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# ON THE ORIENTED COLORING OF THE DISJOINT UNION OF GRAPHS *

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**Abstract.** Let $\overrightarrow{G} = (V, A)$ be an oriented graph and $G$ the underlying graph of $\overrightarrow{G}$. An *oriented k-coloring* of $\overrightarrow{G}$ is a partition of $V$ into $k$ color classes, such that there is no pair of adjacent vertices belonging to the same class and all the arcs between a pair of color classes have the same orientation. The smallest $k$ such that $\overrightarrow{G}$ admits an oriented $k$-coloring is the *oriented chromatic number* $\chi_o(\overrightarrow{G}) = k$ of $\overrightarrow{G}$. The *oriented chromatic number* $\chi_o(G)$ of the undirected graph $G$ is the maximum of $\chi_o(\overrightarrow{G})$ for all orientations $\overrightarrow{G}$ of $G$. Oriented chromatic number of the product of two graphs $G_1, G_2$ was widely studied, but the disjoint union $G_1 \cup G_2$ has not yet been considered. In this article we proved bounds for the oriented chromatic number of any two oriented graphs and we also proved that given two complete graphs $K_n$ and $K_m$ with $n \geq m$, there is a real number $\alpha \in (1, 3)$ such that $\chi_o(K_n \cup K_m) = n + m - \alpha \log_2(m)$. Additionally, we established exact values of the union of one complete graph with one cycle and of one complete graph with a forest.

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## 1. Introduction

For most of the definitions and notations in this article we consider the Bondy and Murty graph theory book [3]. Given a graph $G = (V, E)$, the *orientation* of an edge $e = \{u, v\} \in E$ is one of the two possible ordered pairs $uv$ or $vu$ called *arcs*.

An *oriented graph* $\overrightarrow{G} = (V, A)$ is obtained from $G = (V, E)$, for each $\{u, v\} \in E$ exactly one of the arcs either $uv \in A$, or $vu \in A$. In this case, $\overrightarrow{G}$ is called an *orientation* of $G$, and $G$ is called the *underlying graph* of $\overrightarrow{G}$.

*Keywords.* Oriented graph, oriented chromatic number, disconnected graphs, graph classes, disjoint union of graphs.

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Given an arc $uv \in A$, $v$ is called the *successor* of $u$ and $u$ is called the *predecessor* of $v$. A vertex without predecessors is called *source* and a vertex without successors is called *sink*.

Two graphs are *disjoint* if they have no vertex in common. Let $G$ and $H$ be two vertex disjoint graphs, the *disjoint union* $G \cup H$ of $G$ and $H$ has $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We define the *disjoint union of two oriented graphs* analogously.

Let $C_n$ be a cycle on $n$ vertices and $\overrightarrow{C_n}$ be an orientation of $C_n$. If there is a permutation $(v_1, \ldots, v_n)$ of the vertices of $C_n$, such that, $v_i v_{(i \mod n)+1} \in A(\overrightarrow{C_n})$ then $\overrightarrow{C_n}$ is called *cyclic*, otherwise $\overrightarrow{C_n}$ is called *acyclic*. A *tournament* $\overrightarrow{K}_n$ with $n$ vertices is an orientation of a complete graph $K_n$. A tournament $\overrightarrow{K}_n$ is called *transitive* if and only if whenever $uv, vw \in A(\overrightarrow{K}_n)$, then $uw \in A(\overrightarrow{K}_n)$. The complete bipartite graph $G = K_{1,n}$ is a *star*. Given a forest $F$, the diameter of $F$, $\text{diam}(F)$, is the maximum of the diameters among all the trees in $F$.

Let $\overrightarrow{G}$ be an oriented graph, such that for every $xy, zt \in A(\overrightarrow{G})$ and $C = \{1, 2, \ldots, k\}$ be a set of colors. An *oriented $k$-coloring* of $\overrightarrow{G}$ is a function $c : V(\overrightarrow{G}) \to C$, such that $c(x) \neq c(y)$, and if $c(x) = c(t)$, then $c(y) \neq c(z)$. The *oriented chromatic number* of $\overrightarrow{G}$ denoted by $\chi_o(\overrightarrow{G})$ is the smallest $k$ such that $\overrightarrow{G}$ admits an oriented $k$-coloring. In Figure 1a we offer an example of an oriented graph $\overrightarrow{G}$ with $\chi_o(\overrightarrow{G}) = 4$.

Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be oriented graphs, a *homomorphism* of $\overrightarrow{G}$ into $\overrightarrow{H}$ is a mapping $f : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ such that $f(u)f(v) \in E(\overrightarrow{H})$ for all $uv \in E(\overrightarrow{G})$. When $\overrightarrow{H}$ is an oriented graph on $k$ vertices $V(H) = \{1, 2, \ldots, k\}$, a homomorphism from $\overrightarrow{G}$ into $\overrightarrow{H}$ is an oriented $k$-coloring of $\overrightarrow{G}$. If there is a homomorphism from $\overrightarrow{G}$ into $\overrightarrow{H}$ such that $\chi_o(\overrightarrow{G}) = |V(\overrightarrow{H})|$, then $\overrightarrow{H}$ is called the *color graph* of $\overrightarrow{G}$. In Figure 1b we show the color graph $\overrightarrow{H}$ of the oriented graph $\overrightarrow{G}$ in Figure 1a.

The *oriented chromatic number* $\chi_o(G)$ *of an undirected graph* $G$ is the maximum $\chi_o(\overrightarrow{G})$ for all orientations $\overrightarrow{G}$ of $G$. Given a positive integer number $k$, we denote by $\mathcal{CN}_k$ the class of graphs $G$ such that $\chi_o(G) \leq k$.

As a motivation to the study of oriented chromatic number [20] we consider parallel task scheduling problem, where a set on $n$ *jobs* $V = \{J_1, J_2, \ldots, J_n\}$ plus a *precedence relation* $A \subset \{J_i J_k \mid i, k \in \{1, 2, \ldots, n\}\}$ are given as a input, meaning that once the job $J_i$ is being executed, then job $J_k$ is under execution only after job $J_i$ has ended its execution. The *makespan* of the parallel task is the minimum number of turns necessary to accomplish the scheduling with the execution of each job. The oriented chromatic number $\chi_o(\overrightarrow{G})$ of the oriented graph $\overrightarrow{G} = (V, A)$ is a lower bound to the makespan.

Oriented coloring has been studied by many authors. In 1999, Borodin, Kostochka, Nešetřil, Raspaud and Sopena [4] studied bounds for the oriented chromatic number in graphs with maximum average degree and girth restrictions. Culus and Demange [7] presented an approximation algorithm for oriented coloring. Klostermeyer and MacGillivray [11] proved that oriented chromatic number $\text{OCN}_k$ is NP-complete when $k \geq 4$ and polynomial when $k < 3$. Marshall [12] proved the existence of a planar graph of girth 6 and oriented chromatic number at least 7.

Ochem and Pinlou [13] determined the oriented chromatic number of partial 2-trees for every girth $g \geq 3$. Coelho *et al.* [5] proved that it is NP-complete to decide whether a graph belongs to $\mathcal{CN}_k$ for $k \geq 4$, and polynomial for $k \leq 3$. Sopena [19] did a nice survey on oriented coloring. Sopena [18] determined bounds for the Cartesian product involving particular families of graphs such as the paths, cycles and trees. In spite of the vast amount of literature dedicated to the product of graphs there is no result on oriented chromatic number of disjoint union of graphs.

In contrast with the vertex coloring, the oriented coloring of one of the elements in a disjoint union can interfere with the oriented coloring of another element. For example, consider $\overrightarrow{G} = \overrightarrow{G_1} \cup \overrightarrow{G_2}$ in Figure 2, where $\overrightarrow{G_1}$ is a transitive orientation of $\overrightarrow{K_3}$ and $\overrightarrow{G_2}$ is the orientation a $\overrightarrow{P_4}$ depicted in Figure 2. Sopena [19] proved that $\chi_o(\overrightarrow{G_1}) = \chi_o(\overrightarrow{G_2}) = 3$. One could think that the oriented chromatic number of $\overrightarrow{G}$ is also 3, but this is not the case.

Suppose for a moment that $\chi_o(\overrightarrow{G}) = 3$. Hence, any oriented coloring $\phi$ for $\overrightarrow{G}$ assigns 3 different colors for vertices $a, b, c$, say $\phi(a) = 1$, $\phi(b) = 2$ and $\phi(c) = 3$. If $\overrightarrow{G_2}$ is colored with colors $1, 2, 3$, colors 3 or 1 are not

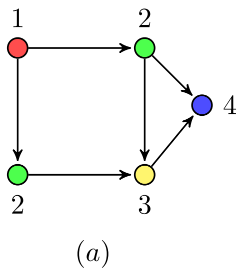


FIGURE 1. (a) An oriented graph $\overrightarrow{G}$ with $\chi_o(\overrightarrow{G}) = 4$. (b) The color graph $\overrightarrow{H}$ of $\overrightarrow{G}$.

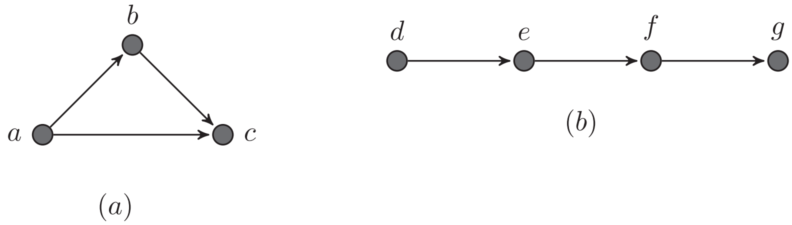


FIGURE 2. Graph $\overrightarrow{G} = \overrightarrow{G}_1 \cup \overrightarrow{G}_2 = \overrightarrow{K_3} \cup \overrightarrow{P_4}$ with $\chi_o(\overrightarrow{G_1}) = \chi_o(\overrightarrow{G_2}) = 3$ and $\chi_o(\overrightarrow{G}) = 4$. (a) $\overrightarrow{G}_1$. (b) $\overrightarrow{G}_2$.

assigned to vertex $e$ or vertex $f$, since vertex $a$ is a source, vertex $c$ is a sink and vertex $e$ and $f$ are neither. Therefore, $\phi(e) = 2$ and $\phi(f) = 2$. We reach a contradiction, since $e$ and $f$ are adjacent. A 4-oriented coloring can be defined to $\overrightarrow{G}$ from $\phi$, assigning an additional color 4 to vertex $g$. Hence, $\chi_o(\overrightarrow{G}) = 4$.

Our work aims to study the oriented chromatic number of the disjoint union of two oriented graphs. We analyze the disjoint union of one complete graph with some of the main classes of graphs. In our study, we have found that the oriented chromatic number of two complete graphs, is related with the size $\nu(n)$ of the maximum oriented transitive tournament contained in every tournament on $n$ vertices – a measure already well studied with extremal results and bounds on the literature [8, 14–16, 21].

In Section 2 we present our main contribution of this paper where the union of any two oriented graphs is considered. We proved that given two oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$, then the oriented chromatic number of $\overrightarrow{G} \cup \overrightarrow{H}$ is the sum of the oriented chromatic numbers of $\overrightarrow{G}$ and $\overrightarrow{H}$ minus the number of colors that can be used to color $\overrightarrow{G}$ and $\overrightarrow{H}$ at the same time.

In Section 3, the union of two complete graphs $K_n$ and $K_m$ is considered. Applying the result of Section 2 to the case of complete graphs we realized that the number of colors that can be used at the same time in the disjoint union of $K_n$ and $K_m$ is the size of the largest transitive tournament that is a subgraph of $K_n$ and $K_m$. This size is defined by the $\nu(n)$ parameter. Utilizing results about the $\nu(n)$ parameter from Stearns [21], Erdos and Moser [8], Sanchez-Flores [15, 16] and Reid and Parker [14] we proved that $\chi_o(K_n \cup K_m) = n + m - \alpha \log_2(m)$ for some real number $\alpha$ in the interval $(1, 3)$. It is important to notice that any improvement in bounding the value of $\nu(n)$ yields the same progress in bounding $\chi_o(K_n \cup K_m)$. In Sections 4 and 5, exact values are determined for the oriented chromatic number, respectively, of the union of a complete graph with a cycle, and of the union of a complete graph with a forest. In Section 6, we present our conclusions and suggestions for further work.

## 2. On the oriented chromatic number of the disjoint union of two graphs

In this section, we initiate the study on the oriented coloring of the disjoint union of graphs. We proved a general result defining an exact value for the oriented chromatic number of the disjoint union of any two oriented graphs.

The main technique of the proof to determine the oriented chromatic number of the disjoint union of two oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$ is to find the largest number of colors $r$ that can be used to color $\overrightarrow{G}$ and $\overrightarrow{H}$ at the same time. To find this value of $r$ we look at the families $\overrightarrow{\mathcal{G}_c}$ and $\overrightarrow{\mathcal{H}_c}$ of color graphs of $\overrightarrow{G}$ and $\overrightarrow{H}$ to find a pair of color graphs $\overrightarrow{G'} \in \overrightarrow{\mathcal{G}_c}$ and $\overrightarrow{H'} \in \overrightarrow{\mathcal{H}_c}$ such that there is a sugraph $\overrightarrow{F_G}$ of $\overrightarrow{G'}$ and a subgraph $\overrightarrow{F_H}$ of $\overrightarrow{H'}$ where $|V(\overrightarrow{F_G})| = |V(\overrightarrow{F_H})|$ and there is a homomorphism from $\overrightarrow{F_G}$ to $\overrightarrow{F_H}$ or a homomorphism from $\overrightarrow{F_H}$ to $\overrightarrow{F_G}$.

We can consider homomorphisms instead of isomorphisms because it is possible to use the colors of these subgraphs in common even if not all the arcs of these subgraphs are used in the coloring of $\overrightarrow{G}$ and $\overrightarrow{H}$.

**Theorem 2.1.** *Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs. Let $\overrightarrow{\mathcal{G}_c}$ and $\overrightarrow{\mathcal{H}_c}$ be the families of color graphs, respectively, of $\overrightarrow{G}$ and $\overrightarrow{H}$. Let $r = \max\{|V(\overrightarrow{F_G})| : \exists\ \overrightarrow{G'} \in \overrightarrow{\mathcal{G}_c}$ and $\exists\ \overrightarrow{H'} \in \overrightarrow{\mathcal{H}_c}$ such that $\overrightarrow{F_G}$ is subgraph of $\overrightarrow{G'}$ and $\overrightarrow{F_H}$ is subgraph of $\overrightarrow{H'}$ with $|V(\overrightarrow{F_G})| = |V(\overrightarrow{F_H})|$ and there is a homomorphism from $\overrightarrow{F_G}$ to $\overrightarrow{F_H}$, or there is a homomorphism from $\overrightarrow{F_H}$ to $\overrightarrow{F_G}\}$, then $\chi_o(\overrightarrow{G} \cup \overrightarrow{H}) = \chi_o(\overrightarrow{G}) + \chi_o(\overrightarrow{H}) - r$.*

*Proof.* Given $\overrightarrow{G'} \in \overrightarrow{\mathcal{G}_c}$ and $\overrightarrow{H'} \in \overrightarrow{\mathcal{H}_c}$, such that $r = |V(\overrightarrow{F_G})| = |V(\overrightarrow{F_H})|$ and assume that there is a homomorphism from $\overrightarrow{F_G}$ to $\overrightarrow{F_H}$. An oriented coloring of $\overrightarrow{G} \cup \overrightarrow{H}$ using $\chi_o(\overrightarrow{G}) + \chi_o(\overrightarrow{H}) - r$ colors can be given by assigning $\chi_o(\overrightarrow{G})$ colors to $\overrightarrow{G}$ and $\chi_o(\overrightarrow{H})$ colors to $\overrightarrow{H}$, such that $\overrightarrow{F_G}$ and $\overrightarrow{F_H}$ share the same $r$ colors that altogether produce an $(\chi_o(\overrightarrow{G}) - r) + (\chi_o(\overrightarrow{H}) - r) + r = \chi_o(\overrightarrow{G}) + \chi_o(\overrightarrow{H}) - r$ oriented coloring for $\overrightarrow{G} \cup \overrightarrow{H}$. Hence $\chi_o(\overrightarrow{G} \cup \overrightarrow{H}) \leq \chi_o(\overrightarrow{G}) + \chi_o(\overrightarrow{H}) - r$.

Consider an optimum oriented coloring of $\overrightarrow{G} \cup \overrightarrow{H}$ using $\chi_o(\overrightarrow{G} \cup \overrightarrow{H})$ and $k \geq 0$ the number of colors occurring in both $\overrightarrow{G}$ and $\overrightarrow{H}$. This coloring defines an $(\chi_o(\overrightarrow{G}) + \chi_o(\overrightarrow{H}) - k)$-oriented coloring for $\overrightarrow{G} \cup \overrightarrow{H}$. Suppose by contradiction that $k > r$. Then, there is an oriented graph $\overrightarrow{G'}$ of $\overrightarrow{\mathcal{G}_c}$ and an oriented graph $\overrightarrow{H'}$ of $\overrightarrow{\mathcal{H}_c}$, such that there is a subgraph $\overrightarrow{F_G}$ of $\overrightarrow{G'}$ and there is a subgraph $\overrightarrow{F_H}$ of $\overrightarrow{H'}$ with $|V(\overrightarrow{F_G})| = |V(\overrightarrow{F_H})|$, where there is a homomorphism from $\overrightarrow{F_G}$ to $\overrightarrow{F_H}$ or there is a homomorphism from $\overrightarrow{F_H}$ to $\overrightarrow{F_G}$ with $|V(\overrightarrow{F_G})| = k$, a contradiction since $k > r$ and $r$ is maximum. Hence, $\chi_o(\overrightarrow{G}) + \chi_o(\overrightarrow{H}) - r \leq \chi_o(\overrightarrow{G} \cup \overrightarrow{H})$. Therefore, $\chi_o(\overrightarrow{G} \cup \overrightarrow{H}) = \chi_o(\overrightarrow{G}) + \chi_o(\overrightarrow{H}) - r$. □

## 3. On the oriented chromatic number of the disjoint union of two complete graphs

Wagner [23] showed the inclusion in several hardness classes of the problem of deciding whether two tournaments are isomorphic. In their article, Babai *et al.* [2] introduced the most efficient algorithm currently known for determining whether two tournaments are isomorphic, with a running time of $O(n^{\log n})$. Hence, determining whether a given sub-tournament is a subgraph of a tournament has proved to be a challenging problem.

In 1959, Stearns [21] introduced the concept of the largest integer number $\nu(n)$ such that every tournament $\overrightarrow{K_n}$ on $n$ vertices has at least one transitive tournament on $\nu(n)$ vertices as a subgraph. Our contribution in Corollary 3.8 on the evaluation of the oriented chromatic number of the union of two complete graphs is the proof that the maximum number $r$ of colors that can be used at the same time in both completes is $\nu(n)$.

We applied Theorem 2.1 in order to prove Theorem 3.7. For this purpose, we considered the largest subgraphs $\overrightarrow{F_G}$ and $\overrightarrow{F_H}$ of the color graphs $\overrightarrow{G'}$ and $\overrightarrow{H'}$, where there was a homomorphism from $\overrightarrow{F_G}$ to $\overrightarrow{F_H}$ or a homomorphism from $\overrightarrow{F_H}$ to $\overrightarrow{F_G}$. We observe that $\overrightarrow{F_G}$ and $\overrightarrow{F_H}$ are induced subgraphs, respectively, of $\overrightarrow{K_p}$ and $\overrightarrow{K_q}$, hence they are tournaments as well. Thus, the largest number $r$ of colors that can be used at the same time to color $\overrightarrow{K_p}$

and $\overrightarrow{K_q}$ is the size of the largest common subgraph of $\overrightarrow{K_p}$ and $\overrightarrow{K_q}$, and we prove that this size is defined by the value of $\nu(q)$ when $p \geq q$.

Stearns obtained the next lower bound for $\nu(n)$.

**Theorem 3.1** (Stearns [21])**.** *Let $n$ be a positive integer number, then $\lfloor \log_2 n \rfloor + 1 \leq \nu(n)$.*

Later in 1964, using the Probabilistic Method, Erdös and Moser showed that given a positive integer number $n$, there is a tournament on $n$ vertices, which the largest transitive sub-tournament has size $2\lfloor \log_2 n \rfloor + 1$ giving the upper bound.

**Theorem 3.2** (Erdös and Moser [8])**.** *Let $n$ be a positive integer number, then $\nu(n) \leq 2\lfloor \log_2 n \rfloor + 1$.*

A combination of the results in [8] and [14] obtains the next three theorems.

**Theorem 3.3** (Erdös and Moser [8], Reid and Parker [14])**.** *Let $n$ be a positive integer number, then*

$$\nu(n) = \begin{cases} 1, & \text{if } \; n = 1, \\ 2, & \text{if } \; 2 \leq n \leq 3, \\ 3, & \text{if } \; 4 \leq n \leq 7, \\ 4, & \text{if } \; 8 \leq n \leq 13, \\ 5, & \text{if } \; 14 \leq n \leq 27. \end{cases}$$

Due the results of Reid and Parker [14] and Sanchez-Flores [15], respectively, when $14 \leq n \leq 54$, and when $n \geq 55$, better lower bounds can be established for $\nu(n)$.

**Theorem 3.4** (Reid and Parker [14])**.** *Let $n$ be a positive integer number, then if $14 \leq n \leq 54$, then $\lfloor \log_2 {}^{16n}/_7 \rfloor \leq \nu(n)$.*

**Theorem 3.5** (Sanchez-Flores [15])**.** *Let $n$ be a positive integer number, then if $n \geq 55$, then $\lfloor \log_2 {}^{n}/_{55} \rfloor + 7 \leq \nu(n)$.*

Sanchez-Flores [16] improved the previous lower bound when $n \geq 28$.

**Theorem 3.6** (Sanchez-Flores [16])**.** *Let $n \geq 28$ be a positive integer number, then $\lfloor \log_2 {}^{n}/_{54} \rfloor + 7 \leq \nu(n)$.*

Theorem 3.7 describes a framework to derive lower and upper bounds for the oriented chromatic number of the disjoint union of two complete graphs from lower and upper bounds for function $\nu(n)$.

**Theorem 3.7.** *Let $p, q$ be a pair of positive integer numbers with $p \geq q$. Let $g$ and $h$ be two functions satisfying that $g(q) \leq \nu(q) \leq h(q)$. Then*

$$p + q - h(q) \leq \chi_o\left(K_p \cup K_q\right) \leq p + q - g(q).$$

*Proof.* Let $T_1 = \overrightarrow{K_p}$ and $T_2 = \overrightarrow{K_q}$ be two tournaments. Let $\overrightarrow{\mathcal{G}_c}$ and $\overrightarrow{\mathcal{H}_c}$ be the families of color graphs, respectively, of $\overrightarrow{K_p}$ and $\overrightarrow{K_q}$ and let $r = \max\{|V(\overrightarrow{F_G})| : \exists\ \overrightarrow{G'} \in \overrightarrow{\mathcal{G}_c}$ and $\exists\ \overrightarrow{H'} \in \overrightarrow{\mathcal{H}_c}$ such that $\overrightarrow{F_G}$ is subgraph of $\overrightarrow{G'}$ and $\overrightarrow{F_H}$ is subgraph of $\overrightarrow{H'}$ with $|V(\overrightarrow{F_G})| = |V(\overrightarrow{F_H})|$ and there is a homomorphism from $\overrightarrow{F_G}$ to $\overrightarrow{F_H}$, or there is a homomorphism from $\overrightarrow{F_H}$ to $\overrightarrow{F_G}\}$.

From the definition of $\nu$ and the fact that $q \leq p$, there are two transitive sub-tournaments in $T_1$ and $T_2$ with size $g(q)$. Hence, $g(q) \leq r$. From Theorem 2.1, $\chi_o(\overrightarrow{K_p} \cup \overrightarrow{K_q}) = p + q - r \leq p + q - g(q)$. Since for every pair of tournaments $K_p$ and $K_q$, we have that $\chi_o(\overrightarrow{K_p} \cup \overrightarrow{K_q}) \leq p + q - g(q)$, by definition $\chi_o(K_p \cup K_q) \leq p + q - g(q)$.

Let $T_1 = \overrightarrow{K_p}$ be the transitive tournament on $p$ vertices and $T_2 = \overrightarrow{K_q}$ be a tournament on $q$ vertices, such that a largest transitive sub-tournament has size exactly $\nu(q)$. Suppose by contradiction that $r > \nu(q)$. Since $\overrightarrow{F_G}$ is a subgraph of $T_1$, $\overrightarrow{F_G}$ is a transitive tournament. Since $\overrightarrow{F_G}$ and $\overrightarrow{F_H}$ are tournaments, there is an isomorphism from $\overrightarrow{F_G}$ to $\overrightarrow{F_H}$. Then $\overrightarrow{F_H}$ is a transitive tournament subgraph of $\overrightarrow{K_q}$ with size $|V(\overrightarrow{F_H})| = r > \nu(q)$, an absurd. Hence, from Theorem 2.1, $p + q - h(q) \leq p + q - \nu(q) = p + q - r = \chi_o(\overrightarrow{K_p} \cup \overrightarrow{K_q}) \leq \chi_o(K_p \cup K_q)$. ◻

Corollary 3.8 is obtained directly from Theorem 3.7 considering $h(q) = g(q) = \nu(q)$.

**Corollary 3.8.** *Let $p, q$ be a pair of positive integer numbers with $p \geq q$. Then $\chi_o(K_p \cup K_q) = p + q - \nu(q)$.*

We have proved on Theorem 3.7 that the oriented chromatic number of the disjoint union of complete graphs is related with the size of the largest transitive tournament in both complete graphs. We use Theorem 3.7 together with previous results to obtain the next 4 corollaries, that are applications of the size of $\nu(n)$, for different values of $n$.

**Corollary 3.9.** *Let $p, q$ be a pair of positive integer numbers with $p \geq q$. If $1 \leq q \leq 27$, then*

$$\chi_o(K_p \cup K_q) = \begin{cases} p+q-1, & \text{if } \ q = 1, \\ p+q-2, & \text{if } \ q = 2, 3, \\ p+q-3, & \text{if } \ 4 \leq q \leq 7, \\ p+q-4, & \text{if } \ 8 \leq q \leq 13, \\ p+q-5, & \text{if } \ 14 \leq q \leq 27. \end{cases}$$

**Corollary 3.10.** *Let $p, q$ be a pair of positive integer numbers with $p \geq q$. Then*

$$p + q - (2\lfloor \log_2 q \rfloor + 1) \leq \chi_o(K_p \cup K_q).$$

**Corollary 3.11.** *Let $p, q$ be a pair of positive integer numbers with $p \geq q$. If $q \geq 28$, then $\chi_o(K_p \cup K_q) \leq p + q - (\lfloor \log_2 q/54 \rfloor + 7)$.*

**Corollary 3.12.** *Let $p, q$ be a pair of positive integer numbers with $p \geq q$. If $q \geq 28$, then $\chi_o(K_p \cup K_q) = p + q - \alpha \log_2 q$, for some $\alpha \in (1, 3)$.*

*Proof.* Suppose that $p \geq q \geq 28$. Observe that $p + q - 3\log_2 q < p + q - (2\lfloor \log_2 q \rfloor + 1)$. Notice that $p + q - (\lfloor \log_2 q/54 \rfloor + 7) \leq p + q - \lfloor \log_2 q/54 \rfloor - \log_2 128 = p + q - \lfloor \log_2 q/54 + \log_2 128 \rfloor \leq p + q - \lfloor \log_2 128q/54 \rfloor \leq p + q - \lfloor \log_2 q + \log_2 128/54 \rfloor \leq p + q - \lfloor 1 + \log_2 q \rfloor = p + q - 1 - \lfloor \log_2 q \rfloor < p + q - \log_2 q$. Hence, $p + q - (\lfloor \log_2 q/54 \rfloor + 7) < p + q - \log_2 q$. From Corollaries 3.10 and 3.11, $p + q - 3\log_2 q < p + q - (2\lfloor \log_2 q \rfloor + 1) \leq \chi_o(K_p \cup K_q) \leq p + q - (\lfloor \log_2 q/54 \rfloor + 7) < p + q - \log_2 q$. □

## 4. The oriented chromatic number of the disjoint union of one complete graph with one cycle

In this section we consider the disjoint union of one complete graph $K_p$ and one cycle $C_q$ with $p \geq 1$ and $q \geq 3$. The proof of Theorem 4.1 defines the family $\overrightarrow{\mathcal{C}_q}$ of color graphs for any orientation of a cycle $C_q$ based on its size $q$. We use these color graphs to establish the oriented chromatic number of the disjoint union of a complete $K_p$ and a cycle $C_q$.

Our strategy for this proof is to find the largest subgraph $\overrightarrow{K'}$ of any orientation of $K_p$ that has a homomorphism to the graph color $\overrightarrow{C'} \in \overrightarrow{\mathcal{C}_q}$ of $C_q$ or that $\overrightarrow{C'}$ has a homomorphism to $\overrightarrow{K'}$.

**Theorem 4.1** (Sopena [19]). *Let $q$ be a integer with $q \geq 3$, then*

$$\chi_o(C_q) = \begin{cases} 3, & \text{if } \ q = 3 \\ 4, & \text{if } \ q = 4 \text{ or } q \geq 6 \\ 5, & \text{if } \ q = 5 \end{cases}$$

Theorems 4.2 and 4.4 establish the required conditions for a tournament to include each of the oriented paths on $n$ vertices. These theorems are essential in determining $\chi_o(K_p \cup C_q)$.

**Theorem 4.2** (Thomason [22]). *Let $n$ be a positive integer number. If $\overrightarrow{K_{n+1}}$ is an oriented tournament on $n+1$ vertices, then $\overrightarrow{K_{n+1}}$ contains each oriented path $\overrightarrow{P_n}$ on $n$ vertices.*

To illustrate Theorem 4.2 we consider the collection of 4 non-isomorphic tournaments on 4 vertices containing each of the 3 non-isomorphic oriented paths on 3 vertices. We notice that the Theorem is tight, since the directed cyclic $\overrightarrow{K_3}$ does not contain 2 of the oriented paths on 3 vertices. Corollary 4.3 is obtained directly from Theorem 4.2.

**Corollary 4.3.** *Let $n$ be a positive integer number. If $\overrightarrow{K_{n+1}}$ is an oriented tournament on $n \geq 5$ vertices, then $\overrightarrow{K_{n+1}}$ contains each oriented path $\overrightarrow{P_4}$ on 4 vertices.*

Before we state Theorem 4.4 we need two definitions. A tournament $\overrightarrow{K_n}$ is regular if every vertex of $\overrightarrow{K_n}$ has the same degree. The Paley tournament $\overrightarrow{QR_7}$ on 7 vertices is the oriented graph with vertex set $V(\overrightarrow{QR_7})\{0, 1, \ldots, 6\}$ in which $ij$ is an arc if and only if $j - i \in \{1, 2, 4\}$.

**Theorem 4.4** (Havet and Thomassé [10]). *If $\overrightarrow{K_n}$ is an oriented tournament on $n$ vertices, then $\overrightarrow{K_n}$ contains each oriented path $\overrightarrow{P_n}$ on $n$ vertices, except when $\overrightarrow{K_n}$ is the directed cycle on $n = 3$ vertices, or $n = 5$ and $\overrightarrow{K_n}$ is regular, and when $\overrightarrow{K_n}$ is the Paley tournament on $n = 7$ vertices.*

In Lemma 4.5 and Theorem 4.6 we examine bounds of the oriented chromatic number of the disjoint union of complete graphs and cycles.

**Lemma 4.5.** *Let $p$ and $q$ be a pair of integers with $p \geq 1$ and $q \geq 3$, then $\chi_o(K_p \cup C_q) \geq p + 1$.*

*Proof.* We consider the possible values of $p$.

(1) If $p \leq 2$, from Theorem 4.1 we have that $\chi_o(K_p \cup C_q) = \chi_o(C_q) \geq 3 = 2 + 1 \geq p + 1$.
(2) If $p \geq 3$. Let $\overrightarrow{K_p}$ be a transitive orientation of $K_p$ and $\overrightarrow{C_q}$ be a cyclic orientation of $C_q$. We start assigning colors $1, 2, \ldots, p$ to the vertices of $\overrightarrow{K_p}$. Since $\overrightarrow{C_q}$ is cyclic, any oriented coloring of $\overrightarrow{C_q}$ using colors of $\{1, 2, \ldots, p\}$ induces a cycle on $\overrightarrow{K_p}$ contradicting the fact that $\overrightarrow{K_p}$ is transitive.

□

**Theorem 4.6.** *Let $p$ and $q$ be a pair of integers with $p \geq 1$ and $q \geq 3$, then*

$$\chi_o(K_p \cup C_q) = \begin{cases} 3, & \text{if } p \leq 2 \text{ and } q = 3, \\ 4, & \begin{cases} \text{if } p \leq 2 \text{ and } q = 4 \text{ or } q \geq 6, \\ \text{if } p = 3 \text{ and } q \neq 5, \end{cases} \\ 5, & \text{if } p \leq 3 \text{ and } q = 5, \\ p + 1, & \text{if } p \geq 4. \end{cases}$$

*Proof.* We consider the possible values of $p$.

(1) If $p \leq 2$, then $\chi_o(K_p \cup C_q) = \chi_o(C_q)$. From Theorem 4.1, $\chi_o(K_p \cup C_3) = 3$, $\chi_o(K_p \cup C_4) = \chi_o(K_p \cup C_q) = 4, q \geq 6$ and $\chi_o(K_p \cup C_5) = 5, p \leq 2$.
(2) Suppose $p = 3$. We consider two cases:
   (a) $q = 5$. From Theorem 4.1, $\chi_o(C_5) = 5$, then 5 colors are necessary to color $K_3 \cup C_5$. We show next that 5 colors are sufficient. Let $\overrightarrow{C_5}$ be an orientation of $C_5$. First we consider that $\overrightarrow{C_5}$ is cyclic with the following orientation with arcs $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1$. Sopena [19] showed that when $\overrightarrow{C_5}$ is cyclic, five different colors are needed to the vertices $v_1, v_2, v_3, v_4, v_5$, say color $i$ is assigned to $v_i$ with $i \in \{1, 2, 3, 4, 5\}$. In this case we assign colors 1, 2 and 4 to the vertices of $K_3$, observe that color 4 is not adjacent to either color 1 or color 2, hence $\chi_o(\overrightarrow{K_3} \cup \overrightarrow{C_5}) \leq 5$, for any orientation $\overrightarrow{K_3}$ of $K_3$. Now we consider that $\overrightarrow{C_5}$ is acyclic.

Sopena [19] showed that when $\overrightarrow{C_5}$ is acyclic, then $\chi_o(\overrightarrow{C_5}) \leq 4$. We observe that, in this case, at most 4 colors are given to vertices $v_1, v_2, v_3, v_4, v_5$, say colors $1,2,3,4$. Then we take the colors of the adjacent vertices $v_1$ and $v_2$ and an additional color 5 to assign to the 3 vertices of $K_3$. Hence $\chi_o(\overrightarrow{K_3} \cup \overrightarrow{C_5}) \leq 5$, for any orientation $\overrightarrow{K_3}$ of $K_3$. Hence, $\chi_o(K_3 \cup C_5) = 5$. Thus, $\chi_o(K_p \cup C_5) = 5, p \leq 3$.

(b) $q \neq 5$. From Theorem 4.1, $\chi_o(C_q) \leq 4$. We consider 2 cases:

(i) $q = 3$. From Corollary 3.9, $\chi_o(K_3 \cup C_3) = 4$.

(ii) $q = 4$ or $q \geq 6$. Sopena [19] proved that if $\overrightarrow{C_q}$ is cyclic, then $\overrightarrow{C_q}$ has an homomorphism to the oriented graph of order 4 with arcs $ab, bc, cd, ca$ and $da$ (Fig. 3). If $\overrightarrow{K_3}$ is transitive, we use the colors $a, c, d$ accordingly to color the vertices of $\overrightarrow{K_3}$. If $\overrightarrow{K_3}$ is cyclic, we use the colors $a, b, c$ suitably to color the vertices of $\overrightarrow{K_3}$. Suppose that $\overrightarrow{C_q}$ is acyclic. Let $V(\overrightarrow{C_q}) = \{v_0, v_1, \ldots, v_{q-1}\}$ be the set of vertices of $\overrightarrow{C_q}$. Since $\overrightarrow{C_q}$ is acyclic, there is a sink vertex in $\overrightarrow{C_q}$. Say that $v_0$ is a sink vertex. Since $\overrightarrow{C_q} - \{v_0\}$ is an oriented path, Sopena [19] proved that $\overrightarrow{C_q} - \{v_0\}$ has a homomorphism to the cyclic $\overrightarrow{C_3}$. Thus, we assign colors $1,2,3$ suitably to $\overrightarrow{C_q} - \{v_0\}$ and assign an additional color 4 to vertex $v_0$. If $\overrightarrow{K_3}$ is cyclic, we use colors $1,2,3$ to color the vertices of $\overrightarrow{K_3}$. If $\overrightarrow{K_3}$ is transitive, we suitably assign the colors $1,2,4$ to the vertices of $\overrightarrow{K_3}$. Hence, $\chi_o(K_3 \cup C_q) = 4$ with $q \neq 5$.

(3) Suppose $p \geq 4$. We consider 3 subcases:

(a) $q = 3$. From Lemma 4.5, $\chi_o(K_p \cup C_3) \geq p + 1$. Hence $p + 1$ colors are necessary to color $K_p \cup C_3$. Let $\overrightarrow{K_p} \cup \overrightarrow{C_3}$ be an orientation of $K_p \cup C_3$. We prove that $p + 1$ colors are sufficient to color $\overrightarrow{K_p} \cup \overrightarrow{C_3}$. We start assigning colors $1, 2, \ldots, p$ to the vertices of $\overrightarrow{K_p}$. Hence, 2 colors (say 1 and 2) of the colored $\overrightarrow{K_p}$ can be used with the additional color $p + 1$ to color $\overrightarrow{C_3}$, hence $\chi_o(\overrightarrow{K_p} \cup \overrightarrow{C_3}) \leq p + 1$. Therefore, $\chi_o(K_p \cup C_3) = p + 1$.

(b) $q = 5$. We consider 2 subcases:

(i) $p = 4$. From Theorem 4.1, $\chi_o(C_5) = 5$. Then 5 colors are necessary to color $K_4 \cup C_5$. Let $\overrightarrow{K_4} \cup \overrightarrow{C_5}$ be an orientation of $K_4 \cup C_5$. We show next that 5 colors are sufficient to color $\overrightarrow{K_4} \cup \overrightarrow{C_5}$. We start assigning colors $1,2,3,4$ to the vertices of $\overrightarrow{K_4}$. Let $\overrightarrow{C_5}$ be an orientation of $C_5$. Let $V(\overrightarrow{C_5}) = \{v_1, v_2, v_3, v_4, v_5\}$ be the set of vertices of $\overrightarrow{C_5}$. Let $\overrightarrow{P} = \overrightarrow{C_5}[\{v_1, v_2, v_3, v_4\}]$ be the induced path by the set of vertices $\{v_1, v_2, v_3, v_4\}$. From Theorem 4.4, the tournament $\overrightarrow{K_4}$ has $\overrightarrow{P}$ as a subgraph. We assign colors $1,2,3,4$ suitably to the vertices in the set $\{v_1, v_2, v_3, v_4\}$. Lastly, we assign an additional color 5 to vertex $v_5$, hence $\chi_o(\overrightarrow{K_4} \cup \overrightarrow{C_5}) \leq 5$. Therefore, $\chi_o(K_4 \cup C_5) = 5$.

(ii) $p \geq 5$. From Lemma 4.5, $\chi_o(K_p \cup C_5) \geq p + 1$. Hence $p + 1$ colors are necessary to color $K_p \cup C_5$. Let $\overrightarrow{K_p} \cup \overrightarrow{C_5}$ be an orientation of $K_p \cup C_5$. We show that $p + 1$ suffices. We start assigning colors $1, 2, \ldots p$ to the vertices of $\overrightarrow{K_p}$. Let $\overrightarrow{C_5}$ be an orientation of $C_5$. Let $V(\overrightarrow{C_5}) = \{v_1, v_2, v_3, v_4, v_5\}$ be the set of vertices of $\overrightarrow{C_5}$. Let $\overrightarrow{P_4} = \overrightarrow{C_5}[\{v_1, v_2, v_3, v_4\}]$ be the induced path by the set of vertices $\{v_1, v_2, v_3, v_4\}$. From Corollary 4.3, it is know that $\overrightarrow{K_p}$ contains $\overrightarrow{P_4}$ as a subgraph. Let $1,2,3,4$ be the colors assigned to this path $\overrightarrow{P_4}$ subgraph of $\overrightarrow{K_p}$. We assign colors $1,2,3,4$ suitably to the vertices $v_1, v_2, v_3, v_4$. Finally, we assign an additional color $p+1$ to the vertex $v_5$, hence $\chi_o(\overrightarrow{K_p} \cup \overrightarrow{C_5}) \leq p+1$. Therefore, $\chi_o(K_p \cup C_5) = p + 1$.

(c) $q = 4$ or $q \geq 6$. From Lemma 4.5, $\chi_o(K_p \cup C_q) \geq p+1$. Hence $p+1$ colors are necessary to color $K_p \cup C_q$. Let $\overrightarrow{K_p} \cup \overrightarrow{C_q}$ be an orientation of $K_p \cup C_q$. We show that $p+1$ colors are sufficient to color $\overrightarrow{K_p} \cup \overrightarrow{C_q}$. We start assigning colors $1, 2, \ldots, p$ to the vertices of $\overrightarrow{K_p}$.

Assume first that $\overrightarrow{C_q}$ is a cyclic orientation of $C_q$. Sopena [19] proved that if $\overrightarrow{C_q}$ is cyclic, then $\overrightarrow{C_q}$ has an homomorphism to the oriented graph of order 4 with arcs $ab, bc, cd, ca$ and $da$ in Figure 3. From Theorem 3.3, the tournament $\overrightarrow{K_p}$, $p \geq 4$, has has a transitive tournament $\overrightarrow{T} = \overrightarrow{K_3}$ as a subgraph of $\overrightarrow{K_p}$. Suppose that the colors assigned to the vertices of $\overrightarrow{T}$ are $1,2,3$. We assign colors $1,2,3$ accordingly

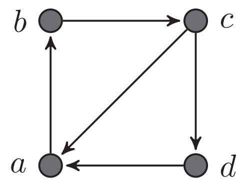


FIGURE 3. The color graph [19] for a cyclic orientation of $\overrightarrow{C_q}$ with $q = 4$ or $q \geq 6$.

to the vertices of $\overrightarrow{C_q}$ mapped by the vertices $a, c, d$. Finally, we assign an additional color $p + 1$ to the vertices of $\overrightarrow{C_q}$ mapped by the vertex $b$. Hence, $\chi_o(\overrightarrow{K_p} \cup \overrightarrow{C_q}) \leq p + 1$. Suppose now that $\overrightarrow{C_q}$ is an acyclic orientation of the cycle $C_q$. Let $V(\overrightarrow{C_q}) = \{v_0, v_1, \ldots v_{q-1}\}$ be the set of vertices of $\overrightarrow{C_q}$. Since $\overrightarrow{C_q}$ is acyclic there is a sink vertex in $\overrightarrow{C_q}$, say that $v_0$ is a sink vertex. Since $\overrightarrow{C_q} - \{v_0\}$ is an oriented path, Sopena [19] proved that there is a homomorphism from $\overrightarrow{C_q} - \{v_0\}$ to the cyclic $\overrightarrow{C_3}$. Suppose that $\overrightarrow{K_p}$ is not transitive. Harary and Moser [9] proved that $\overrightarrow{K_p}$ has at least one cyclic $\overrightarrow{T} = \overrightarrow{K_3}$ as a subgraph of $\overrightarrow{K_p}$. Suppose that the colors assigned to the vertices of $\overrightarrow{T}$ are $1, 2, 3$. We assign colors $1, 2, 3$ suitably to $\overrightarrow{C_q} - \{v_0\}$ and assign an additional color $p + 1$ to $v_0$. Hence, $\chi_o(\overrightarrow{K_p} \cup \overrightarrow{C_q}) \leq p + 1$. If $\overrightarrow{K_p}$ is transitive, we assign colors $1, 2, p + 1$ suitably to $\overrightarrow{C_q} - \{v_0\}$. As $\overrightarrow{K_p}$ is transitive, there is a sink vertex $v$ in $\overrightarrow{K_p}$. We assign the color of $v$, say color $p$, to $v_0$ – the sink vertex of $\overrightarrow{C_q}$. From both cases, $\chi_o(\overrightarrow{K_p} \cup \overrightarrow{C_q}) \leq p + 1$. Therefore, $\chi_o(K_p \cup C_q) = p + 1, p \geq 4$.

□

## 5. THE ORIENTED CHROMATIC NUMBER OF THE DISJOINT UNION OF ONE COMPLETE GRAPH WITH A FOREST

In this section we establish the oriented chromatic number of the disjoint union of one complete graph with a forest. As a byproduct, we characterize the class of graphs $\mathcal{CN}_3 = \{G \mid \chi_o(G) \leq 3\}$.

First we prove in Theorem 5.1 that the disjoint union of a complete $K_p$ with any graph $G$ having the cyclic $\overrightarrow{C_3}$ as a color graph and not having the transitive $\overrightarrow{C_3}$ as a color graph has chromatic number $p + 1$. We use Theorem 5.1 plus two results from Sopena on the oriented chromatic number of a forest stated on Theorems 5.2 and 5.3 in order to establish the oriented chromatic number of the union of a complete $K_p$ and a forest $F$ in Corollary 5.4. Lastly, we applied Corollary 5.4 in order to establish the family $\mathcal{CN}_3$ of graphs with oriented chromatic number at most 3.

**Theorem 5.1.** *If $p \geq 3$, $\chi_o(G) = 3$ and $G$ has an orientation $\overrightarrow{G}$ having a homomorphism to the cyclic $\overrightarrow{C_3}$ and not having a homomorphism to the transitite $\overrightarrow{C_3}$, then $\chi_o(K_p \cup G) = p + 1$.*

*Proof.* We can consider the tournamment $\overrightarrow{K_p}$ transitive and the orientation $\overrightarrow{G}$. We notice that every 3 colors of $\overrightarrow{K_p}$ induce a transitive $\overrightarrow{K_3}$. Hence, no 3 colors of $\overrightarrow{K_p}$ can be used in order to color the vertices of $\overrightarrow{G}$. Two colors of $\overrightarrow{K_p}$ can be used and an additional color $p + 1$ is necessary to color $\overrightarrow{G}$. Since, $p + 1$ suffices, we have that $\chi_o(K_p \cup G) = p + 1$. □

**Theorem 5.2** (Sopena [17]). *If $\overrightarrow{F}$ is an oriented forest, then $\chi_o(F) = \min\{3, \text{diam}(F) + 1\}$.*

**Theorem 5.3** (Sopena [17]). *Every oriented forest $\overrightarrow{F}$ has a homomorphism into a cyclic $\overrightarrow{C_3}$.*

TABLE 1. Oriented chromatic number $\chi_o(G \cup H)$ of the union $G \cup H$, where $G$ is complete graph $K_p$ and $H$ is a cycle $C_q$ or the forest $F$ of diameter $d$.

| G \ H | Forest diameter $d \leq 2$ (Cor. 5.4) | Forest diameter $d \geq 3$ (Cor. 5.4) | $C_q$, $q \geq 3$ (Thm. 4.6) |
|---|---|---|---|
| $K_p$, $p = 2$ | $d + 1$ | 3 | 3, if $\chi_o(C_q) = 3$<br>4, if $\chi_o(C_q) = 4$<br>5, if $\chi_o(C_q) = 5$ |
| $K_p$, $p = 3$ | 3 | 4 | 4, if $\chi_o(C_q) = 3, 4$<br>5, if $\chi_o(C_q) = 5$ |
| $K_p$, $p \geq 4$ | $p$ | $p + 1$ | $p + 1$ |

TABLE 2. Oriented chromatic number bounds of the union of 2 complete graphs $\chi_o(K_p \cup K_q), p \geq q$.

| $q$ | Lower bound | Upper bound |
|---|---|---|
| | $p + q - \nu(q), 1 \leq q \leq 27$ (Cor. 3.9)<br>$p + q - (2\lfloor \log_2 q \rfloor + 1), 28 \leq q$ (Cor. 3.10) | $p + q - \nu(q), 1 \leq q \leq 27$ (Cor. 3.9)<br>$p + q - (\lfloor \log_2 q/54 \rfloor + 7), 28 \leq q$ (Cor. 3.11) |
| $q \leq 2$ | $p$ | $p$ |
| $q = 3$ | $p + 1$ | $p + 1$ |
| $4 \leq q \leq 7$ | $p + q - 3$ | $p + q - 3$ |
| $8 \leq q \leq 13$ | $p + q - 4$ | $p + q - 4$ |
| $14 \leq q \leq 27$ | $p + q - 5$ | $p + q - 5$ |
| $28 \leq q \leq 31$ | $p + q - 9$ | $p + q - 7$ |
| $32 \leq q \leq 53$ | $p + q - 11$ | $p + q - 7$ |
| $54 \leq q \leq 63$ | $p + q - 11$ | $p + q - 8$ |
| $64 \leq q \leq 107$ | $p + q - 13$ | $p + q - 8$ |
| $108 \leq q \leq 127$ | $p + q - 13$ | $p + q - 9$ |
| $128 \leq q \leq 161$ | $p + q - 15$ | $p + q - 9$ |
| $162 \leq q \leq 215$ | $p + q - 15$ | $p + q - 10$ |
| $216 \leq q \leq 257$ | $p + q - 15$ | $p + q - 11$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

**Corollary 5.4.** $\chi_o(K_p \cup F) = \begin{cases} p + 1, & \text{if } p \geq 2 \text{ and } \mathrm{diam}(F) \geq 2 \\ p, & \text{if } p \geq 2 \text{ and } \mathrm{diam}(F) \leq 1 \\ \chi_o(F), & \text{if } p = 1 \end{cases}$

**Corollary 5.5.** $\mathcal{CN}_3 = \{G \mid G$ *is a forest, or* $G$ *is* $C_3$*, or* $G$ *is* $C_3 \cup F$ *where* $F$ *is a forest of stars*$\}$.

## 6. Conclusions and further work

In this paper we study the oriented chromatic number of disconnected graphs. We determine bounds on the oriented chromatic number of the disjoint union of two oriented graphs. We also proved bounds for the disjoint of two complete graphs $K_p$ and $K_q, p \geq q$ as $p + q - (2\lfloor \log_2(q) \rfloor + 1) \leq \chi_o(K_p \cup K_q) \leq p + q - (\lfloor \log_2 q/54 \rfloor + 7)$.

This result derives from the direct relation $\chi_o(K_p \cup K_q) = p + q - \nu(q)$ between the value $\chi_o(K_p \cup K_q)$ and $\nu(q)$, where $\nu(q)$ is the largest size of the transitive sub-tournament contained in every tournament of size $q$. The evaluation of the exact value of $\nu(q)$ is a challenging and interesting problem open for more than 60 years and widely studied by several authors, such as Stearns [21], Erdös and Moser [8], Alon [1], Sanchez-Flores [15] and Reid and Parker [14]. We notice that our result implies that any progress in bounding the value of $\nu(q)$ yields the same progress in bounding $\chi_o(K_p \cup K_q)$.

Additionally, we have established $\chi_o(K_p \cup C_q)$ and $\chi_o(K_p \cup F)$, where $F$ is a forest. We show an application of this later result with the characterization of the class $\mathcal{CN}_3$ of graphs.

Table 1 presents the state-of-art results obtained in this paper regarding to the union of a complete graph with a cycle and a forest. Table 2 presents the results regarding to the bounds for the union of a pair of complete graphs.

For future works we intend to shrink our bounds, expanding the study by including new classes and considering the cases when we have more than 2 connected components.

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