. Alice is now forced to label a vertex in X with m (resp. 0). Alice loses the game since the edge label $m-1$ cannot be created. If $2 \leq \ell \leq m-2$, then, in the second move, Bob labels the last free vertex in X with $\ell - 1$ and wins the game since the edge label m cannot be created.

Now, consider $|Y| = q \geq |X| = p \geq 3$. First, suppose that Alice starts. Without loss of generality, we assume that she labels $v \in X$ with $\ell \in \{1, \ldots, m-1\}$ (Lem. 1). So, Bob assigns label 0 to a vertex in Y, thus forcing Alice to assign label m to a vertex in X.

Next, we consider two cases according to the value of label ℓ.

Case 1. $x < \ell < m - x$, where $x = \lfloor (p-1)/2 \rfloor$. The game proceeds with Bob assigning label α to a vertex of X and, consequently, forcing Alice to assign label $m - \alpha$ to another vertex of X, where α is chosen in ascending order from the set $\{1, 2, \ldots, \lceil (p-2)/2 \rceil\}$. This pattern goes on until the vertices in X are exhausted. If p is odd, then Alice can no longer create the edge label $m - x$. If p is even, then Alice labeled the last free vertex in X with label $m - x$; the only way to create the edge label $m - (x+1)$ is by labeling a vertex in Y with $x + 1$ (if $x + 1$ was not already used by Alice in her first move). However, this move would create a repeated edge label $x = |(x+1) - 1|$. Therefore, in any case, Alice loses the game.

Case 2. $1 \leq \ell \leq x$, where $x = \lfloor (p-1)/2 \rfloor$. Similar to the previous case, the players take turns by assigning labels α (Bob) and $m - \alpha$ (Alice) to vertices of X, where $\alpha \in \{1, 2, \ldots, \lceil (p-2)/2 \rceil\}$, until the game reaches the point where Alice assigns label $m - (\ell - 1)$ to a vertex in X. Bob's next move would be to label a vertex in X with ℓ, but this was Alice's first move in the game. Now, in order to prevent Alice from labeling a vertex in X with $m - \ell$, Bob assigns $m - 2\ell$ to a vertex in Y. This way, if Alice tries to label a vertex in X with $m - \ell$, she creates a second edge label ℓ. Therefore, Bob wins the game since Alice cannot create the edge label $m - \ell$. The remaining cases where $m - x \leq \ell \leq m - 1$ are analogous by complementary labeling.

If Bob starts, then he assigns label 0 to a vertex of Y, which forces Alice to assign label m to a vertex of X. From now on, the players take turns by assigning labels α (Bob) and $m - \alpha$ (Alice) to vertices of X, where $\alpha \in \{1, 2, \ldots, \lfloor p/2 \rfloor\}$, until the vertices of X are exhausted. If $p = |X|$ is even, Bob is the last player to label a vertex in X and he uses $z = p/2$. In this case, Alice can no longer create the edge label $m - z$, therefore losing the game. If p is odd, then Alice labels the last vertex in X with $m - z$, with $z = \lfloor p/2 \rfloor$. The next edge label she must guarantee the existence is $m - (z+1)$. The only way to create the edge label $m - (z+1)$ is by labeling a vertex in Y with $z + 1$. However, this move creates a repeated edge label $z = |(z+1) - 1|$. Therefore, Alice loses the game. □

3.5. Trees and caterpillars

A *caterpillar* $cat(k_0, k_1, \ldots, k_{s-1})$, with $s \geq 1$ and $k_0, k_1, \ldots, k_{s-1} \geq 0$, is a special tree obtained from a path $P_s = (v_0, v_1, \ldots, v_{s-1})$, called *spine*, by joining k_j leaf vertices to v_j, for $j \in \{0, \ldots, s-1\}$. Rosa [17] proved that all caterpillars are graceful. Note that a caterpillar G with m edges and diameter at most two is isomorphic to $K_{1,m}$ and, by Theorem 8, Alice wins the graceful game on G if and only if she is the first player. Theorem 10 settles the remaining cases.

Theorem 10. *Bob has a winning strategy for the graceful game on all caterpillars with diameter at least three.*

Proof. Let G be a caterpillar with diameter at least three. If $G \cong P_n$ or G has diameter greater than 3, then the result follows from Theorem 5 or Lemma 4, respectively. Thus, consider $G \ncong P_n$ and G with diameter

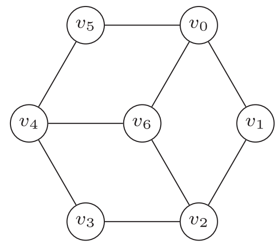

FIGURE 4. Gear graph G_3.

equal to 3. In this case, $G \cong cat(k_0, k_1)$ with $k_0 > 1$ and $k_1 \geq 1$. By Lemma 3, Bob wins the game when he starts since G has at least two vertices of degree 1 with distance 3 apart. Now, suppose that Alice is the first player and labels an arbitrary vertex v with $\ell \in \{1, \ldots, m-1\}$. If $v = v_0$ (resp. v_1), then Bob wins the game labeling vertex v_1 (resp. v_0) with $\ell' \notin \{0, \ell, m\}$. In both cases, Bob prevents Alice from creating edge label m. Let $v \in N(v_0) \setminus \{v_1\}$ be a leaf. Bob labels a leaf $u \in N(v_1) \setminus \{v_0\}$ with 0. Alice is forced to label v_1 with m. If $\ell = 1$ then Bob prevents Alice from creating edge label $m-1$. If $\ell \neq 1$ then again Bob wins by labeling a leaf in $N(v_0)$ with 1. Let $v \in N(v_1) \setminus \{v_0\}$ be a leaf. If $k_1 = 1$ then Bob labels v_0 with $\ell' \notin \{0, m, \ell\}$ and wins since it is not possible to generate label m. Let $k_1 \geq 2$. If $\ell \notin \{1, m-1\}$ then Bob labels v_0 with $m-1$. Now, Alice is forced to assign label 0 to v_1. Bob wins the game assigning label m to $N(v_0) \setminus \{v_1\}$. If $\ell = 1$ (resp. $m-1$) then Bob labels v_1 with 2 (resp $m-2$). Alice is forced to assign label 0 to v_0. Bob wins the game assigning label m to $N(v_1) \setminus \{v_0\}$. □

Corollary 11 follows from Theorem 10 along with the fact that all trees with diameter 3 are caterpillars and from Lemma 4 along with the fact that all trees have at least two leaves.

Corollary 11. *Bob has a winning strategy for the graceful game on all trees with diameter at least three.*

3.6. Gear graphs

A *gear graph* G_n, with $n \geq 3$, is a graph obtained from a cycle with $2n$ vertices $(v_0, v_1, \ldots, v_{2n-1})$ by adding a new vertex v_{2n}, called *central vertex*, and linking it to every vertex v_{2i}, for $0 \leq i \leq n-1$. Figure 4 exhibits the gear graph G_3.

Theorem 12. *Bob has a winning strategy for the graceful game on all gear graphs.*

Proof. Let G_n be a gear graph, $n \geq 3$. Bob starts the game by assigning label 0 to vertex v_1, of degree 2, forcing Alice to assign label m to v_0 or v_2 to create the edge label m. In the next move, Bob assigns label 1 to the free neighbor of v_1, thus winning the game since the edge label $m-1$ cannot be obtained. Now, consider that Alice starts the game. By complementary labeling and by Lemma 1, we assume that Alice assigns a label $\ell \in \{1, \ldots, \lfloor m/2 \rfloor\}$ to an arbitrary vertex u of G_n. There are three cases to analyze, depending on the choice of u.

Case 1. $u = v_{2i+1}$, for $0 \leq i \leq n-1$. In his first turn, Bob assigns label 0 to vertex v_{2i+2} (indices taken modulo n), forcing Alice to assign label m to v_{2n} or v_{2i+3}. If Alice assigns m to v_{2n} (resp. v_{2i+3}), then Bob labels v_{2i+3} (resp. v_{2n}) with $m-2$, if $\ell = 1$, or with 1 otherwise. Bob wins in both cases since it is not possible to create the edge label $m-1$.

Case 2. $u = v_{2i}$, for $0 \leq i \leq n-1$. In his first turn, Bob assigns label 0 to v_{2i+1}, forcing Alice to assign label m to v_{2i+2}. If $\ell = 1$, then Bob wins the game since it is not possible to create the edge label $m-1$; otherwise, Bob assigns label 1 to vertex v_{2i+4}, also winning the game.

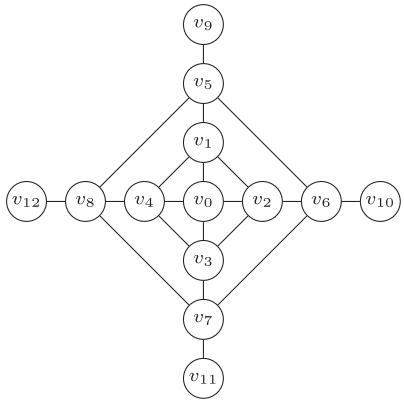

FIGURE 5. Web graph $W(2,4)$.

Case 3. $u = v_{2n}$. Now, Bob assigns label 0 to v_2, which forces Alice to assign label m to v_1 or v_3, say v_3. Next, Bob assigns label 1 to v_1 if $\ell \neq 1$; or he assigns label $m-1$ to v_5 if $\ell = 1$. In both cases, Bob wins the game since it is not possible to create the edge label $m-1$ anymore.

□

3.7. Web graphs

Let $C_n^i = (v_{(i-1)n+1}, v_{(i-1)n+2}, \ldots, v_{(i-1)n+n})$ be a cycle on n vertices such that $i \in \mathbb{N}$. In [6], Gallian defined a *web graph* $W(t,n)$ as a graph formed by t vertex-disjoint n-cycles $C_n^1, \ldots, C_n^t$, where $t \geq 1$ and $n \geq 3$, and more $n+1$ vertices v_0, v_{tn+1}, v_{tn+2}, …, v_{tn+n} such that these last vertices and the vertices of the t n-cycles are linked as follows: (i) v_0 is adjacent to the vertices of the cycle C_n^1; and (ii) for $1 \leq i \leq t$ and $1 \leq j \leq n$, each vertex $v_{(i-1)n+j}$ is adjacent to vertex v_{in+j}. Figure 5 exhibits the web graph $W(2,4)$.

Lemma 13. *Given a web graph $W(t,n)$ with m edges, Alice can label the cycle vertex $v_{(t-1)n+j}$, $1 \leq j \leq n$, in only two cases: when $v_{(t-1)n+j}$'s respective pendent vertex is already labeled or when the labels 0 or m (or both) have already been assigned to a vertex.*

Proof. Suppose that is Alice's turn and the $v_{(t-1)n+j}$'s respective pendent vertex v_{tn+j} has not yet been labeled, and none of the labels 0 and m were assigned to a vertex. Since she cannot assign labels 0 and m (Lem. 1), she labels $v_{(t-1)n+j}$ with an arbitrary label $\ell \in \{1, \ldots, m-1\}$. Now, Bob labels v_{tn+j} with 0 or m, winning the game. □

Theorem 14. *Bob has a winning strategy for the graceful game on all web graphs.*

Proof. The result for every web graph $W(t,n)$, where $n \geq 4$ and $t \geq 1$, follows from Lemma 4 since the vertices $v_{tn+1}, v_{tn+3} \in V(W(t,n))$ both have degree 1 and $d(v_{tn+1}, v_{tn+3}) = 4$. Thus, consider the web graph $W(t,3)$ with $t \geq 1$. By Lemma 3, Bob wins the game on $W(t,3)$ when he starts since $W(t,3)$ has two vertices of degree 1 at distance 3 apart (v_{tn+1} and v_{tn+2}).

Now, suppose that Alice starts the game. We divide the proof into two cases depending on the value of t.

Case 1. $t \geq 2$. By Lemma 13 and by the symmetry of the graph, Alice can choose three types of vertices to label on her first turn: either the center v_0, or a pendent vertex v_{3t+j}, with $j \in \{1, \ldots, 3\}$, or a remaining vertex v_k not adjacent to a pendent vertex. Once Alice makes her choice, she labels the chosen vertex with an arbitrary label $\ell \in \{1, \ldots, m-1\}$. Next, Bob assigns label 0 to a free pendent vertex v_{3t+p}, $p \in \{1,2,3\}$ so that v_{3t+p} is at distance at least three from the vertex labeled by Alice on her last turn. Bob's last move forces Alice to label $v_{3(t-1)+p}$ with m. If $\ell = 1$, then the game is over and Bob is the winner since it is not possible to create the edge label $m-1$. If $\ell \neq 1$, Bob labels any vertex not adjacent to $v_{3(t-1)+p}$ with 1 in order to win.

Case 2. $t = 1$. By Lemma 13, she has only two options on her first turn: to label vertex v_0 or a pendent vertex v_{3+j}, $1 \leq j \leq 3$. First, suppose she labels v_0. By complementary labeling and Lemma 1, we may suppose that Alice labels v_0 with a arbitrary label ℓ, with $1 \leq \ell \leq 4$. Thus, Bob assigns label $m = 9$ to the pendent vertex v_4. Alice is now forced to label v_1 with 0. Bob wins by assigning label $m-1$ to another pendent vertex, since it is not possible to create the edge label $m-1$. Next, suppose that Alice chooses to label a pendent vertex v_{3+j}, $1 \leq j \leq 3$, with an arbitrary label ℓ, $1 \leq \ell \leq 4$. So, Bob assigns 0 to other pendent vertex v_{3+p}, $p \neq j$. This way, Alice is forced to label v_p with m. If $\ell \neq 1$, Bob assigns label 1 to the remaining pendent vertex. In this case and when $\ell = 1$, Bob wins since it is not possible to create the edge label $m-1$.

□

3.8. Prisms

The *Cartesian product* $G_1 \square G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ such that $(u_1, v_1)(u_2, v_2) \in E(G_1 \square G_2)$ if and only if either (1) $u_1 u_2 \in E(G_1)$ and $v_1 = v_2$, or (2) $v_1 v_2 \in E(G_2)$ and $u_1 = u_2$. The *prism* graph $P_{r,2}$, $r \geq 3$, is defined as the Cartesian product $C_r \square P_2$ of a cycle on r vertices and path P_2.

Theorem 15. *Bob has a winning strategy for the graceful game on all prisms $P_{r,2}$, with $r \geq 3$.*

Proof. Let $P_{r,2}$ be a prism with m edges and vertex set $V(P_{r,2}) = \{v_{p,q} \colon 0 \leq p \leq r-1, 0 \leq q \leq 1\}$, where all vertices $v_{p,q}$ with the same index q induce a cycle C_r. W.l.o.g., by symmetry, Alice starts the game by assigning label $\ell \in \{1, 2, \ldots, m-1\}$ to vertex $v_{0,0}$ of $P_{r,2}$ (Lem. 1). Let $r \geq 4$ and $1 \leq \ell \leq m-2$ (the case $\ell = m-1$ is analogous to the case $\ell = 1$ by complementary labeling). Bob assigns label 0 to a neighbor w of $v_{0,0}$, forcing Alice to assign label m to a free neighbor u of w to generate edge label m. If $\ell \neq 1$, then Bob assigns label 1 to the remaining free neighbor of w; otherwise, he assigns label $m-1$ to a remaining free neighbor of $v_{0,0}$ that is not adjacent to u, generating edge label $m-2$. In both cases, Bob wins since the edge label $m-1$ cannot be created. Now, let $r = 3$. By complementary labeling and symmetry, consider that Alice starts the game on $P_{3,2}$ by assigning label $\ell \in \{1, 2, 3, 4\}$ to vertex $v_{0,0}$. We analyze two cases:

Case 1. $3 \leq \ell \leq 4$. Bob is the next player and assigns label 0 to $v_{1,0}$, forcing Alice to assign label m to a neighbor of $v_{1,0}$, either $v_{2,0}$ or $v_{1,1}$. If Alice chooses $v_{2,0}$ (resp. $v_{1,1}$), then Bob assigns label 1 to vertex $v_{1,1}$ (resp. $v_{2,0}$).

Case 2. $1 \leq \ell \leq 2$. Define $\ell^* = m-1$ if $\ell = 1$, or $\ell^* = 1$ if $\ell = 2$. In this case, Bob assigns label 0 to $v_{0,1}$, which forces Alice to assign label m to either $v_{1,1}$ or $v_{2,1}$. If Alice chooses $v_{1,1}$ (resp. $v_{2,1}$), then Bob assigns label ℓ^* to vertex $v_{2,0}$ (resp. $v_{1,0}$) in his next move. In any case, Bob prevents the edge label $m-1$.

Now, suppose that Bob starts the game and assigns label 0 to vertex $v_{0,0}$. Let $r \geq 4$. On the second move, Alice is forced to assign label m to a neighbor u of $v_{0,0}$ to create the edge label m. Bob assigns label 1 to a free neighbor of $v_{0,0}$, creating the edge label 1. Alice assigns label $m-1$ to the last free neighbor of $v_{0,0}$, creating edge label $m-1$. Bob assigns label 2 to a vertex u' such that $d(u', u) = 2$, and wins since it is not possible to create the edge label $m-2$. Now, let $r = 3$. In the second move, Alice is forced to assign label m to a neighbor u of $v_{0,0}$.

Case 1. Alice chooses vertex $u = v_{1,0}$. Bob assigns label 1 to $v_{0,1}$, creating the edge label 1. Bob wins the game because it is not possible to create the edge label $m-1$ without creating a repeated edge label 1.

Case 2. Alice chooses vertex $u = v_{0,1}$. On the third move Bob assigns label 1 to $v_{2,0}$, creating the edge label 1. Alice assigns label $m-1$ to $v_{1,0}$, creating edge labels $m-2$ and $m-1$. In the last move Bob assigns label $m-3$ to $v_{1,1}$, creating edge labels 2 and 3 and preventing the edge label $m-3$.

□

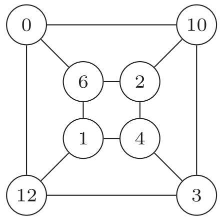

FIGURE 6. A graceful labeling of Q_3.

3.9. Hypercubes

A *hypercube* Q_n is defined recursively in terms of the Cartesian product of two graphs as follows: $Q_1 = K_2$; $Q_n = Q_{n-1}\Box K_2$ for $n \geq 2$. In the next result, we characterize the graceful game for hypercubes (Fig. 6).

Theorem 16. *Bob has a winning strategy for the graceful game on all hypercubes Q_n with $n \geq 2$.*

Proof. Let Q_n be a hypercube with vertex set $V(Q_n) = \{v_1, v_2, \ldots, v_{2^n}\}$, $n \geq 2$, and m edges. Since $Q_2 \cong C_4$, the result for Q_2 follows from Theorem 7. For $n = 3$, the result follows from Theorem 15 since $Q_3 \cong P_{r,2}$. Thus, consider Q_n with $n \geq 4$. Since Q_n is bipartite, we partition $V(Q_n)$ into two disjoint sets X and Y such that $X = \{v_j \colon 1 \leq j \leq 2^{n-1}\}$ and $Y = \{v_j \colon 2^{n-1} + 1 \leq j \leq 2^n\}$. If Alice starts, then w.l.o.g. she labels $v_1 \in X$ with label $\ell \in \{1, \ldots, m-1\}$ (Lem. 1). We consider four different cases depending on the value of ℓ.

Case 1. $\ell = 1$ (resp. $\ell = m-1$). On the next turn, Bob labels a vertex $w \in Y$ that is not adjacent to v_1 with 0 (resp. m). Thus, Alice is forced to label a vertex $u \in X$ adjacent to w with m (resp. 0). Now, Bob labels any vertex in Y with $m-1$ (resp. 1) exhausting Alice's possibilities of creating the edge label $m-1$.

Case 2. $2 \leq \ell \leq \left\lfloor \frac{n}{2} \right\rfloor$. In this case, Bob assigns label 0 to a vertex $w \in Y$ adjacent to v_1. Thus, Alice assigns label m to a vertex adjacent to w. Without loss of generality, suppose that all v_{1+j}, for $1 \leq j \leq n$, are adjacent to w. By the symmetry of the hypercube, we can also suppose that Alice chooses v_2 to label on her second turn. Now, Bob assigns label 1 to a vertex adjacent to w, say v_3. Alice is forced to assign label $m-1$ to a vertex adjacent to w, say v_4. Note that this process repeats until the point where Alice labels a neighbor of w with $m-(\ell-1)$. Now, it is sufficient for Bob to label any vertex in Y with $m-\ell$. This way, Alice cannot create the edge label $m-\ell$, losing the game.

Case 3. $\left\lfloor \frac{n}{2} \right\rfloor < \ell < m - \left\lfloor \frac{n}{2} \right\rfloor$. When n is odd, Bob assigns label 0 to a vertex $w \in Y$ adjacent to v_1. On the other hand, when n is even, Bob assigns label 0 to a vertex $w \in Y$ that is not adjacent to v_1. So, in the next turn, Alice is forced to assign label m to a free neighbor of w to generate edge label m. From now on, Bob applies the same strategy he applied in Case 2: in his next turn, Bob assigns label 1 to a free neighbor of w, which forces Alice to assign label $m-1$ to another free neighbor of w to generate edge label $m-1$, and so on, until the game reaches the point where Bob assigns label $\lfloor n/2 \rfloor$ to the last free neighbor of w. Bob's last move precludes Alice from obtaining the edge label $m - \lfloor n/2 \rfloor$.

Case 4. $m - \left\lfloor \frac{n}{2} \right\rfloor \leq \ell \leq m-2$. Analogous to Case 2 by complementary labeling. When Bob is the first player, he starts by assigning 0 to v_1. Alice is forced to assign label m to a neighbor of v_1. w.l.o.g., suppose that $N(v_1) = \{v_{2^{n-1}+j} \colon 1 \leq j \leq n\}$. By the symmetry of the hypercube, suppose w.l.o.g. that Alice chooses $v_{2^{n-1}+1}$ to label. In his next turn, Bob assigns label 1 to a free neighbor of v_1, which forces Alice to assign label $m-1$ to another free neighbor of v_1 to generate edge label $m-1$, and so on. If n is even, then Bob guarantees his victory since the last free neighbor of v_1 is assigned label $\frac{n}{2}$ by Bob, making it impossible to create edge label $m - \frac{n}{2}$. If n is odd, then the last free neighbor of v_1 is assigned label $m - \lfloor n/2 \rfloor$ by Alice. Now Alice tries to create edge label $m - (\lfloor n/2 \rfloor + 1)$ which can be generated by the pair of values i and $m - (\lfloor n/2 \rfloor + 1) + i$, for $0 \leq i \leq \lfloor n/2 \rfloor + 1$. However, the only pair of labels that is not labeling vertices on the same part is m and $\lfloor n/2 \rfloor + 1$. Bob wins labeling a vertex that is not adjacent to $v_{2^{n-1}+1}$ with $\lfloor n/2 \rfloor + 1$.

□

3.10. 2nd power P_n^2 of a path P_n

The *2nd power of a path* P_n, $n \geq 1$, is the graph P_n^2 that has vertex set $V(P_n^2) = V(P_n)$, with distinct vertices u, v being adjacent in P_n^2 if and only if $d_{P_n}(u,v) \leq 2$. It is known that 2-powers of all paths are graceful [14].

Theorem 17. *Bob has a winning strategy for the graceful game on all P_n^2 with $n \geq 4$. Alice wins on P_3^2.*

Proof. For $n \leq 3$, the result follows from Theorems 5 and 7. Let $n \geq 4$. By the symmetry of $P_n^2 = (v_0, v_1, \ldots, v_{n-1})$, starting by labeling v_j is analogous to start by labeling v_{n-j-1}. So, assume that Alice starts playing on v_j, with $j \in \{\lfloor n/2 \rfloor, \ldots, n-1\}$.

Case 1. P_n^2, $n = 4, 5$. Suppose that Alice labels vertex v_3 (resp. v_2), with an arbitrary label $\ell \in \{1, \ldots, m-1\}$. Bob labels vertex v_0 (resp. v_1) with label $m - \ell$. Now, in order to have a graceful labeling, two adjacent free vertices must be labeled 0 and m. However, this move generates repeated edge labels, for example, $|f(v_0) - f(v_2)| = |f(v_1) - f(v_3)|$ or $|f(v_0) - f(v_2)| = |f(v_3) - f(v_4)|$. If Alice labels vertex v_2 with 0, then Bob labels vertex v_1 with $m-1$. Now, Alice labels v_0 (resp. v_3 or v_4, if $n = 5$) with m. If $n = 5$, Bob labels v_4 (resp. v_0) with $m-3$. In any case, Bob is the winner, since it is not possible to generate label $m-2$ without creating repeated edge labels by the differences in module $|(m-3) - (m-1)| = 2$ e $|(m-2) - m| = 2$.

Case 2. P_n^2 with $n \geq 6$. Suppose that Alice labels v_j with an arbitrary label $\ell \in \{1, \ldots, m-1\}$ (Lem. 1). We consider three subcases:

Case 2.1. $j \geq 5$. First, suppose that $\ell \in \{\frac{m+1}{2}, 1\}$. Bob labels v_0 with m. Alice assigns 0 to either v_1 or v_2. If she chooses v_1, then Bob labels v_2 with $\frac{m-1}{2}$ and wins the game since it is not possible to generate the edge label $m-1$ satisfying the game's rules. If Alice chooses v_2 on the third move, then Bob labels v_3 with $\frac{m-1}{2}$, making it impossible for Alice to assign label $m-1$ to v_4 in order to create the edge label $m-1$. Moreover, if $\ell = 1$, then Alice cannot use label 1 to label v_1. Therefore, we may suppose that $\ell = \frac{m+1}{2}$. So, Alice labels v_1 with 1, creating the edge label $\frac{m-3}{2}$ on v_1v_3. At this point, the only way to create the edge label $m-2$ is by assigning $m-2$ to v_4, thus inducing the repeated edge label $\frac{m-3}{2} = |f(v_3) - f(v_4)|$. Hence, Bob is the winner. Now, suppose that $\ell \neq \frac{m+1}{2}$ and $\ell \neq 1$. Bob labels v_0 with 0. Alice assigns m to either v_1 or v_2. If she chooses v_1, then Bob labels v_2 with $\frac{m+1}{2}$ and wins since it is not possible to create the edge label $m-1$ without repeating the edge label $\frac{m-1}{2}$. If Alice assigns m to v_2 on the third move, then Bob labels v_3 with $\frac{m+1}{2}$, making it impossible for Alice to label v_4 or v_1 with 1 in order to create the edge label $m-1$. If $\ell \in \{\frac{m+1}{2} - 1, \frac{m+1}{2} + 1, m-1\}$, then Alice cannot label v_1 with $m-1$ and Bob wins the game; otherwise, she is forced to label v_1 with $m-1$, creating the edge label $m-1$ on v_0v_1 and $\frac{m-3}{2}$ on v_1v_3. At this point, it is not possible to create the edge label $m-2$ without repeating the edge label $\frac{m-3}{2}$ on the edge v_3v_4. Hence, Bob wins again.

Case 2.2. $j = 4$. If $\ell \in \{3, 4, \ldots, m-2\}$ (resp. $\ell = 2$), then Bob labels v_0 with 0. Alice is thus forced to label v_1 or v_2 with m. Independently of her choice, Bob labels v_3 (resp. v_5) with $\ell - 1$, creating edge label 1 on v_3v_4 (resp. v_4v_5). From now on, any attempt to obtain edge label $m-1$ generates a repeated edge label 1. Therefore, Bob wins. On the other hand, if $\ell = 1$ (resp. $\ell = m-1$), then Bob labels v_0 with m (resp. 0). Alice is now forced to label v_1 or v_2 with 0 (resp. m). If she chooses v_1, then Bob labels v_3 with 2 (resp. $m-2$); otherwise, he labels v_3 with 4 (resp. $m-4$). Bob wins the game since Alice cannot label v_1 or v_2 with $m-1$ (resp. 1) without creating repeated edge labels.

Case 2.3. $j = 3$. In this case, we have P_n^2 with $6 \leq n \leq 7$. If $\ell = 1$ (resp. m), Bob labels v_0 with m (resp. 0), forcing Alice to label either v_1 or v_2 with 0 (resp. m). In any case, Bob labels v_5 with $m-1$ (resp. 1) guaranteeing that Alice cannot create the edge label $m-1$. If $\ell \notin \{1, m-1\}$, then Bob labels v_0 with 0. Thus, Alice is forced to label v_1 (resp. v_2) with m. It can be checked by inspection that, for each value of ℓ, there is at least one label $k \in \{2, \ldots, m-2\}$ that Bob can assign to v_2 (resp. v_1) preventing Alice from creating the edge label $m-1$. Now, consider that Bob is the first player. He starts by assigning label 0 to v_0, giving Alice the options of labeling v_1 or v_2 with m. First, suppose that Alice chooses v_1 in her first move. So, Bob labels v_3 with $m-1$ on his next move, creating the edge label 1. Bob wins the game since it is not possible to create the edge label $m-1$ without generating the repeated edge

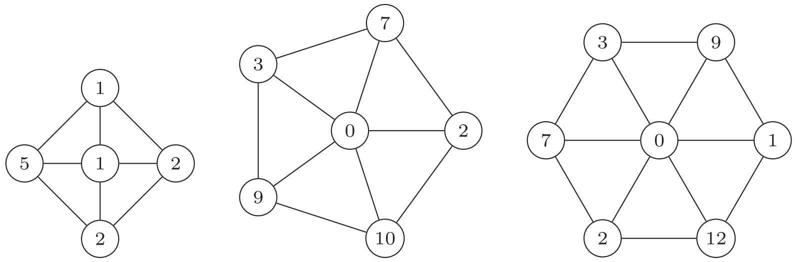

Figure 7. Wheels with graceful labelings.

label 1. Now, suppose that Alice chooses v_2 for her first move. Thus, Bob labels v_3 with $\frac{m+1}{2}$ (m is odd for all n)[1]. This way, Alice cannot label either v_1 or v_4 with 1 since it would generate a repeated edge label $\frac{m-1}{2}$. Thus, her only option to create the edge label $m-1$ is to label v_1 with $m-1$[2]. With the edge labels m and $m-1$ guaranteed, Bob prevents Alice from creating the edge label $m-2$ by labeling v_4 with any label other than 2.

□

3.11. Wheels

A *wheel* W_n is a graph formed by connecting a single vertex v_n to all vertices $v_0, v_1, \ldots, v_{n-1}$ of a cycle C_n, where $n \geq 3$. By the definition, a wheel W_n has $n+1$ vertices and $2n$ edges. In this work, the vertex v_n, which is adjacent to all the other vertices of W_n, is called the *central vertex* of W_n. Figure 7 shows wheel graphs with graceful labelings.

Hoede and Kuiper [13] and Frucht [5] proved that all wheels are graceful. A useful fact about graceful labelings of wheels is that there exist no graceful labeling f of W_n that assigns the label n to its central vertex. Note that if such a labeling existed, the labels 0 and $2n$ should be assigned to two adjacent vertices v_j and v_{j+1} (indices modulo n), $1 \leq j \leq n$, so as to induce the edge label $2n$. However, such an assignment would generate two repeated edge labels with value n, given by $|f(v_n) - f(v_{j+1})| = |n - 2n|$ and $|f(v_n) - f(v_j)| = |n - 0|$. This observation leads to the following result.

Lemma 18 (Frucht [5])**.** *Let v_n be the central vertex of the wheel graph W_n. There exists no graceful labeling f of W_n with $f(v_n) = n$.*

Lemma 18 immediately implies the following result.

Theorem 19. *Bob has a winning strategy for all wheel graphs W_n when he is the first player.*

Proof. Since Bob is the first player, he starts by assigning label n to the central vertex v_n and, by Lemma 18, wins the game. □

Lemma 20. *Bob has a winning strategy for all wheel graphs W_n when Alice is the first player and either starts playing on a vertex $v \neq v_n$ by assigning any label to it, or starts playing on the central vertex v_n by assigning any label $\ell \notin \{0, 2n\}$.*

Proof. Let W_n be a wheel graph. For $n = 3$, the result follows from Theorem 6 since $W_3 \cong K_4$. So, consider W_n with $n \geq 4$ and recall that $|E(W_n)| = 2n$.

[1]When $n = 4$, Bob wins the game at this point since no label can be assigned to v_1 without generating repeated edge labels.

[2]When $n = 5$, Bob wins the game at this point since no label can be assigned to v_4 without generating repeated edge labels.

There are two main cases depending on the vertex Alice chooses to label first. Initially, consider that Alice starts the game labeling a vertex v_j with degree three, for $0 \leq j \leq n-1$. In this case, Alice cannot start the game by assigning a label different from n to v_j since, in the next turn, Bob assigns label n to v_n, winning the game (Lem. 18). Therefore, we assume that she assigns n to v_j. So, Bob assigns label 0 to vertex v_{j-1} (indices taken modulo n). Next, Alice is forced to assign label $2n$ to a neighbor of v_{j-1} (v_n or v_{j-2}) so as to generate the edge label $2n$. Alice cannot assign $2n$ to v_n since this would create a repeated edge label $n = |f(v_n) - f(v_j)| = |f(v_{j-1}) - f(v_j)|$. Hence, Alice assigns label $2n$ to vertex v_{j-2}. Now, in order to preclude Alice from generating the edge label $2n-1$, Bob assigns label $2n-1$ to vertex v_{j-3}, generating the edge label $1 = |f(v_{j-3}) - f(v_{j-2})|$. The only way left for Alice to generate edge label $2n-1$ consists of assigning 1 to the central vertex v_n. However, this move is not allowed since it would generate the repeated edge label $1 = |f(v_n) - f(v_{j-1})|$. Therefore, Bob is the winner.

Now we consider that Alice labels the central vertex v_n. By Lemma 18, she cannot assign label n to v_n. Hence, she uses any label in the set $\{1, \ldots, 2n-1\} \backslash \{n\}$. There are two cases to consider.

Case 1. Alice labels v_n with label $\ell \in \{1, \ldots, n-1\}$. In this case, note that the labels 0 and 2ℓ cannot both be assigned to vertices of W_n because this would generate the repeated edge label ℓ. Thus, in the next turn, Bob assigns label 2ℓ to any vertex, and wins the game.

Case 2. Alice labels v_n with label $\ell \in \{n+1, \ldots, 2n-1\}$. Define $\ell = n+k$ with $k \in \{1, \ldots, n-1\}$. In this case, note that the labels $2n$ and $2k$ cannot both be assigned to vertices of W_n because this would generate the repeated edge label $n-k$. Thus, in the next turn, Bob assigns label $2k$ to any vertex, and wins the game.

□

Lemma 21. *Bob has a winning strategy for wheel graphs W_n, $3 \leq n \leq 6$ when Alice is the first player and starts playing on the central vertex v_n with label* 0*.*

Proof. For $n = 3$, the result follows from Theorem 6 since $W_3 \cong K_4$. Now, we consider W_n with $4 \leq n \leq 6$ and that Alice starts by assigning label 0 to the central vertex v_n. On the second move Bob assigns label n to v_0. We split the proof into two cases depending on Alice's next vertex choice.

Case 1. Alice labels a neighbor of v_0 on her second move. By the symmetry of the graph we can assume she chooses v_1. First, let $n = 4$. Thus, Alice assigns a label $\ell \in \{0, 1, \ldots, 2n\} \setminus \{0, 2, 4, 6, 8\}$ (resp. $\ell = 6$) to v_1. Then, Bob assigns label $2n - \ell$ to v_2 (resp. 1 to v_2). Bob wins the game because Alice cannot create the edge label $2n$. If $n = 5$, then on her second move Alice assigns label $\ell \in \{0, 1, \ldots, 2n\} \setminus \{0, 5, 10\}$ to v_1. Next, Bob assigns label $2n - \ell$ to v_2 and with this move he wins the game because it is not possible to create the edge label $2n$. If $n = 6$, then Alice assigns label $\ell \in \{0, 1, \ldots, 2n\} \setminus \{0, 3, 6, 9, 12\}$ to v_1. Next, Bob assigns label $2n - \ell$ to v_3 and wins the game because it is not possible to create the edge label $2n$. If $\ell = 9$ then, on the fourth move, Bob assigns label 1 to v_3. Now, vertex v_4 is the unique vertex where Alice can assign label $2n$ to create the edge label $2n$. Bob assigns label 8 to v_5 and wins the game because it is not possible to create the edge label $2n - 2$.

Case 2. Alice labels a vertex not adjacent to v_0 on her second move. First, consider $n = 4$. If Alice assigns a label $\ell \in \{0, 1, \ldots, 2n\} \setminus \{0, 8\}$ to v_2, then Bob wins the game, independently of the label assigned by Alice to vertex v_2, because it is not possible to assign the vertex label $2n$ without generating two repeated edge labels n. If Alice assigns label $2n$ to v_2 then, on the fourth move, Bob assigns label 5 to v_1 and wins the game since it is not possible to create the edge label $2n - 1$.

Now, consider $n = 5$. By the symmetry of the graph, we can assume Alice chooses v_2 on the third move. First, consider that Alice assigns label $\ell \in \{0, 1, \ldots, 2n\} \setminus \{0, 5, 10\}$ to v_2. On the fourth move, Bob assigns label $2n - \ell$ to v_3 and wins the game since it is not possible to create the edge label $2n$. If $\ell = 10$, then, on the fourth move, Bob assigns label 1 to v_4 and wins the game since it is not possible to create the edge label $2n - 1$.

Lastly, consider $n = 6$. On the third move, Alice chooses a non-neighbor of v_0. By the symmetry of the graph, we can assume she chooses v_2 or v_3. Firstly, consider that Alice assigns label $\ell \in \{0, 1, \ldots, 2n\} \setminus \{0, 6, 12\}$ to

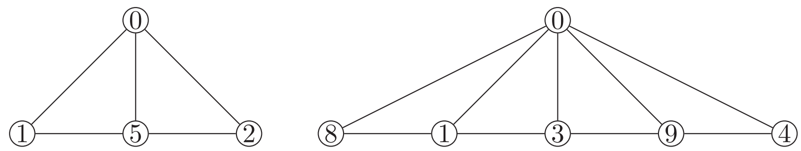

FIGURE 8. Fan graphs with graceful labelings.

v_2. Next, Bob assigns label $2n - \ell$ to v_4. Bob wins the game because it is impossible to create the edge label $2n = 12$. Now, consider that Alice assigns label 12 to v_2. Next, Bob assigns label 5 to v_4. Now, the only way for Alice to generate the edge label $2n - 1 = 11$ is by assigning 1 to v_3 and is what she does. Bob wins the game since it is not possible to generate the edge label $2n - 2 = 10$.

Now, consider that Alice labels vertex v_3 with $\ell \in \{0, 1, \ldots, 2n\} \backslash \{0, 3, 4, 6, 8, 9, 12\}$. Then, Bob assigns label $2n - \ell$ to v_2. Bob wins the game because it is not possible to create the edge label $2n = 12$. If $\ell = 3$, then Bob assigns label 5 to v_2. Alice wants to create the edge label $2n$ and the unique vertex where is possible to assign this label is the vertex v_4. So, this is Alice's next move. At this point of the game Alice loses because it is impossible to create the edge label $2n - 1 = 11$. If $\ell = 4$, Bob assigns label 8 to v_1 and with this move he cancels all Alice's possibilities of create the edge label $2n = 12$. If $\ell = 8$, then Bob assigns label 10 to v_5 and he wins again. If $\ell = 9$, then Bob assigns label 10 to v_2, creating edge labels 10 and 1. On the next move Alice assigns label $2n = 12$ to v_4 which is the unique vertex where is it possible to assign this label. Bob wins the game because it is not possible to create the edge label $2n - 1$. If $\ell = 12$, then Bob assigns label 5 to v_5 and wins the game because it is not possible to generate the edge label 11. □

3.12. Fan graphs

A *fan graph* F_n is a graph formed by connecting a single vertex v_n to all vertices $v_0, v_1, \ldots, v_{n-1}$ of a path P_n, for $n \geq 2$. By the definition, a fan graph F_n has $n + 1$ vertices and $2n - 1$ edges. The vertex v_n, which is adjacent to all the other vertices of F_n, is also called *central vertex*. In 2013, Sudha and Kanniga [21] proved that fan graphs are graceful. Figure 8 illustrate some fan graphs with graceful labelings.

A useful fact about graceful labelings of fan graphs is that, for $1 \leq k \leq n-1$, there exist no graceful labeling f of F_n that assigns the label $n - k$ to its central vertex and labels 0 and $2n - 2k$ to any other of its vertices, since such a labeling would generate two repeated edge labels $n - k$. This observation leads to the following result.

Lemma 22. *There exists no graceful labeling f of F_n with $f(v_n) = n - k$, $f(v_i) = 0$ and $f(v_j) = 2(n - k)$, for $0 \leq i, j \leq n - 1$ and $1 \leq k \leq n - 1$.*

The next result follows immediately from Lemma 22 and the complementary labeling of a graceful labeling.

Lemma 23. *There exists no graceful labeling f of F_n with $f(v_n) = n+k-1$, $f(v_i) = 2n-1$ and $f(v_j) = 2k-1$, for $0 \leq i, j \leq n - 1$ and $1 \leq k \leq n - 1$.*

Theorem 24. *Bob has a winning strategy for all fan graphs F_n when he is the first player.*

Proof. Let F_n be a fan graph. For $n = 2$, the result follows from Theorem 6 since $F_2 \cong K_3$. So, consider F_n with $n \geq 3$ and recall that $|E(F_n)| = 2n - 1$. Bob starts by assigning label $n - 1$ to central vertex v_n. At this point, vertex labels 0 and $2n - 2$ cannot both be in this graph (Lem. 22). Hence, in the next move, Alice is forced to assign label 0 to a vertex v_i of F_n, $0 \leq i \leq n - 1$, and is what she does. In the next move, Bob assigns label $2n - 1$ to any vertex not adjacent to v_i and wins the game since it is not possible to generate the edge label $2n - 1$ anymore. The last move previously described does not work for the case $n = 3$ when the vertex with label 0 is the vertex $v_1 \in V(F_3)$. For this case, on the third move, Bob assigns label 5 to any vertex and wins the game, since all remaining vertex labels generate repeated edge labels. □

Lemma 25. *Bob has a winning strategy for all fan graphs F_n when Alice is the first player and starts playing on the central vertex v_n by assigning any label $\ell \notin \{0, 2n-1\}$.*

Proof. Let F_n be a fan graph with $n \geq 3$. First, suppose that Alice starts the game by assigning a label $\ell \in \{1, \ldots, 2n-2\}$ to the central vertex v_n. Next, Bob chooses vertex v_0 and there are two cases to consider according to the value of ℓ. If $1 \leq \ell \leq n-1$, then Bob assigns label $2l$ to v_0 and Alice loses the game because it is impossible to assign label 0 without creating the repeated edge label $n-k$ (Lem. 22). On the other hand, suppose that $n \leq \ell \leq 2n-2$. In this case, we rewrite $\ell = n+k-1$ for $k \in \{1, \ldots, n-1\}$. Thus, on the second move, Bob assigns label $2k-1$ to v_0 and Alice loses the game because it is impossible to assign label 0 without creating the repeated edge label $n-k$ (Lem. 23). □

Lemma 26. *Bob has a winning strategy for all fan graphs F_n when Alice is the first player and starts playing on a vertex $v \neq v_n$ by assigning any label $\ell \notin \{0, 2n-1\}$.*

Proof. Suppose that Alice starts at a vertex v_j, $0 \leq j \leq n-1$, with label $\ell \in \{1, \ldots, 2n-2\}$. We consider three cases depending on the value of n.

Case 1. F_n with $n = 3$. In the second move, if $\ell = 1$, then Bob assigns label 3 to v_n; if $\ell = 2$, then Bob assigns label 1 to v_n; if $\ell = 3$, then Bob assigns label 4 to v_n; if $\ell = 4$, then Bob assigns label 2 to v_n and wins the game because it is not possible to generate the edge label $2n-1$.

Case 2. F_n with $n = 4$. In the second move, if $\ell = 1$, then Bob assigns label 4 to v_n; if $\ell = \{2, 4, 6\}$, then Bob assigns label $\frac{\ell}{2}$ to v_n; if $\ell = 3$, then Bob assigns label 5 to v_n; if $\ell = 5$, then Bob assigns label 6 to v_n and wins the game since it is not possible to generate the edge label $2n-1$.

Case 3. F_n with $n \geq 5$. In the second move, if $\ell = n$, then Bob assigns label 2 to v_n; if $\ell = n-1$, then Bob assigns label 1 to v_n; otherwise, he assigns label $n-1$ to v_n. By Lemma 22, Bob's last move forces Alice to assign label 0 to a vertex v_i, $i \neq j$. Next, Bob assigns label $2n-1$ to any vertex not adjacent to v_i (such a vertex exists since $n \geq 5$) and wins the game since it is not possible to generate the edge label $2n-1$.

□

Now we are ready to state the following result:

Theorem 27. *Bob has a winning strategy for all fan graphs F_n when Alice is the first player and starts playing: (i) on the central vertex v_n by assigning any label $\ell \notin \{0, 2n-1\}$; (ii) or on a vertex $v \neq v_n$ by assigning any label $\ell \notin \{0, 2n-1\}$.*

4. Concluding remarks

In this work, we study the graceful game introduced by Tuza [22] and contribute to the structural analysis of the game investigating its application in many classic graph classes.

We also developed computational experiments to check the game for some small graphs. We applied the backtracking technique, growing the tree of partial solutions by branching on the set of all possible moves and pruning the branching process as soon as we can decide which player wins the game. We observe that, in this algorithm, at each turn of the game, a player needs to choose one vertex in the set of remaining vertices and a label in the set of remaining labels. Thus, at the k-th turn, the player has $(n-k+1) \cdot (m+1-k+1)$ possible choices. Since the graph is connected, we have $\Omega(n! \times n!)$ possible configurations for the hole game. For this reason, we just considered graphs with at most 13 vertices to compute with our backtracking algorithm.

As we summarize in Table 1, Alice has winning strategies for only few cases, even if she is the first player, say paths, complete graphs K_i, $i \leq 3$, and stars $K_{1,q}$, $q \geq 2$. We also investigate the graceful game for two classes of graphs with small diameter: the wheel graphs W_n and the fan graphs F_n. For these two classes, we have proved that Bob has a winning strategy on all wheels and fan graphs when he is the first player (Thms. 19 and 24) or when Alice starts the game playing on a vertex different from the central vertex or she

TABLE 1. Graph classes and winners: A (Alice) and B (Bob).

Graph classes	First player	
	Alice	Bob
$P_n, n = 1, 2$	A	A
P_3	A	B
$P_n, n \geq 4$	B	B
K_3	A	A
K_4	B	B
$C_n, n \geq 4$	B	B
$K_{1,q}, q \geq 2$	A	B
$K_{p,q}, p, q \geq 2$	B	B
$cat(k_0, \ldots, k_{s-1}), s \geq 2$	B	B
trees, $diam \geq 3$	B	B
$G_n, n \geq 3$	B	B
$W(t, n), r \geq 3$	B	B
$P_{r,2}, t \geq 2, n \geq 3$	B	B
$Q_n, n \geq 2$	B	B
$P_n^2, n \geq 4$	B	B
W_n, $3 \leq n \leq 12$	B	B
F_n, $3 \leq n \leq 12$	B	B

starts by assigning label $\ell \notin \{0, m\}$ to the central vertex (Lems. 20, 25 and 26). The case when Alice is the first player and starts by assigning 0 or m to the central vertex on wheels and fan graphs, where solved by using computational experiments. We determined that Bob wins the graceful game on these graphs with at most 13 vertices. Therefore, based on our results, we pose the following conjecture.

Conjecture 28. Bob has a winning strategy for the graceful game on all fan graphs and wheel graphs.

Acknowledgements. Dedicated to healthcare professionals and essential workers during the COVID-19 pandemic.

Funding. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001 and CAPES-PrInt project number 88881.310248/2018-01, CNPq and FAPERJ.

References

[1] O. Baudon, J. Przybyło, M. Senhaji, E. Sidorowicz, E. Sopena and M. Woźniak, The neighbour-sum-distinguishing edge-colouring game. *Discrete Math.* **340** (2017) 1564–1572.

[2] E. Boudreau, B. Hartnell, K. Schmeisser and J. Whiteley, A game based on vertex-magic total labelings. *Australas. J. Comb.* **29** (2004) 67–73.

[3] M.-L. Chia, H.-N. Hsu, D. Kuo, S.-C. Liaw and Z.-T. Xu, The game L(d,1)-labeling problem of graphs. *Discrete Math.* **312** (2012) 3037–3045.

[4] E. Duchêne, S. Gravier and J. Moncel, New results about impartial solitaire clobber. *RAIRO: Oper. Res.* **43** (2009) 463–482.

[5] R. Frucht, Graceful numbering of wheels and related graphs. *Ann. N.Y. Acad. Sci.* **319** (1979) 219–229.

[6] J.A. Gallian, A dynamic survey of graph labeling. *Electron. J. Comb.* **DS6** (2018) 1–502.

[7] A. Giambrone and E.L.C. King, Vertex-magic edge labeling games on graphs with cycles. *J. Comb. Math. Comb. Comput.* **78** (2011) 75–96.

[8] S.W. Golomb, How to number a graph, in Graph Theory and Computing. Academic Press, New York (1972) 23–37.

[9] R.L. Graham and N.J.A. Sloane, On additive bases and harmonious graphs. *SIAM J. Algebraic Discrete Methods* **1** (1980) 382–404.

[10] J.R. Griggs and R.K. Yeh, Labeling graphs with a condition at distance 2. *SIAM J. Discrete Math.* **5** (1992) 586–595.

[11] B. Hartnell and D. Rall, A vertex-magic edge labeling game. *Congressus Numerantium* **161** (2003) 163–167.
[12] D. Hefetz, M. Krivelevich, M. Stojaković, T. Szabó, Positional Games. Birkhäuser/Springer, Basel (2014).
[13] C. Hoede and H. Kuiper, All wheels are graceful. *Utilitas Mathematica* **14** (1987) 311.
[14] Q.D. Kang, Z.-H. Liang, Y.-Z. Gao and G.-H. Yang, On labeling of some graphs. *J. Comb. Math. Comb. Comput.* **22** (1996) 193–210.
[15] T.K. Mathew Varkey and T.J. Rajesh Kumar, Even graceful labelling of a class of trees. *Int. J. App. Graph Theory Wirel. Ad hoc Networks Sens. Networks* **7** (2016) 1–7.
[16] N. Parvathi and S. Vidyanandini, Graceful labeling of a tree from caterpillars. *J. Inf. Optim. Sci.* **35** (2015) 387–393.
[17] A. Rosa, On certain valuations of the vertices of a graph, in Theory of Graphs, International Symposium, Rome, July 1966. (1967) 349–355.
[18] A. Rosa and A. Kotzig, Magic valuations of finite graphs. *Can. Math. Bull.* **13** (1970) 451–461.
[19] T.J. Schaefer, On the complexity of some two-person perfect-information games. *J. Comput. Syst. Sci.* **16** (1978) 185–225.
[20] G. Sethuraman and P. Ragukumar, Generation of graceful trees from arbitrary trees. *Adv. Stud. Contemp. Math.* **24** (2014) 389–396.
[21] S. Sudha and V. Kanniga, Gracefulness of some new class of graph. *Eng. Sci. Int. Res. J.* **1** (2013) 81–83.
[22] Z. Tuza, Graph labeling games. *Electron. Notes Discrete Math.* **60** (2017) 61–68.
[23] Z. Tuza and X. Zhu, Colouring games, in Topics in Chromatic Graph Theory. Vol. 156 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press (2015) 304–326.