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INVERSE DATA ENVELOPMENT ANALYSIS WITH PRODUCTION TRADE-OFFS

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Abstract. As an important resource allocation and production analysis method, the existing inverse data envelopment analysis (Inverse DEA) methods generally ignore the production trade-offs in the production process. However, in some managerial practice, decision makers (DMs) generally have value judgements concerning the importance of inputs and outputs. These value judgements reflect the production trade-offs of DMs for different inputs and outputs and influence the production process. Therefore, this study investigates the inverse DEA method with production trade-offs. By analyzing the effect of production trade-offs on efficient frontier, this study reveals that the existing methods for identifying the limitations of changed range of inputs and outputs in inverse DEA model under variable returns to scale (VRS) are invalid when considering the production trade-offs, and based on this, the new inverse DEA methods with production trade-offs for resource allocation and production analysis are developed. Moreover, the limitations of changed range of inputs and outputs of the proposed methods are identified, thus avoiding the problem of infeasible solutions. In addition, the application scenarios and practical values of the proposed methods are discussed. Finally, two examples are provided to illustrate the rationality and effectiveness of our approaches.

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1. Introduction

Data envelopment analysis (DEA) is one of the most popular methods for assessing the efficiency of a set of decision-making units (DMUs) with multiple inputs and outputs [4, 5]. DEA methods are based on the underlying production technology characterized by constant returns to scale (CRS) or variable returns to scale (VRS) [25, 26]. There are two mutually dual linear programs, that is, envelopment and multiplier models, for DEA methods under the cases of CRS and VRS. Inverse DEA method [32,35] is an important component of the DEA methodology, which is used to address the problems that how much inputs (outputs) should be provided (can be produced) with a given efficiency level when the targeted outputs (input resources) are adjusted.

In most cases, decision makers (DMs) or the requirements of actual production have the value judgements concerning the perceived managerial importance of inputs and outputs [25]. According to DM's preference

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or pre-determined policies, these value judgements reflect the management reality that regarding the relative importance of different inputs and outputs. In the multiplier models, the value judgements are reflected by additional restrictions on the input and output weights and these restrictions typically improve the discrimination of DEA method. However, these value judgements are referred as production trade-offs in envelopment models. Podinovski [26] pointed out that the value judgement reflected by weights restrictions would result in the expansion of the technology boundary of DEA method (*i.e.*, the boundary of frontier) and this expansion is caused by the dual terms in the envelopment model generated by weight restrictions.

There are many studies discussing the effect of production trade-offs on DEA method. However, few studies have considered production trade-offs in inverse DEA. This causes that existing inverse DEA methods fail to incorporate the value judgements, which affects its application value and scope. Moreover, the core idea of inverse DEA is to calculate the optimal outputs (inputs) of the evaluated DMU with a given efficiency when its inputs (outputs) change by referring to the boundary of production frontier. However, the production trade-offs would change the frontier boundary of inverse DEA. Thus, it is urgent and necessary to explore how the changes of frontier boundary would affect the analysis of inverse DEA method.

Yan *et al.* [33] first introduced a preference cone to reflect that DMs often regard some inputs/outputs as being more valuable or preferable than others in inverse DEA. However, the study of Yan *et al.* [33] only focused on the situation that how much the additional outputs a DMU can produce if its inputs are increased to a specific level under current efficiency level, ignoring the case of diversification of input-output changes. Moreover, their study did not consider the limitations of change range of inputs and outputs in inverse DEA with production trade-offs. This leads to the fact that Yan's approach cannot avoid the problem of infeasible solutions in practice. Meanwhile, although the existing studies [6,36] identified the limitations of changed range of inputs and outputs for inverse DEA under VRS, their methods are invalid in inverse DEA method with production trade-offs. Because the production trade-offs would lead to the expansion of production frontier, this results in that the production frontier may not always be composed of observed DMUs (It is described in detail in Sect. 3.2).

To the best of our knowledge, there is no study discussing this issue. This study thus focuses on the inverse DEA with production trade-offs. First, the effect of production trade-offs on the production frontier in inverse DEA and the reasons for the failure of existing methods are analyzed. Then, two new inverse DEA methods with production trade-offs are developed for resource allocation or production analysis, which can be applicable to the cases of diversification of input-output changes. Furthermore, the limitations of changed range of inputs and outputs for the two new inverse DEA methods are identified, and the problem of infeasible solution is avoided. In general, the main contributions of this study can be summarized as follows: first, the generalized inverse DEA methods with production trade-offs are proposed to apply to diverse decision situations, which provide a powerful tool for resource allocation and production analysis; second, the limitations of adjustment range of inputs and outputs are identified to improve the feasibility of application of the proposed approaches.

The remainder of this study is organized as follows. Section 2 reviews the studies of inverse DEA method. Section 3 introduces the inverse DEA method and DEA method with production trade-offs, and points that the reasons for the failure of existing methods. Section 4 develops two inverse DEA method with production trade-offs for resource allocation and production analysis, and the limitations of changed range of inputs and outputs of the proposed methods are analyzed. Section 5 discusses the application value of the proposed methods. Two examples are provided in Section 6, while the conclusions are summarized in Section 7.

2. Literature review on inverse DEA

Inverse data envelopment analysis (Inverse DEA) is an important component of DEA method [5], which is mainly used to analyze the optimal quantity of inputs (outputs) for reaching the targeted outputs (inputs) with a given efficiency level. The idea of inverse DEA method was initially demonstrated in the study of Zhang *et al.* [35]. Subsequently, Wei *et al.* [32] formulated a generalized inverse DEA method under different scales to returns. Due to its application value, inverse DEA has been widely used in many fields [8], such as resource allocation

[17,37], production analysis [21], organization restructuring or mergers [2,11,12], budgeting and planning [28], CO_2 emissions optimization [9,22,31], business partnerships analysis [1], evaluation of sustainable supply chains performance [24].

The theoretical development in inverse DEA has also made great achievements [8,10], which can be divided into two categories. First category is dedicated to improve the application scope and flexibility of inverse DEA. Yan *et al.* [33] developed an inverse DEA method incorporated the preference of decision makers (DMs). Hadi-Vencheh *et al.* [17] allowed the increase of some inputs (outputs) and the decrease of other inputs (outputs) to improve the flexibility of inverse DEA. Lertworasirikul *et al.* [20] extended inverse DEA method to the case of VRS. Zhang *et al.* [36] accomplished the extension and integration of inverse DEA to enable the estimation of all change relationships. Ghiyasi [13] proposed an inverse DEA method based on cost efficiency and revenue efficiency. Ghobadi [15] extended to the inverse DEA to the framework of dynamic DEA. Ghiyasi *et al.* [14] constructed the inverse DEA models to deal with negative data. Mahla *et al.* [23] proposed an inverse DEA method considering ratio data and preferences of DMs. Zeinodin *et al.* [34] developed an inverse DEA method to deal with merging of DMUs under the inter-temporal dependence assumption. Gerami *et al.* [15] proposed a generalized inverse DEA method for firm restructuring based on value efficiency, rather than technical efficiency. Soleimani-Chamkhorami *et al.* [29] developed a new inverse DEA based method to rank the efficient DMUs. Ghobadi [16] employed inverse DEA concept to deal with the problem of merging DMUs with internal data. Kazemi *et al.* [18] formulated an inverse DEA model for a serially linked two-stage production process under CRS. Soltanifar *et al.* [30] proposed an inverse DEA-R models for merger analysis with negative data. Amin *et al.* [1] developed inverse DEA method to construct diverse business partnerships by using and redistributing each other resources. Second category focuses on analyzing the limitations of change range of inputs and outputs in inverse DEA. Zhang *et al.* [36] accomplished the extension and integration of inverse DEA to enable the estimation of all change relationships. Moreover, their study revealed the reason of infeasible solutions for inverse DEA under VRS. Amin *et al.* [2] pointed out that unreasonable inputs and/or inputs adjustments may cause the DMU to fall outside the current production possibility set (PPS). Subsequently, Amin *et al.* [3] provided a solution to estimate the inputs and/or outputs for the DMU beyond current PPS. Chen *et al.* [6] analyzed the limitations of adjustment range of inputs and outputs for inverse DEA under VRS to avoid the problem of infeasible solutions.

3. Preliminaries

3.1. Inverse DEA method and its geometric interpretation

Assume that there is a set of DMUs and each DMU has m inputs ($x_{ij}, i = 1, \ldots, m$) and s outputs ($y_{rj}, r = 1, \ldots, s$). Suppose that the targeted outputs of DMU_k are increased from y_{rk} to β_{rk} ($r = 1, \ldots, s$). Then, input-oriented inverse DEA model under the assumption of VRS can be obtained as follows [32].

$$
\begin{aligned}
&\text{Min}\, W^T(\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk})^T \\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_k^* \alpha_{ik}, i = 1, \ldots, m, \\
&\quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq \beta_{rk}, r = 1, \ldots, s, \\
&\quad \sum_{j=1}^{n} \lambda_j = 1, \\
&\quad \alpha_{ik} \geq x_{ik}, i = 1, \ldots, m, \\
&\quad \lambda_j \geq 0, j = 1, \ldots, n,
\end{aligned}
\tag{1}
$$

where θ_k^* is the CCR efficiency of DMU_k. β_{rk} is targeted output predetermined by DMs. The idea of model (1) is that assume the outputs of DMU_k are increased from y_{rk} to β_{rk}, then how much additional inputs should

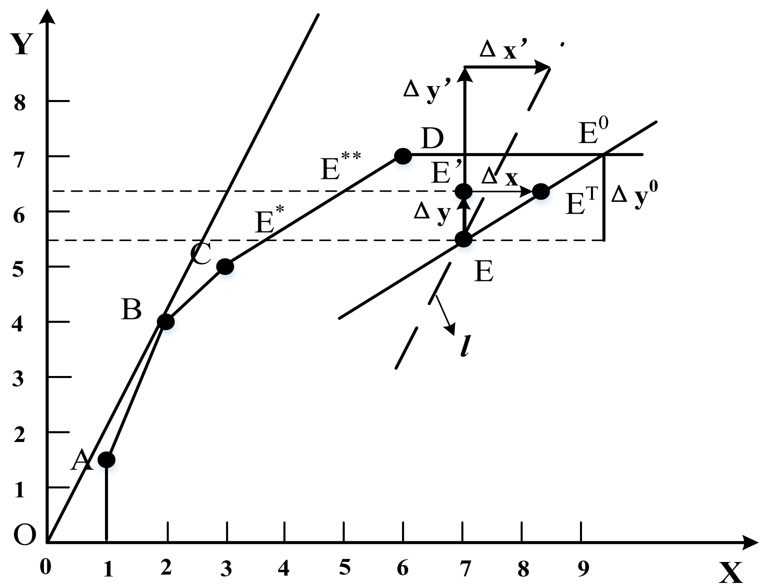

FIGURE 1. Production frontier of a numerical case with one input and one output.

be provided. Thus, the fourth constraint ensures the obtained inputs (α_{ik}) are greater or equal to the original inputs (x_{ik}). $W^T = (w_1, w_2, \ldots, w_m)$ is the weight vector of α_{ik}. The multi-objective programming model (1) can be transformed into a single-objective programming model by setting the value of $W^T = (w_1, w_2, \ldots, w_m)$.

We first introduce a numerical case with one input and one output for illustrating the geometric interpretation of inverse DEA under CRS and VRS. Figure 1 shows the production frontiers (efficient frontiers) of input-oriented DEA model under CRS and VRS.

The idea of inverse DEA is that analyzing the amount of inputs (outputs) adjustments for achieving the level of targeted outputs (inputs) based on DM's preference or development plans while keeping the efficiency values of assessed DMU unchanged. According to Figure 1, consider DMU_E with (x_E, y_E), if the output is increased by Δy according to DM's preference or development goal, the inverse DEA can be applied to explore the optimal changed amount of x_E (*i.e.*, Δx). In Figure 1, the ray OB is the frontier under CRS and the envelope ABCD is the frontier under VRS. The area to the right of ray OB is the production possibility set under CRS (PPS^{CRS}) and the region below envelope ABCD is PPS^{VRS}.

First, we illustrate the idea of inverse DEA under VRS. The line EE^o is parallel to the line CD. The efficiency value of DMU_E is $\theta^*_E = \frac{x^*_E}{x_E}$ under VRS. According to the Theorem of equidistant segments of parallel lines, any points on line EE^o has same efficiency value as the DMU_E. Based on the idea of inverse DEA, it will search for the minimum Δx on line EE^o making $\theta^*_{E^T} = \frac{x^{**}_E}{x_E + \Delta x} = \frac{x_{E^*}}{x_E} = \theta^*_E$ when the targeted output of DMU_E is increased by Δy. This illustrates the process of solving the inverse DEA method and indicates that the production frontier is critical to the implementation of the inverse DEA method. Note that the augment range of output (*i.e.*, Δy) must lie on $[0, \Delta y^o]$, that is, the adjustment of outputs should fall inside PPS^{VRS}. If the augment amount of targeted outputs fall outside PPS^{VRS}, then there is no feasible solution in model (1)[6]. However, the change range of output is not limited in the inverse DEA method under CRS since for any $\Delta y^{'}(\Delta y^{'} \geq 0)$ for evaluated DMU_E in Figure 1, there always exists a $\Delta x^{'}$ in PPS^{CRS} with a given efficiency of evaluated DMU [6]. Therefore, we mainly discuss the inverse DEA method with production trade-offs under VRS in this study, since it is more general to discuss the method under the VRS assumption.

3.2. DEA method with production trade-offs

Consider a set of $DMU_j(j = 1, \ldots, n)$ with m inputs $(x_{ij}, i = 1, \ldots, m)$ and s outputs $(y_{rj}, r = 1, \ldots, s)$. The assessed DMU is denoted (x_{ik}, y_{rk}). Let v_i and u_r represent the multipliers of x_i and y_r, respectively, in the input-oriented multiplier model under VRS. According to the study of Podinovski [26], the general form of DEA method considering weight restrictions is stated as follows.

$$\begin{aligned}
\theta^* = \text{Max} \sum_{r=1}^{s} u_{rk} y_{rk} + u_0 & \\
\text{s.t.} \sum_{i=1}^{m} v_{ik} x_{ik} = 1, k = 1, \ldots, n, & \\
\sum_{r=1}^{s} u_{rj} y_{rj} - \sum_{i=1}^{m} v_{ij} x_{ij} + u_0 \leq 0, j = 1, \ldots, n, & \\
\sum_{r=1}^{s} u_{rt} q_{rt} - \sum_{i=1}^{m} v_{it} p_{it} \leq 0, t = 1, \ldots, K, & \\
u_r, v_i \geq 0; u_0 free\ in\ sign, &
\end{aligned} \tag{2}$$

where u_r,v_i are output and input weights. The third constraint is additional constraints implemented by q_{rt} and p_{it} on u_r and v_i in the multiplier model. According to the above model, the dual envelopment model can be obtained as follows.

$$\begin{aligned}
\theta^* = & \text{Min}\, \theta_k \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq \theta_k x_{ik}, i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq y_{rk}, r = 1, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j, \pi_t \geq 0, \theta_k free\ in\ sign,
\end{aligned} \tag{3}$$

where q_{rt} and p_{it} are referred to as production trade-offs in envelopment model [26].

Similarly, the output-oriented envelopment model with production trade-offs under VRS can be constructed as follows.

$$\begin{aligned}
\vartheta^* = & \text{Max}\, \vartheta_k \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq x_{ik}, i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq \vartheta_k y_{rk}, r = 1, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j, \pi_t \geq 0, \vartheta_k free\ in\ sign,
\end{aligned} \tag{4}$$

Roll *et al.* [27] and Podinovski *et al.* [25] pointed that weight restrictions ($i.e.$, q_r and p_i) imposed on multipliers ($i.e.$, u_r and v_i) would lead to the shift of the production frontier. For example, there is a weight restriction

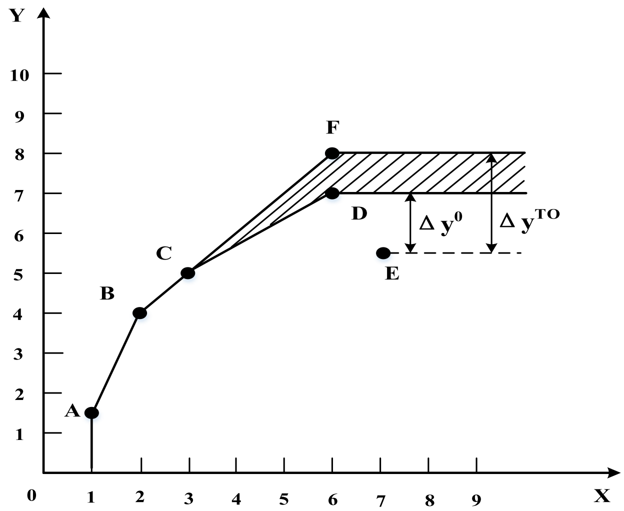

FIGURE 2. Illustration of the effect of weight restrictions on efficiency frontier according to Roll *et al.* [27].

(such as, $u - v \leq 0$) in the numerical case of Section 3.1. Figure 2 illustrates the effect of weight restrictions on the production frontier. Points in Figure 2 represent DMUs. The efficiency frontier without weight restrictions consists of DMU_A, DMU_B, DMU_C, DMU_D, while the efficiency frontier with weight restrictions is composed of DMU_A, DMU_B, DMU_C, DMU_F. According to Chen's study [6], the augment range of output ($i.e.$, Δy) for DMU_E must lie on $[0,\Delta y^o]$. However, the augment range of output can lie on $[0,\Delta y^{TO}]$ when considering the production trade-offs, and the change ranges of input/output cannot be determined by Chen's method [6]. This is because the weight restrictions in multiplier model result in the expansion of the PPS of standard VRS technology by simultaneous changing to the inputs and outputs (shaded area in Fig. 2), which are implemented in proportions $\pi_t(t =, \ldots, K)$. Thus, as depicted in Figure 2, the production frontier referred to as by DMU_E has changed from CD to *CF*. In other words, the production frontier may not necessarily consist of the observed DMUs. This implies that it is difficult to determine production frontier and adjustment range of input/output based on the observed DMUs. To illustrate this phenomenon, the following Remark 3.1 and Definition 3.2 are given.

Remark 3.1. Model (3) identifies the input radial projection of DMU_k on the expanded VRS technology T_{VRS}^{TO} caused by trade-offs instead of on the VRS technology T_{VRS}^{OB} defined by the observed DMUs. Therefore, it is possible that none of the observed DMUs will be on the efficiency frontier. According to Podinovski's study [25], the expanded technology T_{VRS}^{TO} is defined as follows.

Definition 3.2 (Podinovski [25])**.** Expanded technology T_{VRS}^{TO} is the set of all pairs DMUs $(x, y) \in R_{+}^{m+s}$ for which there exist variables λ_j,π_t,s_x,s_y, such that $x_i = \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} + s_x, y_r = \sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} - s_y$, where s_x and s_y represent input excesses and output shortfalls, respectively.

According to Podinovski's study [25], the accuracy of the results is compromised if the efficiency values continue to be calculated with reference to the efficient frontier constructed by the observed DMUs when the production trade-offs are considered. Similarly, as discussed above, the accuracy of the results is also compromised for inverse DEA when the production trade-offs are considered. Therefore, according to Remark 3.1 and Definition 3.2, it is essential to re-explore the possible limited change of resource allocation or production analysis for the application of inverse DEA method with production trade-offs.

4. Inverse DEA method considering production trade-offs

4.1. Inverse DEA based resource allocation method with production trade-offs

First, this subsection develops the input-oriented inverse DEA method with production trade-offs under VRS for analyzing the optimization of resource allocation. As discussed above, the PPS considering trade-offs under VRS is defined as follows.

$$PPS^{TO} = \left\{ \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq x_{ij}; \right.$$
$$\left. \sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq y_{rk}; \sum_{j=1}^{n} \lambda_j = 1; \lambda_j, \pi_t \geq 0 \right\}. \tag{5}$$

where the terms $\sum_{t=1}^{K} \pi_t p_{it}$ *and* $\sum_{t=1}^{K} \pi_t q_{rt}$ are referred to as production trade-offs, which specify simultaneous changes the inputs and outputs by proportions $\pi_t (t = 1, \ldots, K)$.

Based on the defined PPS^{TO}, the input-oriented inverse DEA method considering production trade-offs is developed as follows.

$$\begin{aligned}
\text{Min } & \sum_{k=1}^{n} \sum_{i=1}^{m} w_i \alpha_{ik} \\
\text{s.t. } & \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq \theta_k^* \alpha_{ik}, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq \beta_{rk}, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \alpha_{ik} \geq c_i x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n, \\
& \lambda_j, \pi_t \geq 0,
\end{aligned} \tag{6}$$

where θ_k^* is the optimal solution of model (3). β_{rk} is the targeted outputs determined by DMs or development goals. The targeted outputs (β_{rk}) predetermined by DMs cannot be greater than the upper bound of targeted outputs where the upper bound of targeted outputs can be obtained by the following model (12). The objective function represents the achievement of a specific output target at minimum total cost. In this study, we let $W^T = (w_1, \ldots, w_m) = (1, \ldots, 1)$. Note that the constraint of $\alpha_{ik} \geq x_{ik}$ in model (1) is replaced by $\alpha_{ik} \geq c_i x_{ik}$ in mode (6) where $0 < c_i \leq 1$ since there may be redundant inputs in production. In other words, there may be potential for the evaluated DMUs to produce more outputs with fewer inputs at the current efficiency level. Moreover, this constraint prevents the model from obtaining extreme values in pursuit of an optimal solution. The value of c_i can be determined by DMs according to actual production situation.

Remark 4.1. Assume that the resource allocation of inputs is within current PPS^{TO}. Furthermore, suppose that DMs can modify the production within current PPS according to practical constraints [25]. In this study, the following managerial constraints are considered for representing the limited possible changes of inputs during the planning period.

$$\begin{aligned}
& \alpha_{ik} \leq \tau_i x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n, \\
& \sum_{j=1}^{n} \alpha_{ik} \leq \sigma_i \sum_{j=1}^{n} x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n,
\end{aligned} \tag{7}$$

where τ_i denotes the upper bound for i-th input, which represents that input resources are limited in actual production. This managerial constraint ensures that the attainability of resource allocation. σ_i represent the amount of overall resource i in the next production period. If DMs wish to reallocate the current resource, then $\sigma_i = 1$; if DMs wish to increase the overall resource in the next production period, then $\sigma_i \geq 1$; if DMs wish to decrease the overall resource in the next production period, then $\sigma_i \leq 1$.

According to Remark 4.1, model (6) can be rewritten as follows. Moreover, as discussed above, the overall input resources of DMUs may remain unchanged, increase or decrease for achieving targeted outputs in the next production period.

First, the model (6) is rewritten as follows.

$$
\begin{aligned}
\text{Min } & \sum_{k=1}^{n}\sum_{i=1}^{m} w_i\alpha_{ik} \\
\text{s.t. } & \sum_{j=1}^{n} \lambda_j^k x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq \theta_k^* \alpha_{ik}, \\
& \sum_{j=1}^{n} \lambda_j^k y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq \beta_{rk}, \\
& \sum_{j=1}^{n} \alpha_{ik} \leq \sigma_i \sum_{j=1}^{n} x_{ik}, i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j^k = 1, \\
& \alpha_{ik} \leq \tau_i x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n, \\
& \alpha_{ik} \geq c_i x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n, \\
& \lambda_j^k, \pi_t \geq 0,
\end{aligned}
\tag{8}
$$

where the last two constraints are managerial constraint that ensure that the attainability of resource allocation. However, the value of τ_i cannot be arbitrarily determined by the DMs since it may render the model infeasible. To solve this problem, the following model is developed for determining the value of τ_i.

$$
\begin{aligned}
\text{Min } & \max_i \tau_i \\
\text{s.t. } & \sum_{j=1}^{n} \lambda_j^k x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq \theta_k^* \alpha_{ik}, \\
& \sum_{j=1}^{n} \lambda_j^k y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq \beta_{rk}, \\
& \sum_{j=1}^{n} \alpha_{ik} \leq \sigma_i \sum_{j=1}^{n} x_{ik}, i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j^k = 1, \\
& \alpha_{ik} \leq \tau_i x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n, \\
& \alpha_{ik} \geq c_i x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n, \\
& \lambda_j^k, \pi_t \geq 0,
\end{aligned}
\tag{9}
$$

where the objective function of model (9) minimizes the maximum value of $\tau_i, (i = 1, \ldots, m)$, which devoted to achieving targeted output with minimal input. The multi-objective linear model (9) can be transformed into a single-objective linear model by constructing the following model.

$$\begin{aligned}
\tau^* = & \operatorname{Min} \psi \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j^k x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq \theta_k^* \alpha_{ik}, \\
& \sum_{j=1}^{n} \lambda_j^k y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq \beta_{rk}, \\
& \sum_{j=1}^{n} \alpha_{ik} \leq \sigma_i \sum_{j=1}^{n} x_{ik}, i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j^k = 1, k = 1, \ldots, n, \\
& \alpha_{ik} \leq \tau_i x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n, \\
& \alpha_{ik} \geq c_i x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n, \\
& \tau_i \leq \psi, i = 1, \ldots, m, \\
& \lambda_j^k, \pi_t \geq 0,
\end{aligned} \tag{10}$$

And then, model (8) can be rewritten as follows.

$$\begin{aligned}
\text{Min} & \sum_{k=1}^{n} \sum_{i=1}^{m} w_i \alpha_{ik} \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j^k x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq \theta_k^* \alpha_{ik}, \\
& \sum_{j=1}^{n} \lambda_j^k y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq \beta_{rk}, \\
& \sum_{j=1}^{n} \alpha_{ik} \leq \sigma_i \sum_{j=1}^{n} x_{ik}, i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j^k = 1, \\
& \alpha_{ik} \leq \tau^* x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n, \\
& \alpha_{ik} \geq c_i x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n, \\
& \lambda_j^k, \pi_t \geq 0,
\end{aligned} \tag{11}$$

where τ^* is the optimal solution of model (10).

In addition, it should be noted that due to the production is within the PPS^{TO}, the target-setting of outputs is limited. Improper target-setting of outputs will also lead to the infeasible solutions for model (11). Thus, the following Theorem 4.2 is constructed to determine the feasible range of the targeted outputs.

Theorem 4.2. *To ensure that model (11) is feasible, the targeted outputs* $\beta_k = \{\beta_{1k}, \beta_{2k}, \ldots, \beta_{sk}\}$ *cannot larger than* $\max_k\{y^*_{r1}, y^*_{r2}, \ldots, y^*_{rn}\}, r = 1, \ldots, s,$ *where* $y^*_{rk} = \varphi^*_k y_{rk} + s^*_y = \sum_{j=1}^n \lambda^*_j y_{rj} + \sum_{t=1}^K \pi^*_t q_{rt}$*, which can be obtained by the following model (12).*

$$\begin{aligned}
&\text{Max}\varphi_k\\
\text{s.t.}\ &\sum_{j=1}^n \lambda_j x_{ij} + \sum_{t=1}^K \pi_t p_{it} = x_{ik} - s^x_{ik}, i = 1, \ldots, m; k = 1, \ldots, n,\\
&\sum_{j=1}^n \lambda_j y_{rj} + \sum_{t=1}^K \pi_t q_{rt} = \varphi_k y_{rk} + s^y_{rk}, r = 1, \ldots, s; k = 1, \ldots, n,\\
&\sum_{j=1}^n \lambda_j = 1,\\
&\lambda_j, \pi_t \geq 0,
\end{aligned} \tag{12}$$

where s^x_{ik} and s^y_{rk} are the input excesses and the output shortfalls.

Proof. According to Definition 3.2, if there is $\forall(x_{ik}, y_{rk}) \in T^{TO}_{VRS}$ satisfying that $x_{ik} - s^*_x = \sum_{j=1)}^n \lambda^*_j x_{ij} + \sum_{t=1}^K \pi^*_t p_{it}, \varphi^*_k y_{rk} + s^*_y = \sum_{j=1}^n \lambda^*_j y_{rj} + \sum_{t=1}^K \pi^*_t q_{rt}$ where $(\lambda^*_j, \pi^*_t, s^*_x, s^*_y, \varphi^*_k)$ is the optimal solution of model (12), then $(x^*_{ik} = x_{ik} - s^*_x, y^*_{rk} = \varphi^*_k y_{rk} + s^*_y)$ is on the production frontier. If $\forall\beta_k = \beta_{1k}, \beta_{2k}, \ldots, \beta_{sk}$ is smaller than $\max_k y^*_{rk} = \varphi^*_k y_{rk} + s^*_y = \sum_{j=1}^n \lambda^*_j y_{rj} + \sum_{t=1}^K \pi^*_t q_{rt}, k = 1, \ldots, n$. This means that β_k is within the PPS^{TO}. Moreover, Remark 4.1 assumes that the resource allocation of inputs is within current PPS^{TO} and suppose that DMUs can modify their production in current PPS^{TO} according to managerial constraints. This means that the optimized DMU $(\alpha_{ik}, \beta_{rk})$ is within the PPS^{TO} and satisfies the all constraints of model (11). Thus, if $\forall\beta_k = \{\beta_{1k}, \beta_{2k}, \ldots, \beta_{sk}\}$ is smaller than $\max_k y^*_{rk} = \varphi^*_k y_{rk} + s^*_y = \sum_{j=1}^n \lambda^*_j y_{rj} + \sum_{t=1}^K \pi^*_t q_{rt}, k = 1, \ldots, n,$ model (11) is feasible. □

Note that this study mainly discusses the inverse DEA method with production trade-offs under VRS since the proposed method under the case of CRS can be obtained by omitting the constraint of $\sum_{j=1}^n \lambda_j = 1$ in above models.

4.2. Inverse DEA based production analysis method with production trade-offs

This subsection discusses the output-oriented inverse DEA method with production trade-offs under VRS for formulating the plans of target-settings.

Remark 4.3. Assume that the production analysis of outputs is within current PPS^{TO}. Furthermore, suppose that DMs can modify the production within current PPS^{TO} according to development plans. In this study, the following production constraint is considered for representing the development goals of outputs during the planning period.

$$\beta_{ik} \geq \delta_r y_{rk}, r = 1, \ldots, s; k = 1, \ldots, n, \tag{13}$$

where δ_r denotes the lower bound for r-th output. This production constraint ensures that the attainability of development goals.

According to the Remark 4.3 and PPS^{TO}, the inverse DEA based production analysis model is developed as follows.

$$\begin{aligned}
&\text{Max} \sum_{k=1}^{n}\sum_{r=1}^{s} w_r \beta_{rk} \\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq \alpha_{ik}, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq \vartheta_k^* \beta_{rk}, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\beta_{rk} \geq \delta_r y_{rk}, r = 1, \ldots, s; k = 1, \ldots, n, \\
&\lambda_j, \pi_t \geq 0,
\end{aligned} \tag{14}$$

where ϑ_k^* is the optimal solution of model (4). α_{ik} indicates the amount of resource input for the next production period, which is determined by the DMs based on the organization's actual resource availability. The lower bound of inputs adjustment (α_{ik}) can be obtained by the following model (18). The objective function means that maximizing output as much as possible after adjusting input resources.

Note that the fourth constraint is strongly binding. If the value of δ_r is arbitrarily determined by the DMs may also render the model infeasible. To solve this problem, the following model is developed for determining the value of δ_r.

$$\begin{aligned}
&\text{Max} \min_r \delta_r \\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq \alpha_{ik}, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq \vartheta_k^* \beta_{rk}, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\beta_{rk} \geq \delta_r y_{rk}, r = 1, \ldots, s; k = 1, \ldots, n, \\
&\lambda_j, \pi_t \geq 0,
\end{aligned} \tag{15}$$

where the objective function aims to maximizes the minimum value of $\delta_r, r = 1, \ldots, s$, it means to make the output as increased as possible. Model (15) is a multi-objective linear model; it can be transformed into a single-objective linear model by constructing the following model.

$$\begin{aligned}
\delta_r^* = &\text{Max}\, \rho \\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq \alpha_{ik}, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq \vartheta_k^* \beta_{rk}, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\beta_{rk} \geq \delta_r y_{rk}, r = 1, \ldots, s; k = 1, \ldots, n,
\end{aligned} \tag{16}$$

$$\delta_r \geq \rho, r = 1, \ldots, s,$$
$$\lambda_j, \pi_t \geq 0.$$

Subsequently, model (14) can be rewritten as follows.

$$\begin{aligned}
&\text{Max} \sum_{k=1}^{n} \sum_{r=1}^{s} w_r \beta_{rk} \\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} \leq \alpha_{ik}, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} \geq \vartheta_k^* \beta_{rk}, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\beta_{rk} \geq \delta_r^* y_{rk}, r = 1, \ldots, s; k = 1, \ldots, n, \\
&\lambda_j, \pi_t \geq 0,
\end{aligned} \tag{17}$$

where δ_r^* is the optimal solution of model (16).

DMs generally expect to produce more outputs with minimum inputs. However, due to the production is within the PPS^{TO}, the adjustment of input resource is not without limits. Improper adjustment of input resource may also render the model (17) infeasible. To solve this problem, the following Theorem 4.4 is given to determine the feasible range of the adjustment of input resource, so that DMs can make reasonable adjustments to the input resources.

Theorem 4.4. *To ensure that model (17) is feasible, the resource inputs* $\alpha_k = \{\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk}\}$ *cannot smaller than* $\min\limits_k\{x_{i1}^*, x_{i2}^*, \ldots, x_{in}^*\}, i = 1, \ldots, m,$ *where* $x_{ik}^* = \varepsilon_k^* x_{ik} - s_x^* = \sum_{j=1}^{n} \lambda_j^* x_{ij} + \sum_{t=1}^{K} \pi_t^* p_{it}$*, which can be obtained by the following model (18).*

$$\begin{aligned}
&\text{Min}\, \varepsilon_k \\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{K} \pi_t p_{it} = \varepsilon_k x_{ik} - s_{ik}^x, i = 1, \ldots, m; k = 1, \ldots, n, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{K} \pi_t q_{rt} = y_{rk} + s_{rk}^y, r = 1, \ldots, s; k = 1, \ldots, n, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j, \pi_t \geq 0,
\end{aligned} \tag{18}$$

where s_{ik}^x and $s_r k^y$ are the input excesses and the output shortfalls.

Proof. According to Definition 3.2, if there is $\forall (x_{ik}, y_{rk}) \in T_{VRS}^{TO}$ satisfying that $\varepsilon_k^* x_{ik} - s_x^* = \sum_{j=1}^{n} \lambda_j^* x_{ij} + \sum_{t=1}^{K} \pi_t^* p_{it}, y_{rk} + s_y^* = \sum_{j=1)}^{n} \lambda_j^* y_{rj} + \sum_{t=1}^{K} \pi_t^* q_{rt}$ where $(\lambda_j^*, \pi_t^*, s_x^*, s_y^*, \varepsilon_k^*)$ is the optimal solution of model (18), then $(x_k^* = \varepsilon_k^* x_{ik} - s_x^*, y_k^* = y_{rk} + s_y^*)$ is on the production frontier. If $\forall \alpha_k = \{\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk}\}$ is greater than $\min\limits_k x_{ik}^* = \varepsilon_k^* x_{ik} - s_x^* = \sum_{j=1}^{n} \lambda_j^* x_{ij} + \sum_{t=1}^{K} \pi_t^* p_{it}, i = 1, \ldots, m.$. This means that α_k is below the production frontier. Moreover, Remark 4.3 assumes that the production analysis of outputs is within current PPS^{TO} and suppose that DMUs can modify their production in current PPS^{TO} according to development plans. This means that the optimized DMU $(\alpha_{ik}, \beta_{rk})$ is within the PPS^{TO} and satisfies the all constraints of model (18). Thus, if $\forall \alpha_k = \{\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk}\}$ is greater than $\min_k x^*_{ik} = \varepsilon^*_k x_{ik} - s^*_x = \sum_{j=1}^n \lambda^*_j x_{ij} + \sum_{t=1}^K \pi^*_t p_{it}, i = 1, \ldots, m,$ model (17) is feasible. □

5. Discussion

This section mainly discusses the application scenarios and practical value of the proposed inverse DEA methods with production trade-offs.

In the application of resource allocation, the proposed inverse DEA based resource allocation method is used to analyze the optimization of resource allocation for achieving corresponding output targets under current efficiency level. As shown in Figure 3, DMs first determine the relative importance of different inputs and outputs according to DM's preference or pre-determined policies to reflect the production trade-offs. Then, model (12) is used to identify the limitations of changed range of outputs, which prevents DMs from formulating unreasonable output targets. And model (10) is applied to minimize the maximal value of τ_i in order to achieve output goals with the minimum input cost as much as possible. Finally, the achievement paths of the optimization of resource allocation for achieving the corresponding targeted outputs can be obtained by model (11). This information can provide strong support to DMs for formulating reasonable resource allocation plans to achieve specific output goals.

In the situation of production analysis, the proposed inverse DEA based production analysis method is applied to determine optimal output level when the input resources are adjusted in the next production period while maintaining current efficiency level. As depicted in Figure 3, DMs first determine the relative importance of different inputs and outputs according to DM's preference or pre-determined policies. Then, model (18) is employed to identify the limitations of changed range of inputs to avoid unreasonable adjustment of resource allocation. And model (16) is applied to maximize the minimal value of δ_r in order to achieve maximum output level from the adjustment of resource allocation. Lastly, the optimal outputs level after adjusting the resource allocation can be obtained by model (17). These results can provide clear direction for DMs to formulate reasonable development goals.

6. Illustrative examples

This section illustrates the proposed approach through two examples under VRS assumption, where a numerical example is given to describe the features and advantages of our approach compared with other methods, and an empirical example is used to validate the effectiveness of our method.

6.1. Numerical example

Considering a simple example involving 5 DMUs with two inputs and one output. The data are summarized in Table 1. Assume that to ensure the managerial feasibility, the DM's policy is that the total input resources in the next production period cannot exceed 1.1 times than current available resources for achieving the targeted output (*i.e.*, $\sigma_i = 1.1$). In addition, suppose that the inputs and output incorporate the additional linked weight restriction $u - v_1 + v_2 \leq 0$, it can be transformed as production trade-offs by setting $P_t = (-1, 1), Q_t = 1$ in model (3). The value of c_i is taken as $c_i = 0.1$ in this study. Subsequently, the input-oriented inverse DEA based resource allocation method is applied to illustrate the features and advantages of our approach. Column 5 of Table 1 is the efficiency values with production trade-offs under VRS. The last two columns of Table 1 are the efficiency values without trade-offs obtained by traditional DEA model under CRS and VRS. Note the efficiency values obtained by traditional DEA model under CRS and VRS are equal in this example. Obviously, when production trade-offs are considered, the efficiency values change since the efficient frontier changes, which is consistent with the discussion in Section 3.2.

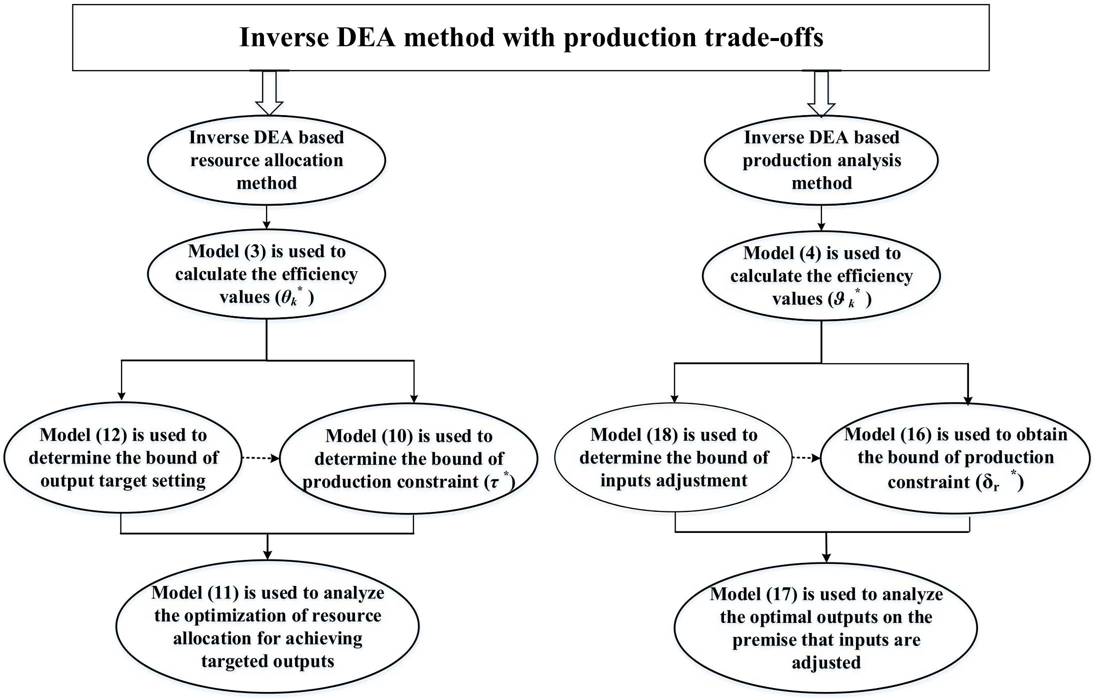

FIGURE 3. Relationship between different models.

TABLE 1. Dataset involving five DMUs with two inputs and one output.

DMU	x_1	x_2	y	θ^{VRS*}	E^{CRS*}	E^{VRS*}
A	2	12	2	1	1	1
B	2	8	2	1	1	1
C	5	5	2.5	1	1	1
D	10	5	2.5	0.667	1	1
E	10	7.5	2.5	0.571	0.667	0.667
Total	29	37.5	11.5	–	–	–

First, Model (12) is used to determine the feasible range of target setting, then model (10) is applied to ensure the attainability of input resource allocation. Finally, the resource allocation plans for achieving targeted output in the next production period can be obtained by model (11). In addition, the inverse DEA method without considering production trade-offs under CRS and VRS are used for comparison analysis. The results are shown in Table 2 where MPVRS denotes the results obtained by the proposed method in this study, TMVRS and TMCRS represent the results obtained by the existing methods under VRS and CRS, respectively. Figure 4 depicts the changes of production frontier when the production trade-offs are considered. Two targets are given for illustrating the differences of the proposed approach and the existing method.

Based on the results for Target 1 reported in the third to eighth rows of Table 2, the following conclusions can be obtained. First, note that $\tau^* = 1.5$ by solving model (10). It means that on the premise of achieving the targeted output in the next production period, the amount of input resources cannot less than 1.5 times than current input level for ensuring the attainability of resource allocation. Second, column 6 and column 10 are the upper bound of targeted output obtained by model (12) and Chen's method [6]. Obviously, the upper

TABLE 2. Results of different methods.

Target	DMUs	MPVRS				TMVRS				TMCRS		
		α_1	α_2	β_{Tar}	$\beta_{\max}$	α_1	α_2	β_{Tar}	$\beta_{\max}$	α_1	α_2	β_{Tar}
Target 1	A	3	7	2	10	Infeasible		2	2.5	2	12	2
	B	3	7	2		Infeasible		2		2	8	2
	C	4.63	5.37	3		Infeasible		3		6	6	3
	D	9	6	3.5		Infeasible		3.5		10	7	3.5
	E	10.5	7	3.5		Infeasible		3.5		10.5	10.5	3.5
	Total	30.13	32.37	14	–	–		14	–	30.5	43.5	14
Target	DMUs	α_1	α_2	β_{Tar}	$\beta_{\max}$	α_1	α_2	β_{Tar}	$\beta_{\max}$	–		
Target 2	A	2.62	7.38	2.5	10	5	12	2.5	2.5	–		
	B	2.62	7.38	2.5		5	12	2.5		–		
	C	4.51	5.49	2.5		5	5	2.5		–		
	D	8.44	6.56	2.5		10	5	2.5		–		
	E	8.52	8.98	2.5		10	7.5	2.5		–		
	Total	26.71	35.79	12.5	–	35	37.5	12.5	–	–		

bounds of targeted output obtained by our approach and Chen's method [6] are significantly different. This means that Chen's method for determining the input-output adjustment bounds in traditional inverse DEA is not applicable in inverse DEA method with production trade-offs. As shown in Table 2, the targeted outputs formulated by our method cannot be achieved by traditional inverse DEA method under VRS. The reason is that considering production trade-offs will expand the production frontier, which makes the upper bound of targeted output change. However, as shown in columns 3–5 of Table 2, if the target setting of output do not exceed the upper bound obtained by Theorem 4.2, the specific achievement paths for realizing the targeted output can be obtained by the proposed model (11). The results show that to achieve the targeted output, the total amount of $x1$ should be increased by $(30.13-29)/29 * 100 = 3.90\%$, while the total amount of x_2 can be reduced by $(37.5-32.37)/37.5 * 100 = 13.68\%$. Note that, as shown in the last three columns of Table 2, the targeted outputs formulated by the proposed method can be achieved by traditional inverse DEA method under CRS. This is because the change range of output is not limited in the inverse DEA method under CRS. This is the reason that we mainly discuss the proposed method under VRS. Therefore, these results show that the proposed method is valid when the production trade-offs are considered.

The relevant results for Target 2 obtained by two methods are shown in the ninth to fifteenth rows of Table 2. As can be seen in Table 2 that to achieve the same targeted output, the total amounts of x_1 and x_2 obtained by our approach are less than that obtained by traditional method. This difference stems from the fact that the inverse DEA approach with production trade-offs needs to consider the relative importance of input-output weights, which in turn leads to the changes in the production frontier. If we ignore the production trade-offs and directly use traditional method for the analysis of resource allocation, it will yield biased results. Therefore, it is essential to develop an inverse DEA approach with production trade-offs and to analyze in depth the production frontier relationships to ensure the feasibility of the model, which is the main objective of this study. Because DMs in practical production generally have preferences on different inputs and outputs depending on the positioning and operation of the organization.

To visually demonstrate the impact of production trade-offs on the efficient frontier, Figure 4 is provided. The envelopes ABCD and ABCF are the production frontiers without and with production trade-offs, respectively. For DMU_D and DMU_E, the corresponding production frontier for them has changed from CD to CF when the production trade-offs are considered. Thus, the adjustment ranges for input and output under current efficiency level are also changed. This is reason that the upper bounds of targeted output for the traditional inverse DEA approach and the inverse DEA approach with production trade-offs are different, and is also the essential difference between them.

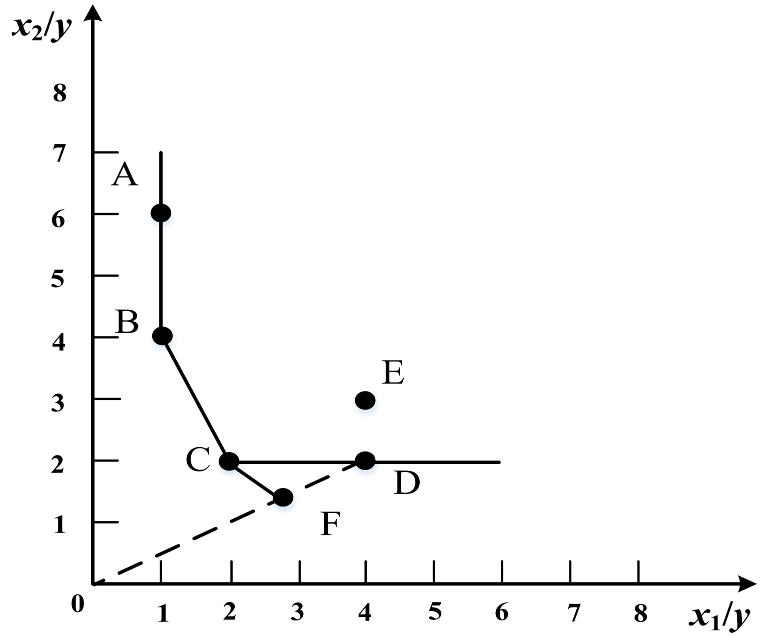

FIGURE 4. Production frontier with trade-offs and without trade-offs.

Note that this subsection mainly illustrate the proposed inverse DEA based resource allocation method (input-oriented) by a numerical example, the proposed inverse DEA based production analysis method (output-oriented) will be examined in the following empirical study.

6.2. Empirical study

This subsection employs an empirical example to illustrate the proposed production analysis approach. The dataset involves 30 banks in Taiwan of China with three inputs and three outputs [7]. The inputs indicators contain the number of staffs (x_1, hundred person), total fixed assets (x_2, ten million NT dollar), and total deposits (x_3, ten million NT dollar). The total loan amount (y_1, ten million NT dollar), total investment (y_2, ten billion NT dollar), and fee and commission income (y_3, ten million NT dollar) are selected as the output. The data source from the study of Chen *et al.* [7], and the specific data are shown in Table 3.

Suppose that the outputs incorporate the additional linked weight restriction $u_1 + 2u_3 - u_2 \leq 0$, that is, DMs places the most emphasis on total investment (y_2) among the three outputs. This weight restriction can be transformed as production trade-offs by setting $P_t = (0, 0, 0), Q_t = (1, -1, 2)$ in model (3). We assume that all DMUs plan to increase the investments of x_1 by 2%, x_2 by 1%, and x_3 by 3%, respectively, in the next production period. The models (16)–(18) are utilized to analyze the development goals planning of Taiwan's 30 banks. First, the model (18) is solved for determining the lower bound of input resources, and thus avoiding inappropriate input resource adjustment. Then model (16) is used to determine the optimal value of δ^*. Finally, model (17) is applied to the production analysis for Taiwan's 30 banks after the adjustment of resource allocation. The results are shown in Table 4.

Columns 2–4 of Table 4 are the minimum input resources with current production technology, which are obtained by model (18). This information can help each DMU adjust its input resources reasonably under current technical conditions. For example, for DMU_3, under current technology level, its first input investment (x_1) cannot be less than $(67.92/70.54) * 100\% = 96.29\%$ of the original input, its second input investment (x_2) cannot be less than $(165.55/171.93) * 100\% = 96.29\%$ of the original input, and its third input investment (x_3) cannot

TABLE 3. Data for Taiwan's 30 banks in 2008.

DMUs	x_1	x_2	x_3	y_1	y_2	y_3
1	63.57	132.65	105.02	97.49	24.37	2.5
2	70.87	171.07	128.73	115.21	34.72	3.51
3	70.54	171.93	131.84	111.44	33.92	3.76
4	6.19	30.3	2.98	7.88	11.76	0.6
5	51.03	195.14	128.93	130.35	26.28	3.96
6	45.54	48.65	39.09	30.96	16.24	2.99
7	20.57	26.35	24.09	20.18	1.55	1.12
8	86.6	149.52	110.02	83.85	46.43	13.25
9	59.1	128.74	102.04	80.96	38.33	3.78
10	62.59	114.41	83.56	75.24	13.59	4.44
11	51.09	112.78	94.54	87.88	14.38	1.32
12	8.87	15.69	13.05	13.62	2.25	0.46
13	33.88	36.36	28.68	17.83	9.99	1.38
14	49.86	99.99	81.13	62.82	19.07	2.12
15	39.56	79.39	62.15	53.28	13.8	2.75
16	28.65	34.58	27.36	23.54	6.3	0.9
17	70.29	92.22	73.02	51.72	8.29	5.11
18	23.27	36.1	28.13	21.05	3.34	1.16
19	26.62	31.42	24.1	21.84	1.54	1.18
20	20.57	28.24	21.48	6.89	4.73	0.74
21	32.64	38.57	32.91	27.89	4.58	1.14
22	24.59	24.48	21.04	16.48	1.41	0.41
23	22.67	60.17	42.2	32.69	20.58	1.57
24	19.74	25.02	21.43	13.62	0.82	0.59
25	9.77	10.38	9.08	8.09	0.45	0.48
26	11.67	10.36	9.34	7.77	0.51	0.46
27	87.92	244.07	199.87	182.39	17.51	2.66
28	82.19	308.73	250.9	198.18	35.41	2.34
29	14.59	17.19	14.62	11.96	0.45	0.32
30	3.98	4.12	3.58	2.69	0.53	0.01
Total	1199.02	2478.62	1914.91	1615.79	413.13	67.01

be less than $(124.45/131.84) * 100\% = 94.39\%$ of the original input. However, for DMU_2, it cannot decrease its inputs investment in the next production period, since its current input investments are the minimum input resources under current production technology. In addition, the optimal value of $\delta^* = 1$ by solving model (16). This indicates that the output level of each DMU can be improved when the input investments of x_1, x_2, and x_3 are increased by 2%, 1%, and 3%, respectively.

Columns 5–7 of Table 4 are the optimal outputs level of DMUs when increasing the input investments in the next production period, which are obtained by model (17). These results can help DMs to formulate reasonable development goals for the next production period. For example, when the DMs plan to increase the corresponding input investments, DMU_1 can formulate output targets of 130, 25, and 3.5 for y_1, y_2, and y_3, respectively, in the next production period. Because the optimal outputs of $y1$, $y2$, and $y3$ for DMU_1 can achieve the level of 133.16, 26.66, and 3.89, respectively, when the DMs plan to increase the corresponding input investments. In addition, it is worth noting that the total amounts of y_1, y_2, and, y_3 are increased by $(1862.94 - 1615.79)/1615.79 * 100\% = 15.30\%$, $(650.27 - 413.13)/413.13 * 100\% = 57.4\%$, and $(104.67-67.01)/67.01 * 100\% = 56.2\%$, respectively. It can be seen that the total amount of y_2 has the largest growth proportion. This is because the outputs incorporate the additional linked weight restriction $u_1 + 2u_3 - u_2 \leq 0$, that is, compared with the outputs y_1 and y_3, DMs place more emphasis on output y_2. This indicates that our approach can effectively reflect the

Table 4. Results of production analysis for 30 banks in Taiwan.

DMUs	Lower bound of inputs			Optimized outputs		
	x_1	x_2	x_3	β_1	β_2	β_3
1	60.09	132.29	104.28	133.16	26.66	3.89
2	70.87	171.07	128.73	146.13	34.72	5.54
3	67.92	165.55	124.45	141.67	33.92	5.18
4	6.19	30.3	2.98	8.21	11.8	0.61
5	51.03	195.14	128.93	131.98	26.63	3.96
6	45.54	48.65	39.09	40.64	26.71	6.06
7	15.73	21.85	17.92	22.18	14.88	2.26
8	86.6	149.52	110.02	83.85	46.43	13.25
9	59.1	128.74	102.04	85.19	38.33	3.85
10	43.4	98.48	71.93	91.88	29.58	6.67
11	50.14	110.69	86.91	109.29	27.39	4.95
12	8.87	15.69	13.05	15.69	12.69	0.81
13	25.56	30.23	21.7	26.09	18.6	3.76
14	43.73	87.69	69.74	84.81	24.75	4.81
15	35.89	72.02	56.38	71.37	21.99	3.82
16	25.02	30.69	24.28	27.86	19.28	3.71
17	47.19	63.16	50.01	51.72	27.01	7.19
18	16.27	25.24	19.67	26.02	13.19	1.98
19	17.26	24.19	18.55	22.25	16.07	2.98
20	8.58	15.02	6.89	14.25	8.73	1.28
21	28.07	33.17	26.86	28.81	20.18	4.18
22	11.23	17.82	14.73	17.27	13.93	2.35
23	22.67	60.17	42.2	34.76	20.58	1.61
24	8.32	14.38	11.98	14.95	10.23	1.51
25	6.04	8.99	7.56	12.23	11.2	0.9
26	5.94	8.75	7.37	10.9	11.72	1.23
27	87.92	244.07	199.87	194.7	35.71	2.66
28	82.19	308.73	250.9	198.18	35.41	2.34
29	7.53	12.53	10.46	14.02	11.01	1.3
30	3.98	4.12	3.58	2.88	0.94	0.03
Total	1048.87	2328.94	1773.06	1862.94	650.27	104.67

preferences of DMs or pre-determined policies. In summary, the results obtained by the proposed approach can provide a reference for DMs to make reasonable development plans.

7. Conclusions

Inverse DEA method is an effective tool for resource allocation or production analysis. Existing inverse DEA methods fail to consider the production trade-offs. However, in some managerial practice, DMs generally have value judgements concerning the importance of inputs and outputs. These value judgements reflect the production trade-offs of DMs for different inputs and outputs and influence the production process. For example, when some resources are scarce, DMs generally want to invest as little of the scarce resource as possible compared with other resources on the premise that the outputs do not be affected. Similarly, when certain outputs are very important, DMs generally expect to produce as much of the critical outputs as possible compared with other outputs after adjusting the input resources in the next production period. These preferences of DMs or organization plan can be reflected in the inverse DEA method through production trade-offs. Therefore, this study focuses on the inverse DEA approach with production trade-offs for expanding the application value of inverse DEA method.

First, this study illustrated the effects of production trade-offs on production frontier in inverse DEA method. Then, we pointed out that the methods for determining the limitations of inputs and outputs in inverse DEA under VRS assumption are not applicable to inverse DEA method with production trade-offs. Since the production trade-offs would change the production frontier of inverse DEA method and make the corresponding models infeasible. On this basis, the new inverse DEA methods with production trade-offs are developed for resource allocation and production analysis. Furthermore, the limitations of inputs and outputs of the new inverse DEA methods with production trade-offs are identified by analyzing the expanded production frontier, and the problem that the proposed models have no feasible solution is avoided. Finally, two examples are employed to illustrate the features and effectiveness of our approaches.

In general, the approaches of this study are effective in avoiding the problem of infeasible solutions when considering production trade-offs and can provide strong support to DMs when allocating resources or formulating development plans. However, this study still has some limitations. This study only focuses on the inverse DEA method with single production process, but there are many DMUs with multiple-stage production processes. For example, the DMUs of high-technology industries usually have a R&D stage and a commercialization stage; the DMUs of industry usually have an economic development stage and an environmental pollution abatement stage. The DMUs in these industries have complex internal structure and may have shared inputs and outputs feedback. Thus, the proposed method may fail to address this issue. In future research, we will fully consider the complexity of the internal structure of DMUs and the intrinsic connection between different stages to further extend the proposed method to the application scenarios with multi-stage production processes, which will further expand the application value and scope of the inverse DEA method with production trade-offs.

Appendix A.

There are a lot of parameters and variables in this study, and their meanings are shown in the following Table A.1.

Table A.1. List of variables.

Variable	Meaning
Subscript j	j-th of DMU
Subscript k	k-th of assessed DMU
n	Number of evaluated DMUs
m	Number of inputs
s	Number of outputs
K	Number of weight restrictions
x_{ij}	i-th input of DMU_j
y_{rj}	r-th output of DMU_j
α_{ik}	i-th optimized input of DMU_k
β_{rk}	r-th targeted output of DMU_k
λ_j	Intensity variable
c_i	Lower bound of inputs adjustment
w_i	Relative importance of i-th input
w_r	Relative importance of r-th output
v_{ij}	Virtual multiplier of i-th input of DMU_j
u_{rj}	Virtual multiplier of r-th output of DMU_j
p_{it}	t-th weight restriction on v_{ij}
q_{rt}	t-th weight restriction on u_{rj}

Table A.1. Continued.

Variable	Meaning
π_t	Dual variable of weight restrictions
σ_i	Production constraint variable for the total amount of i-th input
τ_i	Production constraint variable of i-th input
δ_r	Target-setting constraint variable of r-th output
u_0	Free variables under VRS

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