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ANALYSIS OF A RENEWAL ARRIVAL PROCESS SUBJECT TO GEOMETRIC CATASTROPHE WITH RANDOM BATCH KILLING

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Abstract. This paper considers a population model (system) which is prone to catastrophe that kills individuals in batches. Individuals enter the system in accordance with the renewal process and catastrophe occurs as per the Poisson process. The catastrophe attacks the population in a successive order in batches of random size, each batch of individuals dies with probability ξ. This successive process ends when the whole population is wiped out or a batch of individuals survives with probability $1-\xi$. This type of killing pattern is known as geometric catastrophe. The supplementary variable technique is used to develop the steady-state governing equations. Further using the difference operator, the distributions of population size are evaluated at arbitrary, pre-arrival, and post-catastrophe epochs. In addition to that, a few different measurements of the system's performance are derived. In order to demonstrate the applicability of the model, a number of numerical and graphical outcomes are presented in the form of tables and graphs.

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1. Introduction and literature review

Stochastic population models with mild catastrophes have gained much attention in recent years due to their wide application in a variety of areas such as computer-communication system, medical sciences, etc. When a catastrophe attacks on a population, and some individuals survive the attack, it is called the mild catastrophe, Brockwell *et al.* [4]. Based on the killing pattern, they introduced three types of mild catastrophes viz. geometric, binomial, and uniform. In case of geometric catastrophe, the catastrophe will strike the population in such a way that every individual of the population will die with a probability of ξ, and it will continue to do so in this manner either until a single individual survives or until the whole population perishes.

Artalejo *et al.* [1] studied the basic immigration process subject to binomial and geometric catastrophes in which they evaluated different performance measures such as: the length of time an arbitrary individual survives, the first extinction time, etc. Economou and Gómez-Corral [7] investigated a geometric catastrophe model wherein population increases stochastically according to a batch Markovian arrival process. They studied the semi-regenerative process, which defines the size of the population at random points in time. In their model, the embedded Markov chain (MC) at post-catastrophe epochs resulted in a MC of the $GI/G/1$-type, which is investigated with the help of its R- and G-measures. In addition, they also obtained the distribution of

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the time to extinction. Subsequently, the computational and numerical investigation of this model is recently carried out by Kumar *et al.* [15]. Moreover, Barbhuiya *et al.* [3] considered a batch renewal arrival mechanism that was vulnerable to geometric catastrophes induced by Poisson process. They evaluated the distribution of population size at random point and just before the arrival in explicit form using difference operator technique. For further information on the literature concerning mild catastrophes and their applications one can refer to [5, 6, 10, 12–14, 16, 17, 21] and the references therein.

According to the author's most reliable information, as of today the population model in which catastrophes attack only one individual at a time is discussed in the literature. There is no work available so far wherein individuals are attacked by the catastrophes in batches of variable size. Thus the objective of this work is to analyze a population model under geometric catastrophe with batch killing. The work is motivated by the real life application of the model that can be found in the area of medical science and is described below:
An infection in human blood is caused by the malarial parasite (catastrophe) that leads to sharp decline of the platelet count (population). The parasite gradually kills platelets in batches of random size and the killing process continues until a batch of platelets is immune to the parasite attack, which may happen with the help of antibiotics. This process can be well described by geometric catastrophe model with random batch killing. Mathematically, it can be assumed that catastrophe attacks on a batch of random size such that with probability y_i it attacks a batch of size i or less. In the literature such type of batch mechanism is known as Y-rule; [2, 18]. Similar pattern can also be observed in computer-communication systems where the stored data is affected by some type of viruses; [5].

In this work, we propose a geometric catastrophe model wherein individuals enter the system in accordance to the renewal process while catastrophes occur in accordance with Poisson process. In place of attacking single individual at a time, the catastrophe attacks on batch of individuals with Y-rule. That is, on arrival the catastrophe decides to attack individuals in batches of random size Y with finite support and finite mean $E(Y)$. Mathematically, catastrophe attacks on a batch of size i with $Prob(Y = i) = y_i,\ 1 \leq i \leq B$ where the maximum size of a batch a catastrophe can attack at a time is B.

Further, the catastrophe begins to wipe out each batch of individuals from the population with a probability of ξ in a sequential order until the entire population is wiped out or a batch survives (with probability $1 - \xi$).

The steady-state governing equations of the model is obtained using the supplementary variable technique (SVT). Further analysis is carried out using the difference equation technique, and distributions at arbitrary, pre-arrival, and post-catastrophe epochs are obtained. Finally, the performance measures of the systems are also discussed.

The rest of the work is organized as follows. In the upcoming section, we describe the model by describing the arrival process of individual, catastrophe, and the killing mechanism. In Section 3, we give a detailed analysis of the work, and obtain the distribution of the population size at various epochs viz., arbitrary, pre-arrival and post-catastrophe epochs. Performance measure of the system is provided in Section 4, whereas special cases of the model are given in Section 5. Lastly, a few numerical and graphical results are shown in Section 6 followed by the future work.

2. Model description

This section consists of the description of the model in detail and also describes the notations which are used in the analysis of the model.

- **Arrival process of the individuals:** Individuals enter the system according to a renewal generated process. That is, inter-arrival times of the individuals are independent and identically distributed random variables with cumulative distribution function $A(x)$, probability density function $a(x)$, Laplace-Stieltjes transform $A^*(s)$ and mean inter-arrival time $a_1 = -A^{*(1)}(0) = \frac{1}{\lambda}$ where λ is called the arrival rate and $A^{*(1)}(0) := \frac{d}{ds}A^*(s)\Big|_{s=0}$.

– **Arrival process of the catastrophes:** Catastrophes take place as per the Poisson process, *i.e.*, the inter-occurrence times of successive catastrophes are independent and exponentially distributed with parameter μ.
– **Batch mechanism:** On arrival, catastrophe decides to attack on individuals in batches of random size Y with finite support $\{1, 2, \ldots, B\}$ (where B is denoting the maximum size of the batch a catastrophe can attack at once) and finite mean $E(Y) = \overline{y}$. That is, catastrophe attacks on a batch of size i with probability $y_i = Prob(Y = i),\ 1 \le i \le B$ with $\sum_{i=1}^{B} y_i = 1$. If on arrival, a catastrophe decides to attack on a batch of size i but there are $j(< i)$ individuals in the system then the catastrophe will attack on a batch of j-individuals with probability y_i. In queueing theory, such type of rule is known as Y-rule; [2]. One may note here that for the real-life situations and practical purposes, it is assumed that the batch size distribution has finite support. However, the presented results are also valid for batch size distribution with infinite support.
– **Geometric batch killing:** When a catastrophe occurs, it starts striking the population sequentially and eliminates each batch of individuals with a fixed probability ξ till the whole population destroyed or a batch of individuals survive with probability $1 - \xi$.
– A catastrophe has no effect when there are no individuals in the system, and it is presumed that the process of attacking takes negligible time.
– The arrival process of individuals, the occurrence of catastrophes, and killing pattern are independent of each other.
– Let $\rho = \dfrac{\lambda(1-\xi)}{\mu\xi\overline{y}}$, then $\rho < 1$ ensures the stability of the system.

3. Analysis of the model

Let us define the following random variables:
– $N(t)$ = Population size of the system at time t.
– $U(t)$ = Remaining arrival-time of the next individual at time t.

Let us define,

$$\tilde{p}_n(u,t)du = Prob\{N(t) = n, u < U(t) \le u + du\},\ n \ge 0,\ u \ge 0,$$
$$\tilde{p}_n(t) = Prob\{N(t) = n\},\ n \ge 0.$$

Since our goal is to perform the steady-state investigation of the model, let us define

$$p_n(u) = \lim_{t\to\infty} \tilde{p}_n(u,t) \ \text{ and } \ p_n = \lim_{t\to\infty} \tilde{p}_n(t), \quad n \ge 0. \tag{1}$$

The states of the system can be related by using SVT at time t and $t + \Delta t$, and then taking limit $t \to \infty$, the steady-state equations are evaluated as:

$$-\frac{d}{du}p_0(u) = -\mu p_0(u) + \mu\sum_{i=1}^{B} y_i p_0(u) + \mu\sum_{i=1}^{B} y_i \sum_{k=1}^{\infty}\xi^k\left(\sum_{j=0}^{i-1} p_{ik-j}(u)\right), \tag{2}$$

$$-\frac{d}{du}p_n(u) = -\mu p_n(u) + p_{n-1}(0)a(u) + \mu\sum_{i=1}^{B} y_i \sum_{k=0}^{\infty}\xi^k(1-\xi)p_{ik+n}(u), \quad n \ge 1. \tag{3}$$

Let us define the following transformations for $\Re(s) > 0$ (where $\Re(s)$ denotes the real part of the complex number s):

$$A^*(s) = \int_0^\infty e^{-su}a(u)du. \tag{4}$$

$$P_n^*(s) = \int_0^\infty e^{-su}p_n(u)du, \quad n \ge 0. \tag{5}$$

Taking limit $s \to 0$ in (5), we obtain

$$P_n^*(0) = \int_0^\infty p_n(u)du \ = \ p_n, \quad n \geq 0. \tag{6}$$

Now multiplying (2)–(3) by e^{-su} and then using (4)–(5) after taking integration over the range of u, we finally get

$$(\mu - s)P_0^*(s) = \mu \sum_{i=1}^{B} y_i P_0^*(s) + \mu \sum_{i=1}^{B} y_i \sum_{k=1}^{\infty} \xi^k \left(\sum_{j=0}^{i-1} P_{ik-j}^*(s) \right) - p_0(0), \tag{7}$$

$$(\mu - s)P_n^*(s) = A^*(s)p_{n-1}(0) + \mu \sum_{i=1}^{B} y_i \sum_{k=0}^{\infty} \xi^k (1-\xi) P_{ik+n}^*(s) - p_n(0), \quad n \geq 1. \tag{8}$$

Before proceeding further, we obtain an important result from (7)–(8) which will be used in the further analysis. Adding (7)–(8), throughout the range of n, we obtain

$$\sum_{n=0}^{\infty} P_n^*(s) = \frac{1 - A^*(s)}{s} \sum_{n=0}^{\infty} p_n(0). \tag{9}$$

Setting limit $s \to 0$ in (9), and using $\sum\limits_{n=0}^{\infty} p_n = 1$, we get

$$\sum_{n=0}^{\infty} p_n(0) = \frac{-1}{A^{*(1)}(0)} \ = \ \lambda. \tag{10}$$

This equation indicates that the probability density of the event that the system state is just before the arrival of an individual is equal to the mean arrival rate of individuals, λ; for details see equation (10) of Komota *et al.* [11].

3.1. Population size distribution at arbitrary epoch

Now to obtain $p_n, n \geq 0$, we use the difference operator technique, see [2, 3, 14]. Let us define the difference operator $\mathcal{D}$ such that

$$\mathcal{D}P_n^*(s) = P_{n+1}^*(s) \text{ and } \mathcal{D}p_n(0) = p_{n+1}(0). \tag{11}$$

By introducing the operator $\mathcal{D}$, (8) can be expressed as

$$\left(\mu - s - \mu(1-\xi) \sum_{i=1}^{B} y_i \sum_{k=0}^{\infty} (\xi \mathcal{D}^i)^k \right) P_n^*(s) = \left(A^*(s) - \mathcal{D}\right) p_{n-1}(0), \quad n \geq 1. \tag{12}$$

Setting $s = \mu - \mu(1-\xi) \sum\limits_{i=1}^{B} y_i \sum\limits_{k=0}^{\infty} (\xi \mathcal{D}^i)^k$ in (12), we get

$$\left(A^* \left(\mu - \mu(1-\xi) \sum_{i=1}^{B} y_i \sum_{k=0}^{\infty} (\xi \mathcal{D}^i)^k \right) - \mathcal{D} \right) p_n(0) = 0, \quad n \geq 0. \tag{13}$$

The auxiliary equation corresponding to (13) is:

$$A^* \left(\mu - \mu(1-\xi) \sum_{i=1}^{B} \frac{y_i}{1 - \xi z^i} \right) - z = 0. \tag{14}$$

Using the theory of linear difference equations with constant coefficients, for details see [8,9,20], the solution of (13) is given by

$$p_n(0) = c\alpha^n, \quad n \geq 0, \tag{15}$$

where α is the only root of (14) that lies inside the unit circle (see Appendix Thm. A.1) and c is the corresponding constant which can be obtained by taking the summation of (15) over the range of n from 0 to ∞ and then using (10), that is

$$\sum_{n=0}^{\infty} p_n(0) = \sum_{n=0}^{\infty} c\alpha^n \;=\; \lambda. \tag{16}$$

After simplification (16) gives

$$c = \lambda(1-\alpha). \tag{17}$$

Now using (15) in (12), we get

$$\left(\mu - s - \mu(1-\xi)\sum_{i=1}^{B} y_i \sum_{k=0}^{\infty} (\xi \mathcal{D}^i)^k\right) P_n^*(s) = \left(A^*(s) - \mathcal{D}\right) c\alpha^{n-1}, \quad n \geq 1. \tag{18}$$

One can easily show that the homogeneous solution of (18) is zero; see [3]. Now the particular (and hence general) solution of (18) is given by

$$P_n^*(s) = \frac{\left(A^*(s) - \alpha\right) c\alpha^{n-1}}{\mu - s - \mu(1-\xi)\sum\limits_{i=1}^{B} y_i \sum\limits_{k=0}^{\infty} (\xi\alpha^i)^k}, \quad n \geq 1. \tag{19}$$

Now setting $s = 0$ in (19) and using (17), after some simplification, we get

$$p_n \;=\; P_n^*(0) = \frac{\lambda(1-\alpha)^2\alpha^{n-1}}{\mu - \mu(1-\xi)\sum\limits_{i=1}^{B} \dfrac{y_i}{1-\xi\alpha^i}}, \quad n \geq 1, \tag{20}$$

and using $\sum\limits_{n=0}^{\infty} p_n = 1$, (20) gives

$$p_0 = 1 - \frac{\lambda(1-\alpha)}{\mu - \mu(1-\xi)\sum\limits_{i=1}^{B} \dfrac{y_i}{1-\xi\alpha^i}}\cdot \tag{21}$$

Thus, $p_n, n \geq 0$ can be evaluated from (20)–(21).

3.2. Population size distribution at pre-arrival and post-catastrophe epochs

Let us define p_n^- as the probability of population size be $n(\geq 0)$ just before the arrival of an individual (pre-arrival epoch).

Theorem 3.1. *Population size distribution at pre-arrival epoch, p_n^-, is given by*

$$p_n^- = (1-\alpha)\alpha^n, \quad n \geq 0. \tag{22}$$

Proof. Since $p_n(0)$ is the probability density of the event that there are n individuals in the system and an individual is about to arrive. Therefore the relationship between $p_n(0)$ and p_n^- is given by (for details see [2])

$$p_n^- \propto p_n(0), \quad n \geq 0, \quad \Rightarrow \quad p_n^- = Kp_n(0), \quad n \geq 0, \tag{23}$$

where K is the proportionality constant which can be found by using the normalizing condition $\sum\limits_{n=0}^{\infty} p_n^- = 1$. Thus from (23), we have

$$p_n^- = \frac{p_n(0)}{\sum\limits_{j=0}^{\infty} p_j(0)}, \quad n \geq 0. \tag{24}$$

Using (15) in (24), we finally get

$$p_n^- = (1-\alpha)\alpha^n, \quad n \geq 0, \tag{25}$$

which proves the theorem. □

Now, let us define p_n^+ as the probability of population size be $n(\geq 0)$ just after the occurrence of a catastrophe (post-catastrophe epoch).

Theorem 3.2. *Population size distribution at post-catastrophe epoch, p_n^+, is given by*

$$p_0^+ = 1 - \left(\sum_{j=1}^{B} \frac{y_j}{1-\xi\alpha^j}\right) \frac{\lambda(1-\alpha)(1-\xi)}{\mu - \mu(1-\xi)\sum\limits_{i=1}^{B} \frac{y_i}{1-\xi\alpha^i}}, \tag{26}$$

$$p_n^+ = \left(\sum_{j=1}^{B} \frac{y_j}{1-\xi\alpha^j}\right) \frac{\lambda(1-\alpha)^2(1-\xi)\alpha^{n-1}}{\mu - \mu(1-\xi)\sum\limits_{i=1}^{B} \frac{y_i}{1-\xi\alpha^i}}, \quad n \geq 1. \tag{27}$$

Proof. The relationship between p_n^+ and p_n is given as

$$p_0^+ = \sum_{i=1}^{B} y_i \left(p_0 + \sum_{k=1}^{\infty} \xi^k \sum_{j=0}^{i-1} p_{ik-j}\right), \tag{28}$$

$$p_n^+ = \sum_{i=1}^{B} y_i \sum_{k=0}^{\infty} \xi^k (1-\xi) p_{ik+n}, \quad n \geq 1. \tag{29}$$

After substituting the values of p_n, $n \geq 0$, from (20)–(21) into (28)–(29) and after doing some algebraic elaboration and simplification, we finally get the results (26)–(27). □

With this, the analysis to evaluate the distribution of the population size is completed at arbitrary, pre-arrival, and post-catastrophe epochs.

4. Performance measures of the system

Once the population size distributions are obtained, we can discuss about the system's performance measures such as mean (average), variance, factorial moments of the population size.

Let $L(L^-, L^+)$, $Var(Var^-, Var^+)$, and $FM_n(FM_n^-, FM_n^+)$ be the average, variance, and nth (≥ 1) factorial moment of the number of individuals in the system at arbitrary (pre-arrival, post-catastrophe) epoch, respectively. Then we have

$$L = \sum_{n=1}^{\infty} n p_n, \tag{30}$$

$$Var = \sum_{n=1}^{\infty} n(n-1)p_n + L - L^2, \tag{31}$$

$$FM_n = \sum_{i=n}^{\infty} i(i-1)\dots(i-n+1)p_i, \quad n \geq 1. \tag{32}$$

The performance measures L^-, L^+, Var^-, Var^+, FM_n^-, and FM_n^+ can be obtained from (30)–(32), by simply replacing p_n by p_n^- and p_n^+. Additionally, one can look for the expressions of mean, variance, and factorial moments explicitly as given in the following lemmas.

Lemma 4.1. *The means L, L^-, L^+ and variances Var, Var^-, Var^+, are given by*

$$L = \frac{\lambda}{\Psi}, \qquad Var = \frac{\lambda}{\Psi}\frac{(1+\alpha)}{(1-\alpha)} - L^2, \tag{33}$$

$$L^- = \frac{\alpha}{1-\alpha}, \qquad Var^- = \frac{\alpha}{(1-\alpha)^2}, \tag{34}$$

$$L^+ = \frac{\lambda(1-\xi)\Upsilon}{\Psi}, \qquad Var^+ = \frac{\lambda(1-\xi)\Upsilon}{\Psi}\frac{(1+\alpha)}{(1-\alpha)} - (L^+)^2, \tag{35}$$

where $\Psi = \mu - \mu(1-\xi)\sum_{i=1}^{B}\frac{y_i}{1-\xi\alpha^i}$ *and* $\Upsilon = \sum_{j=1}^{B}\frac{y_j}{1-\xi\alpha^j}$.

Proof. Replacing the values of p_n, p_n^-, and p_n^+ in (30)–(31), and after some simplification we obtain the required results as given in (33)–(35). □

Lemma 4.2. *The nth (≥ 1) factorial moment FM_n, FM_n^-, and FM_n^+ are given by*

$$FM_n = \frac{\lambda}{\Psi}\frac{\alpha^{n-1}n!}{(1-\alpha)^{n-1}}, \tag{36}$$

$$FM_n^- = \frac{\alpha^n n!}{(1-\alpha)^n}, \tag{37}$$

$$FM^+ = \frac{\lambda(1-\xi)\Upsilon}{\Psi}\frac{\alpha^{n-1}n!}{(1-\alpha)^{n-1}}. \tag{38}$$

Proof. Substituting the values of p_i in (32), we obtain

$$FM_n = \frac{\lambda(1-\alpha)^2}{\Psi}\sum_{i=n}^{\infty} i(i-1)\cdots(i-n+1)\alpha^{i-1}, \tag{39}$$

$$FM_n = \frac{\lambda(1-\alpha)^2}{\Psi}\alpha^{n-1}n!\sum_{i=n}^{\infty}\binom{i}{n}\alpha^{i-n}, \tag{40}$$

$$FM_n = \frac{\lambda(1-\alpha)^2}{\Psi}\frac{\alpha^{n-1}n!}{(1-\alpha)^{n+1}}. \tag{41}$$

After simplification (41) gives the required result (36). Similarly, one can get the values of FM_n^- and FM_n^+ as given in (37) and (38), respectively. □

5. Some selected special cases of the model

Here, we discuss a few selected special cases, and compare its results with the existing ones.

5.1. The killing mechanism follows $(1, B)$-rule

Let us assume that catastrophes attack on batches of size $i(=\min\{B, b^*\})$-individuals $(1 \leq i \leq B)$ where b^* is the number of individuals in the system which has not been attacked by the catastrophe. Such type of rule is known as $(1, B)$-rule; [19].

For this, setting $y_B = 1$ and $y_i = 0,\ i \geq 1;\ i \neq B$ in (14), the auxiliary equation becomes:

$$A^*\left(\mu - \frac{\mu(1-\xi)}{1-\xi z^B}\right) - z = 0, \tag{42}$$

which has one root inside the unit circle under $\dfrac{\lambda(1-\xi)}{\mu\xi B} < 1$. Further, from (20), (21), (25), (26) and (27), we obtain the following probability distributions at various epochs:

$$p_n = \frac{\lambda(1-\alpha)^2(1-\xi\alpha^B)\alpha^{n-1}}{\mu\xi(1-\alpha^B)},\ n \geq 1;\quad p_0 \ = \ 1 - \frac{\lambda(1-\alpha)(1-\xi\alpha^B)}{\mu\xi(1-\alpha^B)}, \tag{43}$$

$$p_n^- = (1-\alpha)\alpha^n, \qquad n \geq 0, \tag{44}$$

$$p_n^+ = \frac{\lambda(1-\alpha)^2(1-\xi)\alpha^{n-1}}{\mu\xi(1-\alpha^B)},\ n \geq 1;\qquad p_0^+ \ = \ 1 - \frac{\lambda(1-\alpha)(1-\xi)}{\mu\xi(1-\alpha^B)}. \tag{45}$$

The means and variances at various epochs for $(1, B)$-rule are given by

$$\begin{aligned} L &= \frac{\lambda(1-\xi\alpha^B)}{\mu\xi(1-\alpha^B)}, & Var &= \frac{\lambda(1-\xi\alpha^B)}{\mu\xi(1-\alpha^B)}\frac{(1+\alpha)}{(1-\alpha)} - L^2, \\ L^- &= \frac{\alpha}{1-\alpha}, & Var^- &= \frac{\alpha}{(1-\alpha)^2}, \\ L^+ &= \frac{\lambda(1-\xi)}{\mu\xi(1-\alpha^B)}, & Var^+ &= \frac{\lambda(1-\xi)}{\mu\xi(1-\alpha^B)}\frac{(1+\alpha)}{(1-\alpha)} - (L^+)^2. \end{aligned}$$

5.2. Catastrophes attack single individual at a time

Let us assume that catastrophes attack only one individual at a time, [3]. For this setting $B = 1, i.e., y_1 = 1$ and $y_i = 0,\ i \geq 2$ in (14), the auxiliary equation becomes:

$$A^*\left(\mu - \frac{\mu(1-\xi)}{1-\xi z}\right) - z = 0, \tag{46}$$

which has one root inside the unit circle under $\dfrac{\lambda(1-\xi)}{\mu\xi} < 1$. Further, from (20), (21), (25), (26) and (27), we obtain the following probability distributions:

$$p_n = \frac{\lambda(1-\xi\alpha)(1-\alpha)\alpha^{n-1}}{\mu\xi},\ \ n \geq 1;\quad p_0 \ = \ 1 - \frac{\lambda(1-\xi\alpha)}{\mu\xi}, \tag{47}$$

$$p_n^- = (1-\alpha)\alpha^n, \quad n \geq 0, \tag{48}$$

$$p_n^+ = \frac{\lambda(1-\xi)(1-\alpha)\alpha^{n-1}}{\mu\xi},\quad n \geq 1;\quad p_0^+ \ = \ 1 - \frac{\lambda(1-\xi)}{\mu\xi}. \tag{49}$$

Equations (47)–(48) exactly match with the equations (27)–(29) of Barbhuiya *et al.* [3]. The means and variances at various epochs for $B = 1$ are given by

$$\begin{aligned} L &= \frac{\lambda(1-\xi\alpha)}{\mu\xi(1-\alpha)}, & Var &= \frac{\lambda(1-\xi\alpha)(1+\alpha)}{\mu\xi(1-\alpha)^2} - L^2, \\ L^- &= \frac{\alpha}{1-\alpha}, & Var^- &= \frac{\alpha}{(1-\alpha)^2}, \\ L^+ &= \frac{\lambda(1-\xi)}{\mu\xi(1-\alpha)}, & Var^+ &= \frac{\lambda(1-\xi)(1+\alpha)}{\mu\xi(1-\alpha)^2} - (L^+)^2. \end{aligned}$$

5.3. Population grows in accordance with the Poisson process, and Catastrophes attack single individual at a time

Let us assume that individuals arrive to the system as per the Poisson process with parameter λ, then $A^*(s) = \dfrac{\lambda}{\lambda+s}$. Further, catastrophes attack only one individual at a time,[3], *i.e.*, $B = 1$. Thus, for $B = 1$ ($i.e., y_1 = 1$ and $y_i = 0,\ i \geq 2$) the root of the characteristic equation that lies inside the unit circle is given by

$$\alpha = \frac{\lambda}{\xi(\lambda+\mu)}.$$

Now from (20), (21), (25), (26) and (27), we get the following probability distributions:

$$p_n = p_n^- = \left(1 - \frac{\lambda}{\xi(\lambda+\mu)}\right)\left(\frac{\lambda}{\xi(\lambda+\mu)}\right)^n, \quad n \geq 0, \tag{50}$$

$$p_n^+ = (1-\xi)\left(1 - \frac{\lambda(1-\xi)}{\mu\xi}\right)\left(\frac{\lambda}{\xi(\lambda+\mu)}\right)^n, \quad n \geq 1; \quad p_0^+ = 1 - \frac{\lambda(1-\xi)}{\mu\xi}. \tag{51}$$

Equation (50) exactly match with the equation (30) of Barbhuiya *et al.* [3], and (50)–(51) exactly match with the equations (33)–(35) of Kumar and Gupta [13]. Further, the probability generating function at arbitrary and post-catastrophe epochs obtained from (50) and (51) also matches with the one derived from the equations (40) and (8), respectively, of [7].

In this case, the means and variances are given as

$$\begin{aligned} L = L^- &= \frac{\lambda}{\xi(\lambda+\mu)-\lambda}, & Var = Var^- &= \frac{\lambda\xi(\lambda+\mu)}{(\xi(\lambda+\mu)-\lambda)^2}, \\ L^+ &= \frac{\lambda(1-\xi)(\lambda+\mu)}{\mu(\xi(\lambda+\mu)-\lambda)}, & Var^+ &= \frac{\lambda(1-\xi)(\lambda+\mu)(\xi(\lambda+\mu)^2-\lambda^2)}{\mu^2(\xi(\lambda+\mu)-\lambda)^2}. \end{aligned}$$

6. Numerical and graphical investigation

We have carried out a comprehensive numerical study based on the computational technique discussed in the previous sections. For want of space, we report here only a few selected results. All calculations have been carried out using a laptop with specifications as AMD Ryzen 5 5500U with 8.00 GB of RAM and Radeon Graphics 2.10 GHz using MAPLE 18 and for the sake of brevity the final results are provided up to 6 decimal places.

The numerical analysis has been carried out for three inter-arrival time distributions, viz. Exponential (M), Erlang (E_4), and Hyper-exponential (HE_2) with parameters $\lambda = 10, \mu = 10, \xi = 0.6, y_i = (0.4)^i\Big(\sum\limits_{j=1}^{B}(0.4)^j\Big)^{-1},\ 1 \leq i \leq 5(= B)$, (for HE_2: $a_1 = \frac{1}{3}, a_2 = \frac{2}{3}, \lambda_1 = 5.8578639, \lambda_2 = 15.46918338$ that gives $\lambda = 10$). In all the cases $\rho = 0.41281281 < 1$ which insures the stability of the system. The distribution of

TABLE 1. Population size distribution at various epochs for different inter-arrival timedistributions.

	Exponential (M)			Erlang (E_4)			Hyper-exponential (HE_2)		
n	p_n	p_n^-	p_n^+	p_n	p_n^-	p_n^+	p_n	p_n^-	p_n^+
0	0.25104454	0.25104454	0.50208908	0.19397213	0.29727479	0.49124692	0.27480929	0.23194983	0.50675912
1	0.18802118	0.18802118	0.12499782	0.23961177	0.20890249	0.15123947	0.16820786	0.17814911	0.11440714
2	0.14081949	0.14081949	0.09361780	0.16838123	0.14680105	0.10627979	0.12919208	0.13682745	0.08787042
3	0.10546752	0.10546752	0.07011556	0.11832573	0.10316080	0.07468548	0.09922600	0.10509035	0.06748889
4	0.07899048	0.07899048	0.05251343	0.08315048	0.07249369	0.05248337	0.07621054	0.08071466	0.05183486
5	0.05916035	0.05916035	0.03933022	0.05843194	0.05094314	0.03688139	0.05853352	0.06199291	0.03981177
6	0.04430847	0.04430847	0.02945658	0.04106159	0.03579903	0.02591748	0.04495668	0.04761366	0.03057744
7	0.03318507	0.03318507	0.02206167	0.02885502	0.02515688	0.01821287	0.03452899	0.03656968	0.02348501
8	0.02485414	0.02485414	0.01652321	0.02027715	0.01767838	0.01279864	0.02651999	0.02808735	0.01803766
9	0.01861464	0.01861464	0.01237515	0.01424926	0.01242304	0.00899393	0.02036869	0.02157249	0.01385383
10	0.01394154	0.01394154	0.00926843	0.01001332	0.00872998	0.00632026	0.01564417	0.01656876	0.01064044
20	0.00077423	0.00077423	0.00051471	0.00029406	0.00025637	0.00018561	0.00111750	0.00118355	0.00076007
30	0.00004300	0.00004300	0.00002858	0.00000864	0.00000753	0.00000545	0.00007983	0.00008454	0.00005429
40	0.00000239	0.00000239	0.00000159	0.00000025	0.00000022	0.00000016	0.00000570	0.00000604	0.00000388
50	0.00000013	0.00000013	0.00000009	0.00000001	0.00000001	0.00000000	0.00000041	0.00000043	0.00000028
60	0.00000001	0.00000001	0.00000000	0.00000000	0.00000000	0.00000000	0.00000003	0.00000003	0.00000002
≥70	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
Sum	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
Mean	2.98335689	2.98335689	1.98335689	2.71138993	2.36389103	1.71138993	3.12649814	3.31127718	2.12649814
Var	11.88377524	11.88377524	9.88377524	8.17861525	7.95187184	6.87361305	14.05691144	14.27583377	11.68735335

TABLE 2. Factorial moments of the population size distribution at various epochs for different inter-arrival distributions.

	Exponential (M)			Erlang (E_4)			Hyper-exponential (HE_2)		
n	FM_n	FM_n^-	FM_n^+	FM_n	FM_n^-	FM_n^+	FM_n	FM_n^-	FM_n^+
1	2.9834E+00	2.9834E+00	1.9834E+00	2.7114E+00	2.3639E+00	1.7114E+00	3.1265E+00	3.3113E+00	2.1265E+00
2	1.7801E+01	1.7801E+01	1.1834E+01	1.2819E+01	1.1176E+01	8.0911E+00	2.0705E+01	2.1929E+01	1.4083E+01
3	1.5932E+02	1.5932E+02	1.0592E+02	9.0907E+01	7.9256E+01	5.7379E+01	2.0568E+02	2.1784E+02	1.3990E+02
4	1.9012E+03	1.9012E+03	1.2639E+03	8.5958E+02	7.4941E+02	5.4255E+02	2.7243E+03	2.8853E+03	1.8529E+03
5	2.8360E+04	2.8360E+04	1.8854E+04	1.0160E+04	8.8576E+03	6.4127E+03	4.5105E+04	4.7770E+04	3.0678E+04
6	5.0765E+05	5.0765E+05	3.3749E+05	1.4410E+05	1.2563E+05	9.0953E+04	8.9612E+05	9.4909E+05	6.0950E+05
7	1.0601E+07	1.0601E+07	7.0479E+06	2.3844E+06	2.0788E+06	1.5050E+06	2.0771E+07	2.1999E+07	1.4128E+07
8	2.5302E+08	2.5302E+08	1.6821E+08	4.5093E+07	3.9313E+07	2.8462E+07	5.5023E+08	5.8275E+08	3.7424E+08
9	6.7938E+09	6.7938E+09	4.5165E+09	9.5934E+08	8.3639E+08	6.0552E+08	1.6398E+10	1.7367E+10	1.1153E+10
10	2.0268E+11	2.0268E+11	1.3474E+11	2.2678E+10	1.9771E+10	1.4314E+10	5.4298E+11	5.7507E+11	3.6931E+11

the population size at arbitrary (p_n), pre-arrival (p_n^-), and post-catastrophe (p_n^+) epochs are presented in the Table 1, and the first 10 factorial moments are presented in the Table 2. Further, average (mean) and variance corresponding to every distribution are given in the last two rows of Table 1.

After discussing the special cases and some selected numerical results, we present the effect of different parameters on the performance of the system.

In Figure 1, we have shown the effect of ξ on L for three inter-arrival distributions, viz. M, E_4, HE_2. For this, the selected parameters are: $\lambda = 5, \mu = 15, y_i = (0.4)^i\left(\sum_{j=1}^{B}(0.4)^j\right)^{-1}$, $1 \le i \le 5(= B)$, and ξ varies from 0.2 to 0.85. For each distribution, the value of L decreases as ξ increases since a greater number of individuals died with the increment in ξ which gives a decreased population and hence a low value of L. The decrement in L is substantial as ξ goes from 0.2 to 0.3, after that the decrement in L is very small. Further, for any fix ξ, the value of L is larger for $HE_2(CV = 1.5)$ followed by $M(CV = 1.0)$ and $E_4(CV = 0.5)$ which is due to the coefficient of variation $\left(CV = \frac{\sqrt{variance}}{mean}\right)$ of the distributions. Moreover, for large value of ξ, the inter-arrival distributions have very little impact on L.

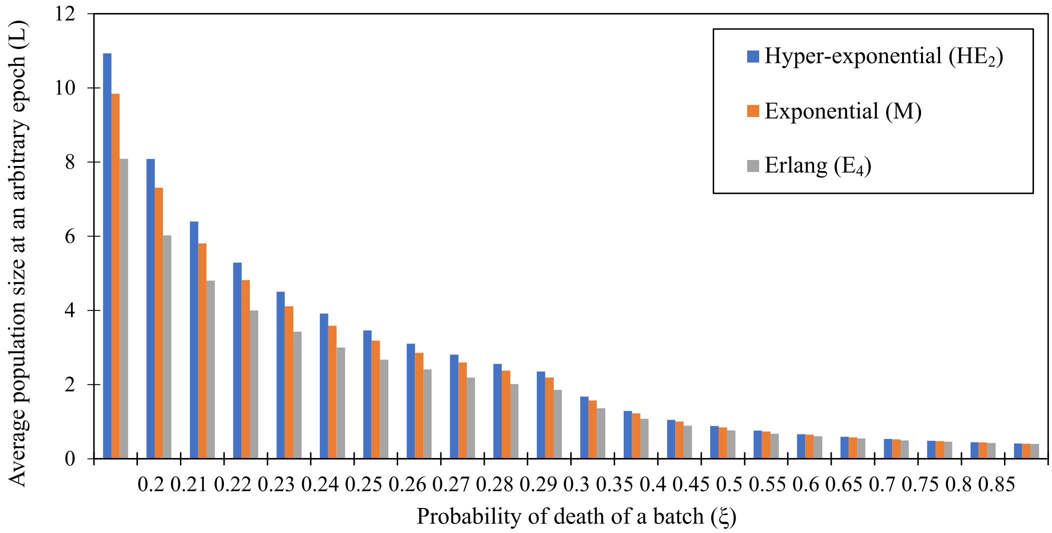

FIGURE 1. L *vs.* ξ for different inter-arrival time distributions.

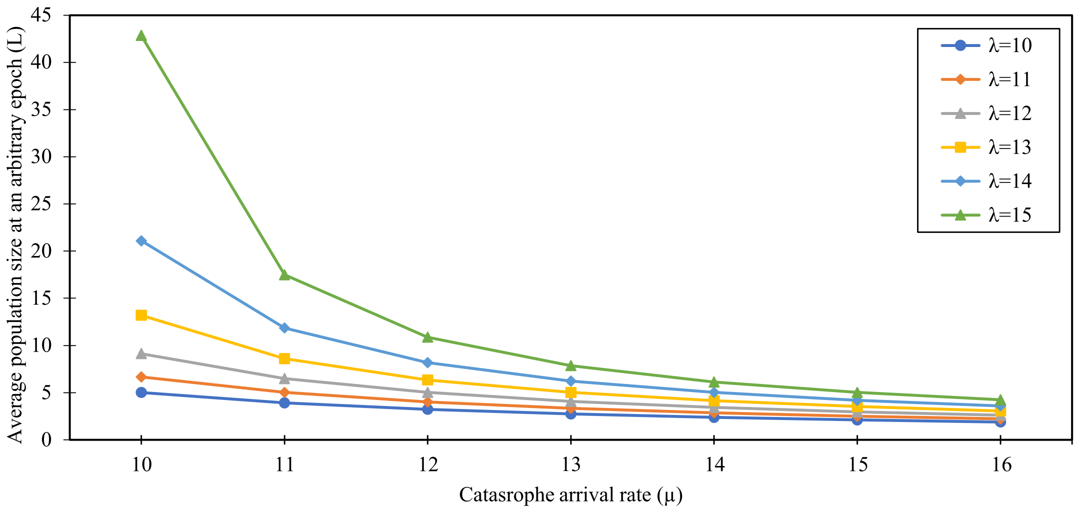

FIGURE 2. L *vs.* μ for different values of λ.

In Figure 2, the impact of μ on L is shown for different values of λ for inter-arrival time distribution as E_4 and parameters are: $B = 5, \xi = 0.5$, λ is varying from 10 to 15, and μ from 10 to 16. It can be seen from the Figure 2 that for any value of μ, population size increases with λ which increases L. Further, for any fixed value of λ, a greater number of catastrophe takes place in the system as μ increases which gives more number of deaths, thus decrement in the value of L.

In Figure 3, the impact of μ on L is presented for different batch size B. Here, we have chosen the inter-arrival time distribution as E_4 with parameters: $\lambda = 5, \xi = 0.5$, $y_i = (0.4)^i \Big(\sum_{j=1}^{B} (0.4)^j \Big)^{-1}$, $1 \le i \le B$, and B takes values from 1 to 8, and μ goes from 10 to 16. It can be seen from Figure 3 that for any value of μ, increment in B gives a greater number of deaths that gives to a reduced value of L. However, for a large value of μ (say 14),

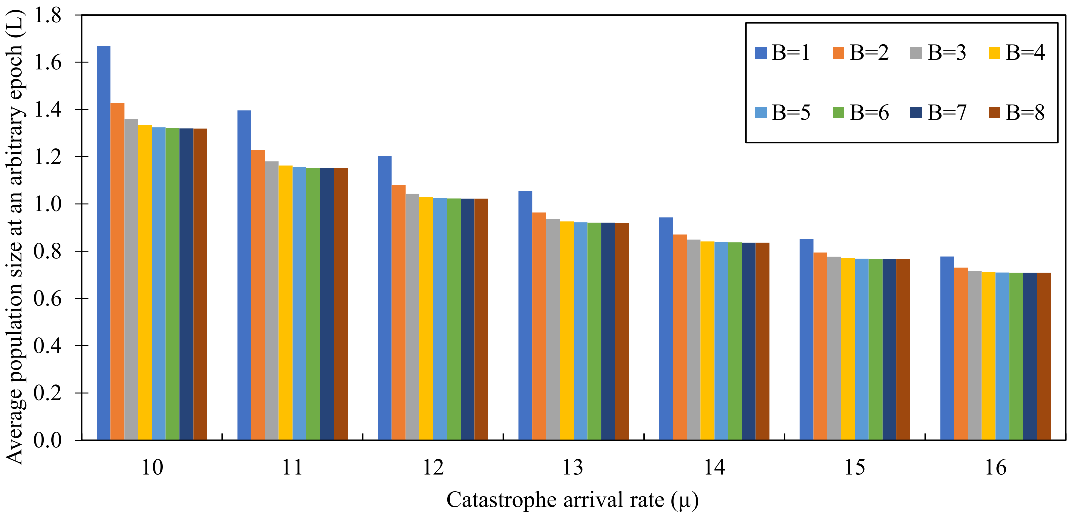

FIGURE 3. L *vs.* μ for different values of B.

there is insignificant effect on L as B increases sufficiently large. Thus, we can conclude that for small values of B it affects the system performance but for a large value of B with large μ, variation in L is insignificant. Hence, one cannot ignore small batch sizes. Further, for any fixed value of B, the value of L decreases as μ increases.

After analyzing the impact of all the parameters on the system's performance, we can conclude that:

– Increment in L takes place as λ increases.
– Decrement in L takes place as μ, ξ, B increases.
– Based on the values of CV, inter-arrival time distributions affects L.

Hence the effect of random batch size and inter-arrival time distributions can not be neglected.

7. Future work

In this work, we have studied a catastrophe model with the Y-rule batch killing mechanism. One might also consider analyzing a population model in which the probability of killing depends on the size of the population. It would be a far more exciting and challenging problem from analytic point of view.

Appendix A.

Theorem A.1. *$A^*\left(\mu - \mu(1-\xi)\sum_{i=1}^{B}\frac{y_i}{1-\xi z^i}\right) - z$ has only one zero in $|z|<1$.*

Proof. Let $g_1(z) = -z$ and $g_2(z) = A^*\left(\mu - \mu(1-\xi)\sum_{i=1}^{B}\frac{y_i}{1-\xi z^i}\right)$. Now on $|z| = 1-\delta$, we have

$$|g_1(z)|_{|z|=1-\delta} = |z|_{|z|=1-\delta} = 1-\delta,$$

and

$$\begin{aligned}
|g_2(z)|_{|z|=1-\delta} &= \left|A^*\left(\mu-\mu(1-\xi)\sum_{i=1}^{B}\frac{y_i}{1-\xi z^i}\right)\right|_{|z|=1-\delta},\\
&\le A^*\left(\left|\mu-\mu(1-\xi)\sum_{i=1}^{B}\frac{y_i}{1-\xi z^i}\right|\right)\Big|_{|z|=1-\delta},\\
&= A^*(0)-\delta A^{*(1)}(0)\left(-\frac{\mu\xi}{1-\xi}\sum_{i=1}^{B}iy_i\right)+o(\delta),\\
&= 1-\delta\left(\frac{\mu\xi}{\lambda(1-\xi)}\sum_{i=1}^{B}iy_i\right)+o(\delta).
\end{aligned}$$

Thus we have $|g_2(z)|<|g_1(z)|$ on $|z|=1-\delta$ if and only if $\dfrac{\lambda(1-\xi)}{\mu\xi\sum_{i=1}^{B}iy_i}<1$. Now using the Rouche's theorem, $A^*\left(\mu-\mu(1-\xi)\sum_{i=1}^{B}\frac{y_i}{1-\xi z^i}\right)-z$ has only one zero in $|z|<1$. □

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