RAIRO-Oper. Res. 58 (2024) 847–864
https://doi.org/10.1051/ro/2023172

RAIRO Operations Research
www.rairo-ro.org

AN EXACT SECOND ORDER CONE PROGRAMMING APPROACH FOR TRAFFIC ASSIGNMENT PROBLEMS

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Abstract. Demographic changes, urbanization and increasing vehicle ownership at unprecedented rates put a lot of strain on cities particularly on urban mobility and transportation and overwhelm transportation network infrastructures and current transportation systems, which are not built to cope with such a fast increasing demand. Traffic congestion is considered as the most difficult challenge to tackle for sustainable urban mobility and is aggravated by the increased freight activity due to e-commerce and on-demand delivery and the explosive growth in transportation network companies and ride-hailing services. There is a need to implement a combination of policies to ensure that increased urban traffic congestion does not lower the quality of life and threaten global climate and human health and to prevent further economic losses. This study aims to contribute to the United Nations (UN) climate action and sustainable development goals in tackling recurring traffic congestion problem in urban areas to achieve a sustainable urban mobility in that it offers a solution methodology for traffic assignment problem. We introduce an exact generalized solution methodology based on reformulation of existing traffic assignment problems as a second order cone programming (SOCP) problem and propose column generation (CG) and cutting plane (CP) algorithms to solve the problem effectively for large scale road network instances. We conduct numerical experiments to test the performance of the proposed algorithms on realistic road networks.

Mathematics Subject Classification. 90B20 and 90C25, 90C46, 90C06.

Received May 7, 2023. Accepted October 24, 2023.

1. Introduction

Demographic changes and urbanization put a lot of strain on cities particularly on urban mobility and transportation creating many challenges among which traffic congestion is considered as the most difficult to tackle [27]. Traffic congestion is defined as "a condition in transport that is characterized by slower speeds, longer trip times, and increased vehicular queuing" [68] and occurs as traffic demand approaches or exceeds limited capacity of transportation network infrastructure.

According to a report by PwC [47], there are other factors that contribute to traffic congestion such as economic expansion, transportation disruption, e-commerce and on-demand delivery [50], and underinvestment in

Keywords. Sustainable development goals, traffic congestion, traffic assignment, column generation, cutting plane, second order cone programming.

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infrastructure. These factors and increasing vehicle ownership at unprecedented rates overwhelm transportation network infrastructures and current transportation systems, which are not built to cope with such a fast increasing demand [29] and due to underinvestment for building extra capacity and for upgrades and maintenance, congestion becomes inevitable.

Sustainable urban mobility is a challenging future goal US, EU and many other countries would like to achieve. It is a challenging but an important one as sustainable urban mobility is linked to social welfare of people, economic growth, and environmental sustainability [10, 65]. The cost of congestion to society is around 270 billion euros [65] and 87 billion dollars [29] per year in EU and in US, respectively. Road transport (passenger and freight) and high congestion levels account for a considerable amount of air pollution and greenhouse gas emissions and lead to serious health issues. Around 40% of transport related CO_2 emissions and 70% of other pollutants are due to urban traffic [49]. It is reported that around 96% of people living in urban areas in EU are exposed to high levels of pollution that can cause health problems, the cost of which is about several hundred billions of euros per year [65]. It is estimated in a report by World Health Organization [67] that around 7 million of premature deaths are due to air pollution caused by urban traffic.

There is a need to implement a combination of policies to ensure that increased urban traffic congestion does not lower the quality of life and threaten global climate and human health and to prevent further economic losses. These policies are classified as near-term or long-term solutions and physical, control, pricing and information-based measures, which must address both demand and supply side of the problem [36, 47]. Experience has shown that some of these measures such as increasing road network capacity, designating high-occupancy vehicle or bus-only lanes may eventually result in higher congestion levels [27, 47]. Likewise, banning cars from streets or promoting alternative means of transportation does not provide a total solution to this problem [21]. These methods can be applied when necessary and where appropriate but may have a higher success probability if implemented in coordination with smart transportation systems and management, which are less costly and quicker to implement [46].

The cities are becoming increasingly instrumented with an ability to collect real-time data through the use of intelligent transportation systems (ITS), internet of things (IoT), GPS systems in mobile telephones, public wireless local area network base stations, sensors, the web, and other similar data-acquisition systems [31, 51, 56]. Transforming these data into decisions in real-time through advanced data analytics methods (*i.e.*, machine learning, optimization) is becoming a reality [33]. Real-time data can be used by traffic management authorities to understand the change in demand patterns, to influence travelers' behavior and choice [64] and to reduce congestion by implementing various low-cost and real-time smart traffic control and management mechanisms [69]. After starting to implement a traffic management system having a real-time data analytics capability, one of the most congested cities in China, Hangzhou, a city with a population of 23 million, has improved travel times by 11% and response time of emergency vehicles by 50% [29].

However, the use of ITS has only partially solved the problem as the information provided to travelers by these systems is used in a selfish and uncoordinated manner to make their choices and to minimize their individual travel times. But through coordination and cooperation of the users that share the road network capacity, it could be possible to distribute traffic across alternative paths and therefore to bridge the gap between UE and System Optimal (SO) solutions and get closer to the SO solution in practice, as well. Cooperation and compliance of users with conventional vehicles to the directions or guidance from a system manager would only be possible if traffic assignment is fair [11, 37, 45]. Advance of autonomous and connected vehicle technology [28, 41] will allow traffic managers to achieve even a higher level of coordination between users. Therefore, a traffic management authority that uses ITS to gather and an analytics capability to analyze data and to guide travelers to alternative acceptable routes could achieve a fair and an efficient solution that considers all users' benefit in the road network and that reduces overall congestion.

This study aims to contribute to the United Nations (UN) climate action and sustainable development goals [62, 63] in tackling recurring traffic congestion problem in urban areas to achieve a sustainable urban mobility in that it offers a solution methodology for traffic assignment problem. Our goal is to propose an exact generalized solution methodology based on reformulation of existing traffic assignment problems as a second order cone programming (SOCP) problem and a column generation (CG) and cutting plane (CP) algorithm to solve the problem effectively for large scale road network instances.

The rest of the paper is organized as follows: In Section 2, we cover the literature on traffic assignment problem and present our contributions. In Section 3, we define the problem setting and introduce a mathematical formulation for it. In Section 4, we propose an exact solution methodology based on SOCP and CG/CP. In Section 5, we present results, based on an extensive computational study. Finally, we conclude in Section 6.

2. Literature review

Traffic assignment models were first introduced by Wardrop [66] with two basic principals, the User Equilibrium (UE, also known as User Optimal or Nash Equilibrium), and the System Optimal (SO). UE is Wardop's [66] first principle and simply states that the travel times on the used routes for any origin–destination (o–d) pair is less than or equal to the travel times on the routes which are not used. Under identical traffic conditions, the experienced travel time in UE is the same for all users, *i.e.*, no traveler in the road network can improve her/his travel time by unilaterally changing routes [53]. In accordance with this principle, travelers act in a selfish manner to minimize their individual travel times not considering the effect of their choice on others in the same road network. Although not very realistic [54], it is also assumed that users have perfect information about the road network structure and the traffic conditions (congestion information) and they are rational in their choice, *i.e.*, they can optimize their decisions. With the proliferation of ITS technologies, the perfect information assumption is no longer unrealistic and urban traffic tends to approach to a state of UE more than ever [17, 45]. Hence, UE can be defined as the collective outcome of selfish choices of fully informed noncooperative travelers in an uncoordinated road network [37].

While UE may be the preferred traffic assignment approach by the travelers, and since everyone acts in her/his own interest, it does not necessarily result in a solution that minimizes the total travel time of the users in the road network. On the contrary, even with the use of ITS, UE traffic assignment is likely to generate congested areas in the road network and everyone might end up with longer travel times. Under fully coordinated and cooperative users assumption, the goal of a centralized system manager would be to distribute traffic across the network to avoid creating congestion and to minimize total travel time of every user in the network. Hence rather than individual benefits, system's benefit is prioritized. This approach is the second principle of Wardrop [66] known as system optimal (SO). In a SO solution, the marginal travel times on the used paths of a given o–d pair is equal and otherwise marginal travel time of an unused path is greater than or equal to those of used paths [53]. Although the resulting solution is an efficient one with the least amount of possible congestion, it may be unfair to some users by assigning them to much longer routes compared to the shortest ones [48] and therefore may not be practical.

Although in theory SO provides the most efficient solution, it may not be possible to achieve it in practice as it would not be possible to influence all drivers to comply with routing directions in the road network and to have a fully coordinated and cooperative traffic assignment under such unfair conditions. However, if individual needs are taken into account, travelers are treated fairly, and they are motivated by high level incentives (such as reducing CO_2 emissions, considering collective good rather than individual gain), they can be influenced through ITS systems and by other means to act in a pro-social manner and comply with routing directions. In a research by Ackermann and Murphy [1], it is stated that individuals in a social network tend to cooperate if they know others also cooperate. Further, with the advance of automated and connected vehicles, it will be easier to enable such coordination between travelers in the road network. A coordinated traffic assignment approach through a centralized system supported by ITS could provide optimized efficient and individually tailored, fair solutions to tackle the traffic congestion problem and to achieve a sustainable urban mobility. Such a centralized traffic assignment approach is called Constrained System Optimal (CSO) or Bounded Rational User Equilibrium (BRUE) and provides a compromise between SO and UE approaches and helps bridge the gap between total travel time in UE and SO. In the remainder of the Literature Review Section, we will focus on studies related to BRUE and CSO traffic assignment approaches. For further details on static and dynamic traffic assignment models, we refer the reader to Bayram [11], Morandi [45] and for a comprehensive review on the use of information systems to influence behavior/choice of travelers to Van Essen *et al.* [64].

2.1. Bounded Rational User Equilibrium

BRUE (also referred to as length constrained user equilibrium), first introduced in the seminal work by Simon [54] and adapted to transportation systems by Mahmassani and Chang [42], is a relaxation of UE in that it assumes that users are willing to take an acceptable route rather than an optimal one, where acceptable is defined based on the length of the shortest route for a given o–d pair and a threshold value or aspiration/indifference level that reflects users' behavior. Under BRUE, no user can improve her/his travel time by switching routes by more than a prespecified threshold value [39]. Following the work of Mahmassani and Chang [42], various studies [22, 39, 43, 58, 74] explored and extended BRUE.

Mahmassani and Chang [42] investigate the existence, uniqueness, and where applicable the stability of BRUE in an idealized transportation system. Chen *et al.* [22] propose a modeling framework for bounded rational interactive decision making, where users follow probabilistic choices using the logit model of discrete choice theory. Mahmassani and Liu [43] incorporate departure time and routing decisions of travelers made at several decision points at the origin and en-route, using a behavioral (multinomial probit) model framework. Szeto and Lo [58] extend BRUE to a dynamic setting and introduce bounded rational dynamic user equilibrium (BRDUE) principle. A route swapping heuristic algorithm is presented to solve the BRDUE problem. Lou *et al.* [39] introduce path-based and link-based representations of BRUE and a formulation with complementarity constraints. They solve the problem using a heuristic algorithm based on penalization and a cutting-plane scheme. Zhou and Li [74] show that the problem can be stated as a convex optimization problem and propose a path-based formulation and column generation based solution methodology combined with the Frank–Wolfe algorithm [30]. They use a constrained shortest path problem to solve a pricing problem to generate new columns.

We refer the reader to Di and Liu [25], Ye and Yang [70] and Szeto *et al.* [59], Morandi [45] for a comprehensive review on the applications of BRUE, behavioral studies considering BRUE in a traffic assignment setting, and review of BRDUE models, respectively.

2.2. Constrained System Optimal

Although BRUE is one of the two alternatives that is used to bridge the gap between UE and SO solutions, the main disadvantage of BRUE compared to CSO is that it just aims to reach an equilibrium state and does not try to minimize total travel time [45]. CSO traffic assignment approach aims to minimize total travel time while honoring individual preferences of users by enforcing additional constraints to assign them to acceptable routes only [35]. The first attempt to apply the idea of finding efficient and fair solutions to traffic assignment problem was made by Jahn *et al.* [34], was extended to a more general setting by Jahn *et al.* [35] and was followed by further studies [4, 5, 7, 12–14, 45, 52, 73]. The quality of such a solution is evaluated based on its efficiency (total travel time) for the centralized system manager (traffic management authority) and its fairness for the users.

Jahn *et al.* [34,35] introduce CSO traffic assignment approach and propose a nonlinear (convex) mathematical formulation for it that aims to achieve fair and efficient solutions by minimizing total travel time. They define a measure of unfairness, which is the ratio of the travel time of the recommended route to that of the shortest possible route a user could have taken. Jahn *et al.* [35] define the normal length of a route which does not depend on the amount of traffic flow and which can be its free flow travel time, geographical distance or travel time in a UE solution. At an optimal solution of a CSO traffic assignment problem, no route carrying a positive flow between an o–d pair is allowed to deviate from the normal length of the route by more than threshold value. They employ a column generation methodology in combination with a modified version of Frank and Wolfe algorithm [30] to solve the problem. Schulz and Stier-Moses [52] study the problem from a theoretical perspective and propose a modified price of anarchy to evaluate efficiency of a given solution by comparing the worst case ratio of the total travel time of a UE to that of a CSO. Li and Zhao [73] propose a game theoretic approach and an integrated equilibrium model with satisfactory degree having two objective functions, that is UE and SO objective functions with user constraints. They solve the problem in an iterative manner using pregenerated alternative paths and Frank and Wolfe algorithm [30]. An application of CSO traffic assignment approach to evacuation planning is presented by Bayram *et al.* [14], Bayram and Yaman [12, 13]. Aiming to minimize the maximum arc utilization (congestion) in the road network and the weighted average experienced travel inconvenience, Angelelli *et al.* [4] propose a linear programming model and a hierarchical solution methodology. They use a metaheuristic to generate acceptable routes and a commercial solver to solve the problem. Angelelli *et al.* [5] extend this study by proposing a heuristic algorithm that generates a subset of feasible paths rather than the complete set to enhance the computational efficiency of the solution methodology. Angelelli *et al.* [6,7] solve the nonlinear CSO traffic assignment problem by using a piecewise linear approximation of the travel time function and employ a matheuristic algorithm to identify promising routes.

Please see Morandi [45] for a comprehensive literature review on studies related to the theory and applications of CSO traffic assignment approach. Due to the nonlinear nature of the BRUE and CSO problems, majority of the proposed solution methodologies that solve them are heuristics or require approximations. Unlike these studies, we do not use approximations and propose exact solution methodologies that can solve CSO traffic assignment problems.

2.3. Our contributions

- We propose a generalized solution methodology that can be used to solve SO, UE, and CSO traffic assignment problems.
- Unlike most of the studies in the literature, we do not use approximations of the original problem or use heuristic methodologies to solve it. We transform the nonlinear traffic assignment formulations into SOCP formulations and solve the problem exactly by using column generation and cutting plane methodologies that employ SOCP duality results. The generality of solution methodology allows for the seamless application of a similar SOCP transformation to a broad spectrum of alternative convex travel time functions.
- Finally, we conduct numerical experiments on realistic large scale road networks to test the effectiveness of our algorithm.

Next, we define the problem and its nonlinear formulation and propose a transformation of the formulation into a SOCP formulation.

3. Problem statement and model development

The mathematical models proposed for UE and SO traffic assignment approaches are first presented by Beckmann *et al.* [16]. In these formulations, positive and increasing convex travel time functions (also referred to as latency or link performance functions) are used, under the assumption that travel time on a segment is only a function of the flow on that road segment only [53]. In this section we define our problem and present UE, SO and CSO optimal traffic assignment models and transform their corresponding nonlinear formulations into a SOCP one.

Consider a directed network $G = (N, A)$, where N is the set of nodes and A is the set of arcs in the network. Each arc a is associated with a convex travel time function t_a. We employ the Bureau of Public Roads (BPR) function [60] to model travel time, where a road segment $a \in A$ is characterized by the parameters $\alpha \geq 0$ and $\beta \geq 0$, which are taken as 0.15 and 4 by the US Department of Commerce Bureau of Public Roads, respectively, its capacity c_a and free flow travel time t_a^0. The set of origin (demand) nodes from where the travelers start their journey and the set of destination nodes for travelers are denoted by O and D, respectively. Let P_{od} be the set of alternative routes from origin $o \in O$ to destination $d \in D$. We define decision variable v_p as the fraction of an o–d pair's demand that uses path $p \in P_{od}^{\lambda}$ from origin $o \in O$ to destination $d \in D$ and x_a as the amount of traffic flow on road segment $a \in A$.

The SO traffic assignment formulation is presented below:

$$
(\mathrm{SO})\begin{cases}
\min \displaystyle\sum_{a\in A} t_a^0\left(1+\alpha\left(\frac{x_a}{c_a}\right)^{\beta}\right)x_a & & (1)\\
\text{s.t.} & & \\
\quad \displaystyle\sum_{p\in P_{od}} v_p = 1, & \forall o\in O,\quad d\in D, & (2)\\
\quad x_a = \displaystyle\sum_{O\in O}\sum_{d\in D}\sum_{p\in P_{od}:a\in p} w_{od}v_p, & \forall a\in A, & (3)\\
\quad v_p \geq 0, & \forall p\in P_{od},\quad o\in O,\quad d\in D, & (4)\\
\quad x_a \geq 0, & \forall a\in A. & (5)
\end{cases}
$$

The objective function (1) minimizes total travel time in the road network. Constraints (2) are traffic assignment constraints and ensure that every user is assigned to a path for every o–d pair. Constraints (3) simply compute the amount of traffic flow on road segment $a \in A$. And finally constraints (4) and (5) define variable domains. Next, we present UE formulation.

$$
(\mathrm{UE})\begin{cases}
\min \displaystyle\sum_{a\in A}\int_0^{x_a} t_a^0\left(1+\alpha\left(\frac{\tau_a}{c_a}\right)^{\beta}\right)d\tau_a & & (6)\\
\text{s.t.} & & \\
\quad \displaystyle\sum_{p\in P_{od}} v_p = 1, & \forall o\in O,\quad d\in D, & (7)\\
\quad x_a = \displaystyle\sum_{O\in O}\sum_{d\in D}\sum_{p\in P_{od}:a\in p} w_{od}v_p, & \forall a\in A, & (8)\\
\quad v_p \geq 0, & \forall p\in P_{od},\quad o\in O,\quad d\in D, & (9)\\
\quad x_a \geq 0, & \forall a\in A. & (10)
\end{cases}
$$

The only difference between UE and SO traffic assignment formulations is the objective function. The objective function (6) of UE is the sum of the integrals of the travel time functions. The constraints of the UE formulation are defined in the same manner as those of SO.

For CSO traffic assignment setting only, a fairness/tolerance level λ is imposed on the normal length of the routes a traveler can be assigned. Based on this tolerance level, a central traffic management authority guarantees for every o–d pair to assign a traveler to a route whose normal length is not more than $(1+\lambda)$ times the normal length of a shortest route connecting the o–d pair. We define $P_{od}^{\lambda} = \{p \in P_{od} : \ell^p \leq (1+\lambda)\ell_{od}^*\}$ as the set of acceptable routes from origin o to destination d of fairness level λ. In this definition, ℓ^p is the length of route p and ℓ_{od}^* is the length of a shortest route from o to d. Please see Table 1 for the notation used to define the problem.

TABLE 1. Description of the notation used.

Sets	
$G = (N, A)$	Traffic network
N	Set of nodes in the network
A	Set of arcs in the network
O	Set of origin nodes
D	Set of destination nodes
P_{od}	Set of alternative routes from origin $o \in O$ to destination $d \in D$
P_{od}^{λ}	Set of acceptable routes from origin o to destination d of fairness level λ
Parameters	
$t_a(x_a)$	Travel time on arc $a \in A$ as a function of traffic flow x_a
t_a^0	Free flow travel time on arc $a \in A$
α	Road characteristic parameter
β	Road characteristic parameter
λ	Fairness/tolerance level
ℓ^p	Length of route p
ℓ_{od}^*	Length of a shortest route from origin o to destination d
Variables	
v_p	Fraction of an o–d pair's demand that uses path $p \in P_{od}^{\lambda}$ from origin $o \in O$ to destination $d \in D$
x_a	Amount of traffic flow on road segment $a \in A$

Below is the CSO traffic assignment formulation:

$$
\text{(CSO)} \begin{cases}
\min \displaystyle\sum_{a \in A} t_a^0 \left(1 + \alpha \left(\frac{x_a}{c_a}\right)^{\beta}\right) x_a & (11) \\
\text{s.t.} & \\
\displaystyle\sum_{p \in P_{od}^{\lambda}} v_p = 1, \qquad \forall o \in O, \quad d \in D, & (12) \\
x_a = \displaystyle\sum_{O \in O} \sum_{d \in D} \sum_{p \in P_{od}^{\lambda} : a \in p} w_{od} v_p, \qquad \forall a \in A, & (13) \\
v_p \geq 0, \qquad \forall p \in P_{od}^{\lambda}, \quad o \in O, \quad d \in D, & (14) \\
x_a \geq 0, \qquad \forall a \in A. & (15)
\end{cases}
$$

The CSO traffic assignment formulation differs from SO traffic assignment formulation in constraints (12). These constraints assign traffic demand for every o–d pair to acceptable set of paths P_{od}^{λ} only. The CSO traffic assignment approach generalizes both the SO and the UE traffic assignment approaches, if UE travel times are used as the normal length. When $\lambda = 0$, the CSO model is the same as the UE traffic assignment model, whereas when $\lambda = \infty$, SO traffic assignment model is obtained.

The formulations presented for UE, SO, and CSO traffic assignment problems are nonlinear due to convex travel time functions used in the objective function. Next, we present a tranformation of these formulations into a SOCP formulation.

4. Reformulation of the problem as a second order cone programming problem

Advances in interior point methodologies have allowed to solve second order cone programming problems almost as efficiently as solving linear programming problems [18]. Due to that, SOCP has been successfully applied to a wide range of convex optimization problems such as portfolio optimization [20], machine-job assignment [2], power distribution system reconfiguration [61], stochastic joint location inventory [8], routing/location in telecommunication/hub-spoke networks [15,32], variable selection in linear regression [44], planning for plug-in electric vehicle fast-charging stations [72], two-player zero sum games [55], and UAV collision avoidance [71]. We refer the reader to Lobo et al. [38], Ben-Tal and Nemirovski [18], Alizadeh and Goldfarb [3] and Benson and Sağlam [19] for an introduction and for more details on the applications on SOCP problems.

In this section, we reformulate UE, SO, and CSO traffic assignment problems as SOCP problems by transferring the nonlinearity in the objective function into the constraint set in the form of quadratic cone constraints. To achieve this, we first reorganize the objective functions of the formulations (SO), (UE), and (CSO) as follows:

$$\text{(SO)} \begin{cases} \min & \displaystyle\sum_{a\in A}\left(t_a^0 x_a + \frac{t_a^0\alpha}{c_a^\beta} x_a^{\beta+1}\right) \\ \text{s.t.} & \\ & (2)\text{–}(5) \end{cases} \tag{16}$$

$$\text{(UE)} \begin{cases} \min & \displaystyle\sum_{a\in A}\left(t_a^0 x_a + \frac{t_a^0\alpha}{(\beta+1)c_a^\beta} x_a^{\beta+1}\right) \\ \text{s.t.} & \\ & (7)\text{–}(10) \end{cases} \tag{17}$$

$$\text{(CSO)} \begin{cases} \min & \displaystyle\sum_{a\in A}\left(t_a^0 x_a + \frac{t_a^0\alpha}{c_a^\beta} x_a^{\beta+1}\right) \\ \text{s.t.} & \\ & (12)\text{–}(15). \end{cases} \tag{18}$$

We take $\beta = 4$ and define auxiliary variables μ_a for each and add the constraints $x_a^5 \leq \mu_a$, for each arc $a \in A$. We represent $x_a^5 \leq \mu_a$ with hyperbolic inequalities of the form,

$$x_a^2 \leq \theta_a h, \qquad \forall a \in A, \tag{19}$$

$$\theta_a^2 \leq u_a x_a, \qquad \forall a \in A, \tag{20}$$

$$u_a^2 \leq \mu_a x_a, \qquad \forall a \in A, \tag{21}$$

$$h = 1, \theta_a, u_a, \mu_a \geq 0, \qquad \forall a \in A. \tag{22}$$

where h, θ_a, u_a, μ_a are auxiliary variables that are used to define hyperbolic inequalities. These hyperbolic inequalities are represented by their respective quadratic cone constraints:

$$\|2x_a, \theta_a - 1\| \leq \theta_a + 1, \qquad \forall a \in A, \tag{23}$$

$$\|2\theta_a, u_a - x_a\| \leq u_a + x_a, \qquad \forall a \in A, \tag{24}$$

$$\|2u_a, \mu_a - x_a\| \leq \mu_a + x_a, \qquad \forall a \in A, \tag{25}$$

$$\mu_a, \theta_a, u_a \geq 0 \qquad \forall a \in A, \tag{26}$$

$$(5).$$

The resulting second order cone programming problem formulations are given below:

$$\text{(SO\_SOCP)} \begin{cases} \min & \sum_{a\in A}\left(t_a^0 x_a + \dfrac{t_a^0\alpha}{c_a^\beta}\mu_a\right) \\ \text{s.t.} & \\ & (2)-(5), (23)-(26) \end{cases} \tag{27}$$

$$\text{(UE\_SOCP)} \begin{cases} \min & \sum_{a\in A}\left(t_a^0 x_a + \dfrac{t_a^0\alpha}{(\beta+1)c_a^\beta}\mu_a\right) \\ \text{s.t.} & \\ & (7)-(10), (23)-(26) \end{cases} \tag{28}$$

$$\text{(CSO\_SOCP)} \begin{cases} \min & \sum_{a\in A}\left(t_a^0 x_a + \dfrac{t_a^0\alpha}{c_a^\beta}\mu_a\right) \\ \text{s.t.} & \\ & (12)-(15), (23)-(26). \end{cases} \tag{29}$$

Note that, our SOCP transformation is generalized in that it can be applied to any convex travel time function. In the following section, we describe how we generate new paths in an iterative manner using column generation or cutting planes since it may not be possible and not efficient to pregenerate all possible paths.

5. A column generation/cutting plane approach

The formulations (SO_SOCP) (27), (2)–(5), (23)–(26), (UE_SOCP) (28), (7)–(10), (23)–(26), and (CSO_SOCP) (29), (12)–(15), (23)–(26) assume that the complete set of candidate paths for each o–d pair, *i.e.*, P_{od} (P_{od}^{λ} for CSO_SOCP), is provided. However, it is impractical to generate and include all possible paths in the problem. In such cases, a column generation (CG) or a cutting plane (CP) approach for the dual of the problem can be used to prevent enumerating all possibilities.

Column generation is a methodology used to solve large scale combinatorial optimization problems. It relies on the Dantzig–Wolfe decomposition [23] of the problem into two problems, namely the master problem and the subproblem. Master problem, which is the original problem, is also called the restricted master problem (RMP) as it only contains a meaningful subset of columns that provides a starting feasible solution. The dual information obtained through the solution of the RMP is used to solve a pricing subproblem to detect any pricing columns (nonbasic variables with eligible reduced costs) or any violated constraints (if the dual of the RMP is solved) and to add them to RMP. This procedure is continued in an iterative manner until the current basic feasible solution of the RMP is optimal, *i.e.*, no columns price out (no violation of corresponding dual constraints). We refer the reader to Barnhart *et al.* [9] and Desaulniers *et al.* [24] for further information on CG and its applications.

To obtain dual information and the dual problem, we transform quadratic cone constraints (23)–(26) into the following structure:

$$q_{1a}^2 + q_{2a}^2 \le q_{3a}^2, \qquad \forall a \in A, \tag{30}$$
$$q_{4a}^2 + q_{5a}^2 \le q_{6a}^2, \qquad \forall a \in A, \tag{31}$$
$$q_{7a}^2 + q_{8a}^2 \le q_{9a}^2, \qquad \forall a \in A, \tag{32}$$
$$-q_{1a} + 2x_a = 0, \qquad \forall a \in A, \tag{33}$$
$$-q_{2a} + \theta_a = 1, \qquad \forall a \in A, \tag{34}$$
$$-q_{3a} + \theta_a = -1, \qquad \forall a \in A, \tag{35}$$
$$-q_{4a} + 2\theta_a = 0, \qquad \forall a \in A, \tag{36}$$

$$-q_{5a} + u_a - x_a = 0, \qquad \forall a \in A, \tag{37}$$
$$-q_{6a} + u_a + x_a = 0, \qquad \forall a \in A, \tag{38}$$
$$-q_{7a} + 2u_a = 0, \qquad \forall a \in A, \tag{39}$$
$$-q_{8a} - x_a + \mu_a = 0, \qquad \forall a \in A, \tag{40}$$
$$-q_{9a} + \mu_a + x_a = 0, \qquad \forall a \in A, \tag{41}$$
$$q_{1a}, q_{3a}, q_{4a}, q_{6a}, q_{7a}, q_{9a}, \theta_a, u_a, \mu_a \geq 0, \qquad \forall a \in A. \tag{42}$$

Constraints (30)–(32) define the three second order quadratic cones. And constraints (33)–(41) are generated by replacing each term of SOCP constraints (23)–(25) by a single auxiliary variable to help derive the reduced costs for path variables in the primal formulations and the duals of (SO_SOCP), (UE_SOCP), and (CSO_SOCP) formulations. Constraints (42) represent variable restrictions. The resulting primal formulations for SO_SOCP, UE_SOCP, CSO_SOCP are as follows:

$$(\text{SO\_SOCP}) \begin{cases} \min & \sum_{a \in A} \left(t_a^0 x_a + \dfrac{t_a^0 \alpha}{c_a^\beta} \mu_a \right) \\ \text{s.t.} & \\ & (2)-(5), (30)-(42) \end{cases} \tag{43}$$

$$(\text{UE\_SOCP}) \begin{cases} \min & \sum_{a \in A} \left(t_a^0 x_a + \dfrac{t_a^0 \alpha}{(\beta+1) c_a^\beta} \mu_a \right) \\ \text{s.t.} & \\ & (7)-(10), (30)-(42) \end{cases} \tag{44}$$

$$(\text{CSO\_SOCP}) \begin{cases} \min & \sum_{a \in A} \left(t_a^0 x_a + \dfrac{t_a^0 \alpha}{c_a^\beta} \mu_a \right) \\ \text{s.t.} & \\ & (12)-(15), (30)-(42). \end{cases} \tag{45}$$

We associate the dual variables z_{od}, ψ_a with constraints (2), (3), (7), (8), (12), (13), and κ_{1a}, κ_{2a}, κ_{3a}, κ_{4a}, κ_{5a}, κ_{6a}, κ_{7a}, κ_{8a}, κ_{9a} for constraints (33)–(41) for (SO_SOCP), (UE_SOCP, and CSO_SOCP) respectively. And the resulting dual formulations for SO, UE, and CSO are as follows:

Dual Constrained System Optimal (DCSO)

$$\max \sum_{o \in O} \sum_{d \in D} z_{od} + \sum_{a \in A} \kappa_{2a} - \sum_{a \in A} \kappa_{3a} \tag{46}$$
$$\text{s.t.} \quad z_{od} - \sum_{a \in A: a \in p} w_{od} \psi_a \leq 0, \qquad \forall o \in O, \quad d \in D, \quad p \in P_{od}^\lambda, \tag{47}$$
$$\psi_a + 2\kappa_{1a} - \kappa_{5a} + \kappa_{6a} - \kappa_{8a} + \kappa_{9a} \leq t_a^0, \qquad \forall a \in A, \tag{48}$$
$$\kappa_{8a} + \kappa_{9a} \leq \frac{t_a^0 \alpha}{c_a^\beta}, \qquad \forall a \in A, \tag{49}$$
$$\kappa_{2a} + \kappa_{3a} + 2\kappa_{4a} \leq 0, \qquad \forall a \in A, \tag{50}$$
$$\kappa_{5a} + \kappa_{6a} + 2\kappa_{7a} \leq 0, \qquad \forall a \in A, \tag{51}$$
$$\kappa_{1a}^2 + \kappa_{2a}^2 \leq \kappa_{3a}^2, \qquad \forall a \in A, \tag{52}$$
$$\kappa_{4a}^2 + \kappa_{5a}^2 \leq \kappa_{6a}^2, \qquad \forall a \in A, \tag{53}$$
$$\kappa_{7a}^2 + \kappa_{8a}^2 \leq \kappa_{9a}^2, \qquad \forall a \in A, \tag{54}$$

$$\kappa_{3a}, \kappa_{6a}, \kappa_{9a} \geq 0, \qquad \forall a \in A. \tag{55}$$

The duals of (SO_SOCP) (DSO) and (UE_SOCP) (DUE) are identically the same as (DCSO) except that constraints (47) are updated in (DSO) and (DUE) as:

$$z_{od} - \sum_{a \in A: a \in p} w_{od}\psi_a \leq 0, \qquad \forall o \in O, \quad d \in D, \quad p \in P_{od},$$

and constraints (49) are updated in DUE as:

$$\kappa_{8a} + \kappa_{9a} \leq \frac{t_a^0 \alpha}{(\beta+1)c_a^\beta}, \qquad \forall a \in A.$$

The (DCSO), (DSO), and (DUE) are also second order cone programming problems. Note that strong duality between (SO_SOCP) – (DSO), (UE_SOCP) – (DUE), and (CSO_SOCP) – (DCSO) problem pairs hold since we can find strictly feasible points for the primal and dual problems, which are also bounded [18], and they attain the same optimal values. This allows us to use both primal and dual versions of the problem.

In order to solve (SO_SOCP), (UE_SOCP), and (CSO_SOCP) by CG and CP, we start with a small set of feasible paths $\bar{P}_{od} \subset P_{od}$ ($\bar{P}^\lambda_{od}$ for (CSO_SOCP)) for each o–d pair, $o \in O, d \in D$ by using a k-shortest path algorithm. A new path $p \in P_{od}$ ($p \in P^\lambda_{od}$ for (CSO_SOCP)) for an o–d pair, $o \in O, d \in D$ can be added to the current RMP if it has negative reduced cost. The reduced cost of a path variable $v_p, p \in P_{od}$ ($p \in P^\lambda_{od}$ for (CSO_SOCP)), denoted by r_p, is calculated as in (56).

$$r_p^{od} = \sum_{a \in A: a \in p} w_{od}\psi_a - z_{od}. \tag{56}$$

To price out candidate paths, a pricing subproblem (PP) is defined and solved for each o–d pair, $o \in O, d \in D$. The PP for an o–d pair seeks a path p from o to d with the most negative r_p^{od} on the subgraph $\mathscr{G}_{od} = (N, A_{od})$ of G, which contains o, d, and all nodes in N and having arc costs $w_{od}\psi_a$. For the origin o the pricing graph contains only its outgoing arcs from the original graph, and for the destination d the pricing graph contains only the incoming arcs. Then, PP translates into solving a resource constrained shortest path problem on $\mathscr{G}_{od}$ for CSO, where the length of the path is the resource bounded by $(1+\lambda)r^*_{od}$. PP is a shortest path problem for SO and UE. Observe that, the cost of path $p \in P_{od}$ ($p \in P^\lambda_{od}$ for (CSO_SOCP)) in the pricing graph $\mathscr{G}_{od}$ is equal to $\sum_{a \in A: a \in p} w_{od}\psi_a$. Then the reduced cost of path variable v_p from origin o to destination d is $\sum_{a \in p} w_{od}\psi_a - z_{od}$. Therefore, in a column generation iteration there exists a path variable with negative reduced cost if and only if the cost of the shortest (cheapest) path having a length (resource) less than or equal to $(1+\lambda)r^*_{od}$ in the pricing graph $\mathscr{G}_{od}$ minus the value of the dual variable z_{od} is negative.

When using the (DCSO), (DSO), and (DUE) formulations, we employ a cutting plane algorithm and use a separation problem to check whether constraints (47) are violated in a similar iterative manner we employ the CG. Please see below the pricing/separation problem for (CSO_SOCP) / (DCSO):

(PP_CSO)

$$r_p^{od} = \min \sum_{a \in A_{od}} w_{od}\pi_a\chi_a - z_{od} \tag{57}$$

$$\text{s.t.} \sum_{a \in \delta^+(i)} \chi_a - \sum_{a \in \delta^-(i)} \chi_a = \begin{cases} 1 & i = o \\ 0 & \forall i \in N \setminus (o,d) \\ -1, & i = d \end{cases} \tag{58}$$

$$\sum_{a \in A_{od}} l_a\chi_a \leq (1+\lambda)r^*_{od}, \tag{59}$$

$$\chi_a \in \{0,1\} \qquad \forall a \in A_{od}. \tag{60}$$

TABLE 2. Transportation networks used in computational study.

Network	#Nodes	#Arcs	#Trips	#o–d pairs
Sioux Falls	24	76	360 600	576
Anaheim	416	914	104 694	1444
Chicago Sketch	933	2950	1 260 907	149 769

TABLE 3. Performance of the proposed column generation and cutting plane algorithms, Sioux Falls network instances.

TA	λ	Opt. Val.	Column generation			Cutting planes		
			#Cols	#Iters	Time (s)	#Cuts	#Iters	Time (s)
SO	∞	71 939.62	712	6	0.87	712	6	1.02
UE	∞	42 313.89	454	4	0.56	454	4	0.66
CSO	0	618 958.58	22	2	0.96	22	2	0.92
CSO	0.01	618 958.58	22	2	0.96	22	2	0.90
CSO	0.02	618 958.58	22	2	0.95	22	2	0.89
CSO	0.03	618 958.58	22	2	0.95	22	2	0.91
CSO	0.04	618 958.58	22	2	0.94	22	2	0.91
CSO	0.05	615 192.56	28	2	0.98	28	2	0.93
CSO	0.1	388 201.91	99	2	0.95	99	2	0.90
CSO	0.15	219 159.31	167	3	1.18	167	3	1.17
CSO	0.2	135 873.96	255	4	1.26	255	4	1.25
		Average	165.91	2.82	0.96	165.91	2.82	0.95

The pricing problem is a resource constrained shortest path problem (also named as elementary shortest path problem with resource constraints) and it is NP-Hard [26]. Therefore it is not reasonable to solve (PP_CSO) at every iteration of the CG/CP algorithm. Instead, we first use a k-shortest path algorithm and check whether these k paths price out as candidate columns in CG or violate constraints (47) in CP algorithm. If no eligible pricing/violating paths are detected, then we employ the pulse algorithm proposed by Lozano *et al.* [40] to solve (PP_CSO).

6. Computational study

In this section, we report the results of extensive computational experiments conducted to test the effectiveness of the proposed SOCP reformulation and CG and CP algorithms using large scale realistic instances from the online library Transportation Networks for Research [57]. We perform tests with small, medium and large scale transportation networks, *i.e.*, Sioux Falls, Anaheim, and Chicago Sketch (see Tab. 2). We use geographical distances as the normal length of arcs. For the k-shortest path algorithm that we employ at the initial phase and during the CG and CP iterations, we take $k = 4$ after some tuning. We perform our computational tests on a notebook with Intel(R) Xeon(R) E-2186M CPU 2.90 GHz Processor, 6 Core(s), 12 Logical Processor(s) and 128 GB RAM by using Java ILOG CPLEX version 12.20.1.

In Tables 3–5 and in Figures 1–3 we compare the computational efficiencies of the CG and CP algorithms for different values of λ for CSO, and for SO, UE traffic assignment problems. For each instance, we report the optimal value, number of columns/cutting planes generated, number of iterations performed and solution times in seconds for CG and CP algorithms, respectively when the problem is solved to optimality. For Sioux Falls and Anaheim networks, the algorithm performs very effectively. While majority of Sioux Falls instances are solved in less than a second, Anaheim instances are solved in around two minutes or less. Chicago Sketch

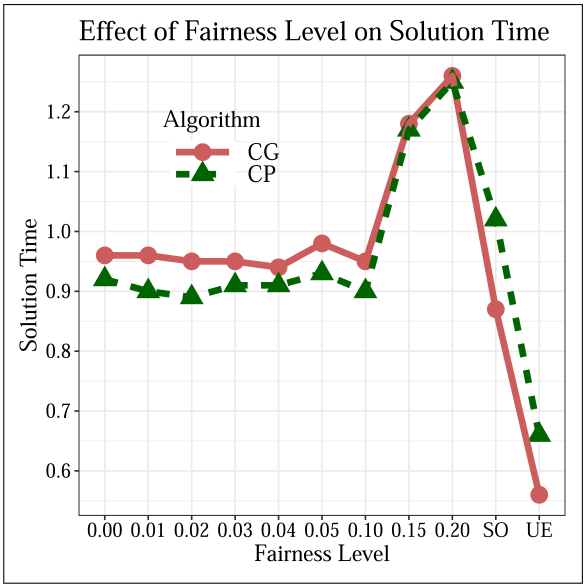
TABLE 4. Performance of the proposed column generation and cutting plane algorithms, Anaheim network instances.

TA	λ	Opt. Val.	Column generation			Cutting planes		
			#Cols	#Iters	Time (s)	#Cuts	#Iters	Time (s)
SO	∞	21 779.75	1644	6	95.54	1636	6	109.01
UE	∞	20 149.99	1307	5	74.39	1304	5	89.19
CSO	0	40 547.55	210	3	85.44	210	3	92.06
CSO	0.01	36 823.19	488	4	108.04	488	4	103.41
CSO	0.02	30 210.77	677	7	122.52	677	7	131.99
CSO	0.03	26 438.97	889	6	126.68	889	6	132.22
CSO	0.04	23 999.69	1031	6	128.47	1030	6	127.32
CSO	0.05	23 827.69	1127	8	158.97	1126	8	168.33
CSO	0.1	22 186.58	1476	7	124.80	1475	7	128.35
CSO	0.15	21 881.02	1617	8	140.58	1610	7	134.37
CSO	0.2	21 794.01	1719	8	125.35	1717	7	128.09
		Average	1107.73	6.18	117.35	1105.64	6.00	122.21

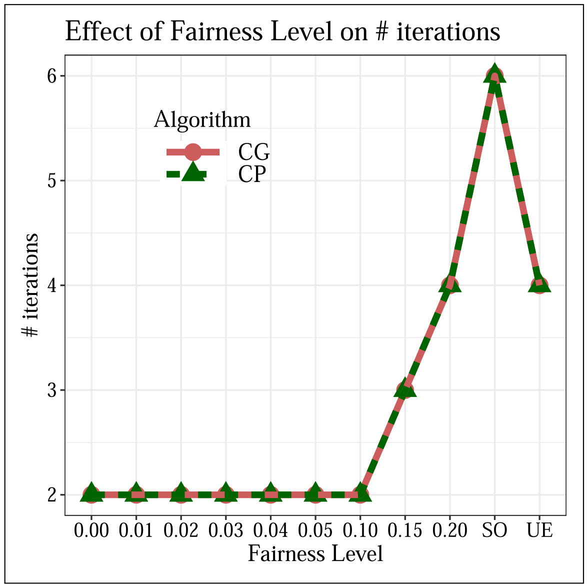
TABLE 5. Performance of the proposed column generation and cutting plane algorithms, Chicago Sketch network instances.

TA	λ	Opt. Val.	Column generation			Cutting planes		
			#Cols	#Iters	Time (s)	#Cuts	#Iters	Time (s)
SO	∞	301 827.97	38 439	6	31 825.99	38 451	7	36 995.59
UE	∞	281 991.69	21 867	5	26 622.11	21 867	5	26 089.12
CSO	0	4 107 544.61	542	3	16 507.15	542	3	16 027.56
CSO	0.01	2 418 388.55	11 098	4	22 306.79	11 097	4	21 432.32
CSO	0.02	2 188 396.11	16 784	4	22 071.15	16 781	4	21 632.71
CSO	0.03	1 797 199.83	20 992	6	32 361.89	20 993	6	32 694.30
CSO	0.04	1 719 550.73	23 895	6	32 832.27	23 885	5	27 897.45
CSO	0.05	1 643 808.68	25 796	6	33 160.87	25 793	6	34 100.05
CSO	0.1	881 531.69	30 455	6	32 707.26	30 449	6	35 316.64
CSO	0.15	730 749.98	32 417	6	32 481.47	32 415	5	28 123.85
CSO	0.2	422 788.14	33 620	6	33 232.17	33 622	6	32 340.05
		Average	23 264.09	5.27	28 737.19	23 263.18	5.18	28 422.69

network is relatively much larger compared to Sioux Falls and Anaheim networks. The number of o–d pairs has a significant effect on the performance of the algorithm as it is related to number of times pricing/separation problems will be solved in a single iteration of CG or CP algorithm. That number for Chicago Sketch network is more than 100 times bigger than that of Anaheim and around 300 times bigger than that of Sioux Falls networks. Further, the number of trips (total demand for all o–d pairs) is way larger than other networks, which has a complicating effect on the problem due to increasing level of congestion in the network and the difficulty of distributing congested traffic across alternative acceptable paths. As λ grows larger, solution times increase, as well. Although the effect of that increase is mild for Sioux Falls and Anaheim networks, it is much bigger for Chicago Sketch network. Solving SO and UE problems is generally more difficult than solving CSO, as a larger number of paths (columns/cuts) need to be generated in a bigger number of iterations. However, the pricing problem (shortest path problem) for SO and UE is much easier than the pricing problem of CSO (resource constrained shortest path problem). Hence at some point, the solution times for CSO becomes worse than those of SO and UE.

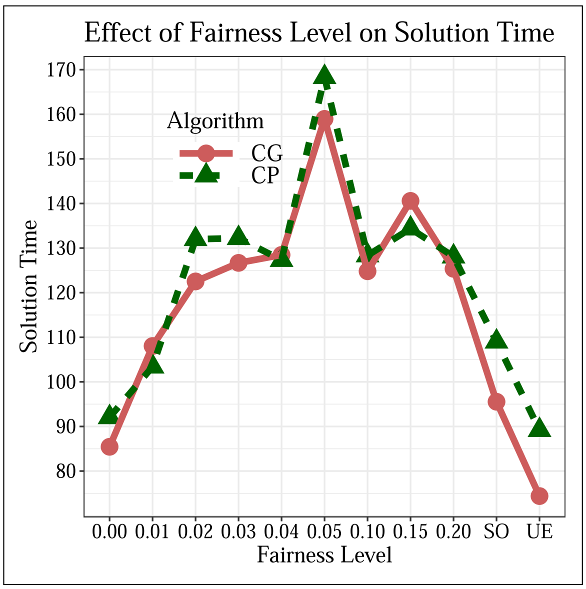


(a)

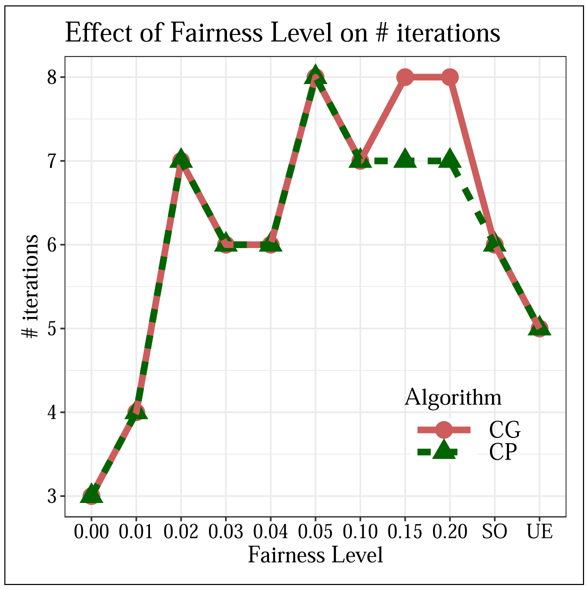


(b)

FIGURE 1. The effect of λ on solution time and # of iterations, Sioux Falls network. (a) The effect of λ on solution time. (b) The effect of λ on # of iterations.



(a)



(b)

FIGURE 2. The effect of λ on solution time and # of iterations, Anaheim network. (a) The effect of λ on solution time. (b) The effect of λ on # of iterations.

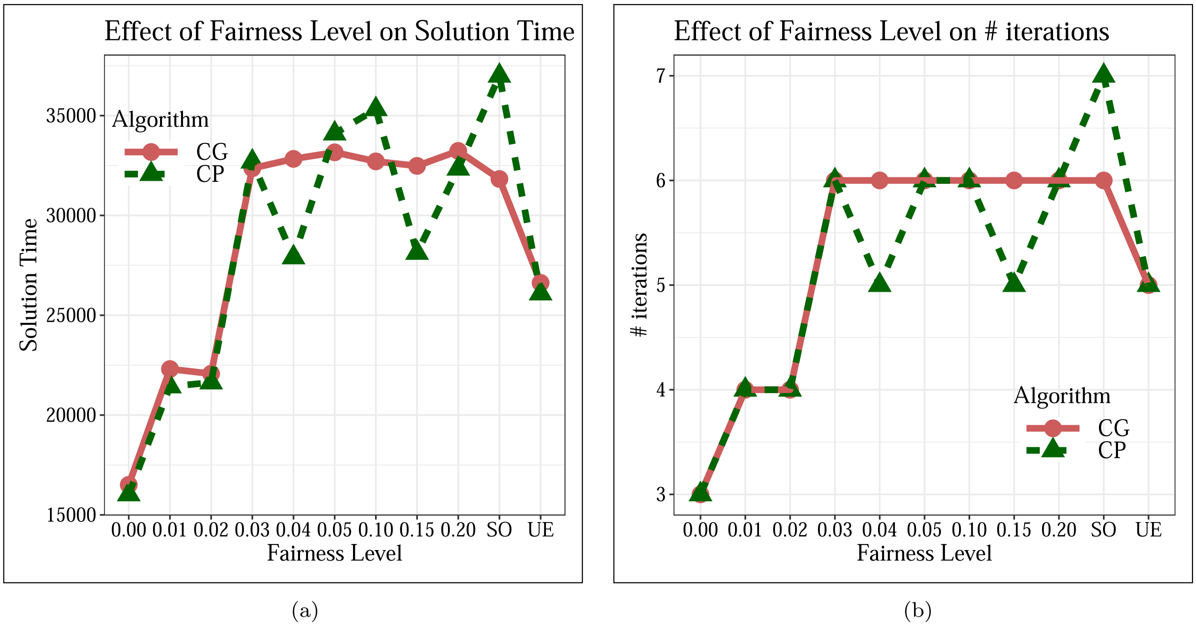


Figure 3. The effect of λ on solution time and # of iterations, Chicago Sketch network. (a) The effect of λ on solution time. (b) The effect of λ on # of iterations.

Although there is no significant difference regarding performance of the CG and CP algorithms, they outperform each other for different instances. In 9 of 11 Sioux Falls instances and 7 of 11 Chicago Sketch instances, CP algorithm has a smaller solution time and in 8 of 11 Anaheim instances CG performs better. The number of iterations performed has a direct effect on the total solution time, specifically for large size networks with a big number of o–d pairs, as each iteration takes considerable amount of time to finalize. Overall, in 19 of 33 instances CP outperforms CG in terms of the number of iterations. On average, CP algorithm has smaller solution times, adds slightly less number of cutting planes (compared to number of columns in CG), and terminates in a smaller number of iterations. We also observe that the number of columns/cutting planes added is not necessarily the same for CG and CP algorithms. The primal and dual versions of the same problem may end up with different alternative solutions and hence different dual values. This may result in different (number of) paths being eligible to enter in the primal or dual restricted master problems.

7. Conclusion

Traffic congestion is considered to be among the most difficult problems to cope with to achieve a sustainable urban mobility. Among the policies that can be used to ensure that increased urban traffic congestion does not lower the quality of life and threaten global climate and human health and to prevent further economic losses, coordinated and cooperative traffic assignment optimized by a central traffic management authority supported by ITS systems and a data analytics capability is the most promising one. Such a traffic assignment approach could achieve a fair and an efficient solution that considers all users' benefit in the road network and that reduces overall congestion.

In this paper, we introduced models and solution methodologies to solve traffic assignment problems exactly. We have introduced a method for transforming nonlinear problems into a SOCP formulation by leveraging the BPR travel time functions, which are convex. The proposed solution methodologies involve using column generation or cutting planes based on SOCP duality results and employ an efficient pulse algorithm to solve the constrained shortest path problem to generate new paths and can be used to solve SO, UE, CSO, and BRUE traffic assignment problems. The approach we propose, along with the solution methodology we employ, possesses a high degree of generality. Specifically, this methodology is applicable to traffic assignment problems with a wide range of convex travel time functions, enabling the utilization of a similar transformation process for such functions. We further conducted extensive numerical experiments to test the performance of the proposed algorithms.

The problem setting can be extended in a variety of ways. In this study, we assumed a deterministic traffic demand between *o*–*d* pairs, road network capacity and traveler behavior. However, there is significant uncertainty that should be taken into account, which can be accommodated with a scenario based two/multi-stage stochastic programming approach. Also, other supply and demand management strategies, use of autonomous vehicles and traveler behavior can be incorporated in the mathematical formulation.

References

[1] K.A. Ackermann and R.O. Murphy, Explaining cooperative behavior in public goods games: how preferences and beliefs affect contribution levels. *Games* **10** (2019) 15.

[2] M.S. Aktürk, A. Atamtürk and S. GüRel, A strong conic quadratic reformulation for machine-job assignment with controllable processing times. *Oper. Res. Lett.* **37** (2009) 187–191.

[3] F. Alizadeh and D. Goldfarb, Second-order cone programming. *Math. Program.* **95** (2003) 3–51.

[4] E. Angelelli, I. Arsik, V. Morandi, M. Savelsbergh and M.G. Speranza, Proactive route guidance to avoid congestion. *Transp. Res. Part B: Methodol.* **94** (2016) 1–21.

[5] E. Angelelli, V. Morandi and M.G. Speranza, Congestion avoiding heuristic path generation for the proactive route guidance. *Comput. Oper. Res.* **99** (2018) 234–248.

[6] E. Angelelli, V. Morandi and M.G. Speranza, Minimizing the total travel time with limited unfairness in traffic networks. *Comput. Oper. Res.* **123** (2020) 105016.

[7] E. Angelelli, V. Morandi, M. Savelsbergh and M.G. Speranza, System optimal routing of traffic flows with user constraints using linear programming. *Eur. J. Oper. Res.* **293** (2021) 863–879.

[8] A. Atamtürk, G. Berenguer and Z.-J. Shen, A conic integer programming approach to stochastic joint location-inventory problems. *Oper. Res.* **60** (2012) 366–381.

[9] C. Barnhart, E.L. Johnson, G.L. Nemhauser, M.W.P. Savelsbergh and P.H. Vance, Branch-and-price: column generation for solving huge integer programs. *Oper. Res.* **46** (1998) 316–329.

[10] S. Battles, Freight management operations, freight analysis framework. U.S. Department of Transportation, Federal Highway Administration, Office of Freight Management and Operations (2007). https://ops.fhwa.dot.gov/freight/freight_analysis/freight_story/congestion.htm.

[11] V. Bayram, Optimization models for large scale network evacuation planning and management: a literature review. *Surv. Oper. Res. Manage. Sci.* **21** (2016) 63–84.

[12] V. Bayram and H. Yaman, Shelter location and evacuation route assignment under uncertainty: a benders decomposition approach. *Transp. Sci.* **52** (2018) 416–436.

[13] V. Bayram and H. Yaman, A stochastic programming approach for shelter location and evacuation planning. *RAIRO-Oper. Res.* **52** (2018) 779–805.

[14] V. Bayram, B.Ç. Tansel and H. Yaman, Compromising system and user interests in shelter location and evacuation planning. *Transp. Res. Part B: Methodol.* **72** (2015) 146–163.

[15] V. Bayram, B. Yıldız and M.S. Farham, Hub network design problem with capacity, congestion, and stochastic demand considerations. *Transp. Sci.* **47** (2023). DOI: 10.1287/trsc.2022.0112.

[16] M. Beckmann, C.B. McGuire and C.B. Winsten, Studies in the economics of transportation. Technical report (1956).

[17] M. Ben-Akiva, A. De Palma and K. Isam, Dynamic network models and driver information systems. *Transp. Res. Part A: Gen.* **25** (1991) 251–266.

[18] A. Ben-Tal and A. Nemirovski, Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications. *MPS-SIAM Series on Optimization*. Society for Industrial and Applied Mathematics (2001).

[19] H.Y. Benson and Ü. Sağlam, Mixed-integer second-order cone programming: a survey, in Theory Driven by Influential Applications. INFORMS (2013) 13–36.

[20] P. Bonami and M.A. Lejeune, An exact solution approach for portfolio optimization problems under stochastic and integer constraints. *Oper. Res.* **57** (2009) 650–670.

[21] R. Busch, Disrupting transportation: we need to think bigger than cars. McKinsey Global Institute, Voices (2017). https://www.mckinsey.com/industries/travel-logistics-and-infrastructure/our-insights/disrupting-transportation-we-need-to-think-bigger-than-cars.

[22] H.C. Chen, J.W. Friedman and J.F. Thisse, Boundedly rational nash equilibrium: a probabilistic choice approach. *Games Econ. Behav.* **18** (1997) 32–54.

[23] G.B. Dantzig and P. Wolfe, Decomposition principle for linear programs. *Oper. Res.* **8** (1960) 101–111.

[24] G. Desaulniers, M.M. Solomon and J. Desrosiers, Column Generation. Springer (2005).

[25] X. Di and H.X. Liu, Boundedly rational route choice behavior: a review of models and methodologies. *Transp. Res. Part B: Methodol.* **85** (2016) 142–179.

[26] M. Dror, Note on the complexity of the shortest path models for column generation in VRPTW. *Oper. Res.* **42** (1994) 977–978.

[27] EU, Sustainable urban mobility in the EU: no substantial improvement is possible without member states' commitment. European Court of Auditors (2020). https://op.europa.eu/webpub/eca/special-reports/urban-mobility-6-2020/en/.

[28] D.J. Fagnant and K. Kockelman, Preparing a nation for autonomous vehicles: opportunities, barriers and policy recommendations. *Transp. Res. Part A: Policy Pract.* **77** (2015) 167–181.

[29] S. Fleming, Traffic congestion cost the US economy nearly $87 billion in 2018. World Economic Forum, Future of the Environment (2019). https://www.weforum.org/agenda/2019/03/traffic-congestion-cost-the-us-economy-nearly-87-billion-in-2018/.

[30] M. Frank and P. Wolfe, An algorithm for quadratic programming. *Nav. Res. Logistics Q.* **3** (1956) 95–110.

[31] C. Harrison, B. Eckman, R. Hamilton, P. Hartswick, J. Kalagnanam, J. Paraszczak and P. Williams, Foundations for smarter cities. *IBM J. Res. Dev.* **54** (2010) 1–16.

[32] H. Hijazi, P. Bonami and A. Ouorou, Robust delay-constrained routing in telecommunications. *Ann. Oper. Res.* **206** (2013) 163–181.

[33] R. Holden, Data-Driven Innovation: Big Data for Growth and Well-being. Organisation for Economic Co-operation and Development, OECD Publishing (2015).

[34] O. Jahn, R.H. Möhring and A.S. Schulz, Optimal routing of traffic flows with length restrictions in networks with congestion, in Operations Research Proceedings 1999. Springer (2000) 437–442.

[35] O. Jahn, R.H. Möhring, A.S. Schulz and N.E. Stier-Moses, System-optimal routing of traffic flows with user constraints in networks with congestion. *Oper. Res.* **53** (2005) 600–616.

[36] P. Jones, P. Anciaes, C. Buckingham, C. Cavoli, T. Cohen, L. Cristea, R. Gerike, C. Halpern and L. Pickup, Urban mobility: preparing for the future, learning from the past. European Union's Horizon 2020 Research and Innovation Programme, grant agreement No 636573 (2018). https://nws.eurocities.eu/MediaShell/media/CREATE-ProjectSummaryReccommendations.pdf.

[37] I. Klein, N. Levy and E. Ben-Elia, An agent-based model of the emergence of cooperation and a fair and stable system optimum using atis on a simple road network. *Transp. Res. Part C: Emerg. Technol.* **86** (2018) 183–201.

[38] M.S. Lobo, L. Vandenberghe, S. Boyd and H. Lebret, Applications of second-order cone programming. *Linear Algebra App.* **284** (1998) 193–228.

[39] Y. Lou, Y. Yin and S. Lawphongpanich, Robust congestion pricing under boundedly rational user equilibrium. *Transp. Res. Part B: Methodol.* **44** (2010) 15–28.

[40] L. Lozano, D. Duque and A.L. Medaglia, An exact algorithm for the elementary shortest path problem with resource constraints. *Transp. Sci.* **50** (2016) 348–357.

[41] H.S. Mahmassani, 50th anniversary invited article – autonomous vehicles and connected vehicle systems: flow and operations considerations. *Transp. Sci.* **50** (2016) 1140–1162.

[42] H.S. Mahmassani and G.L. Chang, On boundedly rational user equilibrium in transportation systems. *Transp. Sci.* **21** (1987) 89–99.

[43] H.S. Mahmassani and Y.H. Liu, Dynamics of commuting decision behaviour under advanced traveller information systems. *Transp. Res. Part C: Emerg. Technol.* **7** (1999) 91–107.

[44] R. Miyashiro and Y. Takano, Mixed integer second-order cone programming formulations for variable selection in linear regression. *Eur. J. Oper. Res.* **247** (2015) 721–731.

[45] V. Morandi, Bridging the user equilibrium and the system optimum in static traffic assignment: how the cooperation among drivers can solve the congestion problem in city networks. Preprint arXiv:2105.05804 (2021).

[46] J. Paniati, Operational solutions to traffic congestion. U.S. Department of Transportation, Federal Highway Administration, Public Roads, FHWA-HRT-05-002. Vol. 68 (2004). https://highways.dot.gov/public-roads/novemberdecember-2004/operational-solutions-traffic-congestion.

[47] PwC, Mobility insights: tackling the growing issue of congestion in urban areas. PricewaterhouseCoopers (2019). https://www.pwc.com/us/en/industrial-products/publications/assets/pwc-mobility-insights-congestion.pdf.

[48] T. Roughgarden, How unfair is optimal routing? in Symposium on Discrete Algorithms: Proceedings of the Thirteenth Annual ACM-SIAM Symposium on Discrete algorithms. Vol. 6 (2002) 203–204.

[49] M. Savelsbergh and T. Van Woensel, City logistics: challenges and opportunities: 50th anniversary invited article. *Transp. Sci.* **50** (2016) 579–590.

[50] B. Schaller, The new automobility: Lyft, uber and the future of american cities. Schaller Consulting (2018). http://www.schallerconsult.com/rideservices/automobility.pdf.

[51] J.L. Schofer and H.S. Mahmassani, Mobility 2050: A Vision for Transportation Infrastructure. AEM (2016).

[52] A.S. Schulz and N.E. Stier-Moses, Efficiency and fairness of system-optimal routing with user constraints. *Networks* **48** (2006) 223–234.

[53] Y. Sheffi, Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods. Prentice-Hall, Inc., Englewood Cliffs, NJ (1985).

[54] H.A. Simon, A behavioral model of rational choice. *Q. J. Econ.* **69** (1955) 99–118.

[55] V.V. Singh and A. Lisser, A second-order cone programming formulation for two player zero-sum games with chance constraints. *Eur. J. Oper. Res.* **275** (2019) 839–845.

[56] M.G. Speranza, Trends in transportation and logistics. *Eur. J. Oper. Res.* **264** (2018) 830–836.

[57] B. Stabler, H. Bar-Gera and E. Sall, Transportation networks for research. Transportation Networks for Research Core Team (2022). https://github.com/bstabler/TransportationNetworks.

[58] W.Y. Szeto and H.K. Lo, Dynamic traffic assignment: properties and extensions. *Transportmetrica* **2** (2006) 31–52.

[59] W.Y. Szeto, Y. Wang and K. Han, Bounded rationality in dynamic traffic assignment, in Bounded Rational Choice Behaviour: Applications in Transport. Emerald Group Publishing Limited (2015) 163–187.

[60] TAM, Traffic Assignment Manual. Bureau of Public Roads, U.S. Department of Commerce (1964).

[61] J.A. Taylor and F.S. Hover, Convex models of distribution system reconfiguration. *IEEE Trans. Power Syst.* **27** (2012) 1407–1413.

[62] UN, Analysis of the transport relevance of each of the 17 SDGs. United Nations, Department of Economic and Social Affairs, Sustainable Development (2015). https://sustainabledevelopment.un.org/content/documents/8656AnalysisoftransportrelevanceofSDGs.pdf.

[63] UNDESA, Sustainable development goals. United Nations Department of Economic and Social Affairs (2015). https://sdgs.un.org/goals.

[64] M. Van Essen, T. Thomas, E. van Berkum and C. Chorus, From user equilibrium to system optimum: a literature review on the role of travel information, bounded rationality and non-selfish behaviour at the network and individual levels. *Transp. Rev.* **36** (2016) 527–548.

[65] H. Van Essen, L. Van Wijngaarden, A. Schroten, D. Sutter, C. Bieler, S. Maffii, M. Brambilla, D. Fiorello, R. Parolin and K. El Beyrouty, Handbook on the external costs of transport. European Commission, Directorate-General for Mobility and Transport (2020). https://data.europa.eu/doi/10.2832/51388.

[66] J.G. Wardrop, Road paper. some theoretical aspects of road traffic research. *Proc. Inst. Civil Eng.* **1** (1952) 325–362.

[67] WHO, 7 million premature deaths annually linked to air pollution. World Health Organization (2014). https://www.who.int/news/item/25-03-2014-7-million-premature-deaths-annually-linked-to-air-pollution.

[68] Wikipedia, Traffic congestion (2022). https://en.wikipedia.org/wiki/Traffic_congestion#cite_note-108.

[69] J. Woetzel, J. Remes, B. Boland, K. Lv, S. Sinha, G. Strube, J. Means, J. Law, A. Cadena and V. Von der Tann, Smart cities: digital solutions for a more livable future. McKinsey Global Institute (2018). https://www.mckinsey.com/business-functions/operations/our-insights/smart-cities-digital-solutions-for-a-more-livable-future.

[70] H. Ye and H. Yang, Rational behavior adjustment process with boundedly rational user equilibrium. *Transp. Sci.* **51** (2017) 968–980.

[71] G. Zhang and X. Liu, UAV collision avoidance using mixed-integer second-order cone programming. *J. Guidance Control Dyn.* **45** (2022) 1732–1738.

[72] H. Zhang, S.J. Moura, Z. Hu, W. Qi and Y. Song, A second-order cone programming model for planning pev fast-charging stations. *IEEE Trans. Power Syst.* **33** (2017) 2763–2777.

[73] L. Zhenlong and Z. Xiaohua, Integrated-equilibrium routing of traffic flows with congestion. *IFAC Proc. Vol.* **17** (2008) 16065–16070.

[74] B. Zhou and X. Li, User equilibrium with length constrained users, in 2012 Fifth International Joint Conference on Computational Sciences and Optimization (CSO). IEEE (2012) 494–499.

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